# Cubic Schrödinger half-wave equation and random initial data

Nicolas Camps, Louise Gassot, Slim Ibrahim

Abstract. We give an overview of the results on the cubic Schrödinger-half-wave equation. This equation serves as a toy model motivated by the study of the long time behavior of solutions to weakly dispersive equations. Indeed, the linear part of the equation is anisotropic, with one direction corresponding to the half-wave operator, which is not dispersive. In particular, the question of local and global well-posedness is a difficult problem. We present a recent probabilistic local well-posedness result below the energy space, which is critical for the Cauchy theory of this equation. Due to the absence of probabilistic smoothing in the second Picard's iteration that rules out the standard approach, we rely on a probabilistic quasilinear iteration scheme adapted from Bringmann's work on the derivative nonlinear wave equation.

Contents



## 1. Introduction

We mention in this note some results on the cubic Schrödinger half-wave equation on $\mathbb{R}^2$

$$i\partial_t u + \left(\partial_{xx}^2 - |D_y|\right) u = \mu\,|u|^2\,u\,, \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}^2\,. \tag{NLS-HW}$$

where $|D_y| := \sqrt{-\partial_{yy}^2}$ and $\mu\in\mathbb{R}$. This equation is motivated by a mathematical interest on the long time behavior and qualitative properties of solutions. It was first introduced in [43] in the defocusing case ($\mu > 0$) on the wave guide $\mathbb{R}_x\times\mathbb{T}_y$ to evidence weak turbulence in the growth of Sobolev norms. Then, in the focusing case ($\mu < 0$), the ground state standing waves and traveling waves on the wave guide $\mathbb{R}_x\times\mathbb{T}_y$ were first constructed in [2], and the stability properties of some standing waves investigated in [3]. We summarize these results in section 3.

Our main concern is to investigate the local Cauchy problem at low regularity. We recall in section 2 the known deterministic well-posednesss and ill-posedness properties

*Keywords:* Nonlinear Schrödinger equation, probabilistic Cauchy theory.
2020 *Mathematics Subject Classification:* 35B40 primary.

for equation (NLS-HW). The main difficulties are caused by the lack of dispersion in the second spatial variable $y$, because equation (NLS-HW) can be decomposed as a coupled system of two transport equations in this variable. Hence we are led to consider generic initial data under the form of random initial data. In this setting, the lack of dispersion causes a lack of probabilistic smoothing of the Duhamel's iterate, hindering the classical approach of Bourgain. However, recent developments enable us to overcome this problem by using a probabilistic quasilinear resolution scheme, which we explain in section 4.

## 2. Cauchy problem

Due to the anisotropy of the equation, the relevant regularity spaces are anisotropic Sobolev spaces $\mathcal{H}^s$ defined as

$$\mathcal{H}^s := L^2_x H^s_y \cap H^{2s}_x L^2_y \,, \quad \dot{\mathcal{H}}^s := L^2_x \dot{H}^s_y \cap \dot{H}^{2s}_x L^2_y \,.$$

Note that as a consequence, the exponents in the Sobolev embedding are the same as the ones in $\mathbb{R}^3$ even if there are only two variables, because the homogeneous dimension is 3. Equation (NLS-HW) is a Hamiltonian system, with a formally conserved energy

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_x u|^2 + \left||D_y|^{\frac{1}{2}} u\right|^2 \, \mathrm{d}x\, \mathrm{d}y + \frac{\mu}{4}\int_{\mathbb{R}^2} |u|^4 \, \mathrm{d}x\, \mathrm{d}y \,.$$

The mass $\|u\|^2_{L^2}$ is also formally conserved by the flow. There is no known conservation law above regularity $\mathcal{H}^{\frac{1}{2}}$ for equation (NLS-HW). Moreover, a Brezis-Gallouët argument does not appear to be sufficient in order to control the norm of high-regularity solutions. Therefore it seems necessary to handle the Cauchy problem below regularity $\mathcal{H}^{\frac{1}{2}}$ in the hope to get global well-posedness. However, we will see that the flow map cannot be $\mathcal{C}^3$ in $\mathcal{H}^s$ when $\frac{1}{4} \leq s < \frac{1}{2}$, and the question of local well-posedness in the energy space and below turns out to be a challenging problem. At supercritical regularities $0 < s < \frac{1}{4}$ a norm-inflation mechanism occurs as expected. One can summarize the state-of-the-art Cauchy theory results for (NLS-HW) in the following diagram.


Figure 1. Deterministic Cauchy theory for equation (NLS-HW).

Let us now detail the properties of the three zones evidenced in this diagram.

### 2.1. Local well-posedness

When $s > \frac{1}{2}$, semilinear well-posedness is obtained using Strichartz estimates with a derivative loss in [2].

**Proposition 2.1** (Local well-posedness above the energy space, [2] Theorem 1.6)**.** *Let $s > \frac{1}{2}$. For every $u_0 \in \mathcal{H}^s(\mathbb{R}^2)$, there exists $T = T(\|u_0\|_{\mathcal{H}^s}) > 0$ such that equation* (NLS-HW) *admits a unique local solution in $\mathcal{C}((-T,T),\mathcal{H}^s)$ with initial data $u_0$.*

It is shown in [2] that local well-posedness actually holds in $L^2_x H^s_y$. The proof follows from a fixed point argument in $L^\infty_t([-T\,,T]; L^2_x H^s_y) \cap L^4_t([-T\,,T]; L^\infty_{x,y}(\mathbb{R}^2))$. Due to the lack of dispersion in the $y$-direction, one needs to trade strictly more than $\frac{1}{2}$ derivatives against integrability (by using Sobolev embedding for instance) in order to control the Strichartz norm $L^4_t([-T\,,T]; L^\infty_{x,y}(\mathbb{R}^2))$. Unfortunately the energy can only control the

$\mathcal{H}^{\frac{1}{2}}$-norm of the solution, and we know no conservation law that controls the $\mathcal{H}^s$-norm when $s > \frac{1}{2}$. For these reasons, global existence for smooth solutions to (NLS-HW) is yet to be proved, even in the defocusing case or for small data.

### 2.2. **Ill-posedness results**

The solutions of equation (NLS-HW) are invariant under the scaling symmetry

$$u \mapsto u_\lambda(t,x,y) = \lambda u(\lambda^2 t, \lambda x, \lambda^2 y)\,. \tag{2.1}$$

This scaling leaves the $\dot{\mathcal{H}}^{\frac{1}{4}}$-norm invariant. In scaling-supercritical regimes $0 < s < \frac{1}{4}$ the flow-map, which is well-defined for smooth initial data, does not extend continuously in $\mathcal{H}^s$ due to a norm inflation mechanism that reflects a low-to-high frequency cascade.

