Minimal types in separably closed fields

Zoé Chatzidakis
Université Paris 7, CNRS

Carol Wood
Wesleyan University

§0. Introduction.

In [1], examples of types of U-rank 1 (i.e., minimal types) in the theories of separably closed fields were constructed, en route to displaying certain dimension phenomena. We construct here additional examples with U-rank 1 and of various transcendence degrees over arbitrary separably closed fields. Our examples include ones which are minimal but of infinite transcendence degree, i.e., not thin. Our interest in building new examples was piqued after seeing the role played by minimal types over separably closed fields in Hrushovski's analysis of abelian varieties. In [2] Delon shows, in the setting of separably closed fields of finite degree of imperfection, that there is a Zariski geometry on the set of realizations of any minimal type. Her proof generalizes to the setting of our examples as well. This article is the result of several working sessions between the authors at Wesleyan University and Paris 7, and was completed during the Model Theory of Fields program at MSRI in 1998. We are grateful for the hospitality and support of all three institutions. We thank Elisabeth Bouscaren and Françoise Delon for reading an earlier version of this paper, providing useful suggestions and corrections.

Background on separably closed fields. Let K be a field of characteristic $p > 0$. Then K^p is a subfield of K, isomorphic to K via the Frobenius map $x \mapsto x^p$. Thus K is naturally a K^p-vector space, of dimension either infinite or p^e for some natural number e. The degree of imperfection is taken to be infinite in the first case, and equal to e otherwise.

Fix a prime number p, and let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$ be the language of rings. For each element e of $\mathbf{N} \cup \{\infty\}$, we take SCF_e to be the $\mathcal{L}$-theory of separably closed fields of characteristic p and degree of imperfection e. Ershov [4] showed that each SCF_e is complete, and that all completions of the theory of separably closed fields are of this form. Furthermore, any complete theory of separably closed fields is stable (Wood [5]), and is superstable only when it is the theory of algebraically closed fields (i.e., when $e = 0$).

Let K be a field of characteristic p. We say that $\mathcal{B} \subseteq K$ is p-independent if the set of all p-monomials in $\mathcal{B}$, i.e., monomials of the form $b_1^{i(1)} \cdots b_n^{i(n)}$ with $b_1, \ldots, b_n \in \mathcal{B}$ and $0 \leq i(1), \ldots, i(n) \leq p-1$, is linearly independent in the K^p-vector space K. A maximal p-independent subset $\mathcal{B}$ of K is called a p-basis, and in this case $K = K^p[\mathcal{B}]$.

For each n fix an enumeration $m_{i,n}(\bar{x})$ of the p-monomials $x_1^{i(1)} \cdots x_n^{i(n)}$ with $0 \leq i(1), \ldots, i(n) \leq p-1$, and define the $(n+1)$-ary functions $\lambda_{i,n} : K^n \times K \to K$ as follows: If the n-tuple $\bar{b}$ is not p-independent, or if the $(n+1)$-tuple $(\bar{b}, a)$ is p-independent, then $\lambda_{i,n}(\bar{b}, a) = 0$. Otherwise, the $\lambda_{i,n}(\bar{b}, a)$ satisfy

$$a = \sum_{i=0}^{p^n-1} \lambda_{i,n}(\bar{b}, a)^p m_{i,n}(\bar{b}).$$

Note that these functions depend on the field K, and that the above properties define them uniquely. These functions are also definable in the pure ring language. Consider the language $\mathcal{L}_\lambda = \mathcal{L} \cup \{\lambda_{i,n} \mid n \in \mathbf{N},\ 0 \leq i < p^n\}$, and let $SCF_{e,\lambda}$ be the theory in the language $\mathcal{L}_\lambda$ obtained by adjoining to SCF_e axioms expressing the defining properties of the functions $\lambda_{i,n}$. Then $SCF_{e,\lambda}$ is complete and eliminates quantifiers but does not eliminate imaginaries (Delon [3]).

Fix a separably closed field K. When the degree of imperfection of K is finite, one may fix a p-basis $\{b_1, \ldots, b_e\}$, and only consider the one variable functions $\lambda_{i,e}(b_1, \ldots, b_e, x)$. Then $Th(K)$ eliminates quantifiers and imaginaries in the language $\mathcal{L} \cup \{b_1, \ldots, b_e, \lambda_{i,e}(b_1, \ldots, b_e, -) : 0 \leq i < p^e\}$ (Delon [3]).

