A PROP structure on partitions

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I. The notion of a PROP

↳ a special kind of symmetric monoidal category

1. Category

Definition : A category $\mathcal{C}$ consists of :

- a class of objects $Ob(\mathcal{C})$: $A, B, C, \dots$
- for each pair of objects A, B a set of morphisms
$$Hom_{\mathcal{C}}(A, B) = \mathcal{C}(A, B)$$
- a composition rule : if $f : A \to B$ and $g : B \to C$ then there is a composition $g \circ f : A \to C$
- an identity morphism for every object A
$$id_A : A \longrightarrow A$$

with two axioms

Associativity : $h \circ (g \circ f) = (h \circ g) \circ f$

Identity law : $id_B \circ f = f = f \circ id_A$

Examples :

1. **Set** objects = sets morphisms = functions
2. **Grp** objects = groups morphisms = group homomorphisms
3. $\mathbf{Vect}_{\mathbb{K}}$ objects = vector spaces over a field $\mathbb{K}$
morphisms = linear maps

4. Surj objects = natural integers $\mathbb{N}$

$[n] = \{1, \dots, n\}$

morphisms = $Surj(m, n)$ is the set of surjections from $[m]$ to $[n]$.

5. Part objects = natural integers $\mathbb{N}$

morphisms = $Part(m, n)$ is the set of partitions of m into n parts.

↳ A partition λ of m into n parts is a sequence of positive integers $\lambda_1 \geq \dots \geq \lambda_n$ such that $\sum \lambda_i = m$.

Remark : $proj : Surj(m, n) \longrightarrow Part(m, n)$

$f \longmapsto$ ordering $|f^{-1}(i)|$ in a decreasing order.

6. R-Mod objects = R-modules for R a commutative ring

morphisms = morphisms of R-modules

Definition : Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor from $\mathcal{C}$ to $\mathcal{D}$ is a mapping that associates

- each object X in $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$,
- each morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that the following two conditions hold :
  - $F(id_X) = id_{F(X)}$
  - $F(g \circ f) = F(g) \circ F(f)$

Examples :

1. $\mathrm{Hom}(A,-)$ for $\mathcal{C}$ a category and A an object

$$\begin{aligned} \mathrm{Hom}(A,-) : \mathcal{C} &\to \mathrm{Set} \\ B &\mapsto \mathrm{Hom}(A,B) \\ f: B \to C &\mapsto \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C) \\ u &\mapsto f \circ u \end{aligned}$$

2. In topology, the fundamental group defines a functor

$$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$$

from the category of pointed topological spaces.

3. Let $\mathcal{C}$ be the category of finitely generated free R-modules.

- $\mathrm{Id} : \mathcal{C} \to R\text{-Mod}$ Forgetful functor
- $T^n : \mathcal{C} \to R\text{-Mod}$ n^{th} tensor product functor
  $G \mapsto G^{\otimes n}$

  ↳ admits an action of $\mathfrak{S}_n$ by permuting the factors
- $S^n : \mathcal{C} \to R\text{-Mod}$ n^{th} symmetric power
  $G \mapsto (T^n(G))_{\mathfrak{S}_n}$

  ↳ coinvariants of T^n by the action of $\mathfrak{S}_n$:

  $$T^n / \sigma \cdot \tau \sim \tau \quad \forall \sigma \in \mathfrak{S}_n$$
- $\Gamma^n : \mathcal{C} \to R\text{-Mod}$ n^{th} divided power
  $G \mapsto (T^n(G))^{\mathfrak{S}_n}$

  ↳ invariants of T^n by the action of $\mathfrak{S}_n$:

  $$\{ \tau \in T^n \mid \sigma \cdot \tau = \tau \ \forall \sigma \in \mathfrak{S}_n \}$$

- $\Lambda^m : \mathcal{C} \to R\text{-Mod}$ mth exterior power

$$G \mapsto \Lambda^m(G) := T^m(G) \big/ \{ g_1 \otimes \cdots \otimes g_m = 0 \mid \exists\, i \neq j \ g_i = g_j \}$$

4. For $k = \mathbb{Z}$ $\mathbf{gr}$ finitely generated free groups

   $\mathbf{ab}$ finitely generated abelian groups

   $\mathfrak{a} : \mathbf{gr} \longrightarrow \mathbf{ab}$ abelianisation functor

$$G \mapsto G \big/ [G, G]$$

where $[G, G]$ is the smallest subgroup containing all the group commutator elements $[g, h] = g^{-1} h^{-1} g h$.