**Theorem 2.2** (Norm inflation [30])**.** *Let $s < \frac{1}{4}$. For every $T > 0$, the solution map cannot be extended as a continuous map from $\mathcal{H}^s$ to $\mathcal{C}([-T,T],\mathcal{H}^s)$. More precisely, there exist a sequence $(t_n)_{n\in\mathbb{N}}$ of positive times tending to zero and a sequence of compactly supported smooth functions $(\psi_n)_{n\gg 1}$ in $\mathcal{C}^\infty(\mathbb{R}^2)$ such that the corresponding smooth solutions $(u_n)_n$ of* (NLS-HW) *given by Proposition 2.1 exist in $\mathcal{H}^s$ on $[0,t_n]$ and inflate:*

$$\lim_{n\to+\infty} \|\psi_n\|_{\mathcal{H}^s} = 0\,,$$
$$\lim_{n\to+\infty} \|u_n(t_n)\|_{\dot{\mathcal{H}}^s} = +\infty\,.$$

In other words, there exist sequences of smooth initial data going to zero in $\mathcal{H}^s$, such that the corresponding solutions experience an inflation of their $\mathcal{H}^s$-norm in arbitrarily short time. Such a norm inflation mechanism was originally exhibited in [31, 32, 33] for the wave equation, then extended to the Schrödinger equation in [18]. Non-uniform continuity of the flow map for $s = \frac{1}{4}$ in equation (NLS-HW) has also been investigated in [30].

To evidence norm inflation in the scaling-supercritical regime, we perform a *small dispersion analysis*. By rescaling an arbitrary compactly supported smooth function, one generates a sequence of smooth initial data $(\psi_n)_n$ converging to zero in $\mathcal{H}^s$ while concentrating its mass in one point. Let $u_n$ be the smooth solution to (NLS-HW) with initial data $\psi_n$. We show that for short times, $u_n$ stays close to the bubble profile $v_n$, which is solution of the dispersionless ODE

$$\begin{cases} i\partial_t v_n = \sigma |v_n|^{p-1} v_n\,, \\ v_n(0) = \psi_n\,. \end{cases} \tag{2.2}$$

The profile $v_n$, which is explicit

$$v_n(t,x) = \mathrm{e}^{-it|\psi_n(x)|}\,\psi_n(x)$$

is oscillating and concentrating, and inflates in $\mathcal{H}^s$ when $0 < s$ along a sequence of times $(t_n)$ that goes to zero. Moreover, when $0 < s_c$, we deduce from a priori energy estimates up to time $t_n$ that the solution $u_n$ of the whole equation stays close to the profile:

$$\sup_n \|u_n(t_n) - v_n(t_n)\|_{H^s} \lesssim 1\,, \tag{2.3}$$

so that the oscillations dominate the dispersion and $u_n$ inflates as $u_n$. In this way, the instability stems from the nonlinear interactions captured by the profile $v_n$.

This proof of Theorem 2.2 was adapted in [42] around any initial data $u(0) \in \mathcal{H}^s$ and not just the zero initial data. However, this indicates that allowing for any sequence of smooth functions $(u_n(0))_n$ to approximate a given initial data $u(0)$ in $\mathcal{H}^s$ always lead to norm inflation. In the following we consider the sequence of initial data $(\rho_{\varepsilon_n} * u(0))_n$ regularized by and approximate identity $\varepsilon_n \to 0$, and describe the class of $u_0$ for which the sequence of regularized solutions experiences norm inflation.

Using the method developed for Schrödinger-type equations in [14] and inspired from [40] for wave equation, we show that the regularization of rough initial data by convolution does not prevent norm inflation in $\mathcal{H}^s$ for a dense class of pathological initial data, which also known to contain a dense $G_\delta$ set in cases where the equation is globally well-posed.

More precisely, we fix $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, valued in $[0,1]$, such that $\int_{\mathbb{R}^2} \rho(x)\,\mathrm{d}x = 1$. Due to the anisotropy, we define an approximate identity $(\rho_\varepsilon)_{\varepsilon>0}$ of the form

$$\rho_\varepsilon(x,y) := \frac{1}{\varepsilon^3}\rho\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon^2}\right).$$

**Theorem 2.3** (Generic ill-posedness for (NLS-HW) [15])**.** *Let $s < \frac{1}{4}$. There exists a dense set $S \subset \mathcal{H}^s$ such that for every $f \in S$, the family of local solutions $u^\varepsilon$ of (NLS-HW) with initial data $\rho_\varepsilon * f$ does not converge as $\varepsilon \to 0$. More precisely, there exist $\varepsilon_n \to 0$ and $t_n \to 0$ such that $u^{\varepsilon_n}(t_n)$ is well-defined and*

$$\lim_{n\to\infty} \|u^{\varepsilon_n}(t_n)\|_{\mathcal{H}^s} = +\infty\,.$$

The idea is to superpose an infinite number of *bubble* solutions in the dispersionless ODE (2.2), in other words, we replace the bump initial data $\psi_n$ by an infinite series of bumps with different scales. Given one convolution parameter $\varepsilon_n$, we prove that only the $n$-th bubble exhibits norm inflation at time $t_n$. Indeed, the bubbles at larger scale would need a bigger time than $t_n$ to start to inflate, whereas the bubbles at smaller scale are flattened by the convolution with $\rho_\epsilon$, which is concentrated at large scale. Finally, a perturbative argument implies that the actual solution to (NLS-HW) still satisfies estimate (2.3).

## 2.3. Semilinear ill-posedness

We have seen that when $s < \frac{1}{4}$, the equation is scaling-supercritical, so that one can evidence some norm-inflation mechanisms of the solutions. Yet, semilinear local well-posedness is only known in $\mathcal{H}^s$ when $\frac{1}{2} < s$. We can show that actually, the flow map cannot be of class $\mathcal{C}^3$ when $\frac{1}{4} < s < \frac{1}{2}$, meaning that (NLS-HW) is *semilinearly ill-posed* for this range of exponents.

**Theorem 2.4** (Semilinear ill-posedness [15])**.** *If there exists a local in time flow map on $\mathcal{H}^s$ with regularity $\mathcal{C}^3$ at the origin, then $s \geq \frac{1}{2}$.*

To establish this result, we note that as a corollary of Remark 2.12 in [10], if there exists a $\mathcal{C}^3$ local in time flow map at the vicinity of the origin in the space $\mathcal{H}^s$, then the following Strichartz estimate holds:

$$\|e^{it(\partial_{xx}^2 - |D_y|)}\phi\|_{L^4([0,1]\times\mathbb{R}^2)} \lesssim \|\phi\|_{\dot{\mathcal{H}}^{\frac{s}{2}}}\,. \tag{2.4}$$

In order to invalidate these Strichartz estimates when $s < \frac{1}{2}$, we consider a one-parameter family of profiles constructed from traveling waves for the one-dimensional Szegő equation. More precisely, we consider a Gaussian distribution $G$, a family of traveling waves profiles for the Szegő equation $K_\rho(y) = \frac{1}{y+i\rho}$ for $\rho \in (0\,,+\infty)$, and set

$$\phi(x,y) = G(x)K_\rho(y)\,.$$

Since $K_\rho$ is a traveling wave for the Szegő equation on the line [38], and in particular a traveling wave for equation (NLS-HW), one can see that for every $t$, there holds

$$\|e^{it|D_y|}K_\rho\|_{L^4(\mathbb{R}_y)} = \|K_\rho\|_{L^4(\mathbb{R}_y)}$$

implying that as $\rho \to 0$,

$$\|e^{it(\partial_{xx}^2 - |D_y|)}\phi\|_{L^4([0,1]\times\mathbb{R}^2_{x,y})} = \|e^{it\partial_{xx}}G\|_{L^4([0,1]\times\mathbb{R}_x)}\|K_\rho\|_{L^4(\mathbb{R}_y)} \sim C\rho^{-\frac{3}{4}}\,.$$

Then we show using the independence of the functions $G$ and $K_\rho$ that

$$\|\phi\|_{\dot{\mathcal{H}}^s} \underset{\rho\to 0}{\sim} C'\rho^{-\frac{1}{2}-\frac{s}{2}}\,.$$

By letting $\rho$ go to zero in (2.4), we see that necessarily $s \geq \frac{1}{2}$ for the inequality $\rho^{-\frac{3}{4}} \lesssim \rho^{-\frac{1}{2}-s}$ to hold true.