Let $A \subseteq K$ be a set. Then the definable closure of A, $dcl(A)$, is the smallest subfield of K containing A and closed under the functions $\lambda_{i,n}$. The algebraic closure of A, $acl(A)$, is the separable closure of $dcl(A)$. (Again, see Delon [3] for details.)

If $K \prec L$ and $\bar{c}$ is a tuple of elements of L we denote the field $dcl(K, \bar{c})$ by $K\langle \bar{c} \rangle$. By quantifier elimination, the type of $\bar{c}$ over K is completely determined by the $\mathcal{L}$-isomorphism type of $K\langle \bar{c} \rangle$ over K.

Let $\mathcal{B} \subset K$ be p-independent, and assume that $c \in L$ is such that $c \in L^{p^n}[\mathcal{B}]$ for every $n \in \mathbf{N}$. Consider the infinite tuple obtained from c by iterating the functions $\lambda_{i,n}(b_1, \ldots, b_n, -)$ for $n \in \mathbf{N}$, $b_1, \ldots, b_n \in \mathcal{B}$, and $0 \leq i < p^n$. This infinite tuple, denoted c_∞, is called the $\mathcal{B}$-tree of c. This tree c_∞ has a natural notion of level, defined by induction: the level of c is 0, and if t is an element of level m, and $n \in \mathbf{N}$, $b_1, \ldots, b_n \in \mathcal{B}$, then for each i, $0 \leq i < p^n$, the element $\lambda_{i,n}(b_1, \ldots, b_n, t)$ has level $m+1$. For $m \in \mathbf{N}$ we also define $c_{\leq m}$ to be the tuple consisting of the elements of c_∞ of level $\leq m$.

Claim. $K\langle c \rangle = K(c_\infty)$.

Proof. It suffices to show that $K(c_\infty)$ is closed under the functions $\lambda_{i,n}$. Extend $\mathcal{B}$ to a p-basis $\mathcal{B}'$ of K. Then the elements of the $\mathcal{B}'$-tree of c all belong to the $\mathcal{B}$-tree of c, and we may therefore assume that $\mathcal{B}$ is a p-basis of K.

By construction of c_∞, for every $n \in \mathbf{N}$ we have: $K(c_{\leq n}) \subseteq (K(c_{\leq n+1}))^p[\mathcal{B}]$. Hence $K(c_\infty) = (K(c_\infty))^p[\mathcal{B}]$, from which we deduce successively: $\mathcal{B}$ is a p-basis of $K(c_\infty)$; L^p and $K(c_\infty)$ are linearly disjoint over $(K(c_\infty))^p$; L is a separable extension of $K(c_\infty)$; $K(c_\infty)$ is an $\mathcal{L}_\lambda$-substructure of L.

Thus the type of c over K is completely described by the $\mathcal{L}$-isomorphism type of c_∞ over K. If we define $tp(c/K)|_m$ to be the subset of $tp(c/K)$ describing the quantifier-free $\mathcal{L}$-type of the tuple $c_{\leq m}$ over K, then $Th(K) \cup \bigcup_m tp(c/K)|_m \vdash tp(c/K)$. We say that a partial type $q(x_1, \ldots, x_n)$ is complete up to level m if q implies a complete quantifier-free $\mathcal{L}$-type over K in the variables $(x_{1,\leq m}, \ldots, x_{n,\leq m})$. By definition, if q is a complete type over K then $q|_m$ is complete up to level m.

§1. Constructions of types of arbitrary transcendence degree.

Conventions. Throughout this section we fix p and we consider types over K a countable model of SCF_e for nonzero e. We further fix an element u of $K \setminus K^p$, and assume L to be a sufficiently saturated elementary extension of K.