2. Symmetric monoidal category

↳ a monoidal category is a category equipped with a way to "tensor" or "combine" objects and morphisms.

Definition : A monoidal category has

- a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- a unit object I ← identity for tensor
- natural isomorphisms :
  - Associator : $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$
  - left and right unitors :

$$\lambda_A : I \otimes A \cong A \qquad \rho_A : A \otimes I \cong A$$

These must satisfy coherence conditions :

Triangle equation

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Pentagon equation

$$\begin{tikzcd}
& (W \otimes (X \otimes Y)) \otimes Z \arrow[dr, "\alpha_{W,X\otimes Y,Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W,X,Y} \otimes 1_Z"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "1_W \otimes \alpha_{X,Y,Z}"] \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[u, "\alpha_{W\otimes X,Y,Z}"] \arrow[rr, "\alpha_{W,X,Y\otimes Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

Definition : A symmetric monoidal category consists of :

- a monoidal category $\mathcal{C}$
- a natural isomorphism called the braiding

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

such that $b_{X,Y} = b^{-1}_{Y,X}$

such that the hexagon equations hold

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "b_{X,Y\otimes Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "b_{X,Y} \otimes 1_Z"] & (Y \otimes X) \otimes Z \arrow[d, "\alpha_{Y,X,Z}"] \\
(Y \otimes Z) \otimes X & Y \otimes (Z \otimes X) \arrow[l, "\alpha^{-1}_{Y,Z,X}"] & Y \otimes (X \otimes Z) \arrow[l, "1_Y \otimes b_{X,Z}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "b_{X\otimes Y,Z}"'] & X \otimes (Y \otimes Z) \arrow[r, "1_X \otimes b_{Y,Z}"] & X \otimes (Z \otimes Y) \arrow[d] \\
Z \otimes (X \otimes Y) & (Z \otimes X) \otimes Y \arrow[l, "\alpha_{Z,X,Y}"] & (X \otimes Z) \otimes Y \arrow[l, "b_{X,Z} \otimes 1_Y"]
\end{tikzcd}$$

Remark: We speak about strict symmetric monoidal category when associativity and unitors are identities instead of isomorphisms.

Examples:

1. Sets with cartesian product

   Tensor : Cartesian product $A \times B$

   Unit : $\{*\}$

   Braiding : swap map $(a,b) \times (b,a)$

2. $\text{Vect}_{\mathbb{K}}$ with tensor product

   Tensor : tensor product of vector spaces $V \otimes W$

   Unit : $\mathbb{K}$

   Braiding : natural iso $V \otimes W \mapsto W \otimes V$

3. Rel objects = sets

morphisms = binary relations

tensor = cartesian product of sets

unit = singleton set $\{*\}$

symmetry = swap of components

4. Hilb objects = finite dim Hilbert spaces

morphisms = linear maps

tensor = Hilbert space tensor product

unit = $\mathbb{C}$

symmetry = $V \otimes W \mapsto W \otimes V$

5. Grp with direct product

Tensor : Direct product of groups $G \times H$

Unit : the trivial group $\{e\}$

Braiding : the swap map $(g,h) \mapsto (h,g)$

↳ Grp with the free product of groups $G * H$ forms a monoidal category which is not symmetric.

⇝ The goal of this talk is to study symmetric monoidal category with objects the natural numbers (like in Surj and Part) and whose symmetric monoidal structure $\otimes$ is given by the sum of integers on objets = PROP.

3. PROP

↳ stands for PROducts and Permutations

↳ arose in the work of Mac Lane

Definition: A PROP is a symmetric monoidal category with objects the natural numbers whose symmetric monoidal structure $\otimes$ is given by the sum of integers on objects.

Remark: In this talk, we consider graded linear PROPs that are PROPs $\mathcal{C}$ such that $\mathcal{C}(m,n)$ is a graded vector space over a field $\mathbb{K}$.

Equivalently, a PROP is a collection $\{\mathcal{C}(m,n)\}_{m,n\in\mathbb{N}}$ of graded vector spaces equipped with a left $\mathfrak{S}_m$-action and a right $\mathfrak{S}_n$-action together with two types of compositions • the horizontal composition

$$\mathcal{C}(m_1,n_1)\otimes\mathcal{C}(m_2,n_2)\longrightarrow\mathcal{C}(m_1+m_2,n_1+n_2)$$

induced by the monoidal product

• the vertical composition

$$\mathcal{C}(m,\ell)\otimes\mathcal{C}(n,m)\longrightarrow\mathcal{C}(n,\ell)$$

given by the categorical composition.