To conclude this paragraph, let us stress out that it follows from such a lack of regularity for the flow-map at the origin that one cannot run a contraction mapping argument to construct solutions in $\mathcal{H}^s$ when $\frac{1}{4} < s < \frac{1}{2}$, since otherwise the flow-map would be analytical.

## 3. Long-time behavior of some special solutions

Given the challenges to overcome in order to get a satisfying global Cauchy theory for equation (NLS-HW), the results on the long-time dynamics are scarce. We mention the existing results regarding modified scattering in the defocusing case ($\mu > 0$) and on the wave guide [43] for a class of smooth and decaying solution, and some stability results [2, 3] for traveling waves in the focusing case ($\mu < 0$), which are conditional to a Cauchy theory in the energy space. Then we compare these dynamics with those observed for the one-dimensional cubic half-wave equation and cubic Schrödinger equation, which are obtained by considering the variables separately.

### 3.1. Modified scattering in the defocusing case

Equation (NLS-HW) was originally introduced by Xu [43] in the defocusing case on the spatial wave guide $\mathbb{R}_x \times \mathbb{T}_y$ to evidence weak turbulence mechanisms in the growth of Sobolev norms. Global existence and modified scattering are obtained for a class of sufficiently smooth and decaying small initial data on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$. In addition, the author shows that the limiting effective dynamics is governed by the Szegő equation on the torus.

In order to formulate the result, we define two spaces $S$ and $S^+$ of sufficiently smooth initial data with enough decay in the spatial variable $x$, mainly, for some fixed $N \geq 13$,

$$\|u_0\|_S = \|u_0\|_{H^N_{x,y}} + \|xu_0\|_{L^2_{x,y}}, \quad \|u_0\|_{S^+} = \|u_0\|_S + \|xu_0\|_S + \|(1-\partial_{xx})^4 u_0\|_S.$$

We also denote by $\Pi_y$ the Szegő projector onto nonnegative Fourier frequencies in the variable $y$:

$$\Pi_y : \sum_{n\in\mathbb{Z}} \widehat{u_n}\, \mathrm{e}^{iny} \in L^2_y(\mathbb{T}) \mapsto \sum_{n\geq 0} \widehat{u_n}\, \mathrm{e}^{iny}\,.$$

**Theorem 3.1** (Modified scattering [43])**.** *There exists $\varepsilon = \varepsilon(N) > 0$ such that if the initial data $u_0 \in S^+$ satisfies $\|u_0\|_{S^+} \leq \varepsilon$, then the corresponding solution $u \in \mathcal{C}(\mathbb{R}_+, S)$ exists globally in $S$. Moreover, there exists $G \in \mathcal{C}(\mathbb{R}_+, S)$ such that*

$$\lim_{t\to+\infty} \|u(t) - e^{it(\partial_{xx}^2 - |D_y|)} G(\pi \ln(t))\|_S = 0\,. \tag{3.1}$$

*The profile $G$ is solution to the following resonant system:*

$$\begin{cases} i\partial_t \widehat{G}_+ = \Pi_y(|\widehat{G}_+|^2 \widehat{G}_+)(\xi, y) \\ i\partial_t \widehat{G}_- = (\mathrm{Id} - \Pi_y)(|\widehat{G}_-|^2 \widehat{G}_-)(\xi, y)\,, \end{cases}$$

*where $G_+ = \Pi_y(G)$ and $G_- = G - G_+$. The Fourier transform $\widehat{G}_\pm$ of $G_\pm$ is the partial Fourier transform in the variable $x$ only, with corresponding Fourier variable $\xi$.*

By considering the Fourier variable $\xi$ as a parameter, the equation for $\widehat{G}_+$ is the cubic Szegő equation on the torus. As a consequence of the work [26] on the growth of Sobolev norms for the Szegő equation, this remark implies the existence of arbitrarily small initial data such that for every $s > \frac{1}{2}$ and $N \geq 1$, the solution $u$ exhibits weak turbulence. It is expected that this result actually holds for a dense $G_\delta$ set of such initial data.

**Corollary 3.2** (Growth of Sobolev norms [43, 26]). *For every $N \geq 13$, for every $\varepsilon > 0$, there exists $u_0 \in S^+$ such that $\|u_0\|_{S^+} \leq \varepsilon$ and the corresponding solution $u$ satisfies:*

$$\forall s > \frac{1}{2}, \quad \limsup_{t\to\infty} \frac{\|u(t)\|_{L^2_x H^s_y}}{\log(t)^N} = \infty, \quad \liminf_{t\to\infty} \|u(t)\|_{L^2_x H^s_y} < \infty.$$

The strategy employed by Xu on $\mathbb{R}_x \times \mathbb{T}_y$ is adapted from the study for the Schrödinger equation on the wave guide $\mathbb{R}_x \times \mathbb{T}^d_y$ from Hani, Pausader, Tzvetkov and Visciglia in [28].

On $\mathbb{R}_x \times \mathbb{R}_y$, modified scattering should also be expected but it would rely on different arguments. Xi [17] constructed wave operators in this setting, and deduced a different type of growth of Sobolev norms in infinite time. Indeed, the resonant behavior is then linked to the cubic Szegő equation on the line, for which there is a transition towards high Fourier frequencies [27]. As a consequence, many solutions have a growth of the following form:

$$\frac{1}{C}\log(t) \leq \|u(t)\|_{L^2_x H^1_y} \leq C\log(t).$$

In comparison, the defocusing Schrödinger equation

$$i\partial_t u + \partial^2_{xx} u = |u|^2 u, \quad x \in \mathbb{R} \text{ or } \mathbb{T}$$

is completely integrable. In particular the conservation laws that control Sobolev norms of arbitrary high regularity imply that the solutions satisfy

$$\|u(t)\|_{H^s} \leq C_s(\|u_0\|_{H^s}),$$

for every $s \geq 1$ and $t \in \mathbb{R}$. However, as noticed in [28], the Schrödinger equation on the wave guide $\mathbb{R}_x \times \mathbb{T}^d_y$ for $d \geq 2$ exhibits modified scattering. As a consequence of the analysis of the resonant system [20], they prove that for every $s \geq 30$, for every $\varepsilon > 0$, there exists a global solution $u \in \mathcal{C}(\mathbb{R}_+, H^s)$ satisfying $\|u_0\|_{H^s} \leq \varepsilon$ and

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty.$$

Concerning the defocusing half-wave equation on the torus ($y \in \mathbb{T}$)

$$i\partial_t u - |D_y| u = |u|^2 u, \tag{3.2}$$

we know that for every $s > 1$, there exist a sequence $(u^n)_n$ of solutions and a sequence of times $t^n \to \infty$ satisfying

$$\|u^n_0\|_{H^s} \to 0, \quad \|u^n(t^n)\|_{H^s} \to +\infty.$$

The proof relies on and the growth of Sobolev norms for the Szegő equation [26], and on the fact that solutions to the half-wave equation initiated from initial with nonnegative frequencies stay close to the corresponding solution to the Szeő equation, until some large finite time [25, 39]. However, the existence of an arbitrary small initial data such that the solution to the half-wave equation satisfies $\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty$ is an open problem.