In the constructions of this section, all types under consideration will be p-dependent on u, i.e., will lie in $L^{p^n}[u]$ for every $n \in \mathbf{N}$. Thus an element will be given by its $\{u\}$-tree. For ease of notation, we write a realization a of such a type as

$$a = \sum_{i=0}^{p-1} a_i^p u^i.$$

In particular, the subscript 0 corresponds to the coefficient of 1 (= u^0) and the index 1 corresponds to the coefficient of u. We concatenate subscripts as we iterate this process, for example

$$a_\sigma = \sum_{i=0}^{p-1} a_{\sigma i}^p u^i.$$

For $n \in \mathbf{N}$, we let p^n denote the set of sequences of length n of elements of $\{0, \dots, p-1\}$, and set $p^{\leq n} = \bigcup_{0 \leq m \leq n} p^m$, $p^{<\omega} = \bigcup_{m \in \mathbf{N}} p^m$. Then $K\langle a \rangle = K(a_\sigma \mid \sigma \in p^{<\omega})$. If $q = tp(a/K)$, then $tr.deg.(q)$ (the transcendence degree of q) denotes the (field) transcendence degree of $K\langle a \rangle$ over K. We denote by $a_{\leq m}$ the tuple of elements defined by stage m, i.e., the tuple $(a_\sigma)_{\sigma \in p^{\leq m}}$, and by $tr.deg.(q|_m)$ the transcendence degree of $K(a_{\leq m})$ over K.
We make repeated use of the following easy consequence of the nature of definable closure:

Fact. If there exist v_0 and $v_1 \in L$ such that $v_0^p + v_1^p u \in K$, then $v_0, v_1 \in K$.

By working in the setting of a fixed u, we see easily that the constructions of [1] are valid for separably closed fields of any degree of imperfection, including the infinite case. Indeed, over any subfield k_0 of L containing u, [1] provides an infinite family of pairwise orthogonal types of U-rank 1 for any (nonzero) value of e.

Lemma 1 (cf. Th. 3.2 of [1]) Assume that $a, b, a_1, b_1 \in L$ satisfy

$$\begin{aligned} a &= a_1^p + b^p u \\ b &= b_1^p + a^p u. \end{aligned}$$

Then
(1) $K(a, b, a_1, b_1)$ is separably algebraic over $K(a_1, b_1)$.
(2) (a, b) and (a_1, b_1) are equi-algebraic over K.
(3) $a \in K \iff b \in K \iff K(a, b, a_1, b_1) = K$.

Proof. (1) The equation $X = a_1^p + (b_1^p + X^p u)^p u$ is separable in X, and has a for solution. Hence a is separable over $K(a_1, b_1)$, and similarly for $b = b_1^p + a^p u$.

(2) Clearly a_1 and b_1 are algebraic over $K(a, b)$. By (1) we know that a and b are algebraic over $K(a_1, b_1)$.

(3) Assume that $a \in K$. By the Fact above, a_1 and b are also in K. Thus b_1 $(= (b - a^p u)^{1/p}) \in K$. Similarly, $b \in K$ implies that a, a_1 and b_1 are in K.

Notation. Fix an integer n, and consider the system of equations $\Sigma(X_{i,j})$, $1 \leq i \leq n$, $1 \leq j \leq 2n-1$, defined by the equations (k) below, for $k = 1, \dots, 2n-2$, where we adopt the convention that $X_{\ell,j}$ stands for the variable $X_{\ell-n,j}$ whenever $\ell > n$:

$$(k) \qquad \begin{aligned} X_{k,k} &= X_{k,k+1}^p + X_{k+1,k}^p u \\ X_{k+1,k} &= X_{k+1,k+1}^p + X_{k,k}^p u, \\ X_{i,k} &= X_{i,k+1}^p && \text{if } i \neq k, k+1, k-n, k-n+1. \end{aligned}$$

Lemma 2. With notation as above,

(1) $\Sigma(X_{i,j})$ is consistent.

(2) If $(a_{i,j})$ is a solution of $\Sigma(X_{i,j})$ in L, then $K(a_{i,j})_{i,j}$ is separably algebraic over $K(a_{1,2n-1},\dots,a_{n,2n-1})$.

(3) If, in addition, $a_{i,1}\in K$ for some i, then all the $a_{i,j}$'s lie in K.

Proof. (1) Let $a_{1,2n-1},\dots,a_{n,2n-1}$ be n elements of L. By reverse induction on k and by Lemma 1, each system (k) is consistent.