Example: Endomorphism PROP End_V

V a vector space

$$\mathrm{End}_V(m,n)=\mathrm{Hom}\left(V^{\otimes m},V^{\otimes n}\right)$$

Horizontal composition = tensor product of linear maps

vertical composition = composition in V

II. A PROP structure on surjections

Theorem : [Vespa '16]

There is a graded linear prop $\mathcal{E}$ such that

$$\mathcal{E}^*(n, m) = \begin{cases} \mathbb{Z}[\mathrm{Surj}(n, m)] & * = n - m \\ 0 & \text{otherwise} \end{cases}$$

Action of the symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$

For $\tau_{k,l} \in \mathfrak{S}_n$ the transposition of k and l

$$[f] \cdot \tau_{k,l} = \prod_{1 \leq i \leq m} \varepsilon\left(\tau_{k,l \,|\, (f \circ \tau_{k,l})^{-1}(i)}\right) [f \circ \tau_{k,l}]$$

For $\tau_{k,l} \in \mathfrak{S}_m$ the transposition of k and l

$$\tau_{k,l} \cdot [f] = (-1)^{(|f^{-1}(k)|-1)(|f^{-1}(l)|-1)} [\tau_{k,l} \circ f]$$

Horizontal composition

It is induced by the disjoint union of sets

Vertical composition

Let $f \in \mathrm{Surj}(m, n)$ and $g \in \mathrm{Surj}(n, l)$ is of the form $g = s \circ \tau_{i,j}$ where s is and $\tau_{i,j} \in \mathfrak{S}_n$

$$[g] \circ [f] = \prod_{k=1}^{l} \varepsilon\left(\overline{\tau_{i,j}}_{\,|\, g^{-1}(k)}\right) (-1)^{(|f^{-1}(i)|-1)(|f^{-1}(j)|-1)} [g \circ f]$$

Remark : The prop freely generated by sCom.

For every prop $\mathcal{C}$, the collection $\{\mathcal{C}(n,1)\}_{n\in\mathbb{N}}$ forms an operad in grVect$_{\mathbb{K}}$. This leads to a forgetful functor from the category of props to the one of operads.

Conversely, to any reduced operad $\mathcal{P}$ one can associate the prop $\Omega\mathcal{P}$ freely generated by $\mathcal{P}$.

Theorem : [E., Hunter, Livernet, Vespa, Zakharevich, 24]

There exists another system of generators having the advantage that the left and right actions of the symmetric groups are by signs and the composition agrees with the set composition of surjections.

$\rightsquigarrow$ One can construct various categories out of the PROP $\mathcal{E}$, defined as subcategories of its Karoubi envelope.

III. On subcategories of the $\underbrace{\text{Karoubi envelope}}$ of a PROP

also called idempotent completion
or pseudo - abelian hull

Karoubi envelope $\rightsquigarrow$ aim to add objects and morphisms to a category so that every idempotent that is every $e_A : A \to A$ $\quad e_A \circ e_A = e_A$ splits : there exists B and $r : A \to B$, $s : B \to A$ such that $s \circ r = e_A$.

Definition : The Karoubi envelope of a given category $\mathcal{C}$ is the category $\mathrm{Kar}(\mathcal{C})$ defined by

- The objects are the pairs (A, e_A) where
  - A is an object in $\mathcal{C}$
  - $e_A : A \to A$ is an idempotent
- A morphism $f : (A, e_A) \to (B, e_B)$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $f = f \circ e_A = e_B \circ f = e_B \circ f \circ e_A$.
- The composition is the one of C.
- The identity on (A, e_A) is e_A.

Construction : Let $\mathcal{C}$ be a prop.

$$\mathbb{K}[\mathfrak{S}_m] = \left\{ \sum_{\sigma \in \mathfrak{S}_m} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{K} \right\}$$

We have a morphism of $\mathbb{K}$-algebras

$$\mathbb{K}[\mathfrak{S}_m] \longrightarrow \mathcal{C}(m, m)$$

It maps any idempotent e in $\mathbb{K}[\mathfrak{S}_m]$ to an idempotent in $\mathcal{C}(m, m)$

Here are idempotents of $\mathbb{K}[\mathfrak{S}_m]$

$$e_{(1^m)} := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma)\sigma \qquad e_{(m)} := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma$$

We denote by $\Lambda\mathcal{C}$ the full subcategory of $\mathrm{Kar}(\mathcal{C})$ whose objects are given by $(m, e_{(1^m)})$ for all $m \in \mathbb{N}$.