### 3.2. Traveling waves in the focusing case

Subsequently, Bahri, Ibrahim and Kikuchi considered in [2, 3] the focusing case $\mu < 0$ for equation (NLS-HW) on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$. They constructed ground state standing waves and traveling waves. Then they obtain orbital stability and transverse instability results for the family of standing waves. More precisely, on $\mathbb{R}_x \times \mathbb{T}_y$, the authors introduce a family of ground state standing wave solutions $u_\omega$ with frequency $\omega > 0$ of the form

$$u_\omega(x, y, t) = e^{i\omega t} Q_\omega(x, y) .$$

The ground states are constructed as minimizers of the energy functional

$$\mathcal{S}_\omega(u) = \frac{1}{2}\|\partial_x u\|_{L^2}^2 + \frac{1}{2}\||D_y|^{1/2}u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^4$$

under the constraint $\mathcal{N}_\omega(u) = 0$ with

$$\mathcal{N}_\omega(u) = \|\partial_x u\|_{L^2}^2 + \||D_y|^{1/2}u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 - \|u\|_{L^4}^4.$$

In [3], the authors establish that for small frequencies $0 < \omega < \omega_*$, the ground state $Q_\omega$ does not depend on the spatial variable $y$. As a consequence, $Q_\omega$ is equal to the line soliton for the Schrödinger equation, which is known to be orbitally stable on the line [16]. On the wave guide for (NLS-HW), however, the Schrödinger line soliton presents transverse instability properties: it is orbitally stable for small frequencies $0 < \omega < \omega_p$ whereas it is orbitally unstable for $\omega > \omega_p$. After showing that $\omega_p \geq \omega_*$, the authors deduce the orbital stability of ground states for small frequencies.

**Theorem 3.3** (Ground state standing waves [3])**.** *There exists $\omega_* > 0$ such that $Q_\omega$ does not depend on $y$ when $0 < \omega \leq \omega_*$, but depends on $y$ when $\omega > \omega_*$. Moreover, the ground state standing wave $u_\omega$ is orbitally stable when $0 < \omega \leq \omega_*$.*

We stress out that the stability results of the grounds states for (NLS-$\mathbb{H}^1$) are conditional to the existence of a good Cauchy theory in the energy space. Unfortunately, such a Cauchy theory is yet to be addressed, since not much is known about the global existence of smooth solutions in Sobolev spaces.

Theorem 3.3 is actually true for any nonlinearity of order $1 < p < 5$ including the cubic nonlinearity $p = 3$. The strategy to study orbital stability relies on the fact that the standing wave only depends in one of the two variables, transferring the problem to the Schrödinger equation on the line. It would be interesting to consider other geometries. For instance one could try to establish some orbital stability or instability property when the two spatial variables lie in $\mathbb{R}_x \times \mathbb{R}_y$, so that the ground state has to depend on both variables, but also on the wave guide $\mathbb{T}_x \times \mathbb{R}_y$ where the dynamics at small frequencies $\omega$ would be governed by the half-wave equation on the line rather than the Schrödinger equation.

## 4. Cauchy problem and random initial data

Given the instabilities that make the equation quasilinear in $\mathcal{H}^s$ when $\frac{1}{4} < s < \frac{1}{2}$, an alternative approach is to study whether well-posedness holds for some initial data distributed on a full-measure set. Such a probabilistic Cauchy theory goes back to the pioneering work of Bourgain on $\mathbb{T}^2$ to prove the invariance of the Gibbs measure [4] and to the general framework developed by Burq and Tzvetkov in [12, 13] for initial data in scaling-supercritical regimes. This approach has been extensively developed since then in many contexts, and we refer to [9] for a survey on this topic. As mentioned in section 4.4, the theory has recently been significantly improved to achieve impressive results beyond the scope of the original semi-linear framework discussed below in section 4.1.

## 4.1. Bourgain and Burq, Tzvetkov's historical approach

In what follows $(g_n(\omega))_n$ is a sequence of normalized independent Gaussian variables with complex values, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $s_c$ be a critical threshold under which instabilities that rule out a deterministic Cauchy theory are known to occur. When $s < s_c$, one can still develop a statistical approach and construct suitable probability measures on $H^s$ on the support of which the initial data in $\mathcal{H}^s$ has better integrability properties in $L^p$ spaces. For these *generic* initial data one can also observe from the dispersive features of the linear part of the equation a nonlinear smoothing effect in the Picard's iterations, which is enhanced by the probabilistic oscillations of the initial data. For instance, when $\phi^\omega$ is distributed according the Gibbs measure associated to NLS on $\mathbb{T}^2$, which is induced by the random variable

$$\omega \in \Omega \mapsto \phi^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{(1+n^2)^{\frac{1}{2}}} \, \mathrm{e}^{inx} \, . \tag{4.1}$$

The series defining $\phi^\omega$ converges in $L^2(\Omega; H^{0-}(\mathbb{T}^2))$ and defines the Gaussian free field on $\mathbb{T}^2$. Due to the Galilean symmetries of the equation [18], the flow map cannot be extended uniformly continuously in $H^s(\mathbb{T}^2)$ when $s < 0$ . Nevertheless, Bourgain was able to solve in [4] the Cauchy problem with data distributed on the support of the Gibbs measure (4.1). He observed from counting estimates and probabilistic considerations that the first Picard's iterate

$$\mathcal{I}^{(1)}(t, \phi^\omega) = -i\mu \int_0^t \mathrm{e}^{i(t-\tau)\Delta} \Big( |\, \mathrm{e}^{i\tau\Delta} \, \phi^\omega|^2 \, \mathrm{e}^{i\tau\Delta} \, \phi^\omega \Big) d\tau$$

is almost-surely in $H^{\frac{1}{2}-}(\mathbb{T}^2)$. In such a favorable situation, the nonlinear Duhamel term is smooth enough to fall on critical or subcritical regimes, and it is possible to run a fixed point argument for the centered solution around the linear evolution of the initial data:

$$v(t,x) = u(t,x) - \mathrm{e}^{it\Delta} \, \phi^\omega(x) \, . \tag{4.2}$$

The centered solution $v$, which is smoother, solves a Schrödinger equation with zero initial data but with some stochastic source terms that come from $\mathrm{e}^{it\Delta} \, \phi^\omega(x)$. The ansatz (4.2) is sometimes called the Da Prato and Debussche's trick.

## 4.2. Randomization procedure for (NLS-HW) on $\mathbb{R}^d$

In compact settings, the relevant probability measures on the phase-space $H^s$ are Gaussian measures constructed from the spectral resolution the Laplace operator, where each mode is decoupled by independent normalized Gaussian variables with complex values. This procedure stems from the consideration of invariant Gibbs measure. However, in the whole Euclidean space $\mathbb{R}^d$ there is no spectral resolution of the Laplace operator, nor nontrivial invariant measure. The standard procedure to generate random initial data $\phi^\omega$ from a given function $\phi \in \mathcal{H}^s$ consists in considering the so-called Wiener unit-scale frequency decomposition $(\phi_n)_n$ of $\phi$ in the frequency space, built from Fourier projectors on translated unit cubes.

