Non-uniqueness of Turbulent Solutions of Navier-Stokes Equation in Dimension $N = 3$

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May 30, 2024

Abstract

In this article, we review the conditional uniqueness property for Navier-Stokes equation, and follow the paper [ABC22] to develop a non-uniqueness theorem of 3-dimensional case with source term. We study the linearized operator of the self-similar equation, then reduce it gradually to a 2-dimensional Euler equation. The construction of the stationary unstable velocity field relies fundamentally on Vishik's unstable profile in [Vis18a] and [Vis18b], which we then lift into 3-dimensional axisymmetric coordinate.

The main improvement compared to the original paper is that we compose the proofs in an uniform way, viewing the linearized operators of perturbations of skew-adjoint operators, and specify the omitted estimates in the original paper. The case without exterior force is still an open problem.

1 Introduction

In this article, we are interested in the following **Navier-Stokes Equation** System with source term in dimension $N = 2, 3$, which describes the evolution of local velocity with respect to time of a viscous non-compressible fluid.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u = f - \nabla p, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ \nabla \cdot u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u|_{t=0} = u_0, & x \in \mathbb{R}^N \end{cases} \tag{N-S}$$

Where $u : \mathbb{R}_{\geq 0} \times \mathbb{R}^N \to \mathbb{R}^N$ is a N-dimensional vector field, representing the local velocity field of a give point in time-space, $p : \mathbb{R}_{\geq 0} \times \mathbb{R}^N \to \mathbb{R}$ is the pressure function, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^N \to \mathbb{R}^N$ is the given external force applied to the fluid, and $u_0 : \mathbb{R}^N \to \mathbb{R}^N$ is the given initial velocity field at time $t = 0$. The parameter $\nu > 0$ is called the viscosity factor, illustrates how much resistance the fluid produces when being deformed at a given rate, the term $-\nu\Delta$ gives the regularity of the system. When there is no viscosity, i.e. $\nu = 0$, then we obtain the **Euler equation** with source term:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = f - \nabla p, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^N \\ \nabla\cdot u = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^N \\ u|_{t=0} = u_0, & x\in\mathbb{R}^N \end{cases} \tag{Euler}$$

Observe that by proper rescaling:

$$u'(t,x) := \nu u(\nu t,x), \quad p'(t,x) := \nu^2 p(\nu t,x), \quad f'(t,x) := \nu^2 f(\nu t,x)$$

To ease the notion, in the following sections, we assume that $\nu \equiv 1$, and the inequality constant may vary from line to line.

Since u is divergence free, we apply the divergence operator to the first equation of system (N-S):

$$\Delta p = \nabla\cdot f - \nabla\cdot((u\cdot\nabla)u) \tag{1.1}$$

This equation draws the dependence of p on u and f, so, we only need to resolve u in the original equation.

Before starting the main discussion, we have to introduce some notions:

1 For two vectors of the same dimension N,

$$u^{(\alpha)} = (u_1^{(\alpha)}, u_2^{(\alpha)}, \cdots, u_N^{(\alpha)}) \in \mathbb{R}^N, \qquad \alpha = 1,2$$

We define the tensor product of these two vectors as a $N\times N$ matrix :

$$u^{(1)}\otimes u^{(2)} := (u^{(1)})^T(u^{(2)}) = (u_i^{(1)}u_j^{(2)})_{1\le i,j\le N} \in \mathbb{R}^{N\times N}$$

2 For two $N\times N$ matrices

$$A^{(\alpha)} = (A_{ij}^{(\alpha)})_{1\le i,j\le N} \in \mathbb{R}^{N\times N}, \qquad \alpha = 1,2$$

We define the contraction product of them:

$$A^{(1)} : A^{(1)} := \sum_{1\le i,j\le N} A_{ij}^{(1)}A_{ij}^{(2)}$$

With the notion above, we observe that for all $u \in C^\infty(\mathbb{R}^N)^N$ such that $\nabla\cdot u = 0$, we have:

$$(u\cdot\nabla)u = \nabla\cdot(u\otimes u) \tag{1.2}$$

Finally, for

$$u \in C_w([0,T], L^2(\mathbb{R}^N)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^N))$$

We define the kinetic energy:

$$E_u(t) := \frac{1}{2}\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \nu\|u\|_{L^2([0,t],\dot{H}^1(\mathbb{R}^N))}^2, \quad \forall\ t\in[0,T] \tag{1.3}$$

We follow the approach of Jean Leray's paper [Ler34], and firstly define the solution we want to consider:

Definition 1.1 (Turbulent Solution)**.** *For all times* $T > 0$*, let* $u_0 \in L^2(\mathbb{R}^N)^N$*, such that* $\nabla \cdot u_0 = 0$*,* $f \in L^1([0,T], L^2(\mathbb{R}^N)^N)$*, we say that* $u \in \mathcal{S}'([0,T] \times \mathbb{R}^N)^N$ *is a turbulent solution of equation (N-S) on the time interval* $[0,T]$*, if:*

$$u \in C_w([0,T], L^2(\mathbb{R}^N)^N) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^N)^N)$$

For all functions $\psi \in \mathcal{D}([0,T] \times \mathbb{R}^N)$*, for all* $t \in [0,T]$*, we have:*

$$\int_{\mathbb{R}^N} \nabla \psi(t,x) \cdot u(t,x) \mathrm{d}x = 0$$

And for all function $\phi \in C^1([0,T], H^1(\mathbb{R}^N)^N)$*,such that* $\nabla \cdot \phi = 0$*, we have:*

$$\begin{aligned}\int_{\mathbb{R}^N} u(T,x)\phi(T,x)\mathrm{d}x &- \int_0^T \int_{\mathbb{R}^N} u(s,x) \cdot \partial_t \phi(s,x) \mathrm{d}x\mathrm{d}s \\ &= -\nu \int_0^T \int_{\mathbb{R}^N} \nabla u(s,x) : \nabla \phi(s,x) \mathrm{d}x\mathrm{d}s + \int_{\mathbb{R}^N} u_0(x)\phi(0,x)\mathrm{d}x \\ &\quad + \int_0^T \int_{\mathbb{R}^N} u \otimes u(s,x) : \nabla \phi(s,x) \mathrm{d}x\mathrm{d}s + \int_0^T \int_{\mathbb{R}^N} f(s,x) \cdot \phi(s,x) \mathrm{d}x\mathrm{d}s\end{aligned} \tag{1.4}$$

Finally, the solution satisfies the following Energy Inequality:

$$E_u(t) \leq \frac{1}{2} \|u_0\|^2_{L^2(\mathbb{R}^N)} + \int_0^t \int_{\mathbb{R}^N} f \cdot u \ \mathrm{d}x\mathrm{d}s \qquad \forall\ t \in [0,T] \tag{1.5}$$

Remark 1.1. *1 In fact, regarding the space where u belongs to, we could define turbulent solution for*

$$f \in L^1([0,T], L^2(\mathbb{R}^N)^N) + L^2([0,T], H^{-1}(\mathbb{R}^N)^N)$$

2 The Energy Inequality describes that a part of the kinetic energy is transformed into internal energy by viscosity.

Applying Cauchy-Schwarz Inequality, we have a weaker version of energy inequality:

$$E_u(t) \lesssim_T \|u_0\|^2_{L^2(\mathbb{R}^N)} + \|f\|^2_{L^1([0,t], L^2(\mathbb{R}^N))} \qquad \forall\ t \in [0,T] \tag{1.6}$$

[Ler34] have provided the existence of turbulent solution:

Proposition 1.1. *Let* $N = 2, 3$*,* $u_0 \in L^2(\mathbb{R}^N)^N$*,* $f \in L^1([0,T], L^2(\mathbb{R}^N)^N) + L^2([0,T], H^{-1}(\mathbb{R}^N)^N)$*, then for all* $T > 0$*, there exists a turbulent solution* u *of the system (N-S) on time interval* $[0,T]$*.*

Given a turbulent solution u of system (N-S), with initial data u_0 and source term f, the pressure p can be derived by formula (1.1). For $\lambda > 0$, we consider a proper rescaling for the system:

$$\begin{aligned} u_\lambda(t,x) &:= \lambda u(\lambda^2 t, \lambda x), \qquad & f_\lambda(t,x) &:= \lambda^3 f(\lambda^2 t, \lambda x) \\ (u_0)_\lambda(x) &:= \lambda u_0(\lambda x) & p_\lambda(t,x) &:= \lambda^2 p(\lambda^2 t, \lambda x) \end{aligned} \tag{1.7}$$

Also gives a solution of the system (N-S), and the uniqueness of solution relies fundamentally on its behavior under rescaling. We call the norms which are invariant under the proper rescaling (1.7) to be **critical norms**, these norms draws a line between the **subcritical** cases where we work on less weaker norms and could expect well-posedness results and the **supercritical** cases where the norms are more regular, potentially lead to non-uniqueness.

For the uniqueness of the turbulent solution, [LP59] has proved it for dimension $N = 2$, which essentially relies on the fact that proper rescaling does not change the corresponding $L^\infty_t L^2_x$ and $L^2_t \dot{H}^1_x$ norm of the solution. We will cover it in the latter section.

The problem for the uniqueness for general turbulent solutions in dimension $N = 3$ has remained long open. Leray's work has proven the "strong-weak" uniqueness of the turbulent solution, i.e. if the solution is strong, which satisfies a certain regularity condition, then all turbulent solutions coincide.

In this article, we consider the equation (N-S) with source term not regular enough to admit a strong solution, then, the "strong-weak" principle cannot be applied. We follow the lines of [ABC22], to construct a source term which gives a counter-example for the uniqueness in dimension $N = 3$:

Theorem 1.1. *Assume $\nu = 1$, then there exists a time $T > 0$, an external source term $f \in L^1([0,T], L^2(\mathbb{R}^N)^N)$, and two distinct turbulent solutions u and $\tilde{u}$ to the system (N-S) on time interval $[0,T]$ with initial condition $u_0 \equiv 0$.*

2 Previous Results: A Revisit

In this section, we want to review the uniqueness properties and their conditions of the turbulent solution in dimension $N = 2, 3$. The core difference lies in the Sobolev embedding and norm invariance under proper rescaling.

2.1 Case 2D: Unconditional Uniqueness

Proposition 2.1. *Given initial data $u_0 \in L^2(\mathbb{R}^2)^2$, $f \in L^1([0,T], \dot{H}^{-1}(\mathbb{R}^2)^2)$, then there exists an unique turbulent solution u of the system (N-S) defined on time interval $[0,T]$.*

Proof. Assume that there exists two turbulent solution (u, p_u) and (v, p_v), where p denotes the associated pressure function, we define:

$$w := u - v \in L^\infty([0,T], L^2(\mathbb{R}^2)^2) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^2)^2)$$

Then, w satisfies the system in sense of distribution:

$$\begin{cases} \partial_t w - \Delta w = -\nabla \cdot (u \otimes u - v \otimes v) - \nabla(p_u - p_v) \\ \nabla \cdot w = 0 \\ w|_{t=0} \equiv 0 \end{cases}$$

We want to eliminate the pressure term, thus consider the projection operator onto the divergence null subspace:

$$\begin{aligned}\mathbb{P}: \quad & \mathcal{S}'(\mathbb{R}^N)^N && \to && \mathcal{S}'(\mathbb{R}^N)^N \\ & f = (f_1, \cdots, f_N) && \mapsto && \left(\delta_{i,j} - \frac{D_i D_j}{|D|^2}\right)_{i,j=1}^N f\end{aligned} \tag{2.1}$$

Where we use the Fourier multiplier convention:

$$\left(\delta_{i,j} - \frac{D_i D_j}{|D|^2}\right)_{i,j=1}^N f = \mathcal{F}^{-1}\left[\left(\delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2}\right)_{i,j=1}^N \hat{f}\right]$$

Then, applying the projection operator on to the equation, we obtain:

$$\begin{cases}\partial_t w - \Delta w = -\mathbb{P}(\nabla \cdot (u \otimes u - v \otimes v)) \\ \nabla \cdot w = 0 \\ w|_{t=0} \equiv 0\end{cases} \tag{2.2}$$

Evidently a heat equation with source term and initial condition.

Estimate the norm of the non-linear term, we sightly abuse the notions:

$$\begin{aligned}\|\mathbb{P}(\nabla \cdot (u \otimes u - v \otimes v))\|_{L_t^2 \dot{H}_x^{-1}} &\leq \|u \otimes u - v \otimes v\|_{L_t^2 L_x^2} \\ &\leq \|u \otimes (u - v)\|_{L_t^2 L_x^2} + \|(u - v) \otimes v\|_{L_t^2 L_x^2} \\ &\leq (\|u\|_{L_t^4 L_x^4} + \|v\|_{L_t^4 L_x^4})\|w\|_{L_t^4 L_x^4} \\ &\lesssim (\|u\|_{L_t^4 \dot{H}_x^{\frac{1}{2}}} + \|v\|_{L_t^4 \dot{H}_x^{\frac{1}{2}}})\|w\|_{L_t^4 \dot{H}_x^{\frac{1}{2}}}\end{aligned}$$

And by interpolation inequality, for any function f satisfying

$$f \in L^\infty([0, T], L^2(\mathbb{R}^2)) \cap L^2([0, T], \dot{H}^1(\mathbb{R}^2))$$

We have,

$$\begin{aligned}\|f\|_{L_t^4 \dot{H}_x^{\frac{1}{2}}} &\lesssim \left(\int_0^T \|f(s)\|_{L^2(\mathbb{R}^2)^2}^2 \|f(s)\|_{\dot{H}^1(\mathbb{R}^2)^2}^2 \mathrm{d}s\right)^{\frac{1}{4}} \\ &\lesssim \|f\|_{L^\infty([0,T], L^2(\mathbb{R}^2)^2)}^{\frac{1}{2}} \|f\|_{L^2([0,T], \dot{H}^1(\mathbb{R}^2)^2)}^{\frac{1}{2}}\end{aligned}$$

Thus,

$$\mathbb{P}(\nabla \cdot (u \otimes u - v \otimes v)) \in L^2([0, T], \dot{H}^{-1}(\mathbb{R}^2)^2)$$

By examine the remain terms, according to the definition (1.1),

$$u, v \in L^\infty([0, T], L^2(\mathbb{R}^2)) \cap L^2([0, T], \dot{H}^1(\mathbb{R}^2)) \hookrightarrow L^2([0, T], H^1(\mathbb{R}^2)^2)$$

Thus,

$$\Delta w \in L^2([0, T], \dot{H}^{-1}(\mathbb{R}^2)^2), \qquad w \in L^2([0, T], H^1(\mathbb{R}^2)^2)$$

Consider the equation (2.2), we obtain:

$$\partial_t w \in L^2([0,T], \dot{H}^{-1}(\mathbb{R}^2)^2) \hookrightarrow L^2([0,T], H^{-1}(\mathbb{R}^2)^2)$$

Apply Lions-Magenes Lemma to the space embedding sequence

$$H^1(\mathbb{R}^2)^2 \hookrightarrow L^2(\mathbb{R}^2)^2 \hookrightarrow H^{-1}(\mathbb{R}^2)^2$$

We have:

$$w \in C^0([0,T], L^2(\mathbb{R}^2)^2)$$

Thus, by convolution, there exists a sequence of test functions w_m, such that,

$$w_m \underset{m\to+\infty}{\longrightarrow} w \quad in \quad L^\infty([0,T], L^2(\mathbb{R}^2)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^2))$$

For all $t \in (0,T]$, slightly modify test functions by multiplying a cut off function supported on $[0, T+\frac{1}{m}]$ and equals to 1 on $[0, T-\frac{1}{m}]$ and passing to the limit $m \to +\infty$, we obtain:

$$\begin{aligned}
&\frac{1}{2}\|w(t)\|^2_{L^2(\mathbb{R}^2)^2} + \int_0^t \|w(s)\|^2_{\dot{H}^1(\mathbb{R}^2)^2}\mathrm{d}s \\
&\quad = -\int_0^t \langle \mathbb{P}(\nabla\cdot(u\otimes u - v\otimes v))(s), w(s)\rangle_{\dot{H}^{-1}\times\dot{H}^1}\,\mathrm{d}s \\
&\quad \lesssim \int_0^t (\|u(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2} + \|v(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2})\|w(s)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)^2}\|w(s)\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^2)^2}\mathrm{d}s
\end{aligned}$$

To ease notion, we assume:

$$c_{u,v}(t) = \|u(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2} + \|v(s)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)^2}$$

Using AM-GM inequality, absorb the $\|w(s)\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^2)^2}$ term,

$$\begin{aligned}
&\|w(s)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)^2}\|w(s)\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^2)^2} \\
&\qquad\qquad \le \frac{1}{2}\left(3\times\frac{1}{c_{u,v}(s)}\|w(s)\|^2_{\dot{H}^1(\mathbb{R}^2)^2} + c^3_{u,v}(s)\|w(s)\|^2_{L^2(\mathbb{R}^2)^2}\right)
\end{aligned}$$

Thus,

$$\frac{1}{2}\left(\|w(t)\|^2_{L^2(\mathbb{R}^2)^2} + \int_0^t \|w(s)\|^2_{\dot{H}^1(\mathbb{R}^2)^2}\mathrm{d}s\right) \lesssim \int_0^t c^4_{u,v}(t)\|w(s)\|^2_{L^2(\mathbb{R}^2)^2}\mathrm{d}s$$

Since $c^4_{u,v} \in L^1([0,T])$, apply Grönwall's inequality, finally we have:

$$\|w(t)\|^2_{L^2(\mathbb{R}^2)^2} + \int_0^t \|w(s)\|^2_{\dot{H}^1(\mathbb{R}^2)^2} \equiv 0$$

It directly leads to $u = v$. □

Remark 2.1. *We could see that in the proof, all the regularity parameters are "on the edge", this is tightly associated with the fact that the rescaling of equation (N-S) does not change the norm in the definition (1.1).*

We could latter see that this is not the case for case 3D.