Proposition: The category $\Lambda\mathcal{C}$ is equivalent to the category $\mathcal{C}_\Lambda$ where $\mathcal{C}_\Lambda(m,n)$ is the quotient of $\mathcal{C}(m,n)$ by the relation

$$f \sim \varepsilon(\sigma)\varepsilon(\tau)\, \tau \cdot f \cdot \sigma \quad \text{for } \tau \in \mathfrak{S}_n \;\; \sigma \in \mathfrak{S}_m .$$

For $[f] \in \mathcal{C}_\Lambda(m,n)$ and $[g] \in \mathcal{C}_\Lambda(\ell, m)$

$$[f] * [g] = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma)\, [f \circ \varphi(\sigma) \circ g] .$$

Remark:
- $\mathrm{Kar}(\mathcal{C})$ inherits a prop structure from that of $\mathcal{C}$.
- However, it does not induce a prop structure at the level of a given subcategory

IV. A graded linear prop spanned by partitions

Theorem : [E., H., d.V., Z. 24]

For all $n, m \in \mathbb{N}$, there is an isomorphism

$$\mathcal{E}_{\Lambda}^{\bullet}(m, n) \cong \begin{cases} \mathbb{K}[\mathrm{Part}(m,n)] & \bullet = m - n \quad m \geq n \\ 0 & \text{otherwise} \end{cases}$$

There is a particular monoidal product $\otimes$ which turn $\mathcal{E}_{\Lambda}^{\bullet}$ into a prop.

Let us distinguish the following partitions :

$$P_{m,m} = \{1, \dots, 1\} \in \mathrm{Part}(m, m)$$

$$P_{m,m-1} = \{2, 1, \dots, 1\} \in \mathrm{Part}(m, m-1)$$

$$P_{m,n} = P_{n+1,n} * \dots * P_{m,m-1} \in \mathrm{Part}(m, n)$$

For $\alpha \in \mathrm{Part}(m, n)$ and $\beta \in \mathrm{Part}(m', n')$

$$[\alpha] \otimes [\beta] = \pm P_{m+m', n+n'}.$$

V. Relation with functor homology on free groups.

Let $\mathcal{F}(\mathcal{C}, \mathbb{K})$ be the category of functors $\mathcal{C} \to \mathbb{K}\text{-Mod}$

Definition: For $F, G \in \mathcal{F}(\mathcal{C}, \mathbb{K})$ we define:

$$\mathrm{Ext}^i_{\mathcal{F}(\mathcal{C},\mathbb{K})}(G, F) = H_i\left(\mathrm{Hom}_{\mathcal{F}(\mathcal{C},\mathbb{K})}(P_\bullet, F)\right)$$

where $P_\bullet$ is a projective resolution of G.

Given $K, H \in \mathcal{F}(\mathcal{C}, \mathbb{K})$, we consider the Yoneda product

$$\mathrm{Ext}^m_{\mathcal{F}(\mathcal{C},\mathbb{K})}(F, G) \otimes \mathrm{Ext}^n_{\mathcal{F}(\mathcal{C},\mathbb{K})}(G, H) \longrightarrow \mathrm{Ext}^{m+n}_{\mathcal{F}(\mathcal{C},\mathbb{K})}(F, H)$$

and the External product

$$\mathrm{Ext}^m_{\mathcal{F}(\mathcal{C},\mathbb{K})}(F, G) \otimes \mathrm{Ext}^n_{\mathcal{F}(\mathcal{C},\mathbb{K})}(H, K) \longrightarrow \mathrm{Ext}^{m+n}_{\mathcal{F}(\mathcal{C},\mathbb{K})}(F \otimes H, G \otimes K)$$

Theorem: [Vespa 18]

Consider the category $\mathcal{E}_T$ whose objects are $\mathbb{N}$ and whose morphisms are

$$\mathcal{E}_T(n, m) = \mathrm{Ext}^\bullet_{\mathcal{F}(\mathcal{C},\mathbb{K})}(T^n \circ a, T^m \circ a)$$

with composition given by the Yoneda product. Together with the external product, this category forms a prop which is exactly the prop $\mathcal{E}$.

Theorem : [E., H., L., V., Z. 24]

Consider the category $\mathcal{E}_\Lambda$ whose objets are $\mathbb{N}$ and whose morphisms are

$$\mathcal{E}_\Lambda(m, n) = \mathrm{Ext}^{\bullet}_{\mathcal{F}(\mathbb{C}, \mathbb{K})}(\Lambda^m \circ a, \Lambda^n \circ a)$$

with composition given by the Yoneda product.
This category is the category $\Lambda\mathcal{E}$ and inherits the prop structure.