(2) By Lemma 1, for $0\le k\le 2n-2$ the elements $a_{i,k}$, $i=1,\dots,n$, are separably algebraic over $K(a_{1,k+1},\dots,a_{n,k+1})$. Hence the tuple $(a_{i,j})$ is separably algebraic over $K(a_{1,2n-1},\dots,a_{n,2n-1})$.

(3) Assume that $a_{i,1}\in K$. Then the systems of equations (1) through $(i-2)$ give: $a_{i,i-1}=a_{i,1}^{1/p^{i-2}}\in K$.

By Lemma 1 applied to the equations $(i-1)$, we deduce from $a_{i,i-1}\in K$ that $a_{i-1,i-1},a_{i-1,i}$ and $a_{i,i}$ are in K. Proceeding in the same manner with the equations $(i),\dots,(2n-2)$, we obtain successively that for $k=i-1,\dots,2n-2$, the elements $a_{k+1,k},a_{k,k+1}$ and $a_{k+1,k+1}$ are in K. For k ranging from $i-1$ to $2n-2$, we have $a_{k,k+1}\in K$. (Recall our convention that $a_{\ell,j}$ stands for $a_{\ell-n,j}$ when $\ell>n$). Therefore $a_{k,2n-1}=a_{k,k+1}^{1/p^{2n-k-2}}\in K$. All possible first subscripts occur in the range of values we have for k, and so $a_{1,2n-1},\dots,a_{n,2n-1}\in K$. Since K is separably closed, assertion (2) suffices to show that the $a_{i,j}$'s lie in K.

Definition. For n as above and $1\le j\le 2n-1$, let $\eta_j\in p^{j-1}$ be the sequence of zeros of length $j-1$ (so $\eta_1=\emptyset$), and let $\Phi_n(z_1,\dots,z_n)$ be the partial type given by: the tuple $(z_{i,\eta_j})_{1\le i\le n,1\le j\le 2n-1}$ solves the system $\Sigma(X_{i,j})$.

The resulting Φ_n is consistent, and becomes complete up to level $2n-2$ once one requires that $z_1,\dots,z_n$ be algebraically independent over K.

Lemma 3. Let $p|_m$ be a 1-type complete up to level m, and assume that $\sigma(1),\dots,\sigma(k)\in p^m$ are such that if a realizes p then $\{a_{\sigma(1)},\dots,a_{\sigma(k)}\}$ is a separating transcendence basis of $K(a_{\le m})$ over K.

Given a k-type $q|_n$ complete up to level n and realizable by a k-tuple of algebraically independent elements, consider the partial type $r(x)=p|_m(x)\cup q|_n(x_{\sigma(1)},\dots,x_{\sigma(k)})$. Then r is consistent, and complete up to level $m+n$.

Proof. Let $b_1,\dots,b_k\in L$ realize $q|_n$. Choose $a_{\le m}$ (in some algebraic closure of L) such that $a_{\sigma(j)}=b_j$ for $j=1,\dots,k$, and such that for any c realizing $p|_m$, the tuples $a_{\le m}$ and $c_{\le m}$ have the same quantifier-free $\mathcal{L}$-type over K.

Then $a_{\le m}$ is separably algebraic over $K(b_1,\dots,b_k)$, and therefore is in L. Hence, $a_{\le m+n}\in K(b_{1,\le n},\dots,b_{k,\le n},a_{\le m})$, which gives the result.

Theorem 1. For any $d\ge 1$, there exists a type over K of U-rank 1 and of transcendence degree d.

Proof. By results of [1], we may choose $p_1,\dots,p_d$ to be pairwise orthogonal 1-types over K, each of U-rank 1 and transcendence degree 1, and such that all elements definable over realizations of $p_1, \dots, p_d$ are p-dependent on u. For $d = 1$ there is nothing new to be proved. We assume $d > 1$.

Let $\bar{Y}$ be a countably infinite tuple of variables, and let $X_\infty = \{X_\sigma \mid \sigma \in p^{<\omega}\}$, $X_{\leq n} = \{X_\sigma \mid \sigma \in p^{\leq n}\}$. Fix an enumeration f_n, $n \in \mathbf{N}$, of the elements of $K[X_\infty, \bar{Y}]$ such that $f_n \in K[X_{\leq n}, \bar{Y}]$ for every n.

