The basic model theory of valued fields

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12 May 2008

partially supported by MRTN-CT-2004-512234 and ANR-06-BLAN-0183

Languages

A language is a collection $\mathcal{L}$, finite or infinite, of symbols. These symbols are of three kinds:
– *function* symbols,
– *relation* symbols,
– *constant* symbols.

Example. The language of ordered (abelian) groups:

$$\mathcal{L}_{og} = \{+, -, 0, <\}.$$

$+$ is a binary function, $-$ a unary function, 0 a constant symbol, and $<$ a binary relation.

$\mathcal{L}$-structures

Interpretation of the language symbols in a *universe*

Examples

1. $(\mathbb{Z}, +, -, 0, <)$, the natural structure on the additive group of the integers,
2. $(\mathbb{R}, +, -, 0, <)$, the natural structure on the additive group of the reals,
3. (multiplicative notation)
   $(\mathbb{R}^{>0}, \cdot, ^{-1}, 1, <)$ the multiplicative group of the positive reals.

Formulas and definable sets

Formulas are built up using the languages symbols, as well as $=$, $\wedge$, $\vee$, $\rightarrow$, $\neg$, variable symbols, parentheses, and *quantifiers* $\exists$, $\forall$.
Below we'll discuss them in a language $\mathcal{L}_{og} \cup \{c_1, \ldots, c_n\}$ ($c_1, \ldots, c_n$ new symbols of constants)

Terms: built up from functions, variables, and constants: $x + x$, $x + x + x, \ldots, nx$, $-nx$ ($n \in \mathbb{N}$), $c_1 + c_2$, $2c_3$.
General form for term $t(x_1, \ldots, x_m)$:

$$\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{n} \ell_j c_j,$$

where the n_i, ℓ_j belong to $\mathbb{Z}$.

Qf-formulas: apply relations and Boolean connectives to terms: $\bar{x} = (x_1, \ldots, x_m)$, $t_1(\bar{x}), \ldots, t_4(\bar{x})$ terms:

$$\big(t_1(\bar{x}) = t_2(\bar{x}) \wedge t_3(\bar{x}) < t_4(\bar{x})\big) \vee \quad \big(t_1(\bar{x}) < t_2(\bar{x})\big)$$

Formulas with quantifiers, satisfaction, definable sets

Quantify over *free* variables. Satisfaction is what it should be.
More precisely, if M is an $\mathcal{L}$-structure, then each term defines a function from some cartesian power of M to M. Then, if $\bar{a}$, $\bar{b}$ are tuples in M, and t_1, t_2 are terms, we will have

$$M \models t_1(\bar{a}) = t_2(\bar{b})$$

if and only if the evaluations of the terms t_1 at $\bar{a}$ and t_2 at $\bar{b}$ give the same element.
Satisfaction is then defined by induction on the complexity of the formulas. Definable set = set of tuples satisfying a formula.

Allowing parameters

One can also allow parameters: this can be viewed as for instance looking at fibers of definable sets under a projection. E.g., have formula $\varphi(\bar{x}, \bar{y})$, $\bar{x}$ and $\bar{y}$ tuples of variables, and in a model look at family of definable sets defined by $\varphi(\bar{x}, \bar{a})$ as $\bar{a}$ varies within the model.

Example of formula

$$\varphi(x, y) := x < y \wedge \forall z \; x < z \rightarrow y = z \vee y < z.$$

Says that y is a successor of x for the ordering.
The elements a, b of the ordered group G satisfy φ (notation: $G \models \varphi(a, b)$) iff b is a successor of a in the ordering.
So, φ defines in $(\mathbb{Z}, +, -, 0, <)$ the graph of the successor function.
But in $(\mathbb{R}, +, -, 0, <)$, it defines the emptyset, since the ordering is dense.

Theory

Theory = set of formulas with no free variables (called *sentences*; think of *axioms*). Hopefully consistent (= is satisfied by some structure). Sometimes *complete*: given a sentence φ, either φ or $\neg\varphi$ (but not both) is a consequence of the theory.

Examples

1. The theory of abelian ordered divisible groups (complete): the obvious axioms.
2. The theory of ordered Z-groups (complete): axioms for an ordered abelian group, with a unique smallest positive element (denoted 1); for all $n > 1$, the axiom

$$\forall x \bigvee_{i=0}^{n-1} \exists y \; ny = x - i.$$

Language of rings: $\{+, -, \cdot, 0, 1\}$. Usual interpretation in a ring.
Language of ordered rings: add the binary symbol $<$.