2.2 Case 3D: Strong-Weak Principle

Proposition 2.2. *Given initial data* $u_0 \in L^2(\mathbb{R}^3)^3$*,* $f \in L^2([0,T], \dot{H}^{-1}(\mathbb{R}^3)^3)$*, then if the system (N-S) admits a turbulent solution* u *defined on* $[0,T]$*, furthermore satisfies the extra regularity below:*

$$u \in L^4([0,T], \dot{H}^1(\mathbb{R}^3)^3)$$

Then, every turbulent solution v *defined on* $[0,T]$ *coincide with* u*.*

Proof. Similarly, we consider the difference of two solutions:

$$w := u - v \in C_w([0,T], L^2(\mathbb{R}^3)^3) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^3)^3)$$

Then, we have the following estimate of energy:

$$\begin{aligned}
E_w(t) &= \frac{1}{2}\|u(t)-v(t)\|^2_{L^2(\mathbb{R}^3)^3} + \|u-v\|^2_{L^2([0,t],\dot{H}^1(\mathbb{R}^3)^3)} \\
&= E_u(t) + E_v(t) - \langle u(t), v(t)\rangle_{L^2(\mathbb{R}^3)^3} - 2\int_0^t \langle \nabla u(s), \nabla v(s)\rangle_{L^2(\mathbb{R}^3)^9} \mathrm{d}s \\
&\leq \|u_0\|_{L^2(\mathbb{R}^3)^3} + \int_0^t \int_{\mathbb{R}^N} f \cdot (u+v)\ \mathrm{d}x\mathrm{d}s \\
&\qquad\qquad - \langle u(t), v(t)\rangle_{L^2(\mathbb{R}^3)^3} - 2\int_0^t \langle \nabla u(s), \nabla v(s)\rangle_{L^2(\mathbb{R}^3)^9} \mathrm{d}s
\end{aligned} \tag{2.3}$$

We now examine each of the terms appearing in the weak formulation (1.4):

1 For the term $u \otimes u$, by Sobolev embedding,

$$\|u \otimes u\|_{L^2(\mathbb{R}^3)^3} \leq \|u\|_{L^4(\mathbb{R}^3)} \lesssim \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)^3} \|u\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^3)^3}$$

Thus,

$$\|u \otimes u\|_{L^2([0,T]),L^2(\mathbb{R}^3)^3} \lesssim \|u\|^{\frac{1}{2}}_{L^\infty([0,T],L^2(\mathbb{R}^3)^3)} \|u\|^{\frac{3}{2}}_{L^3([0,T],\dot{H}^1(\mathbb{R}^3)^3)} < +\infty$$

2 Similarly, for the term $v \otimes v$, we have:

$$\|v \otimes v\|_{L^{\frac{4}{3}}([0,T],L^2(\mathbb{R}^3)^3)} \lesssim \|v\|^{\frac{1}{2}}_{L^\infty([0,T],L^2(\mathbb{R}^3)^3)} \|v\|^{\frac{3}{2}}_{L^2([0,T],\dot{H}^1(\mathbb{R}^3)^3)} < +\infty$$

The extra regularity of $u \otimes u$ guarantees that $\partial_t u \in L^2([0,T], \dot{H}^{-1}(\mathbb{R}^3)^3)$, by Lions-Magenes Lemma, we have:

$$u \in C^0([0,T], L^2(\mathbb{R}^3)^3)$$

And the equation (N-S) for u holds in canonical $L^2([0,T],\dot{H}^{-1}(\mathbb{R}^3)^3)$ sense, thus, we could couple it with v,

$$\int_0^t \int_{\mathbb{R}^3} \partial_t u \cdot v + (\nabla u - u \otimes u) : \nabla v \ \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{R}^3} f \cdot v \ \mathrm{d}x\mathrm{d}s \tag{2.4}$$

By prolonging u with u_0 in $[-\epsilon, 0]$, with $u(t)$ in $[T, T+\epsilon]$, $\epsilon > 0$, then apply convolution in time variable, we could construct a mollified sequences

$$(\phi_n)_{\mathbb{N}} \in C^1([0,T], H^1(\mathbb{R}^3)^3)$$

Satisfying:

$$\nabla \cdot \phi_n = 0$$

$$\phi_n \underset{n\to+\infty}{\longrightarrow} u \quad in \quad C^0([0,T],L^2(\mathbb{R}^3)^3) \cap L^4([0,T],\dot{H}^1(\mathbb{R}^3)^3)$$

$$\partial_t \phi_n \underset{n\to+\infty}{\longrightarrow} \partial_t u \quad in \quad L^2([0,T],\dot{H}^{-1}(\mathbb{R}^3)^3)$$

Then, apply ϕ_n to (1.4) with respect to v, pass to the limit $n \to +\infty$, we obtain:

$$\begin{aligned} \langle v(t), u(t)\rangle_{L^2(\mathbb{R}^3)^3} - \int_0^t \int_{\mathbb{R}^3} v \cdot \partial_t u + (v \otimes v - \nabla v) : \nabla u \ \mathrm{d}x\mathrm{d}s \\ = \|u_0\|^2_{L^2(\mathbb{R}^3)^3} + \int_0^t \int_{\mathbb{R}^3} f \cdot u \ \mathrm{d}x\mathrm{d}s \end{aligned} \tag{2.5}$$

Combining (2.4), (2.5) and (2.3), we have:

$$E_w(t) \le \int_0^t \int_{\mathbb{R}^3} u \otimes u : \nabla v + v \otimes v : \nabla u \ \mathrm{d}x\mathrm{d}s$$

Regarding that for any $a, b \in H^1(\mathbb{R}^3)^3$, a divergence null, we have:

$$\int_{\mathbb{R}^3} a \otimes b : \nabla b \ \mathrm{d}x = -\int_{\mathbb{R}^3} \nabla \cdot (a \otimes b) \cdot b = -\int_{\mathbb{R}^3} |b|^2 \nabla \cdot a + a \otimes b : \nabla b \ \mathrm{d}x = 0$$

Thus, we have:

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} u \otimes u : \nabla v + v \otimes v : \nabla u \ \mathrm{d}x\mathrm{d}s &= -\int_0^t \int_{\mathbb{R}^3} w \otimes u : \nabla w \ \mathrm{d}x\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla) u \cdot w \ \mathrm{d}x\mathrm{d}s \end{aligned}$$

To estimate the term above, we have:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (w \cdot \nabla) u \cdot w \ \mathrm{d}x \right| &\le \|w\|^2_{L^4(\mathbb{R}^3)^3} \|u\|_{\dot{H}^1(\mathbb{R}^3)^3} \\ &\lesssim \|w\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)^3} \|w\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^3)^3} \|u\|_{\dot{H}^1(\mathbb{R}^3)^3} \end{aligned}$$

Finally, we have a constant $C > 0$, apply AM-GM inequality,

$$\begin{aligned} E_w(t) &\leq C \int_0^t \|w(s)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)^3} \|w(s)\|^{\frac{3}{2}}_{\dot{H}^1(\mathbb{R}^3)^3} \|u(s)\|_{\dot{H}^1(\mathbb{R}^3)^3} \, \mathrm{d}s \\ &\leq \frac{1}{2} \int_0^t \|w(s)\|_{\dot{H}^1(\mathbb{R}^3)^3} + C' \int_0^t \|u(s)\|^4_{\dot{H}^1(\mathbb{R}^3)^3} \|w(s)\|^2_{L^2(\mathbb{R}^3)^3} \, \mathrm{d}s \end{aligned}$$

Regarding the definition (1.3) of E_w,

$$\|w(t)\|^2_{L^2(\mathbb{R}^3)^3} \lesssim \int_0^t \|u(s)\|^4_{\dot{H}^1(\mathbb{R}^3)^3} \|w(s)\|^2_{L^2(\mathbb{R}^3)^3} \, \mathrm{d}s$$

Recall that $u \in L^4([0,T], \dot{H}^1(\mathbb{R}^3)^3)$, we conclude by Grönwall's inequality.

□

3 Non-Uniqueness: Review and Foreseen

In this section, we are going to briefly review the history of non-uniqueness problem of Navier-Stokes equation, and give a symmetrical refined version of the main theorem (1.1).

3.1 A Brief History

Non-Uniqueness of Navier-Stokes equation has been a long question in different contexts, less regular the solution is, we could expect more of the non-uniqueness.

In the weakest sense, Buckmaster and Vicol [BV19] have proven that in the class of finite kinetic energy distributions, for any given initial datum, the solution is always non-unique. However, their solution lies in the finite energy class $L^\infty_t L^2_x$ but is not regular enough to be a Leray solution.

But in the context of Leray's turbulent solution, the problem remains open. The method adopted by [ABC22], which would be presented in this paper, is partially inspired by Vishik's study of non-uniqueness of Euler's equation with external force in [Vis18a] and [Vis18b]. The main idea is to consider a unstable steady state, and apply an external force as perturbation in $L^1_t L^p_x$. We would utilize his method of constructing an **unstable vortex solution**.

How about removing the external force? The introduction of external force has provided us with more flexibility, and it's still an open problem of the uniqueness of (N-S) without external force. But in the context of Euler equation, Bressan, Murray, and Shen have proposed an potential initial data which perhaps could lead to non-unique solutions without external force in [BM20] and [BS21]. We still need further exploration for similar conclusions on Navier-Stokes equation.

3.2 Strategy of Refining

We want to work on exactly the critical case, where the amplitude becomes of vital importance: for $a > 0$, if

$$u_0(x) \sim \frac{a}{|x|}$$

Then,

$$\Delta u_0(x) \sim \frac{a}{|x|^3}, \quad u_0 \cdot \nabla u_0 \sim \frac{a^2}{|x|^3}$$

We could observe that for $a \ll 1$, the viscose term Δu_0 is dominating, which give us regularity. For $a \gg 1$, then the nonlinear convection term is leading, and we could search for non-uniqueness.

The schema is described below: The term **unstable** refers that the linearized operator has an unstable eigenvalue, and the corresponding perturbation term turns to 0 while evolution backward, it's hard to treat the question on positive time $\mathbb{R}^+$, thus we have to stretch it into $\mathbb{R}$.

Furthermore, we would like to add additional symmetrical structure to the solution, which we hope to lies in the critical case. A family of this kind of solutions that is wildly studied are called **self-similar solutions**, as the name suggests, are invariant under the rescaling (1.7).

To realise these two goals in one time, we introduce the **similarity variables** to convert the positive time $\mathbb{R}^+$ to whole real line $\mathbb{R}$ with 0 corresponding to $-\infty$:

$$\xi := \frac{x}{\sqrt{t}}, \quad \tau := \log t \in \mathbb{R} \tag{3.1}$$

$$u(t,x) =: \frac{1}{\sqrt{t}} U(\tau, \xi), \quad f(t,x) =: \frac{1}{t^{\frac{3}{2}}} F(\tau, \xi) \tag{3.2}$$

Here the lower case letters denote the functions in physical space, and the upper case letters denote functions with similarity variables. The Navier-Stokes equation becomes:

$$\begin{cases} \partial_\tau U - \frac{1}{2}(1 + \xi \cdot \nabla_\xi) U - \Delta U + U \cdot \nabla U = F - \nabla P, & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot U = 0, & (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3 \end{cases} \tag{3.3}$$

A self-similar solution to the Navier-Stokes equation (N-S) is 1-1 corresponding by the change of variables (3.1) to a steady state of equation (3.3) with similarity variables.

We call a solution $\tilde{U}$ to (3.3) is **linear unstable**, if the linearized operator $\mathbf{L}_{\mathrm{ss}}$ at $\tilde{U}$ defined below has eigenvalue with positive real part.

$$\mathbf{L}_{\mathrm{ss}}(U) := \frac{1}{2}(1 + \xi \cdot \nabla_\xi) U + \Delta U + \mathbb{P}(\tilde{U} \cdot \nabla U + U \cdot \nabla \tilde{U}) \tag{3.4}$$

Where $\mathbb{P}$ is the Leray projector define in (2.1). We would study the **unstable manifold** associated with the most unstable eigenvalue λ_0, i.e. the eigenvalue with the largest real part which we denote as $a = \Re(\lambda_0) \in \mathbb{R}$. Then, the solutions on this manifold satisfy the following asymptotic behavior:

$$U = \tilde{U} + U_{\text{lin}} + O(e^{2\tau a}), \qquad \tau \to -\infty$$

Where U^{lin} is a non-trivial solution to the linearized equation $\partial_t U^{\text{lin}} = \mathbf{L}_{\text{ss}} U^{\text{lin}}$ on $\mathbb{R} \times \mathbb{R}^3$ corresponding to an eigenfunction of λ_0. Since U^{lin} decays at the rate $e^{\tau a}$ as $\tau \to -\infty$, thus,

$$\lim_{\tau \to -\infty} U \sim \tilde{U}$$

Since we could view $\tau = -\infty$ as $t = 0$, and by the formula (3.2), the corresponding velocity field in physical space satisfies:

$$\lim_{t \to 0^+} \|u(t)\|_{L^2(\mathbb{R}^3)^3} = \lim_{\tau \to -\infty} e^{\frac{1}{2}\tau} \|U(\tau)\|_{L^2(\mathbb{R}^3)^3} = 0 \tag{3.5}$$

The same holds true for $\tilde{u}$, this correspond to the non-uniqueness of $t = 0$ in time variable with initial value $u_0 = 0$.

To sum up all the arguments above, we have the following refined version of our main theorem (1.1):

Theorem 3.1. *There exists a smooth, compactly supported stationary real-valued velocity $\tilde{U}$ which does not depend on self-similar time τ, and a smooth, compactly supported source term*

$$\tilde{F} := -\frac{1}{2}(1 + \xi \cdot \nabla_\xi)\tilde{U} - \Delta \tilde{U} + \tilde{U} \cdot \nabla \tilde{U}$$

satisfying the following properties:

1 *The linearized operator $\mathbf{L}_{\text{ss}}$ defined by (3.4) has an unstable eigenvalue λ with a non-zero eigenfunction $\eta \in H^k(\mathbb{R}^3)$ for all $k \geq 0$. Denote one of the solutions to the linearized equation $\partial_t U^{\text{lin}} = \mathbf{L}_{\text{ss}} U^{\text{lin}}$ as below:*

$$U^{\text{lin}}(\tau) := \Re(e^{\lambda\tau}\eta)$$

2 *There exists a time-like upper bound $T \in \mathbb{R}$ and a velocity field U^{per}, satisfying the following smallness condition. For all $k \in \mathbb{N}$, we have:*

$$\|U^{\text{per}}(\tau)\|_{H^k(\mathbb{R}^3)} \lesssim_k e^{2\tau a}, \quad \forall \tau \in (-\infty, T]$$

And in the view of (3.5), $\tilde{u}$,$\tilde{u}+u^{\text{lin}}+u^{\text{per}}$ are two distinct turbulent solutions to the system (N-S) with initial value $u_0 \equiv 0$ and external force $\tilde{f}$.