We will build the type q by induction on n. We assume that at stage n we have built a type $q_n = q_n|_{N(n)}$ complete up to level $N(n)$, and satisfying:

(a) $tr.deg.(q_n) = d$ and q_n extends q_{n-1},

(b) for any tuple $\bar{c}$ from L and a realizing q_n, if $f_{n-1}(a_{\leq n-1}, \bar{c}) = 0$, then either the polynomial $f_{n-1}(X_{\leq n-1}, \bar{c})$ vanishes on all realizations of q_n, or $a \in acl(K, \bar{c})$.

For $n = 0$, the condition is vacuous. Choose $N(0)$ so that $p^{N(0)} \geq d$. Let $\sigma(1), \dots, \sigma(d)$ be distinct elements of $p^{N(0)}$, and consider the partial type q_0 which says that $x_{\sigma(1)}, \dots, x_{\sigma(d)}$ are algebraically independent over K, and that $x_\sigma = 0$ for $\sigma \in p^{N(0)} \setminus \{\sigma(1), \dots, \sigma(d)\}$. Then q_0 is complete up to level $N(0)$.

Assume that we have built q_n satisfying the induction hypothesis. We will now build q_{n+1}, in two steps. Choose $\tau(1), \dots, \tau(d) \in p^{N(n)}$ such that if a is a realization of q_n then $\{a_{\tau(1)}, \dots, a_{\tau(d)}\}$ is a separating transcendence basis of $K(a_{\leq N(n)})$ over K.

Step 1.

Consider $f_n(X_{\leq n}, \bar{Y})$. By our choice of $\tau(1), \dots, \tau(d)$, we know that $a_{\leq N(n)}$ is separably algebraic over $K(a_{\tau(1)}, \dots, a_{\tau(d)})$. Working modulo the theory of fields and using compactness, there are polynomials $h_j \in K[Z_1, \dots, Z_d, \bar{Y}]$, $j = 1, \dots, m$, such that if a realizes q_n and $\bar{c}$ is any tuple such that $f_n(X_{\leq n}, \bar{c})$ does not vanish on all realizations of q_n, then

$$f_n(a_{\leq n}, \bar{c}) = 0 \to \bigvee_{j=1}^{m} (h_j(a_{\tau(1)}, \dots, a_{\tau(d)}, \bar{c}) = 0 \wedge h_j(Z_1, \dots, Z_d, \bar{c}) \not\equiv 0).$$

Consider now the d-type $p_1(z_1) \cup \dots \cup p_d(z_d)$. Since the p_i's are pairwise orthogonal and of rank 1, for any tuple $\bar{c}$ in L we have

$$p_1(z_1) \cup \dots \cup p_d(z_d) \cup \{\bigvee_{j=1}^{m} (h_j(z_1, \dots, z_d, \bar{c}) = 0 \wedge h_j(Z_1, \dots, Z_d, \bar{c}) \not\equiv 0)\}$$
$$\to \bigvee_{i=1}^{d} \text{“} z_i \in acl(K, \bar{c}) \text{”}.$$

Note that “$z \in acl(K, \bar{c})$” is equivalent modulo $Th(K)$ to an infinite disjunction of formulas. By compactness, there is an integer M and a formula $\theta(z, \bar{y})$, with parameters in K, such that for every tuple $\bar{c}'$ in L, the formula $\theta(z, \bar{c}')$ defines a finite (possibly empty) set, and such that

$$p_1|_M(z_1)\cup\dots\cup p_d|_M(z_d)\cup\{\bigvee_{j=1}^{m}(h_j(z_1,\dots,z_d,\bar y)=0\wedge h_j(Z_1,\dots,Z_d,\bar y)\not\equiv 0)\}$$
$$\to\bigvee_{i=1}^{d}\theta(z_i,\bar y).$$

Define $q_n''=q_n\cup p_1|_M(x_{\tau(1)})\cup\dots\cup p_d|_M(x_{\tau(d)})$. By Lemma 3, this partial type is consistent and complete up to level $N(n)+M$. For $i=1,\dots,d$, choose $\rho(i)\in p^{(N(n)+M)}$ such that a_σ is separably algebraic over $K(a_{\rho(i)})$ whenever a realizes q_n'' and $\sigma\in p^{(N(n)+M)}$ extends $\tau(i)$. Note that this implies that $a_{\tau(i)}$ is separably algebraic over $K(a_{\rho(i)})$, and that $\{a_{\rho(1)},\dots,a_{\rho(d)}\}$ is a separating transcendence basis of $K(a_{\le N(n)+M})$ over K .