To mimic the compact setting, each mode is decoupled by the independent Gaussian variables $(g_n)$:

$$\omega \in \Omega \mapsto \phi^\omega \sim \sum_n g_n(\omega)\phi_n \, , \quad \text{where } \phi \sim \sum_n \phi_n \in \mathcal{H}^s \, .$$

Then, for *many* initial data $\phi^\omega$ in a statistical ensemble $\Sigma \subset \mathcal{H}^s$ that has full measure, one expects to observe a *probabilistic smoothing effect* for the centered solution around the linear evolution thanks to the combination of space-time oscillations (dispersion) and probabilistic oscillations (randomization). Namely, the goal is to show that there exists $\nu > \frac{1}{2}$ such that for all $\phi^\omega \in \Sigma$,

$$v(t) := u(t) - \mathrm{e}^{it(\partial_{xx} - |D_y|)}\,\phi^\omega \in \mathcal{C}([-T\,,T];\mathcal{H}^\nu)\,. \tag{4.3}$$

Then $v$ solves the original equation perturbed by stochastic terms stemming from the linear evolution $\mathrm{e}^{it(\partial_{xx}^2 - |D_y|)}\,\phi^\omega$:

$$i\partial_t v + (\partial_{xx}^2 - |D_y|)v = \mu|v + \mathrm{e}^{it(\partial_{xx}^2 - |D_y|)}\,\phi^\omega|^2(v + \mathrm{e}^{it(\partial_{xx}^2 - |D_y|)}\,\phi^\omega).$$

In this case, if $\nu > \frac{1}{2}$, the centered solution $v$ is obtained from a fixed point argument at subcritical regularities in $\mathcal{H}^\nu$. This strategy has been successful in many contexts for the Schrödinger equation

$$i\partial_t u + \Delta u = \mu|u|^2 u\,.$$

The fixed point argument relies on nonlinear smoothing properties ensured by bilinear estimates [8, 7] or by local smoothing estimates [34]. Concerning the wave equations, smoothing properties are obtained directly from the Duhamel's integral formula that gains one derivative [12, 13, 35].

In contrast, when we replace the Laplace operator $\Delta$ by the anisotropic Schrödinger - half wave operator $\partial_{xx}^2 - |D_y|$, we will see that the first Picard's iterate $\mathcal{I}^{(1)}(t,\phi^\omega)$ does not gain any regularity for equation (NLS-HW), so that the probabilistic semilinear ansatz (4.2) fails.

### 4.3. Lack of nonlinear smoothing effect for the dispersioneless half-wave equation

Let us understand why probabilistic smoothing does not occur for (NLS-HW). In order to observe a probabilistic smoothing effect, we exploit dispersive properties of the equation to gain decay and Strichartz estimates without trading regularity. Equation (NLS-HW), however, is constructed so that there is no dispersion in the $y$-direction. Therefore, in the low $x$-frequency regimes, the Strichartz estimates come with a derivative loss so that we have neither usable bilinear estimates nor local smoothing estimates at our disposal. A manifestation of this lack of dispersion is that the second Picard iteration of the randomized initial data does not have a better regularity than the initial data: there is no probabilistic smoothing at the nonlinear level.

Nevertheless, the terms of the equation that prevent probabilistic smoothing are specific, and come from *high-low-low* type frequency interactions. For simplicity, we assume that there is no dependence in the variable $x$ (in practice, we rather restrict the solution to low $x$-frequency). In the first Picard iteration $\mathcal{I}^{(1)}(t,\phi^\omega)$, the high-low-low type frequency interactions involve the product of $\phi^\omega$ projected at high $y$-frequencies $|\eta| \gg 1$ and the square of $\phi^\omega$ projected at low $y$-frequencies $|\eta| \lesssim 1$. These singular contributions read

$$\int_0^t \mathrm{e}^{i(t-\tau)|D_y|}\left(e^{-i\tau|D_y|}P_{|\eta|\gg 1}\phi^\omega\right)\left(\overline{e^{-i\tau|D_y|}P_{|\eta|\leq 1}\phi^\omega}\right)\left(e^{-i\tau|D_y|}P_{|\eta|\leq 1}\phi^\omega\right)\,\mathrm{d}\tau\,.$$

In this case, the linear operator $e^{-i\tau|D_y|}$ is not dispersive as it acts as a transport equation on each term $P_{|\eta|\gg 1}\phi^\omega$ and $P_{|\eta|\leq 1}\phi^\omega$ after separating between positive and negative frequencies. Hence derivatives of the high-low-low interaction term all fall at the same time onto the first term $P_{|\eta|\gg 1}\phi^\omega$. This implies that the high-low-low interaction cannot receive more derivatives than the data. We mention the work of Oh [36] on the Szegő equation posed on the circle $\mathbb{T}$, who rigorously proved that the first nontrivial Picard's iterate $\mathcal{I}^{(1)}$ does not gain regularity compared to the initial data.

In this sense, the equation is not semilinear since the linear evolution is not a good approximation of the solution. As a consequence, one needs to resort to a refined resolution scheme that has to be quasilinear.

### 4.4. A refined probabilistic ansatz

In order to prove that a derivative wave equation is almost-surely well-posed at low regularity, Bringmann [5] developed a refined probabilistic ansatz in a quasilinear setting. In [15], we adapt this strategy to overcome the obstruction discussed in the previous paragraph, and show that (NLS-HW) is almost surely well-posed in $\mathcal{H}^s(\mathbb{R}^2)$ below the energy space, for some $s < \frac{1}{2}$. The main idea is to construct the solution by induction on the frequencies. At each step, the classical probabilistic ansatz (4.3) is refined by replacing the linear correction $e^{it(\partial_{xx}^2 - |D_y|)}\phi^\omega$ with a colored linear evolution, which solves a paracontrolled linear equation that encapsulate the singular high-low-low type frequency interactions. As discussed below, the probabilistic independence between the high frequencies and the low frequencies of the initial is a key probabilistic structure that is exploited in a remarkable way.

In the periodic case $\mathbb{T}^2$, Deng, Nahmod and Yue constructed in breakthrough papers [22, 23] dynamics on the support of the Gibbs measure for NLS with an arbitrary renormalized nonlinearity, introducing at the same time powerful methods such as the random averaging operators and the random tensors. More recently, Bringmann, Deng, Nahmod and Yue [6] pushed even further the paracontrolled approach for dispersive PDE in the probabilistic setting to solve the $\phi_3^4$ problem for the wave equation. The new developments incorporate tools ranging from random matrix theory to sophisticated counting estimates.