3. The theory of algebraically closed fields: axioms for commutative fields, and for all $n > 1$, the axiom

$$\forall x_0, \ldots, x_n, \exists y \ (x_n = 0 \vee \sum_{i=0}^{n} x_i y^i = 0).$$

   (incomplete: one needs to specify the characteristic).

4. The theory of real closed fields (language of rings): axioms for commutative fields; $\forall x \exists y \; y^4 = x^2$; for all $n \geq 1$, the axiom:

$$\forall x_0, \ldots, x_{2n+1}, \exists y \; (x_n = 0 \vee \sum_{i=0}^{2n+1} x_i y^i = 0).$$

(complete)

5. The theory of real closed fields (language of ordered rings): axioms for commutative ordered fields; $\forall x \; x > 0 \rightarrow \exists y \; y^2 = x$; every polynomial of odd degree has a root. (complete)

Quantifier-elimination

We fix a theory T (= set of axioms). As the name indicates, T *eliminates quantifiers* iff every formula is equivalent, modulo T, to a formula without quantifiers.

Other formulation: In every model M of T, if $D \subset M^{n+1}$ is quantifier-free definable, and $\pi : M^{n+1} \to M^n$ is the projection, then $\pi(D)$ is quantifier-free definable.

Examples

1. Algebraically closed fields (note: we do not mention the characteristic).
2. Real closed fields in the language of ordered rings.

3. Divisible abelian groups in the language of ordered abelian groups.
4. Ordered Z-groups in the language of Pressbürger:

$$\{+, -, 0, 1, <, \equiv_n\}_{n \geq 2}$$

where $\equiv_n$ is defined by the axiom $x \equiv_n y \leftrightarrow \exists z \; nz + x = y$, and 1 is the smallest positive element.

Valued fields - definition

Recall that a valued field is a field K, with a map $v : K^\times \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, and satisfying the following axioms:

- $v(x) = \infty \iff x = 0$,
- $\forall x, y \ v(xy) = v(x) + v(y)$
- $\forall x, y \ v(x + y) \geq \min\{v(x), v(y)\}$.

By convention, ∞ is greater than all elements of Γ.

Languages

Several natural languages.
1. Maybe the most natural (used in the definition) is the *two-sorted* language with a sort for the valued field and one for the value group; each sort has its own language (the language of rings for the valued field sort, and the language of ordered abelian groups with an additional constant symbol ∞; there is a function v from the field sort to the group sort. Thus our structure is

$$\big((K,+,-,\cdot,0,1),(\Gamma\cup\{\infty\},+,-,0,\infty,<),v\big).$$

Formulas are built as in classical first-order logic, except that variables come with their sort. Thus for instance, in the three defining axioms, all variables are of the field sort. To avoid ambiguity, one sometimes write $\forall x\in K$, or $\forall x\in\Gamma$. Or one uses a different set of letters.

2. Another natural language is the language $\mathcal{L}_{\mathrm{div}}$ obtained by adding to the language of rings a binary relation symbol $|$, interpreted by

$$x|y \iff v(x) \leq v(y).$$

Note that the valuation ring $\mathcal{O}_K$ is quantifier-free definable, by the formula $1|x$, and that the group Γ is isomorphic to $K^\times/\mathcal{O}_K^\times$, the order been given by the image of $|$. Hence the ordered abelian group Γ is *interpretable* in $(K, +, -.\cdot, 0, 1, |)$.

In both languages, the residual field k_K, as well as the residue map $\mathcal{O}_K \to k_K$, are interpretable: k_K is the quotient of $\mathcal{O}_K$ by the maximal ideal of $\mathcal{O}_K$.

Variations on this are:
the field K in the language of rings with a (unary) predicate for the valuation ring;
$\mathcal{O}_K$ in the language of rings with a binary function symbol Div interpreted by $\mathrm{Div}(x, y) = xy^{-1}$ if $v(y) \leq v(x)$, 0 otherwise.

EQ for algebraically closed valued fields

Theorem
The theory of algebraically closed valued fields eliminates quantifiers in the language $\mathcal{L}_{\mathrm{div}} = \{+, -, \cdot, 0, 1, |\}$.