Step 2.

Let $N(n+1)=N(n)+M+2d-2$, and $q_{n+1}=q_n''\cup\Phi_d(x_{\rho(1)},\dots,x_{\rho(d)})$. By Lemma 3, q_{n+1} is consistent and complete up to level $N(n+1)$.

We need to show that q_{n+1} satisfies the induction hypothesis. Let a realize q_{n+1} and assume that $f_n(a_{\le n},\bar c)=0$ for some tuple $\bar c$ such that $f_n(X_{\le n},\bar c)$ does not vanish on all realizations of q_{n+1}. Then for some $j=1,\dots,m$, $h_j(a_{\tau(1)},\dots,a_{\tau(d)},\bar c)=0$ and $h_j(Z_1,\dots,Z_d,\bar c)\not\equiv 0$. By our choice of M, there is i such that $\theta(a_{\tau(i)},\bar c)$ holds. Thus $a_{\tau(i)}$ lies in $acl(K,\bar c)$, and therefore so does $a_{\rho(i)}$, since $a_{\tau(i)}$ and $a_{\rho(i)}$ are equi-algebraic over K. From $\Phi_d(a_{\rho(1)},\dots,a_{\rho(d)})$ and Lemma 2 we then deduce that $a_{\rho(1),\le 2d-2},\dots,a_{\rho(d),\le 2d-2}\in acl(K,\bar c)$. Thus $a_{\le N(n+1)}\subseteq acl(K,\bar c)$.

Assume now that a realizes $q=\bigcup_n q_n$. Then $tr.deg.(K\langle a\rangle/K)=d$. Assume that $\bar b$ is a tuple of elements of L, and that $tp(a/K\langle\bar b\rangle)$ forks over K. Then there is a polynomial $f\in K[X_\infty,\bar Y]$ and a tuple $\bar c$ from $K\langle\bar b\rangle$ such that $f(a_\infty,\bar c)=0$ and $f(X_\infty,\bar c)$ does not vanish identically on the set of realizations of q. Then $f=f_n$ for some n, and by the definition of q_{n+1}, we obtain that $a\in acl(K,\bar b)$. This shows that $U(q)=1$.

Theorem 2. There exists a type of U-rank 1 and of infinite transcendence degree over K.

Proof. Conventions and notation are as before. By results of [1], we may select an infinite sequence $(p_n)_{n\in\mathbf{N}}$ of types of U-rank and transcendence degree 1, which are pairwise orthogonal.

Let $\bar Y$ be a countably infinite tuple of variables, and fix an enumeration f_n, $n\in\mathbf{N}$, of the elements of $K[X_\infty,\bar Y]$ such that $f_n\in K[X_{\le n},\bar Y]$ for every n.

We will build the type q by induction on n. We assume that at stage n we have built a type $q_n=q|_{N(n)}$ complete up to level $N(n)$, and satisfying:

(a) $tr.deg.(q_n)=n$ and q_n extends q_{n-1},

(b) for any tuple $\bar c$ from L and a realizing q_n, if $f_{n-1}(a_{\le n-1},\bar c)=0$, then either the polynomial $f_{n-1}(X_{\le n-1},\bar c)$ vanishes on all realizations of q_n, or $a\in acl(K,\bar c)$.

For $n=1$, let $N(1)=0$, and q_1 the type of a transcendental element. Then q_1 satisfies trivially the induction hypothesis.

Assume that we have built q_n satisfying the condition. We will now build q_{n+1}, in three steps.

Step 1. Increasing the transcendence degree.