Let us explain how the refined probabilistic ansatz applies to equation (NLS-HW). We do not need to introduce a randomization along the Schrödinger variable $x$ since the Schrödinger equation is sufficiently dispersive. Hence, given $\phi \in \mathcal{H}^s(\mathbb{R}^2)$ we decompose it using partial Fourier projectors (in the half wave direction $y$) on an interval of length two centered around $k \in \mathbb{Z}$, denoted $P_{1,k}$:

$$\phi = \sum_{k\in\mathbb{Z}} P_{1,k}\phi, \quad \operatorname{supp}(\mathcal{F}_{y\to\eta}P_{1,k}\phi) \subseteq [k-1, k+1]\,.$$

We consider a sequence of independent normalized Gaussian variables $(g_k(\omega))_{k\in\mathbb{Z}}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and define the Wiener randomization of $\phi$ as

$$\omega \in \Omega \mapsto \phi^\omega := \sum_{k\in\mathbb{Z}} g_k(\omega) P_{1,k}\phi\,. \tag{4.4}$$

The relevant probability measure on $\mathcal{H}^s$ is the measure induced by this random variable. For $T > 0$ and $s, \sigma \in \mathbb{R}$, we establish the convergence of approximate local solutions towards a local solution to (NLS-HW) in the mixed Lebesgue space

$$X_{T_0}^s := \mathcal{C}_t\big([-T_0\,, T_0]\,; \mathcal{H}^s(\mathbb{R}^2)\big) \cap L_t^8\left([-T_0\,, T_0]\,; L_x^4 W_y^{\sigma,\infty}(\mathbb{R}^2)\right)\,,$$

where $(8,4)$ is a Strichartz admissible pair for the Schrödinger equation on the line:

$$\|\, \mathrm{e}^{it\partial_{xx}^2} u_0\|_{L_t^8(\mathbb{R};L_x^4(\mathbb{R}))} \lesssim \|u_0\|_{L_x^2(\mathbb{R})}\,.$$

We also denote the truncated initial data at $y$-frequency less than $n$ as

$$P_{\leq n}\phi^\omega := \sum_{|k|\leq n} g_k(\omega)P_{1,k}\phi\,.$$

The main result from [15] can be stated as follows.

**Theorem 4.1** (Probabilistic local well-posedness [15]). *Let $s \in (13/28, 1/2]$, $\phi \in \mathcal{H}^s$ and the corresponding random initial data $\omega \mapsto \phi^\omega \in L^2(\Omega; \mathcal{H}^s)$ defined in (4.4). There exist $T_0 > 0$ and a full measure set $\Sigma \subset \mathcal{H}^s$ such that for any $\phi^\omega \in \Sigma$ the following holds. There exists a uniform random time $T^\omega \in (0, T_0]$ such that for all $n \in \mathbb{N}$, there exists a function $u_n \in C([-T_0, T_0], \mathcal{H}^\infty)$ which is the unique solution on $[-T^\omega, T^\omega]$ to (NLS-HW) with smooth initial data $P_{\leq n}\phi^\omega$:*

$$\begin{cases} i\partial_t u_n + (\partial_{xx}^2 - |D_y|)u_n = |u_n|^2 u_n\,, \quad (t,x,y) \in [-T^\omega\,,\,T^\omega] \times \mathbb{R}^2\,, \\ u_n|_{t=0} = P_{\leq n}\phi^\omega\,. \end{cases}$$

*Moreover, the sequence $(u_n)_{n\geq 1}$ converges in $L^2_\omega\left(\Omega\,; X^{s,\sigma}_{T_0}\right)$ for some $0 < \sigma < s$ to a limiting object $u$ which is solution to (NLS-HW) on $[-T^\omega, T^\omega]$ with initial data $\phi^\omega$.*

The idea behind the refined probabilistic ansatz is the following. We first construct the solution $u_N$ for dyadic $N \in 2^{\mathbb{N}}$ by induction on $N$. The solution at step $N$ is constructed from the solution at step $\frac{N}{2}$ following the ansatz

$$u_N = u_{\frac{N}{2}} + F_N + w_N\,. \tag{4.5}$$

On the one hand, the probabilistic term $F_N$ isolates the problematic rough *high-low-low* frequency interactions, and is called the *adapted linear evolution* or the *colored Gaussian.* On the other hand, the nonlinear remainder term $w_N$ exhibits a nonlinear smoothing effect and lies in a subcritical space $\mathcal{H}^s(\mathbb{R}^2)$, $s > \frac{1}{2}$.

Let $0 < \gamma < 1$ be some parameter to be optimized at the end of the analysis. We see the cubic term as a trilinear interaction denoted by

$$\mathcal{N}(u) = |u|^2 u\,, \quad \mathcal{N}(u_1, u_2, u_3) = \overline{u_1}u_2u_3 + u_1\overline{u_2}u_3 + u_1u_2\overline{u_3}\,.$$

The *adapted linear evolution* $F_N$ is solution to the paracontrolled linear equation

$$\begin{cases} i\partial_t F_N + (\partial_{xx}^2 - |D_y|)F_N = \mathcal{N}(F_N, P_{\leq N^\gamma}u_{\frac{N}{2}}, P_{\leq N^\gamma}u_{\frac{N}{2}})\,, \\ F_N(0) = P_N\phi^\omega\,. \end{cases} \tag{4.6}$$

It encapsulates the *high-low-low* interactions at scale $N$. The high frequencies are carried by the solution $F_N$, whose initial data is the projection $P_N\phi^\omega = P_{\leq N}\phi^\omega - P_{\leq \frac{N}{2}}\phi^\omega$ of $\phi^\omega$ at frequency $N$, so that we expect this property to stay true at least for small times. The low frequencies are carried by the projection of the solution $u_{\frac{N}{2}}$ constructed at step $\frac{N}{2}$ onto the low frequencies $|\eta| \leq N^\gamma$, with $0 < \gamma < 1$.

The nonlinear remainder $w_N$ is solution to (NLS-HW) with a stochastic forcing terms and zero initial condition:

$$\begin{cases} i\partial_t w_N + (\partial_{xx}^2 - |D_y|)w_N = \mathcal{N}\big(u_N\big) - \mathcal{N}\big(u_{\frac{N}{2}}\big) - \mathcal{N}\big(F_N, P_{\leq N^\gamma}u_{\frac{N}{2}}, P_{\leq N^\gamma}u_{\frac{N}{2}}\big)\,, \\ w_N(0) = 0\,. \end{cases}$$

Since we removed the high-low-low interactions from the stochastic forcing term, we expect that $w_N$ exhibits probabilistic nonlinear smoothing.

We then proceed as follows. By definition of the induction scheme (4.5), the solution at step $N$ can be written as a series

$$u_N = u_{N_0} + \sum_{L=2N_0}^{N} \left(w_L + F_L^\omega\right)\,.$$

We need to prove that smooth remainder terms $(w_N)_{N\geq N_0}$ exist on a uniform time interval $[-T_0, T_0]$, and that the series with general term $w_N$ converges almost-surely in a subcritical space $\mathcal{C}([-T_0\,, T_0]; \mathcal{H}^\nu(\mathbb{R}^2))$, for some $\nu > \frac{1}{2}$. On the other hand, the series with general term $(F_N)_{N\geq N_0}$ composed of the probabilistic corrections converges almost-surely in the space of rough regularity $\mathcal{C}([-T_0\,,T_0];\mathcal{H}^s(\mathbb{R}^2))$. Then, we prove that there exists a random time $T^\omega > 0$ such that the limit of $(u_N)_N$ solves (NLS-HW) in $\mathcal{C}\big([-T^\omega\,,T^\omega];\mathcal{H}^s(\mathbb{R}^2)\big)$. Finally, we use an argument from [41] to show that the result for dyadic frequencies $N$ extends to the general approximation with arbitrary integer frequencies $n$.