Furthermore, we could make these observations:

1 Firstly, since we admit an external force in the system, any smooth function with enough decay could be seen as a potential steady state of the equation, this gives us a large range of choice.

2 Secondly, by simply modifying the amplitude of the steady state $\tilde{U}$, we could view the terms arisen from nonlinear convection as the leading term, and the other terms, including viscosity, acts as a small perturbation.

Thus, it is reduced to find a smooth unstable steady state with enough decay for the 3-dimensional Euler equation with external force. Since in 3 dimensions, the well-studied steady solutions for Euler equation are *sheer flows* and *vortex columns*, which are constant along one axis, thus do not satisfy the decay property. Fortunately, it's natural to impose extra symmetry and lower the dimension to case 2D.

In 2 dimensions, a frequently considered family of steady state functions are so-called **vortices**, which write in form of polar coordinate as:

$$\bar{u}(r,\theta) = \bar{u}^{\theta}(r)e_{\theta}, \quad (r,\theta) \in \mathbb{R}_{+} \times (\mathbb{R}\backslash 2\pi\mathbb{Z})$$

Where e_θ denotes the clockwise tangent vector at $(1,\theta)$ of unit circle. Vishik has constructed an unstable vortex with power-law decay as $r \to +\infty$. Our method is lifting Vishik's vortex in a axisymmetric without swirl way into 3 dimensions, of the form

$$u(r,\theta,z) = u^{r}(r,z)e_{r} + u^{z}(r,z)e_{z}$$

Where $e_r,\ e_z$ are respectively the unit vector in the r and z-direction. The corresponding vorticity and stream function are of following form:

$$\omega := \nabla \times u = -\omega^{\theta}(r,z)e_{\theta} \quad \varphi = -\Delta^{-1}\omega = \varphi^{\theta}(r,z)e_{\theta} \tag{3.6}$$

Then, with these variables, apply curl operator on the Euler equation (Euler) with $f = 0$, together with the definition of ω, φ, deduces the following system:

$$\begin{cases} \partial_t \omega^{\theta} + u \cdot \nabla \omega^{\theta} - \dfrac{u^r}{r}\omega^{\theta} = 0 \\ w^{\theta} = \left(\partial_r^2 + \dfrac{1}{r}\partial_r - \dfrac{1}{r^2}\right)\varphi^{\theta} \\ u = -\partial_z \varphi^{\theta} e_r + \left(\partial_r + \dfrac{1}{r}\right)\varphi^{\theta} e_z \end{cases} \tag{3.7}$$

As $r \to +\infty$, all the terms of order r^{-1} and r^{-2} could be neglected, reduce to the following 2-dimensional Euler-like system:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ w = \Delta\varphi, \quad u = \nabla \times \varphi \end{cases} \tag{3.8}$$

Hence, we could lift Vishik's unstable 2D vortex into a 3D **hollow vortex ring** with large diameter in order to ease the influence of those higher order terms. This suggests us a possible way to construct the unstable steady state.

4 Instability: From 2D to Axisymmetric Case

In this section, we are going to develop the instability property of the 3-dimensional axisymmetrical Euler system (3.8). It relies fundamentally on the construction of Vishik's unstable vortex for 2-dimensional Euler's equation in [Vis18a] and [Vis18b].

4.1 2D Instability: Vishik's Vortex

As mentioned in the refining strategy, we recall the **unstable vortex** constructed by Vishik in [Vis18b].

Firstly, we define the following family of **m-fold rotational symmetrical** functions:

Definition 4.1. *For integer $m \geq 2$, define:*

$$L^2_m := \{f \in L^2(\mathbb{R}^2, \mathbb{C}) | f(R_{\frac{2\pi}{m}} x) = e^{i\frac{2\pi}{m}} f(x) \quad for\ a.e.\ x \in \mathbb{R}^2\}$$

where $R_\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation by angle ϕ.

By using Fourier series in 2D polar coordinates (B.1), for $f \in L^2(\mathbb{R}^2)$,

$$f(r, \theta) = \sum_{k \in Z} f_k(r) e^{-ik\theta}$$

Then, by the uniqueness of the decomposition above, we have:

$$f \in L^2_m \quad \Leftrightarrow \quad f_k = 0, \ \ \forall m \nmid k$$

Thus, we have the following decomposition of L^2_m:

$$L^2_m = \bigoplus_{k \in Z} U_k, \quad U_k := \{g(r) e^{-imk\theta} | g \in L^2(\mathbb{R}_+, r\mathrm{d}r)\} \tag{4.1}$$

where U_k is closed and mutually orthogonal subspaces.

Given a vorticity function, we want to construct its corresponding velocity profile, the **Biot–Savart law** stated below provides us with an useful tool:

Consider an compressible 2-dimensional flow u and its vorticity ω, satisfying:

$$\nabla \cdot u = 0, \quad \nabla \times u = \omega$$

By the first equation, we could introduce a vectorial potential φ satisfying:

$$\nabla \times \varphi = u$$

Note that the choice of φ is not unique, we could further assume that,

$$\nabla \cdot \varphi = 0$$

Thus, we obtain the vectorial Poisson equation:

$$w = \nabla \times u = \nabla \times (\nabla \times \varphi) = \nabla(\nabla \cdot \varphi) - \Delta\varphi = -\Delta\varphi$$

Formally, we have the identity below:

$$u = -\nabla \times \Delta^{-1}\omega$$

which is unique up to adding a harmonic function.

Since the flow we consider lies in $L^2([0,T], \dot{H}^1(\mathbb{R}^2)^2)$, we could define the following operator:

$$\begin{array}{rcc} \mathrm{BS}_{2d} : L^2(\mathbb{R}^2) & \to & \dot{H}^1(\mathbb{R}^2)^2 \\ \omega & \mapsto & \nabla \times \Delta^{-1}\omega \end{array} \tag{4.2}$$

We could write the image above into the convolution form:

$$\nabla \times \Delta^{-1}\omega = K_{\mathrm{BS}_{2d}} * \omega \tag{4.3}$$

where $K_{\mathrm{BS}_{2d}} = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$ is the convolution kernel of the Fourier operator BS_{2d}, where for $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$. We could easily verify that the image of an element in L^2_m is vectorial m-fold rotational symmetrical.

Remark 4.1. *Follow the same procedure, we could define the Biot–Savart operator* $\mathrm{BS}_{3d} : L^2(\mathbb{R}^3)^3 \to \dot{H}^1(\mathbb{R}^3)^3$ *in 3-dimensional case, where the corresponding convolution form writes:*

$$\mathrm{BS}_{3d}[\omega](x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x-y)\mathrm{d}y \tag{4.4}$$

with the corresponding convolution kernel

$$K_{\mathrm{BS}_{3d}}(x) := -\frac{1}{4\pi}\frac{x}{|x|^3} \in L^{\frac{3}{2}}_w(\mathbb{R}^3) \tag{4.5}$$

Let us consider the behavior of linearized operator associated with Euler-like equation (3.8) around a steady state. Given a smooth, real-valued ,divergence-free vector field with sufficient decay at $r \to +\infty$, with the form:

$$\tilde{u}(r,\theta) = \zeta(r)e_\theta$$

Set the corresponding vorticity which is a radial function:

$$\tilde{\omega} = \nabla \times \tilde{u} = \partial_r\zeta + \frac{1}{r}\zeta$$

We define the linearized operator of the term $\tilde{u} \cdot \nabla\tilde{w}$:

Definition 4.2. *Let:*

$$\begin{array}{ccc} \mathbf{A} : \mathrm{D}(\mathbf{A}) & \to & L^2_m \\ \omega & \mapsto & -(u \cdot \nabla\tilde{\omega} + \tilde{u} \cdot \nabla\omega) \end{array} \tag{4.6}$$

where $u = \mathrm{BS}_{2d}[\omega]$*, the domain of definition is:*

$$\mathrm{D}(\mathbf{A}) := \{\omega \in L^2_m | \tilde{u} \cdot \nabla\omega \in L^2_m\} \subset L^2_m$$

In fact, first part of the operator is compact if given $\nabla\tilde{\omega}$ with sufficient decay:

$$\begin{array}{ccccc} \mathbf{K} : L^2_m & \to & \dot{H}^1(\mathbb{R}^2)^2 & \to & L^2_m \\ \omega & \mapsto & u = \mathrm{BS}_{2d}[\omega] & \mapsto & u \cdot \nabla\tilde{w} \end{array}$$

whose compactness essentially comes from Rellich's embedding theorem. Thus, we could view $\mathbf{A}$ as a compact perturbation of the operator $\mathbf{A}-\mathbf{K} : \omega \mapsto \tilde{u}\cdot\nabla\omega$. Formally, we have:

$$\langle \tilde{u} \cdot \nabla\omega, \phi\rangle_{L^2} = \langle \nabla\omega, \tilde{u}\phi\rangle_{L^2} = -\langle \omega, \nabla \cdot (\tilde{u}\phi)\rangle_{L^2} = -\langle \omega, \tilde{u} \cdot \nabla\phi\rangle_{L^2}$$

The operator $\mathbf{A} - \mathbf{K}$ is skew-adjoint, therefore its spectrum belongs to the imaginary axis. By the reasoning (A.1), the essential spectrum also lies in the imaginary axis, and the remainder of the spectrum is consisted of countable eigenvalues of finite dimensional eigenspace.

We now state a version of Vishik's unstable vortex constructed in [Vis18a] and [Vis18b].

Theorem 4.1. *There exists a symmetric fold number* $m \geq 2$*, and a smooth, real, radially symmetrical vorticity function* $\tilde{\omega} = \tilde{\omega}(r)$*, with its corresponding smooth velocity field* $u(r, \theta) = \zeta(r)e_\theta$*, satisfying the following decay property:*

$$|\tilde{\omega}| + r|\partial_r\tilde{\omega}| \lesssim (1 + r^2)^{-1}$$

and the corresponding linearized operator $\mathbf{A}$ *has an unstable eigenvalue* λ*, i.e.* $\Re(\lambda) > 0$*.*

Remark 4.2. *By the convolution form of Biot-Savart law (4.3), we have the decay property of* ζ*:*

$$\zeta(r) \lesssim (1 + r^2)^{-\frac{1}{2}}$$

We would expect the unstable state is still unstable under proper truncation. Let the truncation $\phi \in C^\infty_0(\mathbb{R}^2)$ be a radical function, satisfying:

$$\phi|_{B_0(\frac{1}{2})} \equiv 1, \quad \mathrm{supp}(\phi) \subset B_0(1)$$

For $R \geq 0$, we use the following notions:

$$\phi_R(x) := \phi\left(\frac{x}{R}\right), \quad \tilde{u}_R := \phi_R\tilde{u}, \quad \tilde{\omega}_R := \nabla \times \tilde{u}_R$$

And consider the associated truncated operator $\mathbf{A}_R$ which is obtained by change $(\tilde{u}, \tilde{\omega})$ to $(\tilde{u}_R, \tilde{\omega}_R)$ in (4.2).

Here states the instability of properly truncated operator:

Proposition 4.1. *Let* $\tilde{\omega}$ *be the unstable vortex in (4.1),* λ_∞ *be an unstable eigenvalue of* $\mathbf{A}$*, then for all* $\epsilon \in (0, \Re(\lambda_\infty))$*, there exists* $R_0 = R_0(\tilde{\omega}, \lambda_\infty, \epsilon)$ *such that, for all* $R \geq R_0$*,* $\mathbf{A}_R$ *has an unstable eigenvalue* λ_R *satisfying* $|\lambda_R - \lambda_\infty| < \epsilon$*.*

Proof. Let us make the first observation: every subspace $U_k \subset L^2_m$ defined in (4.1) is an invariant subspace of $\mathbf{A}$ and $\mathbf{A}_R$. Furthermore, the restrictions $\mathbf{A}^{(k)} := \mathbf{A}|_{U_k}$, $\mathbf{A}_R^{(k)} := \mathbf{A}_R|_{U_k}$ are continuous.

In fact, assume $\omega(r,\theta) = g(r)e^{-imk\theta}$, we have the following calculation:

$$\mathbf{A}(w) = \frac{imk}{r}\left(\zeta g - f\partial_r\tilde{\omega}\right)e^{-imk\theta}$$

$$\mathbf{A}_R(w) = \frac{imk}{r}\left(\phi_R\zeta g - f\partial_r\tilde{\omega}_R\right)e^{-imk\theta}$$

where $\Delta^{-1}\omega = f(r)e^{-imk\theta} \in \dot{H}^2(\mathbb{R}^2)$, which is the potential function constructed in Biot-Savart law, satisfying:

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2k^2}{r^2}\right)f(r) = g(r)$$

In fact, we have the following explicit expression:

$$f(r) = -\frac{1}{2mk}\left(r^{mk}\int_r^\infty s^{1-mk}g(s)\mathrm{d}s + r^{-mk}\int_0^r s^{1+mk}g(s)\mathrm{d}s\right)$$

Apply Hölder's inequality, we obtain:

$$\|\frac{f(r)}{r}\|_{L^\infty(R_+)} \lesssim_{km} \|g\|_{L^2(\mathbb{R}_+, r\mathrm{d}r)} \lesssim \|\omega\|_{L^2(\mathbb{R}^2)}$$

Since ζ is smooth with $\zeta(0) = 0$, thus $\frac{\zeta(r)}{r} \in L^\infty(\mathbb{R}_+)$, by the L^2 integrability of $\tilde{\omega}$ and $\tilde{\omega}_R$, we have the continuity of $\mathbf{A}^{(k)}$ and $\mathbf{A}_R^{(k)}$. Furthermore, by the decay property of ζ and ω, we have the following convergence in operator norm:

$$\mathbf{A}_R^{(k)} \underset{R\to\infty}{\longrightarrow} \mathbf{A}^{(k)} \tag{4.7}$$

By projection on to the subspaces U_k, the unstable eigenvalue λ_∞ could be associated to a non-zero eigenfunction η belonging to U_{k_0}, with $k_0 \in \mathbb{Z}$. Since $\mathbf{A}^{(0)}$ maps every element in U_0 to 0, thus $k_0 \neq 0$.

Since $A_R^{(k)}$ is a compact perturbation of a self-adjoint operator, thus suit the reasoning (A.1). We want to study the behavior of resolvents under truncation. Let K be a compact subset of $\mathrm{res}(\mathbf{A}) \cap \{\Re(z) > 0\}$, for all $\lambda \in K$, we have:

$$\begin{aligned}\mathbf{A}_R^{(k)} - \lambda &= (\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)}) + (\mathbf{A}^{(k)} - \lambda)\\ &= (\mathbf{A}^{(k)} - \lambda)(\mathrm{Id} + (\mathbf{A}^{(k)} - \lambda)^{-1}(\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)}))\end{aligned} \tag{4.8}$$

Since K is compact,

$$\sup_{\lambda\in K} \|(\mathbf{A}^{(k)} - \lambda)^{-1}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} < +\infty$$

due to the operator convergence, we could choose $R_0 > 0$ depending on K, such that for all $R \geq R_0$,

$$\sup_{\lambda\in K} \|(\mathbf{A}^{(k)} - \lambda)^{-1}(\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)})\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} < \frac{1}{2}$$

Thus, $\mathbf{A}_R^{(k)} - \lambda$ is invertible, and its inverse is:

$$(\mathbf{A}_R^{(k)} - \lambda)^{-1} = (\mathrm{Id} + (\mathbf{A}^{(k)} - \lambda)^{-1}(\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)}))^{-1}(\mathbf{A}^{(k)} - \lambda)^{-1}$$

Furthermore, observe that,

$$\begin{aligned}&(\mathrm{Id} + (\mathbf{A}^{(k)} - \lambda)^{-1}(\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)}))^{-1} - \mathrm{Id}\\ &\quad= (\mathrm{Id} + (\mathbf{A}^{(k)} - \lambda)^{-1}(\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)}))^{-1}(\mathbf{A}^{(k)} - \lambda)^{-1}(\mathbf{A}_R^{(k)} - \mathbf{A}^{(k)})\end{aligned} \tag{4.9}$$

we have the following convergence in operator norm uniformly for $\lambda \in K$:

$$(\mathbf{A}_R^{(k)} - \lambda)^{-1} \underset{R\to\infty}{\longrightarrow} (\mathbf{A}^{(k)} - \lambda)^{-1} \tag{4.10}$$

Since λ_∞ is a isolated eigenvalue, we could choose a disc $C \subset \{\Re(z) > 0\}$ centering at λ_∞, with radius less that ϵ, satisfying $C \cap \mathrm{res}(\mathbf{A}) = \{\lambda_\infty\}$. Denote its counterclockwise oriented boundary as $\vec{c} \subset \mathrm{res}(\mathbf{A}) \cap \{\Re(z) > 0\}$ is a compact subset. Consider the projector onto the eigenspace corresponding to λ_∞:

$$\mathrm{Pr}_R^{(k_0)} := \frac{1}{2\pi i}\int_{\vec{c}} (\lambda - \mathbf{A}_R^{(k_0)})^{-1} \mathrm{d}\lambda$$

and we similarly define $\mathrm{Pr}^{(k_0)}$. By assumption, λ_∞ is an eigenvalue of $\mathbf{A}^{(k_0)}$, thus $\mathrm{Pr}^{(k_0)}$ is non trivial, and let $K = \vec{c}$, by the uniform convergence (4.10), we have the following convergence in operator norm:

$$\mathrm{Pr}_R^{(k_0)} \underset{R\to\infty}{\longrightarrow} \mathrm{Pr}^{(k_0)}$$

Then, for R sufficiently large, depending on $\mathbf{A}$ and K, $\mathrm{Pr}_R^{(k_0)}$ is non-trivial. Since it could only admit discrete spectrum in $\{\Re(z) > 0\}$, thus it must have an eigenvalue λ_R in C, and so must $\mathbf{A}_R$. By the choice of disc C, we have $|\lambda_R - \lambda| < \epsilon$, this conclude the proof. □

Thus, we could construct an unstable smooth profile with compactly support. Choose $\epsilon < \frac{\Re(\lambda_\infty)}{2}$ and $\bar{R} \geq R_0$, define the truncated smooth vortex

$$\bar{u} := \tilde{u}_{\bar{R}}, \qquad \bar{\omega} := \nabla \times \bar{u}$$

Note that $\bar{u}$ is still divergence free due to the truncation function ϕ is radial. Then by the proposition (4.1) above, the associated operator $\bar{\mathbf{A}} := \mathbf{A}_{\bar{R}}$ has an unstable eigenvalue.