Let $\sigma(1), \ldots, \sigma(n) \in p^{N(n)}$ be such that whenever a realizes q_n then $\{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}$ is a separating transcendence basis of $K(a_{\leq N(n)})$ over K. Consider the type

$$q'_n = q_n \cup \{x_{\sigma(1)} = x^p_{\sigma(1)\,0}, \ldots, x_{\sigma(n-1)} = x^p_{\sigma(n-1)\,0}, x_{\sigma(n)} = x^p_{\sigma(n)\,0} + x^p_{\sigma(n)\,1}u\}.$$

By Lemma 3, this type is consistent and complete up to level $N(n)+1$. Furthermore, if a realizes it then $\{a_{\sigma(1)\,0}, \ldots, a_{\sigma(n-1)\,0}, a_{\sigma(n)\,0}, a_{\sigma(n)\,1}\}$ is a separating transcendence basis of $K(a_{\leq N(n)+1})$ over K.

Steps 2 and 3. Apply Steps 1 and 2 of the preceding construction to the partial type q'_n, with $d = n+1$, to get a type q_{n+1} satisfying the induction hypothesis.

We conclude as in the preceding construction that $U(q) = 1$ for $q = \bigcup_n q_n$, while Step 1 of the present construction guarantees that q has infinite transcendence degree.

§2. Additional remarks on the case of infinite degree of imperfection.

Let $\{v_n \colon n \in \mathbf{N}\}$, be a p-independent subset of K, with $v_0 = u$. The construction in Theorem 2 can easily be modified to give a type q of U-rank 1 over K, and such that if a realizes q, then all the v_i's are definable over $\{a, u\}$. In order to incorporate the infinitely many p-independent elements in this type of U-rank 1, we no longer assume, as in §1, that definable elements in our types be p-dependent on u. It suffices to change step 1 in Theorem 2 as follows: define

$$q'_n = q_n \cup \{x_{\sigma(1)} = x^p_{\sigma(1)\,0} + v^p_n u, x_{\sigma(n)} = x^p_{\sigma(n)\,0} + x^p_{\sigma(n)\,1}u, x_{\sigma(i)} = x^p_{\sigma(i)\,0} \text{ for } i = 2, \ldots, n-1\}.$$

Then apply steps 1 and 2 of the first construction to the type q'_n and the variables $x_{\sigma(1)\,0}, \ldots, x_{\sigma(n-1)\,0}, x_{\sigma(n)\,0}, x_{\sigma(n)\,1}$ (with $d = n+1$).

This construction shows how infinitely many p-basis elements from K can be defined over $\{a, u\}$, where a realizes a minimal type over K. In contrast, the next Lemma shows one can never define an element p-independent over K via a minimal type.

Lemma 4. Let a be an element of U-rank 1 over K. Then $K\langle a\rangle \subseteq KL^{p^n}$ for every n, where $K\langle a\rangle$ is the definable closure of a over K.

Proof. If this were false, then there would be an element $b \in K\langle a\rangle$, such that b is p-independent from K over L^p. Let $c \in L^{p^\infty}$ be transcendental over $K\langle b\rangle = K(b)$, and consider $tp(b/K(bc))$. This type forks over K, but is non-algebraic over K, which shows that $tp(b/K)$ does not have U-rank 1. Since b is definable over $K(a)$, a cannot have U-rank 1 either.

Delon's work showing that the set of realizations of a minimal type has a Zariski geometry generalizes to the infinite invariant case. We will indicate below the changes which are needed; the interested reader should then be able to follow the arguments given by Delon [2] in this broadened context.

We fix a model K of SCF_∞ with a countable p-basis $\mathcal{B}$, and a saturated elementary extension L of K. We also fix a minimal type q over K, and let Q be the set of its realizations in L. By Lemma 4, $Q \subset KL^{p^n}$ $(= L^{p^n}[\mathcal{B}])$ for every $n \in \mathbf{N}$. By the discussion in §0 about quantifier elimination, the type over K of an n-tuple $(a_1, \dots, a_n)$ of elements of Q is given by the $\mathcal{L}$-isomorphism type over K of the associated n-tuple of $\mathcal{B}$-trees $(a_{1,\infty}, \dots, a_{n,\infty})$.

We want to define a topology on the cartesian powers of Q. Observe that in general an equation of the type $\lambda_{i,n}(a_1, \dots, a_n, x) = 0$ defines a set containing all elements which are not in $L^p[a_1, \dots, a_n]$, and thus is much too large. Here Lemma 4 allows us, in working inside Q^n, to circumvent this problem. Fix an enumeration $\mathcal{B} = \{b_1, b_2, \dots\}$. Since for every $m \geq 0$ and $a \in Q$, the tuple $a_{\leq m}$ contains only finitely many distinct elements, we may fix natural numbers $n(1) \leq n(2) \leq \dots$ such that $Q \subset L^{p^m}[b_1, \dots, b_{n(m)}]$ for every $m > 0$.