Corollary. Every formula in the variables $\bar{x} = (x_1, \ldots, x_n)$ is equivalent, modulo the theory of algebraically closed valued fields (ACVF), to a Boolean combination of formulas of the form

$$v(f(\bar{x})) \leq v(g(\bar{x})), \qquad h(\bar{x}) = 0,$$

where f, g, h are polynomials over $\mathbb{Z}$.

Three sorts?

The proof of eq of ACVF puts in evidence a trichotomy of valued field extensions. Namely, given a subfield A of an algebraically closed field K, one can reduce the study of an extension B/A to the study of extensions of the following type:

a. B/A immediate (same value group, same residue field),
b. B/A purely residual,
c. B/A totally ramified.

The proof of quantifier elimination is done using a back-and-forth argument: we are given two $\aleph_1$-*saturated* algebraically closed valued fields K and L, two countable substructures A and B of K, L respectively, and an $\mathcal{L}_{\mathrm{div}}$-isomorphism $f : A \to B$. We also have some $c \in K$ and want to extend f to $A(c)$. We let C be the algebraic closure of $A(c)$. It then suffices to extend f to C.

One extends f in three stages, to

- the subfield A_0 generated by A,
- a purely residual extension A_1 of A_0 contained in C and having same residue field as C,
- a totally ramified extension A_2 of A_1 contained in C and having same value group as C,
- the immediate extension C/A_2.

The language of Pas-Denef

This splitting of cases is also apparent in the results of Ax-Kochen-Ershov, and in their proof. This suggest passing to three sorts: the valued field, the value group, and the residue field, with additional maps the valuation and the residue map. It turns out that for quantifier-elimination results this is not quite enough. One language, which is quite convenient, is the language $\mathcal{L}_{\mathrm{Pas}}$:

- It has three sorts: the valued field, the value group and the residue field.
- The language of the field sort is the language of rings.
- The language of the value group is any language containing the language of ordered abelian groups (and ∞).
- The language of the residue field is any language containing the language of rings.
- In addition, we have a map v from the field sort to the value group (the valuation), and a map $\overline{\mathrm{ac}}$ from the field sort to the residue field (*angular coefficient map*).

The angular coefficient map

It is a multiplicative map $\overline{\mathrm{ac}} : K \to k_K$, which is multiplicative, sends 0 to 0, and on $\mathcal{O}_K^\times$ coincides with the residue map. It therefore suffices to know this map on a set of representatives of the value group.

In all natural examples, there is a natural coefficient map (because there is a natural section of the value group):

- On the valued field $k((t))$, ($v(t) = 1$, v trivial on k), define $\overline{\mathrm{ac}}(0) = 0$, and $\overline{\mathrm{ac}}(t) = 1$. Thus, if $a_j \neq 0$ then
$$\overline{\mathrm{ac}}(\sum_{i \geq j} a_i t^i) = a_j.$$
- On $\mathbb{Q}_p$, define $\overline{\mathrm{ac}}$ by $\overline{\mathrm{ac}}(p) = 1$.

More on $\overline{\mathrm{ac}}$

In most cases we do strengthen the language by adding this $\overline{\mathrm{ac}}$ map.

However, note that it is definable in the field $\mathbb{Q}_p$: indeed, $\overline{\mathrm{ac}}$ equals 1 on the $(p-1)$-th-powers. It therefore suffices to specify the values of $\overline{\mathrm{ac}}$ on a system of generators of the finite group $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times\, p-1}$.

It is not true that every valued field has an angular component map. However, every valued field K has an *elementary extension* K^* which has an angular component map.

Relative qe for Henselian fields of residue characteristic 0

Theorem

Let (K, Γ_K, k_K) be an $\mathcal{L}_{\mathrm{Pas}}$-structure, where K is a Henselian valued field, and k_K has characteristic 0. Then every formula $\varphi(x, \xi, \bar{x})$ (x, ξ, $\bar{x}$, tuples of variables of the valued field, valued group, residue field sort) of the language is equivalent to a Boolean combination of formulas

$$\varphi_1(x) \wedge \varphi_2(v(f(x)), \xi) \wedge \varphi_3(\overline{\mathrm{ac}}(f(x)), \bar{x}),$$

where $f(x)$ is a tuple of elements of $\mathbb{Z}[x]$, φ_1 is a **quantifier-free** *formula of the language of rings, φ_2 is a formula of the language of the group sort, and φ_3 is a formula of the language of the residue field sort.*

Example

Consider the natural $\overline{\mathrm{ac}}$ map on $k((t))$, where k is a field of characteristic 0, and look at the $\mathcal{L}_{\mathrm{Pas}}$-structure

$$(K, \mathbb{Z} \cup \{\infty\}, k)$$

where the language of the group sort is the Pressbürger language. Then in the above the formula φ_2 will be a formula without quantifiers.