We want to show that in fact the operator defined above admits an eigenfunction with better property. Consider the restriction of $\bar{\mathbf{A}}$ on a subspace of L^2 with sufficient decay, and generalize it to non m-fold rotational symmetrical case in order to lift it to 3-dimensional axisymmetric case.

Firstly, let us introduce our desire weighted space, where we use coordinate (r, z) in $\mathbb{R}^2$ corresponding to the axisymmetric 3D case.

$$L^2_\gamma := L^2(\mathbb{R}^2, \gamma \mathrm{d}r\mathrm{d}z) \tag{4.11}$$

where the smooth weight function $\gamma > 0$ satisfies:

$$\gamma|_{B_0(\bar{R})} \equiv 1, \quad (1 + r^2 + z^2)^{50} \lesssim \gamma \lesssim (1 + r^2 + z^2)^{50} \tag{4.12}$$

Now we consider the operator $\mathbf{L}_\infty$ defined below:

$$\begin{aligned} \mathbf{L}_\infty : \mathrm{D}(\mathbf{L}_\infty) \quad & \to \quad L^2_\gamma \\ \omega \quad & \mapsto -(u \cdot \nabla\bar{\omega} + \bar{u} \cdot \nabla\omega) \end{aligned} \tag{4.13}$$

where $u = \mathrm{BS}_{2d}[\omega]$, and the domain of definition

$$\mathrm{D}(\mathbf{L}_\infty) = \{\omega \in L^2_\gamma | \bar{u} \cdot \nabla\omega \in L^2_\gamma\}$$

. Observe that by the decaying property of $\omega \in L^2_\gamma$, use the convolution form (4.3), and apply Hölder's inequality, we in fact have

$$u = \mathrm{BS}_{2d}[\omega] \in L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$$

Using the fact that

$$\mathrm{supp}(\bar{u}) \cup \mathrm{supp}(\bar{\omega}) \subset B_0(\bar{R}) \tag{4.14}$$

Recall the smoothness and compact support of $\bar{\omega}$, the compactness of mapping $\omega \mapsto u \cdot \nabla\bar{\omega}$ is directly driven from Rellich theorem. Thus, by the same reasoning, $\mathbf{L}_\infty$ could be viewed as a skew-adjoint operator perturbed by a compact operator, thus its essential spectrum $\mathrm{ess}(\mathbf{L}_\infty) \subset \{\Im(z) = 0\}$, and the remainder of the spectrum is consisted of countable discrete eigenvalues of finite algebraic multiplicity.

Since $\mathbf{L}_\infty$ is just a slight modification of $\bar{\mathbf{A}}$, we could expect similar instability result in weighted space:

Proposition 4.2. *The operator $\mathbf{L}_\infty$ has an unstable eigenvalue.*

Proof. Assume λ is an unstable eigenvalue of $\bar{\mathbf{A}}$, then, there exists a corresponding eigenvector $\eta \in L^2_m$, satisfying:

$$\bar{\mathbf{A}}\eta = \lambda\eta$$

By (4.14), we have:

$$\operatorname{supp}(\eta) = \operatorname{supp}(\bar{\mathbf{A}}\eta) \subset B_0(\bar{R})$$

Since $\gamma|_{B_0(\bar{R})} \equiv 1$, thus $\eta \in L^2_\gamma$, thus:

$$\mathbf{L}_\infty \eta = \lambda \eta$$

$\square$

4.2 Axisymmetric Instability

We want to lift the previous argument to **3-dimensional axisymmetric-without-swirl case**, with coordinate written as $(r', z) \in \mathbb{R}_+ \times \mathbb{R}$, the function u defined on this coordinate is independent of the angle θ, of the form:

$$u(r', z) = u^r(r', z)e_r + u^z(r', z)e_z$$

with its vorticity:

$$\omega := \nabla \times u = -\omega^\theta(r, z)e_\theta$$

Thus, we could represent the profile and vorticity with a function $\mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}^2$ and $\mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ respectively. In the discussion below, we would **use 2D Euclidean coordinate to represent 3D behavior**, be ware that the operators are different from the original 2D operators!

As mentioned in the strategy subsection, to ease the influence of r^{-1} and r^{-2} term, the lifted vortex should be hollow. To realize "hollowness", we work on the shifted variable $r := r' - l \in \mathbb{R}_{\geq -l}$, where $l > 2\bar{R}$.

From now on, we use the following convention:

Convention 1. *All the operators with subscript l are the 2-dimensional operators induced by the corresponding 3-dimensional operators applied to the axisymmetric-without-swirl variable $(r', z) = (r + l, z)$.*

Denote our 2-dimensional unstable vortex in euclidean coordinate (r, z) as:

$$\bar{u}(r, z) = \bar{u}^r(r, z)e_r + \bar{u}^z(r, z)e_z$$

Notice that by the construction, $\bar{u}$ is radial and purely swirl, thus divergence-free in 2D normal coordinate:

$$\nabla \cdot \bar{u} = (\partial_r, \partial_z) \cdot \bar{u} = 0$$

By the discussion above, it could be lifted to a 3-dimensional axisymmetric-without-swirl profile in the coordinate (r, z), denoted also by $\bar{u}$.

Notice now, $\bar{u}$ is not correspond to a 3D divergence-free profile, since:

$$\nabla_l \cdot \bar{u}(r, z) = \left(\partial_r + \frac{1}{r + l}\right)\bar{u}^r + \partial_z \bar{u}^z = \frac{1}{r + l}\bar{u}^r$$

Observe that the error term is decaying as $l \to \infty$, we would correct the error with the lemma below with the correction term also decaying:

Lemma 4.1. *For all $l \geq 2\bar{R}$, there exists a real correction $v_l \in C_0^\infty(B_0(\bar{R}), \mathbb{R}^2)$, satisfying:*

$$\nabla_l \cdot (\bar{u} + v_l) = 0$$

and the following decay property:

$$v_l \xrightarrow[l\to\infty]{} 0 \quad in \quad C^k(B_0(\bar{R})) \quad \forall k \geq 0$$

Proof. Observe that the divergence operator associated with shifting parameter l could be written into the following form:

$$\nabla_l \cdot u = \frac{1}{r+l}(\partial_r, \partial_z) \cdot [(r+l)u]$$

Thus, the correction term must solve the following equation:

$$(\partial_r, \partial_z) \cdot [(r+l)v_l] = -\bar{u}^r$$

Recall that $\bar{u}^r \in C_0^\infty(B_0(\bar{R}))$ with $\int_{\mathbb{R}^2} \bar{u}^r(r,z)\mathrm{d}r\mathrm{d}z = 0$ inherited form its rotational symmetry, apply the Bogovskii's operator (B.2) in case $N = 2$, we obtain a solution V for the equation:

$$(\partial_r, \partial_z) \cdot V = -\bar{u}^r$$

with $V \in C_0^\infty(B_0(\bar{R}), \mathbb{R}^2)$ independent of l. Simply let $v_l := \frac{1}{r+l}V$, and we obtain the desire conclusion. ☐

By the lemma above, for any $l > 2\bar{R}$, we could define the corrected background profile:

$$\bar{u}_l := \bar{u} + v_l \in C_0^\infty(B_0(\bar{R}), \mathbb{R}^2), \quad \bar{\omega}_l := \nabla_l \times \bar{u}_l \in C_0^\infty(B_0(\bar{R}))$$

As the flow is axisymmetric without swirl, *we could identify the vorticity with its angular component as a scalar function:*

$$\bar{\omega}_l = -\bar{\omega}_l^\theta e_\theta, \quad \bar{\omega}^\theta = -\partial_z \bar{u}_l^r + \partial_r \bar{u}_l^z$$

We consider the linearized operator of equation (3.7) around the background profile $(\bar{u}_l, \bar{\omega}_l)$. Firstly, we have to define the proper functional space we work on. As before, we could expect that the functions are decaying sufficiently fast:

$$L^2_{\gamma,l} := L^2(\mathbb{R}_{r\geq -l} \times \mathbb{R}_z, \gamma \mathrm{d}r\mathrm{d}z)$$

where γ is the same weight function defined in (4.12).

Define the linearized operator:

$$\begin{array}{rcl} \mathbf{L}_l : \mathrm{D}(\mathbf{L}_l) & \to & L^2_{\gamma,l} \\ \omega & \mapsto & \mathbf{L}_l \omega \end{array} \tag{4.15}$$

where,

$$-\mathbf{L}_l\omega := \bar{u}_l \cdot \nabla\omega + \mathrm{BS}_l[\omega] \cdot \nabla\bar{\omega}_l - \frac{\bar{u}_l^r}{r+l}\omega - \frac{(\mathrm{BS}_l[\omega])^r}{r+l}\bar{\omega}_l \tag{4.16}$$

$\nabla = (\partial_r, \partial_z)$ is the normal gradient operator, BS_l is the operator induced by 3-dimensional Biot-Savart operator BS_{3d} applied to the axisymmetric coordinate $(r', z) = (r+l, z)$, defined by the following process:

1 Firstly, lift the vorticity function to 3-dimensional coordinate (r', z):

$$\omega_l(r', z) := \omega(r'-l, z)e_\theta \in L^2(\mathbb{R}^3, \mathbb{R}^3)$$

2 Secondly, calculate the stream function φ_l:

$$\varphi_l := \Delta^{-1}\omega_l = \varphi_l^\theta(r', z)e_\theta \in \dot{H}^2(\mathbb{R}^3, \mathbb{R}^3)$$

3 Then, apply curl operator to the stream function, obtain the profile:

$$u_{l,3d} := \mathrm{BS}_{3d}[\omega_l] = \nabla \times \phi_l = u^r_{l,3d}(r', z)e_r + u^z_{l,3d}(r', z)e_z \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$$

4 Finally, reduce it to the coordinate (r, z):

$$\mathrm{BS}_l[\omega](r, z) := u^r_{l,3d}(r+l, z)e_r + u^z_{l,3d}(r+l, z)e_z : \mathbb{R}^2 \to \mathbb{R}^2$$

Slight abuse the notion, we could view the function φ_l as a scalar function on $\mathbb{R}_{\geq -l} \times \mathbb{R}$. The functions defined in the steps satisfy the following system of equations:

$$\begin{aligned} \Delta_l\varphi_l := \left(\partial_r^2 + \frac{1}{r+l}\partial_r - \frac{1}{(r+l)^2} + \partial_z^2\right)\varphi_l &= \omega \quad in \quad \mathbb{R}_{r\geq -l} \times \mathbb{R} \\ \partial_r\varphi_l|_{r=-l} &= 0 \\ \mathrm{BS}_l[\omega] := -\partial_z\varphi_l e_r + &\left(\partial_r + \frac{1}{r+l}\right)\varphi_l e_z \end{aligned} \tag{4.17}$$

Note that the boundary condition is due to the fact that $r = -l$ is corresponding to $r' = 0$, i.e. the axis, and φ should be axisymmetric.

The domain of definition is:

$$\mathrm{D}(\mathbf{L}_l) := \{\omega \in L^2_{\gamma,l} | \bar{u}_l \cdot \nabla\omega \in L^2_{\gamma,l}\}$$

We observe the following decomposition of the operator $\mathbf{L}_l$:

$$\mathbf{L}_l = \mathbf{M}_l + \mathbf{K}_l + \mathbf{S}_l$$

as the names suggest, we have the main skew-symmetric term:

$$-\mathbf{M}_l\omega := \bar{u}_l \cdot \nabla\omega + \frac{1}{2}\omega\nabla \cdot v_l$$

the compact term:

$$-\mathbf{K}_l\omega := \mathrm{BS}_l(\omega)\cdot\nabla\bar{\omega}_l$$

and the small term:

$$-\mathbf{S}_l\omega := -\frac{\bar{u}_l^r}{r+l}\omega - \frac{(\mathrm{BS}_l(\omega))^r}{r+l}\bar{\omega}_l - \frac{1}{2}\omega\nabla\cdot v_l$$

Remark 4.3. *Note that the additional term $\frac{1}{2}\omega\nabla\cdot v_l$ is to make $\mathbf{M}_l$ a skew-adjoint operator.*

Here are the basic properties of the compact and small operators:

Lemma 4.2. *$\mathbf{K}_l, \mathbf{S}_l : L^2_{\gamma,l} \to L^2_{\gamma,l}$ are two bounded operators. Moreover, $\mathbf{K}_l$ is a compact operator, and*

$$\|\mathbf{S}_l\|_{L^2_{\gamma,l}\to L^2_{\gamma,l}} \xrightarrow[l\to\infty]{} 0$$

Proof. We follow the notion of the process above. Since we are considering the behavior with l varying, all the inequality constants are independent of l.

By the polynomial growth property of weight function γ, we have:

$$\|\omega_l\|^2_{L^2(\mathbb{R}^3)^3} = 2\pi\int_{\mathbb{R}}\int_0^\infty (r+l)|\omega(r,z)|^2\mathrm{d}r\mathrm{d}z \lesssim l\|\omega\|^2_{L^2_{\gamma,l}}$$

then, by the standard estimate of BS_{3d}, and apply the Sobolev embedding:

$$\|u_{l,3d}\|_{L^6(\mathbb{R}^3)^3} \lesssim \|u_{l,3d}\|_{\dot{H}^1(\mathbb{R}^3)^3} \lesssim \|\omega_l\|_{L^2(\mathbb{R}^3)^3} \lesssim l^{\frac{1}{2}}\|\omega\|_{L^2_{\gamma,l}}$$

Thus, by the formulation of $\mathrm{BS}_l[\omega]$, we have:

$$\begin{aligned}
&\|\mathrm{BS}[\omega]\|^6_{L^6(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)}\\
&\quad= \int_0^\infty\int_{\mathbb{R}}(|u_l^r(r+l,z)|^6 + |u_z^r(r+l,z)|^6)(r+l)\mathrm{d}r\mathrm{d}z\\
&\quad\lesssim \int_0^\infty\int_0^{2\pi}\int_{\mathbb{R}}(\sin^6(\theta)+\cos^6(\theta))|u_l^r(r+l,z)|^6\\
&\qquad\qquad\qquad\qquad + u_z^r(r+l,z)|^6)(r+l)\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\\
&\quad\lesssim \|u_{l,3d}\|^6_{L^6(\mathbb{R}^3)^3}
\end{aligned}\tag{4.18}$$

Recall that $\mathrm{supp}(\bar{u})\cup\mathrm{supp}(\bar{\omega})\in B_0(\bar{R})$, for the derivative of $\mathrm{BS}_l[\omega]$:

$$\begin{aligned}
\int_{B_0(\bar{R})}|\nabla\mathrm{BS}_l[\omega](r,z)|^2\mathrm{d}r\mathrm{d}z &\lesssim \frac{1}{l}\int_{B_0(\bar{R})}|\nabla u_{l,3d}(r+l,z)|^2(r+l)\mathrm{d}r\mathrm{d}z\\
&\lesssim \frac{1}{l}\|u_{l,3d}\|^2_{\dot{H}^1(\mathbb{R}^3)^3} \lesssim \|\omega\|_{L^2_{\gamma,l}}
\end{aligned}$$

The Compactness of $\mathbf{K}_l$ comes from the smoothness of $\nabla\bar{\omega}_l$ and Rellich theorem.