The local existence of the smooth solution $u_n$ is guaranteed by the local well-posedness result from Theorem 1.6 in [2]. However, the time of existence in say $\mathcal{H}^\nu$, for some $\nu > \frac{1}{2}$, depends on the $\mathcal{H}^\nu$-norm of $u_n$. In particular it depends on $n$ and on $\omega$ in an intricate way. The strategy is first to truncate the equation for $u_n$ to show the convergence of $(u_n)_{n\in\mathbb{N}}$, on a time interval $[-T_0, T_0]$ which does not depend on $n$. In order to get convergence on a fixed time interval, we follow [5] by making use of the *truncation method* from De Bouard and Debussche [21], which consists in truncating the terms involved in the nonlinearity so that they stay bounded by one in suitable space-time functional spaces. Then, we prove that on some random time interval $[-T^\omega, T^\omega]$, the limit of $(u_n)_n$ solves the equation (NLS-HW) without truncation in $\mathcal{C}\big([-T^\omega\,,T^\omega];\mathcal{H}^s(\mathbb{R}^2)\big)$.

To obtain improved Strichartz estimates for the adapted linear evolution $F_N$ one has to decompose it into elementary pieces whose frequency localization and stochastic structure are well understood. Then, the *probabilistic Strichartz estimates* for small times follows from Bernstein estimates and probabilistic decoupling. These estimates control the $L^\infty$ norm of $F_N$ with a loss $N^{\frac{\gamma}{2}-\sigma}$ ($\frac{\gamma}{2} - \sigma + \epsilon$ derivatives) instead of the expected loss $N^{\frac{1}{2}}$ ($\frac{1}{2} + \epsilon$ derivatives).

To conclude this paragraph, we stress out that we only implement the probabilistic scheme in the half-wave variable $y$, whereas a traditional deterministic analysis is performed in the Schrödinger variable $x$, respectively.

We also point out that we used a $TT^*$-type argument to control the frequency localization of $F_N$, whereas Bringmann implemented in [5] Gronwall inequalities and sophisticated energy estimates for wave-type equation. For this reason, we think that our analysis in the context of Schrödinger-type equations is more flexible and adapts to other contexts.

## 5. Perspectives

To conclude this note, let us mention some possible developments linked to equation (NLS-HW) and to the probabilistic quasilinear resolution scheme.

(1) Reaching a global well-posedness theory for equation (NLS-HW) in the defocusing case, whether in the deterministic or the probabilistic case. The first difficulty is that one cannot use a Yudovich argument to get local existence in the energy space $\mathcal{H}^{\frac{1}{2}}$, because the $L^q$-norms are not controlled by the energy when $q > 6$. The second difficulty is that when $s > \frac{1}{2}$, the Brezis-Gallouët estimate fails to extend smooth solutions globally in time. The reason is that $\mathcal{H}^s$ is not an algebra when $s < \frac{3}{4}$, and there is no conservation law that controls the $\mathcal{H}^{\frac{3}{4}}$-norm. This is in contrast with the half-wave equation or the Szegő equation on the line.

From the point of view of random initial data, it would be challenging but interesting to understand the long-time behavior of the probabilistic solutions to equation (NLS-HW) generated by the paracontrolled resolution scheme. Enven if we constructed in Theorem 4.1 a probabilistic solution in the presence of a conserved energy, this conservation law is not sufficient to globalize the solutions. The reason is that the specific probabilistic structure of the initial data is needed to do a local existence result. To iterate the local existence result, one has to understand how this information is transported by the flow on longer time scales. To achieve such a goal, one could try to prove quasi-invariance of the probability measure. Unfortunately, the current techniques strongly relies on the dispersive properties of the equation, and quasi-invariance fails for dispersionless models see for instance [37].

(2) It would be very relevant to improve the lower bound on the regularity we get in Theorem 4.1, in order to cover the whole quasilinear regime. Our lower bound on $s$ is far from being optimal, and new insigths are needed to go all the way down to $s_c = \frac{1}{4}$.

(3) Probabilistic local well-posedness for other non-dispersive PDE on degenerate geometries.

The refined probabilistic ansatz developed in [15] for equation (NLS-HW) also applies to the half-wave equation on the line (3.2) and Szegő equation, by removing the variable $x$ all throughout the paper. We believe that this strategy will also be successful to investigate other Schrödinger-type equations arising in degenerate geometries. One example is the Schrödinger equation on the Heisenberg group in the radial case

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \tag{NLS-$\mathbb{H}^1$}$$

where $\mathbb{H}^1$ is parameterized by three real coordinates $(x, y, s) \in \mathbb{H}^1$ for which the sub-Laplacian for radial functions reads

$$\Delta_{\mathbb{H}^1} = \partial_{xx} + \partial_{yy} + (x^2 + y^2)\partial_{ss}.$$

This equation is a totally non dispersive equation [1]. As a consequence, the study of the Cauchy problem at low regularity is a delicate issue. Properties of the flow map are similar to equation (NLS-HW), and can be summarized in the following diagram in the scale of Sobolev spaces associated to the operator $\Delta_{\mathbb{H}^1}$.

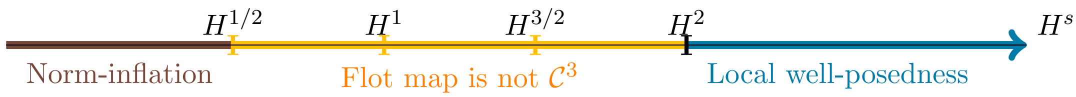


FIGURE 2. Deterministic Cauchy theory for equation (NLS-$\mathbb{H}^1$).

(4) Modified scattering for defocusing (NLS-HW) on $\mathbb{R}_x \times \mathbb{R}_y$, global solutions for smooth and decaying initial data, following [17].

(5) Growth of Sobolev norms for the defocusing half-wave equation (3.2). In particular it is open whether there exist a solution satisfying $\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty$.

(6) Orbital stability or instability property for traveling waves of focusing (NLS-HW) on different geometries than the wave guide $\mathbb{R}_x \times \mathbb{T}_y$, for instance when the two spatial variables lie in $\mathbb{R}_x \times \mathbb{R}_y$, or in the wave guide $\mathbb{T}_x \times \mathbb{R}_y$.

## Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1929284 while the authors were in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the program "Hamiltonian Methods in Dispersive and Wave Evolution Equations". The authors would like to thank the organizers of the program, and all the staff at ICERM for their great hospitality. N. Camps is supported by the region "Pays de la Loire" through the project MasCan. S Ibrahim is supported by the NSERC grant No. 371637-2019.

## Bibliography


[1] H. Bahouri, P. Gérard, and C.-J. Xu. Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg. *Journal d'Analyse Mathématique*, 82(1):93–118, 2000.

[2] Y. Bahri, S. Ibrahim, and H. Kikuchi. Remarks on solitary waves and Cauchy problem for Half-wave-Schrödinger equations. *Communications in Contemporary Mathematics*, page 2050058, 2020.