A definable (with parameters) function $K \to \Gamma$ will therefore be locally defined by expressions of the form

$$(\sum_i m_i v(f_i(x)) + \alpha)/N$$

where the f_i are polynomials over K, $\alpha \in \Gamma_K$, and N is some integer.

Aside on the p-adics

One of the language in which the field of p-adic numbers eliminates quantifiers is the language of Macintyre, $\mathcal{L}_{\mathrm{Mac}}$, which is obtained by adding to $\mathcal{L}_{\mathrm{div}}$ predicates P_n, $n > 1$, which are interpreted by

$$P_n(x) \leftrightarrow \exists y \; y^n = x \wedge x \neq 0.$$

In fact, the relation $|$ is unnecessary, as it is quantifier-free definable in $\mathbb{Q}_p$: for instance, if $p \neq 2$, we have:

$$v(x) \leq v(y) \iff y = 0 \vee P_2(x^2 + py^2).$$

The definition however depends on p, and for uniformity questions it is better to include $|$ in the language.

$\mathbb{Q}_p$ with angular component maps

If one wishes to study the p-adics in a three-sorted language with angular components, one is obliged to add angular components of higher order, namely, for each n, a multiplicative map $\overline{\mathrm{ac}}_n : K \to \mathbb{Z}/p^n\mathbb{Z}$, which on $\mathbb{Z}_p$ coincides with the usual $\mod p^n$ reduction. We also require that $\overline{\mathrm{ac}}_n = \overline{\mathrm{ac}}_{n+1} \mod p^n$. Note that this requires introducing many new sorts.

Axioms for the p-adics

The $\mathcal{L}_{\mathrm{div}}$ theory of the valued field $\mathbb{Q}_p$ is axiomatised by expressing the following properties:

K is a Henselian valued field of characteristic 0, with residue field $\mathbb{F}_p$. Its value group is a Z-group, with $v(p)$ the smallest positive element.

Elementary extension

An extension K^* of our field K, which has the same elementary properties as K: if $\varphi(\bar{x})$ is a formula, and $\bar{a}$ a tuple in K, then

$$K \models \varphi(\bar{a}) \iff K^* \models \varphi(\bar{a}).$$

Notation: $K \prec K^*$.

Examples.

- Let K be a subfield of $\mathbb{Q}_p$, relatively algebraically closed in $\mathbb{Q}_p$. Then $K \prec \mathbb{Q}_p$.
- If a theory T eliminates quantifiers, and $M_1 \subset M_2$ are two models of T, then $M_1 \prec M_2$.
- If $K \subset L$ are algebraically closed [valued] fields, then $K \prec L$.

Saturated extensions

Saturated models: K is $\aleph_1$-saturated if whenever $A \subset K$ is countable, and $\Sigma(\bar{x})$ is a collection of formulas with parameters in A and which is finitely satisfiable in K, then there is a tuple $\bar{a}$ in K which satisfies all formulas in $\Sigma(\bar{x})$.

In particular an $\aleph_1$-saturated valued field will have the following properties:

- No countable set is cofinal in its value group
- Every countable pseudo-convergent sequence has a pseudo-limit.
- The valuation map has a cross-section (and therefore there is an angular component map)

Every structure has an $\aleph_1$-saturated elementary extension.

Criterion for quantifier-elimination

Let T be a theory, and Δ a set of formulas, which is closed under Boolean combinations. In order to show that every formula is equivalent modulo T to a formula in Δ, it suffices to show the following:
Whenever M and N are two $\aleph_1$-saturated models of T, A, B are countable substructures of M, N respectively, and $f : A \to B$ is a bijection which preserves all formulas in Δ, i.e., for a tuple a in A and a formula $\varphi(x) \in \Delta$,

$$M \models \varphi(a) \iff N \models \varphi(f(a)),$$

if $c \in M$, then f extends to a bijection with domain containing c and which preserves formulas in Δ.