To give an estimate on $\mathbf{S}_l$, firstly we have the estimate on $\nabla\cdot v_l$:

$$\nabla\cdot v_l = -\frac{1}{r+l}\bar{u}^r - \frac{1}{r+l}v_l^r \xrightarrow[l\to\infty]{} 0 \quad \textit{in } L^\infty(\mathbb{R}_{\geq -l}\times\mathbb{R})$$

Combine all the estimates above, we have:

$$\left\| \frac{\bar{u}_l^r}{r+l}\omega + \frac{1}{2}\omega\nabla\cdot v_l \right\|_{\mathbf{L}^2_{\gamma,l}} \lesssim \left(\frac{\|\bar{u}_l^r\|_{L^\infty}}{r+l} + \|\nabla\cdot v_l\|_{L^\infty} \right) \|w\|_{\mathbf{L}^2_{\gamma,l}}$$

where the bound satisfies:

$$\frac{\|\bar{u}_l^r\|_{L^\infty}}{r+l} + \|\nabla\cdot v_l\|_{L^\infty} \xrightarrow[l\to\infty]{} 0$$

For the remaining term, regarding $\mathrm{supp}(\bar{\omega}_l) \in B_0(\bar{R})$,

$$\begin{aligned} \left\| \frac{(\mathrm{BS}_l(\omega))^r}{r+l}\bar{\omega}_l \right\|_{\mathbf{L}^2_{\gamma,l}} &\lesssim \frac{1}{l-\bar{R}} \|\mathrm{BS}_l(\omega)\|_{L^6(B_0(\bar{R}))} \|\bar{\omega}_l\|_{L^3(B_0(\bar{R}))} \\ &\lesssim l^{-\frac{7}{6}} \|\mathrm{BS}_l(\omega)\|_{L^6((r+l)\mathrm{d}r\mathrm{d}z)} \|\bar{\omega}_l\|_{L^3(B_0(\bar{R}))} \\ &\lesssim l^{-\frac{2}{3}} \|w\|_{L^2_{\gamma,l}} \|\bar{\omega}_l\|_{L^3(B_0(\bar{R}))} \end{aligned}$$

In the last step, we apply the estimate (4.18), and since the decaying property of v_l, we have that $\|\bar{\omega}_l\|_{L^\infty}$ is uniformly bounded with respect to l, thus give us the decay property of $\mathbf{S}_l$. □

With the decomposition above, we could make the following claim: *The essential spectrum of $\mathbf{L}_l$ lies in the left half-plane $\{\Re(z) \le o_{l\to\infty}(1)\}$, in the remaining half-plane, the spectrum is consisted of discrete eigenvalues, with finite algebraic multiplicity.* We could always take l sufficiently large, such that $o_{l\to\infty}(1) < \frac{1}{2}$.

Since we are considering l large enough, as the small term turn to zero, only need to consider the formal limits of the main and compact term:

$$\begin{aligned} \mathbf{K}_\infty : \quad & L^2_\gamma & \to \quad & L^2_\gamma \\ & \omega & \mapsto \quad & -\mathrm{BS}_{2d}[\omega]\cdot\nabla\bar{\omega} \\ \mathbf{M}_\infty : \mathrm{D}(\mathbf{M}_\infty) \subset\ & L^2_{\gamma,l} & \to \quad & L^2_{\gamma,l} \\ & \omega & \mapsto \quad & -\bar{u}\cdot\nabla\omega \end{aligned}$$

where the domain of definition:

$$\mathrm{D}(\mathbf{M}_\infty) = \{\omega \in L^2_\gamma | \bar{u}\cdot\nabla\omega \in L^2(B_0(\bar{R}))\}$$

We could observe that:

$$\mathbf{M}_\infty + \mathbf{K}_\infty = \mathbf{L}_\infty$$

which, by the discussion of the previous subsection, has an unstable eigenvalue. Thus, we could expect that for l large enough, the operator $\mathbf{L}_l$ is unstable:

Proposition 4.3. *Let λ_∞ be an unstable eigenvalue of $\mathbf{L}_\infty$, then, for all $\epsilon \in (0, \Re(\lambda_\infty))$, there exist $l_0 = l_0(\bar{u}, \lambda_\infty, \epsilon) \ge 2\bar{R}$, such that for all $l \ge l_0$, the operator $\mathbf{L}_l$ has an unstable eigenvalue λ_l with $|\lambda_l - \lambda_\infty| < \epsilon$*

Before beginning the proof, we firstly make some observation. Since l might vary, it could be more convenient to consider all operators defined on the same space L^2_γ rather than $L^2_{\gamma,l}$. Thus, consider the following projection operator:

$$\begin{array}{rcl} \mathrm{P}_l : L^2_\gamma & \to & L^2_{\gamma,l} \\ \omega & \mapsto & \omega|_{r\geq -l} \end{array}$$

By composing with $\mathbf{L}_l$, we obtain:

$$\mathbf{L}_l\mathrm{P}_l = \mathbf{M}_l\mathrm{P}_l + \mathbf{K}_l\mathrm{P}_l + \mathbf{S}_l\mathrm{P}_l : \mathrm{P}_l^{-1}[\mathrm{D}(\mathbf{L}_l)] \subset L^2_\gamma \to L^2_\gamma$$

and it's easy to see that for $l \geq 2\bar{R}$,

$$L^2_\gamma = L^2(\mathbb{R}_{r<-l} \times \mathbb{R}, \gamma \mathrm{d}r\mathrm{d}z) \oplus L^2_{\gamma,l}$$

Thus for $z \neq 0$, we have:

$$z \in \sigma(\mathbf{L}_l) \Longleftrightarrow z \in \sigma(\mathbf{L}_l\mathrm{P}_l) \tag{4.19}$$

The original problem could be transformed into the generalized lemma below:

Lemma 4.3. *Let* $(\mathbf{M}_n)_{n\in\mathbb{N}}$, $\mathbf{M}_\infty : \mathrm{D}(\mathbf{M}_\infty) \subset \mathcal{H} \to \mathcal{H}$ *closed normal operators,* $(\mathbf{K}_n)_{n\in\mathbb{N}}$, $\mathbf{K}_\infty : \mathcal{H} \to \mathcal{H}$ *compact operators,* $(\mathbf{S}_n)_{n\in\mathbb{N}} : \mathcal{H} \to \mathcal{H}$ *bounded operators, satisfying the following properties:*

1. *There exist* $(\mu_n)_{n\in\mathbb{N}\cup\{\infty\}} \in \mathbb{R}$, *such that* $\lim_{n\to\infty} \mu_n = \mu_\infty$, *and:*

$$\sigma(\mathbf{M}_n) \subset \{z \in \mathbb{C} | \Re(z) \leq \mu_n\}, \quad \forall n \in \mathbb{N} \cup \{\infty\}$$

2. *For all* $f \in \mathcal{H}$,

$$(\lambda - \mathbf{M}_n)^{-1} f \underset{n\to\infty}{\longrightarrow} (\lambda - \mathbf{M}_\infty)^{-1} f \quad \textit{in } \mathcal{H} \tag{4.20}$$

locally uniformly on $\lambda \in \{z \in \mathbb{C} | \Re(z) > \mu_\infty\}$

3. $\mathbf{K}_n \longrightarrow \mathbf{K}_\infty$ *in operator norm.*

4. *The smallness of* $\{\mathbf{S}_n\}_{n\in\mathbb{N}}$*:*

$$\lim_{n\to\infty} \|\mathbf{S}_n\|_{\mathcal{H}\to\mathcal{H}} = 0 \tag{4.21}$$

Let λ_∞ *be an isolated eigenvalue of* $\mathbf{M}_\infty + \mathbf{K}_\infty$, *such that* $\Re(\lambda_\infty) > \mu_\infty$, *and let* $V \subset \{z \in \mathbb{C} | \Re(z) > \mu_\infty\}$ *such that* $V \cap \sigma(\mathbf{M}_\infty + \mathbf{K}_\infty) = \{\lambda_\infty\}$, *then, for* n *big enough, there exist an eigenvalue of* $\mathbf{M}_n + \mathbf{K}_n + \mathbf{S}_n$ *in* V.

Proof. To ease the notion, denote $\mathbf{S}_\infty = 0$, and for $n \in \mathbb{N} \cup \{\infty\}$, we denote:

$$\mathbf{C}_n := \mathbf{M}_n + \mathbf{K}_n + \mathbf{S}_n$$

We start by showing that: for all $C \subset \text{res}(\mathbf{C}_\infty) \cap \{z \in \mathbb{C} | \Re(z) > \mu_\infty\}$ compact, $f \in \mathcal{H}$, we have the following convergence for $\lambda \in C$ uniformly:

$$(\lambda - \mathbf{C}_n)^{-1} f \underset{n\to\infty}{\longrightarrow} (\lambda - \mathbf{C}_\infty)^{-1} f \quad \textit{in } \mathcal{H} \tag{4.22}$$

For n sufficiently large, such that $\{z \in \mathbb{C} | \Re(z) \le \mu_n\} \cap C = \varnothing$, we have the following identity:

$$\lambda - \mathbf{M}_n - \mathbf{K}_\infty = (\lambda - \mathbf{M}_n)(\text{Id} - (\lambda - \mathbf{M}_n)^{-1}\mathbf{K}_\infty)$$

By corollary (A.1), with the assumption, we have the following convergence uniformly on C in operator norm:

$$(\lambda - \mathbf{M}_n)^{-1}\mathbf{K}_\infty \underset{n\to\infty}{\longrightarrow} (\lambda - \mathbf{M}_\infty)^{-1}\mathbf{K}_\infty$$

By the assumption (4.20) we made on $\lambda \in C$, we have:

$$1 \in \text{res}((\lambda - \mathbf{M}_\infty)^{-1}\mathbf{K}_\infty)$$

thus, by the same reasoning in (4.8), for n sufficiently large, we have:

$$1 \in \text{res}((\lambda - \mathbf{M}_n)^{-1}\mathbf{K}_\infty) \Rightarrow C \subset \text{res}(\mathbf{M}_n + \mathbf{K}_\infty)$$

and the inversion identity:

$$(\lambda - \mathbf{M}_n - \mathbf{K}_\infty)^{-1} = (\text{Id} - (\lambda - \mathbf{M}_n)^{-1}\mathbf{K}_\infty)^{-1}(\lambda - \mathbf{M}_n)^{-1}$$

By the similar argument of (4.9), we have the following convergence in operator norm uniformly for $\lambda \in C$:

$$(\text{Id} - (\lambda - \mathbf{M}_n)^{-1}\mathbf{K}_\infty)^{-1} \underset{n\to\infty}{\longrightarrow} (\text{Id} - (\lambda - \mathbf{M}_\infty)^{-1}\mathbf{K}_\infty)^{-1}$$

combine the two formulas above and the assumption (4.20), for all $f \in \mathcal{H}$, we have the following convergence for $\lambda \in C$ uniformly:

$$(\lambda - \mathbf{M}_n - \mathbf{K}_\infty)^{-1} f \underset{n\to\infty}{\longrightarrow} (\lambda - \mathbf{M}_\infty - \mathbf{K}_\infty)^{-1} f \quad \textit{in } \mathcal{H} \tag{4.23}$$

and by (A.3), we have $(\lambda - \mathbf{M}_n - \mathbf{K}_\infty)^{-1}$ is uniformly bounded for n large.

Observe the following expression:

$$\lambda - \mathbf{M}_n - \mathbf{K}_n = (\lambda - \mathbf{M}_n - \mathbf{K}_\infty)(\text{Id} - (\lambda - \mathbf{M}_n - \mathbf{K}_\infty)^{-1}(\mathbf{K}_\infty - \mathbf{K}_n))$$

and for the same reasoning as (4.9), plus the uniform bound of $(\lambda - \mathbf{M}_n - \mathbf{K}_\infty)^{-1}$, we have the following convergence in operator norm for $\lambda \in C$ uniformly:

$$(\text{Id} - (\lambda - \mathbf{M}_n - \mathbf{K}_\infty)^{-1}(\mathbf{K}_\infty - \mathbf{K}_n))^{-1} \underset{n\to\infty}{\longrightarrow} \text{Id}$$

thus, for $f \in \mathcal{H}$, we have the following convergence for $\lambda \in C$ uniformly:

$$(\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1} f \underset{n\to\infty}{\longrightarrow} (\lambda - \mathbf{M}_\infty - \mathbf{K}_\infty)^{-1} f \quad \textit{in } \mathcal{H}$$

and $(\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1}$ is uniformly bounded for n large.

Finally,

$$\lambda - \mathbf{C}_n = (\lambda - \mathbf{M}_n - \mathbf{K}_n)(\mathrm{Id} - (\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1}\mathbf{S}_n)$$

by the uniform boundedness of $(\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1}$ and assumption (4.21), we have for n large enough, $\lambda - \mathbf{C}_n$ is invertible, and we have the following inversion formula:

$$(\lambda - \mathbf{C}_n)^{-1} = (\mathrm{Id} - (\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1}\mathbf{S}_n)^{-1}(\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1}$$

Apply again the argument of (4.9), we have the following convergence in operator norm for $\lambda \in C$:

$$(\mathrm{Id} - (\lambda - \mathbf{M}_n - \mathbf{K}_n)^{-1}\mathbf{S}_n)^{-1} \underset{n\to\infty}{\longrightarrow} \mathrm{Id}$$

Combine the result above, we obtain the claim (4.22).

Let $\vec{\Gamma} \subset V$ be a counterclockwise contour containing λ_∞, for $n \in \mathbb{N} \cup \{\infty\}$ sufficiently large, consider the Riesz projectors:

$$\mathrm{Pr}_{\vec{\Gamma}}(\mathbf{C}_n) := \frac{1}{2\pi i}\int_{\vec{\Gamma}} (\lambda - \mathbf{C}_n)^{-1} \mathrm{d}\lambda$$

By hypothesis, the operator $\mathrm{Pr}_{\vec{\Gamma}}(\mathbf{C}_\infty)$ is non-trivial, and by the convergence we have proven just before, for any $f \in \mathcal{H}$

$$\mathrm{Pr}_{\vec{\Gamma}}(\mathbf{C}_n)f \underset{n\to\infty}{\longrightarrow} \mathrm{Pr}_{\vec{\Gamma}}(\mathbf{C}_\infty)f \quad \textit{in } \mathcal{H}$$

Thus for n large enough, $\mathrm{Pr}_{\vec{\Gamma}}(\mathbf{C}_n)$ is non-trivial, $\sigma(\mathbf{C}_n) \cap V \neq \varnothing$. The only thing left to study is the spectrum of $\mathbf{M}_n + \mathbf{S}_n$

For $\Re(\lambda) > \mu_n$, we have the following identity:

$$\lambda - \mathbf{M}_n + \mathbf{S}_n = (\lambda - \mathbf{M}_n)(\mathrm{Id} - (\lambda - \mathbf{M}_n)^{-1}\mathbf{S}_n)$$

by the norm estimate (A.3), we have:

$$\|(\lambda - \mathbf{M}_n)^{-1}\mathbf{S}_n\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{\|\mathbf{S}_n\|_{\mathcal{H}\to\mathcal{H}}}{\Re(\lambda) - \mu_n}$$

Thus, we have:

$$\sigma(\mathbf{M}_n + \mathbf{S}_n) \subset \{z \in \mathbb{C} | \Re(z) \leq \mu_n + \|\mathbf{S}_n\|_{\mathcal{H}\to\mathcal{H}}\}$$

Thus, by the reasoning (A.1), the essential spectrum of $\mathbf{C}_n$ also lies in the half-plane defined above, thus there is a eigenvalue of $\mathbf{C}_n$ in V. □

Proof. To complete the proof of the original proposition, we proof by contradiction. If proposition (4.3) fails to be true, then, there exists $\epsilon \in (0, \Re(\lambda_\infty))$ and a sequence of $(l_n)_{n\in\mathbb{N}} \to \infty$, satisfying the operator $\mathbf{L}_{l_n}$ has no eigenvalue in $B_{\lambda_\infty}(\epsilon)$.