[3] Y. Bahri, S. Ibrahim, and H. Kikuchi. Transverse Stability of Line Soliton and Characterization of Ground State for Wave Guide Schrödinger Equations. *Journal of Dynamics and Differential Equations*, pages 1–43, 2021.

[4] J. Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Commun. Math. Phys.*, 176(2):421–445, 1996.

[5] B. Bringmann. Almost sure local well-posedness for a derivative nonlinear wave equation. *Internat. Math. Res. Notices*, 2021(11):8657–8697, 2021.

[6] B. Bringmann, Y. Deng, A. R. Nahmod, and H. Yue. Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation. *arXiv preprint arXiv:2205.03893*, 2022.

[7] Á. Bényi, T. Oh, and O. Pocovnicu. On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$. *Trans. Am. Math. Soc., Ser. B*, 2:1–50, 2015.

[8] A. Bényi, T. Oh, and O. Pocovnicu. *Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS*, pages 3–25. Springer International Publishing, 2015.

[9] A. Bényi, T. Oh, and O. Pocovnicu. *On the Probabilistic Cauchy Theory for Nonlinear Dispersive PDEs*, pages 1–32. Springer International Publishing, 2019.

[10] N. Burq, P. Gérard, and N. Tzvetkov. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces. *Invent. Math.*, 159:187–225, 2005.

[11] N. Burq, P. Gérard, and N. Tzvetkov. Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. 38(2):255–301, 2005.

[12] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations I: local theory. *Invent. Math.*, 173(3):449–475, 2008.

[13] N. Burq and N. Tzvetkov. Random data Cauchy theory for supercritical wave equations II: a global existence result. *Invent. Math.*, 173(3):477–496, 2008.

[14] N. Camps and L. Gassot. Pathological set of initial data for scaling-supercritical nonlinear Schrödinger equations. *Int. Math. Res. Not.*, 2022.

[15] N. Camps, L. Gassot, and S. Ibrahim. Refined probabilistic local well-posedness for a cubic Schr\" odinger half-wave equation. *arXiv preprint arXiv:2209.14116*, 2022.

[16] T. Cazenave. *Semilinear Schrödinger equations.* Courant lecture notes. American Mathematical Society, Providence, RI, 2003.

[17] X. Chen. Existence of modified wave operators and infinite cascade result for a half wave Schrödinger equation on the plane, *arXiv preprint arXiv:2302.12067*, 2023.

[18] M. Christ, J. Colliander, and T. Tao. Ill-posedness for nonlinear Schrodinger and wave equations. *arXiv preprint arXiv:math/0311048*, 2003.

[19] J. Colliander, S. Ibrahim, M. Majdoub, and N. Masmoudi Energy critical NLS in two space dimensions. *J. Hyperbolic Differ. Equ.* 6:3 (2009), 549–575.

[20] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Inventiones mathematicae*, 181(1):39–113, 2010.

[21] A. de Bouard and A. Debussche. A stochastic nonlinear Schrödinger equation with multiplicative noise. *Comm. Math. Phys.*, 205(1):161–181, 1999.

[22] Y. Deng, A. Nahmod, and H. Yue. Invariant gibbs measures and global strong solutions for nonlinear schrödinger equations in dimension two. *arXiv:1910.08492*, 2019.

[23] Y. Deng, A. Nahmod, and H. Yue. Random tensors, propagation of randomness, and nonlinear dispersive equations. *Invent. Math.*, 228:539–686, 2022.

[24] L. Gassot and M. Latocca. Probabilistic local well-posedness for the Schrödinger equation posed for the Grushin Laplacian. *arXiv:2103.03560*, 2021.

[25] P. Gérard and S. Grellier. Effective integrable dynamics for a certain nonlinear wave equation. *Analysis and PDE*, 5, 10 2012.

[26] P. Gérard and S. Grellier. The cubic Szegő equation and Hankel operators. *Astérisque, Société mathématique de France, Paris*, 389, 2017.

[27] P. Gérard and A. Pushnitski. Unbounded Hankel operators and the flow of the cubic Szegő equation. *arXiv:2206.11543*, 2022.

[28] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia. Modified scattering for the cubic Schrödinger equation on product spaces and applications. In *Forum of mathematics, Pi*, volume 3. Cambridge University Press, 2015.

[29] S. Ibrahim, M. Majdoub, and N. Masmoudi Well- and ill-posedness issues for energy supercritical waves. In *Anal. PDE* 4(2): 341-367 (2011). DOI: 10.2140/apde.2011.4.341

[30] I. Kato. Ill-posedness for the Half wave Schrödinger equation. *arXiv:2112.10326*, 2021.

[31] G. Lebeau. Non linear optic and supercritical wave equation. *Bull. Soc. R. Sci. Liege*, 2001.

[32] G. Lebeau. Perte de régularité pour les équations d'ondes sur-critiques. *Bull. SMF*, 133(1):145–157, 2005.

[33] H. Lindblad. A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations. *Duke Math. J.*, 72:503–539, 1993.

[34] B. Dodson, J. Lührmann, and D. Mendelson. Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation. *Adv. Math.*, 347:619–676, 2019.

[35] B. Dodson, J. Lührmann, and D. Mendelson. Almost sure scattering for the $4D$ energy-critical defocusing nonlinear wave equation with radial data. *Am. J. Math.*, 142(2):475–504, 2020.

[36] T. Oh. Remarks on nonlinear smoothing under randomization for the periodic KdV and the cubic Szegö equation. *Funkcialaj Ekvacioj*, 54(3):335–365, 2011.

[37] T. Oh, P. Sosoe, and N. Tzvetkov. An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *Journal de l'Ecole Polytechnique Mathématiques*, 5:793–841, 2018.

[38] O. Pocovnicu. Traveling waves for the cubic Szegő equation on the real line. *Analysis and PDE*, 4(3):379–404, 2011.

[39] O. Pocovnicu. First and second order approximations for a nonlinear wave equation. *Journal of Dynamics and Differential Equations*, 25(2):305–333, 2013.

[40] C. Sun and N. Tzvetkov. Concerning the pathological set in the context of probabilistic well-posedness. *Comptes Rendus. Math.*, 358(9-10):989–999, 2020.

[41] C. Sun and N. Tzvetkov. Refined probabilistic global well-posedness for the weakly dispersive NLS. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 213:91, 2021.

[42] B. Xia. Generic Ill-posedness for wave equation of power type on three-dimensional torus. *Int. Math. Res. Not.*, 20:15533–15554, 2021.

[43] H. Xu. Unbounded Sobolev trajectories and modified scattering theory for a wave guide nonlinear Schrödinger equation. *Mathematische Zeitschrift*, 286(1):443–489, 2017.



Nicolas Camps, Louise Gassot, Slim Ibrahim
Laboratoire de Mathématiques Jean Leray, Nantes Université, UMR CNRS 6629, 2 rue de la Houssinière, 44322 Nantes Cedex 03, France
Université de Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
Department of Mathematics and Statistics University of Victoria, 3800 Finnerty Road, Victoria BC V8P 5C2, Canada
Pacific Institute for Mathematical Sciences, 4176-2207 Main Mall, Vancouver, BC V6T 1Z4, Canada
nicolas.camps@univ-nantes.fr
louise.gassot@normalesup.org
ibrahims@uvic.ca