We define by induction on d the notion of a $\{Q, \mathcal{B}\}$-allowed term (in the variable x) of level d as follows: x is $\{Q, \mathcal{B}\}$-allowed of level 0; if $t(x)$ is a $\{Q, \mathcal{B}\}$-allowed term of level d and $0 \leq i < p^{n(d+1)}$, then $\lambda_{i,n(d+1)}(b_1, \dots, b_{n(d+1)}, t(x))$ is $\{Q, \mathcal{B}\}$-allowed of level $d+1$.

Claim. Let $D \subset L^n$ be a K-definable set (in the language $\mathcal{L}$). Then $D \cap Q^n$ is a Boolean combination of sets defined by equations of the form $f(t_1(x_1), \dots, t_n(x_n)) = 0$, where f is a polynomial over K and each $t_i(x_i)$ is a tuple of $\{Q, \mathcal{B}\}$-allowed terms in the variable x_i.

Proof. If $a_1, \dots, a_n \in Q$, the type of $(a_1, \dots, a_n)$ over K is determined by the $\mathcal{L}$-isomorphism type over K of the n-tuple of $\mathcal{B}$-trees $(a_{1,\infty}, \dots, a_{n,\infty})$. Every element of $a_{i,\infty}$ is either 0, or the evaluation of some $\{Q, \mathcal{B}\}$-allowed term at a_i. The result follows, using compactness and the saturation of L, and noting that the $\mathcal{L}$-isomorphism type over K of a tuple is given by the set of all polynomial equalities and inequalities with coefficients in K which are satisfied by the tuple.

Now to define the topology. Fix $n > 0$. A closed subset of Q^n is a set defined by a (possibly infinite) conjunction of formulas of the form $f(t_1(x_1), \dots, t_n(x_n)) = 0$, where f is a polynomial over K and each $t_i(x_i)$ is a tuple of $\{Q, \mathcal{B}\}$-allowed terms in the variable x_i.

We then define $K[X_\infty]$ to be the polynomial ring over K in the variables X_t for t a $\{Q, \mathcal{B}\}$-allowed term in x. For $m \in \mathbf{N}$, we define $K[X_{\leq m}]$ to be the polynomial ring over K in the variables X_t for t a $\{Q, \mathcal{B}\}$-allowed term in x of level $\leq m$. Similarly for $K[X_{1,\infty}, \dots, X_{n,\infty}]$. Note that for every $m \geq 0$, $K[X_{1,\leq m}, \dots, X_{n,\leq m}]$ is finitely generated over K.

Theorem 3. Let K be a separably closed field. The set of realizations of a minimal type over K forms a Zariski geometry.

This theorem is due to Hrushovski when the type has finite transcendence degree over K (" thin types") and when e is finite. Delon's proof extends the finite e case to the non-thin setting, i.e, allowing the type to have infinite field transcendence degree over K. With no additional ideas or definitions beyond what we have given, and with only very minor modifications, the proofs of Section 5 of Delon [2] go through to give Theorem 3 for the infinite invariant case. We leave the details to the interested reader.

References.

[1] Z. Chatzidakis, G. Cherlin, S. Shelah, G. Srour, C. Wood, Orthogonality of types in separably closed fields, in: Classification theory, Proc. of the 1985 Chicago Conference, Springer Verlag, Lecture Note 1292.

[2] F. Delon, Separably closed fields, in Model Theory and Algebraic Geometry, E. Bouscaren ed., Springer Verlag, Lecture Note 1696, Berlin 1998.

[3] F. Delon, Idéaux et types sur les corps séparablement clos, Supplément au Bull. de la S.M.F, Mémoire 33, Tome 116, 1988.

[4] Y. Ershov, Fields with a solvable theory, Doklady 174 (1967), pp. 19-20; English translation in Soviet Mathematics 8 (1967), pp. 575-576.

[5] C. Wood, Notes on the stability of separably closed fields, Journal of Symbolic Logic 44 (1979), pp. 412-416.