We apply Lemma (4.3) to the contradicting triplet $(\mathbf{M}_{l_n},\ \mathbf{K}_{l_n},\ \mathbf{S}_{l_n})$, it remains to verify the hypotheses.

1 Directly deduced by the skew-adjointness of $(\mathbf{M}_l \mathrm{P}_l)_{l \geq 2\bar{R}}$ and $\mathbf{M}_\infty$

2 We consider a series of auxiliary operators:

$$-\mathbf{M}'_l \omega := \bar{u}_l \cdot \nabla \omega$$

then, by the calculation done in (4.2), we have:

$$\lim_{l \to \infty} \|\mathbf{M}_l - \mathbf{M}'_l\|_{L^2_{\gamma,l} \to L^2_{\gamma,l}} = 0$$

For $\lambda \in \mathbb{C}$, satisfying $\Re(\lambda) > 0$, we have the following identity:

$$\lambda - \mathbf{M}'_l \mathrm{P}_l = (\lambda - \mathbf{M}_l)(\mathrm{Id} - (\lambda - \mathbf{M}_l \mathrm{P}_l)^{-1}(\mathbf{M}_l - \mathbf{M}'_l)\mathrm{P}_l)$$

since $\mathbf{M}_l \mathrm{P}_l$ is skew-adjoint, thus normal, by proposition (A.3), we have:

$$\|(\mathrm{Id} - (\lambda - \mathbf{M}_l \mathrm{P}_l))^{-1}\|_{L^2_\gamma \to L^2_\gamma} = \Re(\lambda)^{-1}$$

thus, the bound is uniform for $\lambda \in C$. As a consequence,

$$\sigma(\mathbf{M}'_l \mathrm{P}_l) \subset \{z \in \mathbb{C} | |\Re(z)| \leq \|(\mathbf{M}_l - \mathbf{M}'_l)\mathrm{P}_l\|_{L^2_\gamma \to L^2_\gamma}\}$$

and for $\lambda \in \mathbb{C}$, such that $\Re(\lambda) > \|(\mathbf{M}_l - \mathbf{M}'_l)\mathrm{P}_l\|_{L^2_\gamma \to L^2_\gamma}$, we have:

$$(\lambda - \mathbf{M}'_l \mathrm{P}_l)^{-1} = (\mathrm{Id} - (\lambda - \mathbf{M}_l \mathrm{P}_l)^{-1}(\mathbf{M}_l - \mathbf{M}'_l)\mathrm{P}_l)^{-1}(\lambda - \mathbf{M}_l)^{-1}$$

By the same argument of (4.9), we have the following convergence in operator norm for $\lambda \in C$ uniformly:

$$(\mathrm{Id} - (\lambda - \mathbf{M}_l \mathrm{P}_l)^{-1}(\mathbf{M}_l - \mathbf{M}'_l)\mathrm{P}_l)^{-1} \underset{l \to \infty}{\longrightarrow} \mathrm{Id} \tag{4.24}$$

So we have:

$$\|(\lambda - \mathbf{M}_l \mathrm{P}_l)^{-1} - (\lambda - \mathbf{M}'_l \mathrm{P}_l)^{-1}\|_{L^2_\gamma \to L^2_\gamma} = 0$$

To ease the notion, we denote

$$\bar{u} = \bar{u}_\infty \quad \mathbf{M}_\infty \mathrm{P}_\infty = \mathbf{M}'_\infty$$

For $l \in (2\bar{R}, \infty]$, $\mathbf{M}'_l \mathrm{P}_l$ are operators of the form $\omega \mapsto -\bar{u}_l \cdot \nabla \omega$, we could give a explicit formula of $(\lambda - \mathbf{M}'_l \mathrm{P}_l)^{-1}$:

Consider the flow $X_l(t, x)$ induced by vector field $\bar{u}_l$, satisfying:

$$\begin{cases} \partial_t X_l(t, x) = \bar{u}_l(X_l(t, x)) \\ X_l(0, x) = x \end{cases}$$

Then, by the property of transport equation, we have:

$$\partial_t X_l(t, x) = \bar{u}_l(x) \cdot \nabla_x X_l(t, x)$$

Then, for $l \in (2\bar{R}, \infty]$, $f \in L^2_\gamma$, define the following integration:

$$\mathbf{A}^{(l)}_\lambda f(x) := -\int_0^\infty e^{-\lambda t} f(X_l(t,x))\mathrm{d}t$$

then, we have the following computation:

$$\begin{aligned}
-\mathbf{M}'_L \mathrm{P}_l(\mathbf{A}^{(l)}_\lambda f)(x) &= \int_0^\infty e^{-\lambda t}\bar{u}_l \cdot \nabla(f(X_l(t,x)))\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t}\bar{u}_l \cdot \nabla_x X_l(t,x)\nabla f(X_l(t,x))\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t}\partial_t X_l(t,x)\nabla f(X_l(t,x))\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t}\mathrm{d}f(X_l(t,x)) = f(x) - \lambda \mathbf{A}^{(l)}_\lambda f(x)
\end{aligned}$$

where in the last step we apply integration by parts.

Thus, for $\lambda \in \mathrm{res}(\mathbf{M}'_L \mathrm{P}_l)$, we have:

$$(\lambda - \mathbf{M}'_L \mathrm{P}_l)^{-1} = \mathbf{A}^{(l)}_\lambda$$

by the explicit expression of $\bar{u}_l$, we have $X_l(t,x) \equiv x$, thus:

$$\begin{aligned}
&\|[(\lambda - \mathbf{M}'_L \mathrm{P}_l)^{-1} - (\lambda - \mathbf{M}_\infty)^{-1}]f\|_{L^2_\gamma} \\
&\qquad \leq \int_0^\infty e^{-\Re(\lambda)t}\|f(X_l) - f(X_\infty)\|_{L^2(B_0(\bar{R}))}\mathrm{d}t
\end{aligned}$$

By the smallness of v_l in lemma (4.1), we have X_l converges locally uniformly to X_∞ on $[0,\infty) \times B_0(\bar{R})$ as $l \to \infty$, thus the right hand side converges to zero uniformly for $\lambda \in C$. Combine with (4.24), we obtain hypothesis 2.

3 To ease notions, for $p > 1$, $R > 0$ we denote:

$$\|f\|_{L^p_R} := \|f\|_{L^p(B_0(R))}$$

Since we have $\nabla\bar{\omega}_l \in C^\infty_0(B_0(\bar{R}))$, only need to prove that for all $\omega \in L^2_\gamma$,

$$\|\mathrm{BS}_{2d}[\omega] - \mathrm{BS}_l[\mathrm{P}_l\omega]\|_{L^2_{\bar{R}}} \leq o_{l\to\infty}(1)\|\omega\|_{L^2_\gamma}$$

We follow the notion of (4.17), observe that if $l = \infty$, it is identical with the normal 2-dimensional Biot-Savart operator BS_{2d}.

Recall that the images satisfy the following equations:

$$\mathrm{BS}_{2d}[\omega] = -\partial_z\varphi e_r + \partial_r\varphi e_z$$

$$\mathrm{BS}_l[\omega] = -\partial_z\varphi_l e_r + (\partial_r\varphi_l + \frac{1}{r+l}\varphi_l)e_z$$

and the supporting functions satisfy:

$$\Delta_l \varphi_l = \left(\partial_r^2 + \frac{1}{r+l}\partial_r - \frac{1}{(r+l)^2} + \partial_z^2\right)\varphi_l = \omega, \quad \partial_r \varphi_l|_{r=-l} = 0$$

$$\Delta \varphi = \left(\partial_r^2 + \partial_z^2\right)\varphi = \omega$$

Thus, we have:

$$\|\mathrm{BS}_{2d}[\omega] - \mathrm{BS}_l[\mathrm{P}_l\omega]\|_{L^2_{\bar{R}}} \le \|\frac{1}{r+l}\varphi_l\|_{L^2_{\bar{R}}} + \|\nabla(\varphi - \varphi_l)\|_{L^2_{\bar{R}}} =: I + II$$

Now we estimate the term I and II separately.

For the term I, recall that the 3D Biot-Savart kernel defined in (4.5), apply Young's inequality for weak L^p norms (B.1), for $p \in (1,3)$, we have:

$$\|(\partial_r, \partial_z)\varphi_l\|_{L^{p^*}(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)} \lesssim_p \|\omega\|_{L^p(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. And apply Sobolev embedding, we have:

$$\|\varphi_l\|_{L^q(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)} \lesssim_p \|\omega\|_{L^p(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{2}{3}$.

Observe that for $p \in (1,2]$, due to the decay of γ, apply Hölder's inequality, we have:

$$\|\omega\|_{L^p(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)} \lesssim_p l^{\frac{1}{p}}\|\omega\|_{L^2_\gamma}$$

Thus, take $p = \frac{6}{5}$, $q = 6$, we have:

$$\|\frac{1}{r+l}\varphi_l\|_{L^2_{\bar{R}}} \lesssim \|\frac{1}{r+l}\varphi_l\|_{L^6_{\bar{R}}} \lesssim l^{-\frac{7}{6}}\|\varphi_l\|_{L^6(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)} \lesssim l^{-\frac{1}{3}}\|\omega\|_{L^2_\gamma}$$

Thus we obtain the control for the first term.

For the term II, recall that the 2D Biot-Savart kernel $K_{\mathrm{BS}_{2d}} \in L^2_w(\mathbb{R}^2)$, thus for $p \in (1,2)$, we have:

$$\|(\partial_r, \partial_z)\varphi\|_{L^{p^*}(\mathbb{R}^2)} \lesssim \|\omega\|_{L^p(\mathbb{R}^2)} \lesssim_p \|\omega\|_{L^2_\gamma}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$. Then, apply Sobolev inequality, we have:

$$\|\varphi\|_{\dot{C}^{2-\frac{2}{p}}} \lesssim \|\omega\|_{L^p(\mathbb{R}^2)} \lesssim_p \|\omega\|_{L^2_\gamma}$$

where $\|\cdot\|_{\dot{C}^{2-\frac{2}{p}}}$ denotes the homogeneous Hölder norm. Since φ is defined up to a constant, and adding a constant does not change the value of term II, thus, we could assume that $\varphi(0) = 0$, and for $R > 0$, $q \ge 1$, $p \in (1,2)$, we have:

$$\|\varphi\|_{L^q(B_0(R))} \lesssim_p R^{2-\frac{2}{p}+\frac{2}{q}}\|\omega\|_{L^2_\gamma}$$

To estimate $\nabla(\varphi_l - \varphi)$, we introduce a cutoff function to extend the region into whole space. Let $\chi \in C_0^\infty(\mathbb{R}^2)$, satisfying:

$$\chi|_{B_0(\frac{1}{4})} \equiv 1, \quad \chi|_{B_0^C(\frac{1}{2})} \equiv 0$$

For $l \geq 8\bar{R}$, take $\chi_l(x) := \chi(\frac{x}{l})$, then:

$$\Delta[(\varphi_l - \varphi)\chi_l] = \frac{1}{(r+l)^2}\varphi_l\chi_l - \frac{1}{r+l}\partial_r\varphi_l\chi_l + 2\nabla(\varphi_l - \varphi)\cdot\nabla\chi_l + (\varphi_l - \varphi)\Delta\chi_l$$

By Hölder's inequality and Sobolev embedding, we have:

$$\|\nabla(\varphi_l - \varphi)\|_{L^2_{\bar{R}}} \lesssim \|\nabla[(\varphi_l - \varphi)\chi_l]\|_{L^6(\mathbb{R}^2)} \lesssim \|\Delta[(\varphi_l - \varphi)\chi_l]\|_{L^{\frac{3}{2}}}$$

and since $\nabla\chi_l \lesssim \frac{1}{l}$, $\Delta\chi_l \lesssim \frac{1}{l^2}$, we have:

$$\begin{aligned}&\|\Delta[(\varphi_l - \varphi)\chi_l]\|_{L^{\frac{3}{2}}_{\frac{l}{2}}}\\ &\qquad\lesssim \frac{1}{l^2}(\|\varphi_l\|_{L^{\frac{3}{2}}_{\frac{l}{2}}} + \|\varphi\|_{L^{\frac{3}{2}}_{\frac{l}{2}}}) + \frac{1}{l}(\|\nabla\varphi_l\|_{L^{\frac{3}{2}}_{\frac{l}{2}}} + \|\nabla\varphi\|_{L^{\frac{3}{2}}_{\frac{l}{2}}})\end{aligned}$$

Thus, we apply the estimates above,

$$\|\varphi_l\|_{L^{\frac{3}{2}}_{\frac{l}{2}}} \lesssim l^{\frac{7}{12}}\|\varphi_l\|_{L^4(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)} \lesssim l^{\frac{3}{2}}\|\omega\|_{L^2_\gamma}$$

Let $p = \frac{12}{11}$, $q = \frac{3}{2}$,

$$\|\varphi\|_{L^{\frac{3}{2}}_{\frac{l}{2}}} \lesssim l^{\frac{3}{2}}\|\omega\|_{L^2_\gamma}$$

Similarly,

$$\|\nabla\varphi_l\|_{L^{\frac{3}{2}}_{\frac{l}{2}}} \lesssim l^{-\frac{1}{6}}\|\nabla\varphi_l\|_{L^2(\mathbb{R}^2,(r+l)\mathrm{d}r\mathrm{d}z)} \lesssim l^{\frac{2}{3}}\|\omega\|_{L^2_\gamma}$$

$$\|\nabla\varphi\|_{L^{\frac{3}{2}}_{\frac{l}{2}}} \lesssim l^{\frac{8}{15}}\|\nabla\varphi\|_{L^{\frac{5}{2}}_{\frac{l}{2}}} \lesssim l^{\frac{8}{15}}\|\omega\|_{L^2_\gamma}$$

thus, we obtain the decay with respect to l for term II.

4 Directly from Lemma (4.2).

Thus, Lemma (4.3) yields Proposition (4.3). □

5 Instability: From Euler to Navier-Stokes

As mentioned in the strategy part, the linearized operator of Navier-Stokes equation formally approaches the linearized operator of Euler equation while the amplitude of the velocity profile is sufficiently large. In this section, we will give a rigorous statement of the idea above.

5.1 Vorticity Instability

For l sufficiently large, we fix $\bar{U}$ to be the 3-dimensional axisymmetical-no-swirl velocity profile associated with $\bar{u}_l$ with respect to coordinate (r', z):

$$\bar{U}(r', z) := \bar{u}_l^r(r' - l, z)e_r + \bar{u}_l^z(r' - l, z)e_z$$

and $\bar{\Omega}$ is its corresponding vorticity field

Let L^2_{aps} denote the set of L^2 integrable axisymmetrical-pure-swirl vector field on $\mathbb{R}^3$:

$$L^2_{\text{aps}} = \{\Omega^\theta(r', z)e_\theta | \Omega^\theta \in L^2(\mathbb{R}^3)\}$$

We consider the base profile with relative amplitude β, i.e. the operator is linearized around $\beta\bar{U}$, take curl on the operator L_{ss} defined in (3.4) and restrict it on the set of L^2_{aps}, we obtain the following operator:

$$\begin{array}{ccc} \mathbf{L}^{(\beta)}_{\text{vor}} : \mathrm{D}(\mathbf{L}^{(\beta)}_{\text{vor}}) & \to & L^2_{\text{aps}} \\ \Omega & \mapsto & \dfrac{1}{2}\left(2 + \xi \cdot \nabla_\xi\right)\Omega + \Delta\Omega - \beta([\bar{U}, \Omega] + [U, \bar{\Omega}]) \end{array}$$

where the domain of definition:

$$\mathrm{D}(\mathbf{L}^{(\beta)}_{\text{vor}}) := \{\Omega \in L^2_{\text{aps}} \cap H^2(\mathbb{R}^3) | \xi \cdot \nabla_\xi \Omega \in L^2(\mathbb{R}^3)\}$$

the lifted velocity field $U := \mathrm{BS}_{3d}[\Omega]$, and the bracket:

$$[f, g] := (f \cdot \nabla)g - (g \cdot \nabla)f$$

We could decompose the operator into the following terms and a multiplication by constant:

$$\begin{aligned} \mathbf{D}\Omega &= (\frac{3}{4} + \frac{\xi}{2} \cdot \nabla_\xi)\Omega + \Delta\Omega & \quad \mathbf{M}\Omega &= -\bar{U} \cdot \nabla\Omega \\ \mathbf{S}\Omega &= \bar{\Omega} \cdot \nabla U + \Omega \cdot \nabla\bar{U} & \quad \mathbf{K}\Omega &= -U \cdot \nabla\bar{\Omega} \end{aligned} \tag{5.1}$$

D is the **diffusion** term, **M** is the **main** term, **S** is the **small** term, and **K** is the **compact** term. Observe that $\mathrm{D} - \Delta$ is a skew-adjoint operator.

Finally, we define:

$$\mathbf{T}_\beta := \frac{1}{\beta}\mathbf{D} + \mathbf{M} + \mathbf{S} + \mathbf{K} = \frac{1}{\beta}\left(\mathbf{L}^{(\beta)}_{\text{vor}} - \frac{1}{4}\right)$$

and we could define the formal limit when $\beta \to \infty$:

$$\mathbf{T}_\infty := \mathbf{M} + \mathbf{S} + \mathbf{K}$$

where the domain of definition is

$$\mathrm{D}(\mathbf{T}_\infty) = \{\Omega \in L^2_{\text{aps}} | \bar{U} \cdot \nabla\Omega \in L^2\}$$

We could view $\mathbf{T}_\infty$ as a lift of $\mathbf{L}_l$ into 3-dimensions, and the part $\mathbf{S}$ is exactly a part of $\mathbf{S}_l$ whose smallness could be obtained by slightly modifying the proof of Lemma (4.2). Thus, we could fix l sufficiently large, such that:

$$\alpha := \sup_{\lambda \in \sigma(\mathbf{T}_\infty)} \Re(\lambda) > \mu := \|\mathbf{S}\|_{L^2_{\text{aps}} \to \mathbf{L}^2(\mathbb{R}^3)} \tag{5.2}$$

From now on, we consider l and $\bar{U}$ as fixed and satisfying the estimate (5.2).

Now we state the instability of self-similar Navier-Stokes equation:

Theorem 5.1. *Let λ_∞ be an unstable eigenvalue of $\mathbf{T}_\infty$ such that $\Re(\lambda_\infty) > \mu$, then for all $\epsilon \in (0, \Re(\lambda_\infty) - \mu)$, there exists $\beta_0 > 0$, such that for all $\beta > \beta_0$, $\mathbf{T}_\beta$ has an unstable eigenvalue λ_β such that $|\lambda_\beta - \lambda_\infty| < \epsilon$, and $\mathbf{L}^{(\beta)}_{\text{vor}}$ has an unstable eigenvalue defined by*

$$\tilde{\lambda}_\beta := \beta\lambda_\beta + \frac{1}{4}$$

We could see that the theorem above is analogous to Proposition (4.3), also of type (4.3), thus, we need the following lemma concerning the resolvent set and pointwise convergence to ensure condition (1) and (2):

Lemma 5.1. *For all $\beta > 0$ and $\lambda \in \{\Re(\lambda) > \mu\}$, we have:*

$$\|(\lambda - \beta^{-1}\mathbf{D} - \mathbf{M} - \mathbf{S})^{-1}\|_{L^2_{\text{aps}} \to L^2_{\text{aps}}} \leq (\Re(\lambda) - \mu)^{-1}$$

and for all $\Omega \in L^2_{\text{aps}}$, we have:

$$(\lambda - \beta^{-1}\mathbf{D} - \mathbf{M} - \mathbf{S})^{-1}\Omega \underset{\beta\to\infty}{\longrightarrow} (\lambda - \mathbf{M} - \mathbf{S})^{-1}\Omega \quad \text{in } L^2(\mathbb{R}^3)$$

locally uniformly on $\lambda \in \{\Re(\lambda) > \mu\}$.

Proof. Consider Laplace transform to give the explicit formulation of resolvent:

Given $\Omega \in C_0^\infty(\mathbb{R}^3)^3 \cap L^2_{\text{aps}}$, consider the Cauchy problem below:

$$\begin{cases} \partial_\tau \Omega_\beta - \beta^{-1}\left(\Delta + \dfrac{3}{4} + \xi \cdot \nabla_\xi\right)\Omega_\beta + \bar{U} \cdot \nabla\Omega_\beta - \mathbf{S}\Omega_\beta = 0 \\ \Omega_\beta(0, \cdot) = \Omega \end{cases}$$

Since all $\bar{U}$, $\bar{\Omega}$ and the initial value Ω is smooth and compactly supported, the solution is smooth with sufficient decay in space, and satisfies the equation in classical sense. by the standard energy estimate, we have:

$$\frac{1}{2}\partial_\tau \|\Omega_\beta(\tau)\|^2_{L^2} + \beta^{-1}\|\Omega_\beta(\tau)\|^2_{\dot{H}^1} \leq \|\mathbf{S}\Omega_\beta(\tau)\|_{L^2}\|\Omega_\beta(\tau)\|_{L^2} \leq \mu\|\Omega_\beta(\tau)\|^2_{L^2}$$

Use Grönwall's inequality, we have:

$$\|\Omega_\beta(\tau)\|_{L^2} \leq e^{\mu\tau}\|\Omega\|_{L^2}$$

Then, the following integral is well-defined. By integration by parts , we have:

$$(\lambda - \beta^{-1}\mathbf{D} - \mathbf{M} - \mathbf{S})^{-1}\Omega = \int_0^\infty e^{-\lambda\tau}\Omega_\beta(\tau,\cdot)\mathrm{d}\tau$$

apply Minkowski inequality, we have:

$$\|(\lambda - \beta^{-1}\mathbf{D} - \mathbf{M} - \mathbf{S})^{-1}\Omega\|_{L^2} \leq \int_0^\infty e^{-\lambda\tau}\|\Omega_\beta(\tau)\|_{L^2}\mathrm{d}\tau \leq (\Re(\lambda) - \mu)^{-1}$$

To examine the convergence, we consider similar schema for the non-diffusion case $\beta = \infty$:

$$\begin{cases}\partial_\tau\Omega_\infty + \bar{U}\cdot\nabla\Omega_\infty - \mathbf{S}\Omega_\infty = 0\\ \Omega_\infty(0,\cdot) = \Omega\end{cases}$$

which could be viewed as a transport equation whose coefficient function is smooth and compact supported, with a compact supported perturbation term, thus the solution Ω is compactly supported in space and smooth in time and space. Similarly, we have the same energy inequality, and:

$$(\lambda - \mathbf{M} - \mathbf{S})^{-1}\Omega = \int_0^\infty e^{-\lambda\tau}\Omega_\infty(\tau,\cdot)\mathrm{d}\tau$$

The error function $\Omega_\beta - \Omega_\infty$ satisfies the following equation:

$$\begin{aligned}\left(\partial_\tau - \beta^{-1}\left(\Delta + \frac{3}{4} + \xi\cdot\nabla_\xi\right) + \bar{U}\cdot\nabla - \mathbf{S}\right)(\Omega_\beta - \Omega_\infty)\\ = \beta^{-1}\left(\Delta + \frac{3}{4} + \xi\cdot\nabla_\xi\right)\Omega_\infty =: \beta^{-1}F\end{aligned}$$

where F is smooth and compactly supported in space. Then, standard energy estimate tells us:

$$\begin{aligned}\frac{1}{2}\partial_\tau\|\Omega_\beta - \Omega_\infty\|_{L^2}^2 + &\beta^{-1}\|\Omega_\beta - \Omega_\infty\|_{\dot{H}^1}^2\\ &\leq \mu\|\Omega_\beta - \Omega_\infty\|_{L^2}^2 + \beta^{-1}\langle F, \Omega_\beta - \Omega_\infty\rangle_{L^2}\\ &\leq \mu\|\Omega_\beta - \Omega_\infty\|_{L^2}^2 + \beta^{-1}\left(\|\Omega_\beta - \Omega_\infty\|_{\dot{H}^1}^2 + \frac{1}{4}\|F\|_{\dot{H}^{-1}}^2\right)\end{aligned}$$

absorb the $\|\cdot\|_{\dot{H}^1}$ term, we have:

$$\frac{1}{2}\partial_\tau\|\Omega_\beta - \Omega_\infty\|_{L^2}^2 \leq \mu\|\Omega_\beta - \Omega_\infty\|_{L^2}^2 + \frac{1}{4\beta}\|F\|_{\dot{H}^{-1}}^2$$

by Grönwall's inequality, we have:

$$\Omega_\beta - \Omega_\infty \underset{\beta\to\infty}{\longrightarrow} 0 \quad in\ L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^3)^3) \tag{5.3}$$

Split the integral into two parts, and apply Minkowski inequality:

$$\begin{aligned}&\|(\lambda-\beta^{-1}\mathbf{D}-\mathbf{M}-\mathbf{S})^{-1}\Omega-(\lambda-\mathbf{M}-\mathbf{S})^{-1}\Omega\|_{L^2}\\&\leq\int_0^T e^{-\lambda\tau}\|(\Omega_\beta-\Omega_\infty)(\tau)\|_{L^2}\mathrm{d}\tau+\int_T^\infty e^{-\lambda\tau}(\|(\Omega_\beta)(\tau)\|_{L^2}+\|(\Omega_\infty)(\tau)\|_{L^2})\mathrm{d}\tau\end{aligned}$$

The convergence of first part comes form (5.3), and the smallness of second part comes form the energy estimate for $\lambda\in\{\Re(\lambda)>\mu\}$ locally uniformly. ☐

By the reasoning (A.1), we have that:

$$\sigma_{\text{ess}}(\mathbf{T}_\beta)\in\{\Re(\lambda)\leq\mu\}$$

and the remainder of its spectrum is consisted of discrete eigenvalues with finite algebraic multiplicity.

Now we set ourselves in the setting of Lemma (4.3), let:

$$\mathbf{M}_\beta=\beta^{-\mathbf{1}}\mathbf{D}+\mathbf{M}+\mathbf{S},\quad\mathbf{K}_\beta=\mathbf{K},\quad\mathbf{S}_\beta=0$$

by the similar argument of extracting subsequence by contradiction, we get the conclusion of theorem (5.1).

5.2 Return to Velocity Form

In the following discussion, it's more convenient to work on the original linearized operator (3.4) which acts on the velocity fields, we introduce the amplified operator $L_{\text{ss}}^{(\beta)}$

$$\begin{aligned}\mathbf{L}_{\text{ss}}^{(\beta)}:\mathrm{D}(\mathbf{L}_{\text{ss}})&\to L^2_\sigma\\U&\mapsto\frac{1}{2}(1+\xi\cdot\nabla_\xi)U+\Delta U-\beta\mathbb{P}(\bar{U}\cdot\nabla U+U\cdot\nabla\bar{U})\end{aligned}$$

where

$$L^2_\sigma=\{u\in L^2(\mathbb{R}^3)^3|\nabla\cdot u=0\}$$

and the domain of definition is:

$$\mathrm{D}(\mathbf{L}_{\text{ss}})=\{U\in L^2_\sigma\cap H^2(\mathbb{R}^3)^3|\xi\cdot\nabla_\xi U\in L^2(\mathbb{R}^3)^3\}$$

The following corollary identifies the unstable eigenvalue of $\mathbf{L}_{\text{vor}}^{(\beta)}$ and $\mathbf{L}_{\text{ss}}$

Corollary 5.1. *In the setting of Theorem (5.1), for $\beta\geq\beta_0$, the unstable eigenvalue $\tilde{\lambda}_\beta$ is also an unstable eigenvalue of $\mathbf{L}_{\text{ss}}^{(\beta)}$.*

Proof. By choose β_0 sufficiently large, we could assure that $\Re(\tilde{\lambda}_\beta)>1$, denote the corresponding eigenfunction as $\Omega_\beta\in\mathrm{D}(\mathbf{L}_{\text{vor}}^{(\beta)})$, then we have:

$$(\tilde{\lambda}_\beta-1)\Omega_\beta-\frac{\xi}{2}\cdot\nabla_\xi\Omega_\beta-\Delta\Omega_\beta=-\beta([\bar{U},\Omega_\beta]+[U_\beta,\bar{\Omega}]):=F\tag{5.4}$$

we could see that $\mathrm{BS}_{3d}[\Omega_\beta]$ is the eigenvalue corresponding to $\tilde{\lambda}_\beta$ of $\mathbf{L}_{\mathrm{ss}}^{(\beta)}$ as long as $\mathrm{BS}_{3d}[\Omega_\beta] \in \mathrm{D}(\mathbf{L}_{\mathrm{ss}}^{(\beta)})$.

Firstly, we have $\mathrm{BS}_{3d}[\Omega_\beta] \in \dot{H}^1(\mathbb{R}^3)^3 \cap \dot{H}^3(\mathbb{R}^3)^3$ and $\nabla \cdot \mathrm{BS}_{3d}[\Omega_\beta] = 0$, thus we only need to prove that $\mathrm{BS}_{3d}[\Omega_\beta] \in L^p(\mathbb{R}^3)^3$ for some $p \leq 2$, then, by interpolation argument, we could obtain that $\mathrm{BS}_{3d}[\Omega_\beta] \in H^2(\mathbb{R}^3)^3$, and $\xi \cdot \nabla_\xi \mathrm{BS}_{3d}[\Omega_\beta] \in L^2(\mathbb{R}^3)^3$ comes from the equation

$$\left(\tilde{\lambda}_\beta - \frac{1}{2} - \frac{\xi}{2} \cdot \nabla_\xi - \Delta\right) \mathrm{BS}_{3d}[\Omega_\beta] = \mathrm{BS}_{3d}[F]$$

and the fact that $F \in L^2(\mathbb{R}^3)^3$ is compactly supported, thus $\mathrm{BS}_{3d}[F] \in L^2(\mathbb{R}^3)^3$.

By Sobolev embedding $\dot{W}^{1,1}(\mathbb{R}^3) \hookrightarrow L^{\frac{3}{2}}(\mathbb{R}^3)$, it is reduced to prove that $\Omega_\beta \in L^1(\mathbb{R}^3)$. We apply the inversed self-similar transform:

$$h(t,x) := t^{\tilde{\lambda}_\beta - 1} \Omega_\beta \left(\frac{x}{t^{\frac{1}{2}}}\right), \quad g(t,x) := t^{\tilde{\lambda}_\beta - 2} F\left(\frac{x}{t^{\frac{1}{2}}}\right)$$

then, we have the following heat equation:

$$\begin{cases} \partial_t h(t,x) + \Delta h(t,x) = g(t,x) \\ h(t,x) \equiv 0 \end{cases}$$

the initial data comes from the fact that

$$\|h(t,\cdot)\|_{L^2(\mathbb{R}^3)^3} = t^{\Re(\tilde{\lambda}_\beta) - \frac{1}{4}} \|\Omega_\beta\|_{L^2(\mathbb{R}^3)^3} \underset{t\to 0}{\longrightarrow} 0^+$$

we could represent Ω_β in the following Duhamel form:

$$\Omega_\beta = h(t) = \int_0^1 e^{(1-s)\Delta} g(s) \mathrm{d}s$$

notice that

$$\|g(s)\|_{L^1(\mathbb{R}^3)^3} = t^{\Re(\tilde{\lambda}_\beta) - \frac{1}{2}} \|F(s)\|_{L^1(\mathbb{R}^3)^3} \leq \|F(s)\|_{L^1(\mathbb{R}^3)^3} \quad \forall t \in [0,1]$$

and for $t > 0$,

$$e^{t\Delta} f = \mathcal{F}^{-1}(e^{-t|\xi|^2}) * f = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}} * f$$

is a convolutional operator whose $L^1 \to L^1$ norm is uniformly bounded by 1, thus $\Omega_\beta \in L^1(\mathbb{R}^3)^3$. ☐

6 Nonlinear Instability: Conclusion

In the previous chapters, we have studied the instability property of the linearized operator $\mathbf{L}_{\mathrm{ss}}$ defined in (3.4) with a given background real, compactly supported velocity profile $\beta\bar{U}$ with β sufficiently large. Now, we want to apply this property into our self-similar equation (3.3), and obtain the asymptotic behavior when $\tau \to -\infty$.

We use the notion introduced in (3.1) and (3.2) to separate the physical space and self-similar space.

We would like to have the following schema:

given a real compactly supported velocity profile $\bar{U}$ *for which the associated linearized operator* $\mathbf{L}_{\text{ss}}$ *has an unstable eigenvalue* λ*. We would like to choose it to be the eigenvalue with largest real part.*

Choose $U^{\text{lin}}(\tau, x) = \Re(e^{\lambda\tau}\eta(x)) \neq 0$*, where* η *is the corresponding divergence-free eigenfunction associated with* λ*, with non-degenerate real part. Then, we have:*

$$\partial_\tau U^{\text{lin}} = \mathbf{L}_{\text{ss}} U^{\text{lin}}$$

we could write it into "$U^{\text{lin}} = e^{\tau \mathbf{L}_{\text{ss}}}[U^{\text{lin}}|_{\tau=0}]$", where $e^{\tau \mathbf{L}_{\text{ss}}}$ is the semigroup we will study later on.

we would expect the perturbed solution of the self-similar equation (3.3) is of the form

$$U = \bar{U} + U^{\text{lin}} + U^{\text{per}}$$

where the perturbation term U^{per} *is small of order* $o(e^{\Re}(\lambda)\tau)$ *as* $\tau \to -\infty$*. The extra smallness ensures that* $\bar{U}$ *and* U *are not identical*

We formulate the argument as follows:

Theorem 6.1. *Given* $\bar{U} \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ *to be a real, smooth, compactly supported, divergence free velocity profile. Suppose that the associated linearized operator* $\mathbf{L}_{\text{ss}} : L^2_\sigma \to L^2_\sigma$ *has an unstable eigenvalue* λ*, then,* λ *could be chosen to be maximally unstable, that is,*

$$\Re(\lambda) = \sup_{z \in \sigma(\mathbf{L}_{\text{ss}})} \Re(z) > 0$$

and statement (2) in theorem (3.1) holds.

In the following sections, we will introduce the semigroup generated by the operator $\mathbf{L}_{\text{ss}}$ and finally construct U^{per} via fix point argument.

6.1 Study of the Linearized Operator

In this subsection, we study the spectrum of the linearized operator $\mathbf{L}_{\text{ss}} : \mathrm{D}(\mathbf{L}_{\text{ss}}) \to L^2_\sigma$, which is closed and densely defined. In the light of Lemma (A.3), the spectrum of $\mathbf{L}_{\text{ss}}$ is bounded in the right and the norm of its resolvent is of order $O(\Re(z)^{-1})$. Apply Hille-Yosida's theorem, the operator generates a semigroup $e^{\tau \mathbf{L}_{\text{ss}}} : L^2_\sigma \to L^2_\sigma$. For detailed proof, please refer to [JS13, Chapter 2].

We define the *right spectral bound* of $\mathbf{L}_{\text{ss}}$ as:

$$s(\mathbf{L}_{\text{ss}}) := \sup\{\Re(\lambda) | \lambda \in \sigma(\mathbf{L}_{\text{ss}})\} > 0$$

and the growth bound for the semigroup:

$$\omega_0(\mathbf{L}_{\text{ss}}) := \inf\left\{\omega \in \mathbb{R} | \exists M > 0,\ \|e^{\tau \mathbf{L}_{\text{ss}}}\|_{L^2_\sigma \to L^2_\sigma} \leq M e^{\omega\tau}\right\}$$

It is intuitive that $s(\mathbf{L}_{\mathrm{ss}}) \leq \omega_0(\mathbf{L}_{\mathrm{ss}})$, for detailed proof, please refer to [EN01, Proposition 2.2]. In fact, by [Kat66, Corollary 2.11] and [JS13, Lemma 2.7], we have $s(\mathbf{L}_{\mathrm{ss}}) = \omega_0(\mathbf{L}_{\mathrm{ss}})$.

We conclude the spectral property by the proposition below:

Proposition 6.1. *We have $s(\mathbf{L}_{\mathrm{ss}}) < \infty$, and there exists an eigenvalue $\lambda = s(\mathbf{L}_{\mathrm{ss}})+ib \in \sigma(\mathbf{L}_{\mathrm{ss}})$ and its corresponding eigenfunction $\eta = \eta_1+i\eta_2$, $\eta_i \in \mathrm{D}(\mathbf{L}_{\mathrm{ss}})$ such that $\mathbf{L}_{\mathrm{ss}}\eta = \lambda\eta$.*

Moreover, for $\delta > 0$, there exists $M(\delta) > 0$, such that:

$$\|e^{\tau\mathbf{L}_{\mathrm{ss}}}\|_{L^2_\sigma\to L^2_\sigma} \leq M(\delta)e^{(s(\mathbf{L}_{\mathrm{ss}})+\delta)\tau}$$

As there is a dispersion term in the formula of $\mathbf{L}_{\mathrm{ss}}$, we could expect parabolic regularity:

Proposition 6.2. *For any integer $\sigma_2 \geq \sigma_1 \geq 0$ and $\delta > 0$, then there exists $M = M(\sigma_1, \sigma_2, \delta)$, for all $U_0 \in L^2_\sigma \cap H^{\sigma_1}(\mathbb{R}^3)^3$, we have:*

$$\|e^{\tau\mathbf{L}_{\mathrm{ss}}}U_0\|_{H^{\sigma_2}} \leq \frac{M}{\tau^{\frac{\sigma_2-\sigma_1}{2}}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\delta)\tau}\|U_0\|_{H^{\sigma_1}}$$

Proof. To ease notion, we denote $U(\tau,\cdot) := e^{\tau\mathbf{L}_{\mathrm{ss}}}U_0 \in C(\mathbb{R}_+, L^2_\sigma)$. Instant to see that for small τ, $\tau^{-\frac{\sigma_2-\sigma_1}{2}}$ is dominating, for large τ, $e^{(s(\mathbf{L}_{\mathrm{ss}})+\delta)\tau}$ is dominating.

We start by studying its short-time behavior, since the linearized equation is more neat, we return to the physical space, starting at $\tau = 0$:

$$u(t,x) := \frac{1}{\sqrt{t+1}}U(\log(t+1), \frac{x}{\sqrt{t+1}}), \quad \bar{u}(t,x) := \frac{1}{\sqrt{t+1}}\bar{U}(\frac{x}{\sqrt{t+1}})$$

Observe that $u(0,x) = U(0,x) =: U_0(x)$, then, we have the following equation:

$$\begin{cases} \partial_t u - \Delta u = -\mathbb{P}(\bar{u}\cdot\nabla u + u\cdot\nabla\bar{u}), & (t,x) \in [0,\infty)\times\mathbb{R}^3 \\ u(0,x) = U_0(x) \end{cases}$$

We observe that for $t \in \mathbb{R}_+$ locally, $\bar{u}$ is smooth and uniformly compact supported, thus the source term lies in $C(\mathbb{R}_+, H^{-1})$. Thus there exists a solution in $C(\mathbb{R}_+, L^2_\sigma)\cap L^2_{loc}(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3)^3)$, by the uniqueness of Duhamel's formulation, we have that u lies in this space also.

Here we apply an bootstrap argument, for almost everywhere $t_0 \in \mathbb{R}_+$, we have $u(t_0) \in H^1(\mathbb{R}^3)^3$, then, consider solving the same equation with initial condition at t_0, since the source term lies in $L^2_{loc}(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$, deducing that:

$$u \in C([t_0,+\infty, \dot{H}^1)) \cap L^2_{loc}([t_0,+\infty), \dot{H}^2(\mathbb{R}^3)^3)$$

let $t_0 \to 0$, we have:

$$u \in C((0,+\infty), H^1(\mathbb{R}^3)^3)$$

Thus, we write the solution into the following Duhamel form:

$$u(t) = e^{t\Delta}U_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\bar{u}\cdot\nabla u + u\cdot\nabla\bar{u})\mathrm{d}s$$

recall the parabolic regularity for the heat semigroup, which could be easily deduced form the convolution form by induction:

$$t^{\frac{s}{2}}\|e^{t\Delta}f\|_{\dot{H}^s} \lesssim \|f\|_{L^2}$$

Thus, for $t \in [0,T]$ we have:

$$\|u(t)\|_{L^2} \le \|U_0\|_{L^2} + C(\bar{U},T)\int_0^t \Big(\|u(s)\|_{L^2} + (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^2}\Big)\,\mathrm{d}s$$

since $t^{-\frac{1}{2}}$ is locally integrable, apply Grönwall's inequality, we have:

$$\|u(t)\|_{L^2} \le C(\bar{U},T)\|U_0\|_{L^2}$$

Similarly, for $\dot{H}^1$ norm,

$$\|u(t)\|_{\dot{H}^1} \le t^{-\frac{1}{2}}\|U_0\|_{L^2} + C(\bar{U},T)\int_0^t \Big(\|u(s)\|_{\dot{H}^1} + (t-s)^{-\frac{1}{2}}\|u(s)\|_{\dot{H}^1}\Big)\,\mathrm{d}s$$

apply again Grönwall's inequality,

$$t^{\frac{1}{2}}\|u(t)\|_{\dot{H}^1} \le C(\bar{U},T)\|U_0\|_{L^2}$$

By differentiating the equation, we could similarly prove that, for $k \in \mathbb{N}$,

$$t^{\frac{1}{2}}\|u(t)\|_{\dot{H}^{k+1}} \le C(\bar{U},T,k)\|U_0\|_{\dot{H}^k}$$

for $k \ge m \in \mathbb{N}$, dividing the time interval $[0,t]$ into $k-m$ pieces of same length, apply the estimate above to each sub-interval, we have:

$$t^{\frac{k-m}{2}}\|u(t)\|_{\dot{H}^k} \le C(\bar{U},T,k,m)\|U_0\|_{\dot{H}^m}$$

which yields that, in self-similar variables, for $\tau \in [0, \log(T+1)]$,

$$\frac{(e^\tau - 1)^{\frac{k-m}{2}}}{e^{\frac{k-2}{2}\tau}}\|U(\tau)\|_{\dot{H}^k} \le C(\bar{U},T,k,m)\|U_0\|_{\dot{H}^m}$$

We restrict ourselves to $T = e^2 - 1$, then, by Taylor series, $(e^\tau - 1)^{\frac{k-m}{2}} \sim \tau^{\frac{k-m}{2}}$, and the denominator is bounded, thus

$$\tau^{\frac{k-m}{2}}\|U(\tau)\|_{\dot{H}^k} \le C(\bar{U},k,m)\|U_0\|_{\dot{H}^m} \quad \forall \tau \in [0,2]$$

then, by a simple interpolation argument, we could replace $\dot{H}^k$ norm with H^k norm.

Then, we study the long-time behavior: for $\tau > 2$, $k > m$, we have:

$$\begin{aligned}
\|U(\tau)\|_{H^k} = \|e^{\mathbf{L}_{\mathrm{ss}}}(e^{(\tau-1)\mathbf{L}_{\mathrm{ss}}})U_0\|_{H^K} &\le C(\bar{U},k)\|e^{(\tau-1)\mathbf{L}_{\mathrm{ss}}}U_0\|_{L^2} \\
&\le C(\bar{U},k,\delta)e^{(\tau-1)(s(\mathbf{L}_s s)+\delta)}\|U_0\|_{L^2} \\
&\le C(\bar{U},k,m,\delta)\frac{e^{\tau(s(\mathbf{L}_s s)+\delta)}}{\tau^{\frac{k-m}{2}}}\|U_0\|_{H^m}
\end{aligned}$$

where the second last inequality is deduced from Proposition (6.1), and the last one comes from the fact that exponential is growing faster than any polynomial.

Combine the two cases above, we obtain the desire conclusion. $\square$

As a corollary of the proposition above, we have the eigenfunction η corresponding to λ satisfies $\eta \in H^k(\mathbb{R}^3)^3$ for all $k \in \mathbb{N}$, and,

$$\|U^{\mathrm{lin}}(\tau)\|_{H^k} = \|\Re(e^{\tau \mathbf{L}_{\mathrm{ss}}}\eta)\|_{H^k} = e^{s(\mathbf{L}_{\mathrm{ss}})\tau}\|\Re(e^{ib\tau}\eta)\|_{H^k} \sim e^{s(\mathbf{L}_{\mathrm{ss}})\tau} \tag{6.1}$$

6.2 Construction of the Nonlinear Profile

In this subsection, we study the equation satisfied by the perturbation term in the self-similar variables:

$$\partial_\tau U^{\mathrm{per}} + \mathbf{L}_{\mathrm{ss}} U^{\mathrm{per}} = -\mathbb{P}[((U^{\mathrm{per}} + U^{\mathrm{lin}}) \cdot \nabla)(U^{\mathrm{per}} + U^{\mathrm{lin}})] \tag{6.2}$$

and the initial condition is formally:

$$U^{\mathrm{per}}(-\infty) = 0$$

We state the existence and decay of U^{per} as the proposition below:

Proposition 6.3. *Assume that $N > \frac{5}{2}$ is an integer, then, there exists lifespan $T = T(\bar{U}, U^{\mathrm{lin}}, N) \in \mathbb{R}$, $\epsilon_0 = \epsilon_0(s(\mathbf{L}_{\mathrm{ss}})) > 0$, such that (6.2) admits a solution $U^{\mathrm{per}} \in C((-\infty, T], H^N(\mathbb{R}^3)^3)$, and we have the following estimate:*

$$\|U^{\mathrm{per}}(\tau)\|_{H^N} \le e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)\tau} \quad \forall \tau \le T$$

To ease notion, we assume all constants depend on $\bar{U}$, U^{lin} and N, which are now fixed. Let $T \in \mathbb{R}$ and $\epsilon_0 > 0$ to be two undetermined constants.

Consider the norm with desired decay in time:

$$\|U\|_X := \sup_{\tau \le T} e^{-(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)\tau}\|U\|_{H^N}$$

and the associated Banach space:

$$X := \{U \in C((-\infty, T], H^N(\mathbb{R}^3)^3) | \|U\|_X < +\infty\}$$

We study the following functional:

$$\mathcal{T}U(\tau) := -\int_{-\infty}^{\tau} e^{(\tau - s)\mathbf{L}_{\mathrm{ss}}}\mathbb{P}[((U + U^{\mathrm{lin}}) \cdot \nabla)(U + U^{\mathrm{lin}})(s)]\mathrm{d}s$$

We want to show that the map is indeed a contraction:

Proposition 6.4. *There exists $T = T(\bar{U}, U^{\mathrm{lin}}, N) \in \mathbb{R}$ and $\epsilon_0 = \epsilon_0(s(\mathbf{L}_{\mathrm{ss}})) > 0$, such that:*

$$\mathcal{T} : \{U \in X | \|U\|_X \le 1\} \to \{U \in X | \|U\|_X \le 1\}$$

is a contraction.

Proof. Recall that for $M > \frac{3}{2}$, $H^M(\mathbb{R}^3)$ is a Banach algebra, i.e. for $f, g \in H^N(\mathbb{R}^3)$, we have:

$$\|fg\|_{H^M} \lesssim_M \|f\|_{H^M}\|g\|_{H^M}$$

use the inequality above, we estimate the Duhamel form term by term:

1 The quadratic term:

$$\begin{aligned}\|e^{(\tau-s)\mathbf{L}_{\mathrm{ss}}}\mathbb{P}[(U(s)\cdot\nabla)U(s)]\|_{H^N}&\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau-s)}\|(U(s)\cdot\nabla)U(s)\|_{H^{N-1}}\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau-s)}\|U(s)\|_{H^N}\|U(s)\|_{H^{N-1}}\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau+s)}\|U\|_X^2\end{aligned}$$

2 The linear terms:

$$\begin{aligned}&\|e^{(\tau-s)\mathbf{L}_{\mathrm{ss}}}\mathbb{P}[(U^{\mathrm{lin}}(s)\cdot\nabla)U(s)+(U(s)\cdot\nabla)U^{\mathrm{lin}}(s)]\|_{H^N}\\ &\quad\lesssim\frac{e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau-s)}}{(\tau-s)^{\frac{1}{2}}}\|(U^{\mathrm{lin}}(s)\cdot\nabla)U(s)+(U(s)\cdot\nabla)U^{\mathrm{lin}}(s)\|_{H^{N-1}}\\ &\quad\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)\tau+s(\mathbf{L}_{\mathrm{ss}})s}\|U\|_X\end{aligned}$$

3 The constant term:

$$\begin{aligned}\|e^{(\tau-s)\mathbf{L}_{\mathrm{ss}}}\mathbb{P}[(U^{\mathrm{lin}}(s)\cdot\nabla)U^{\mathrm{lin}}(s)]\|_{H^N}&\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau-s)}\|(U^{\mathrm{lin}}(s)\cdot\nabla)U^{\mathrm{lin}}(s)\|_{H^{N-1}}\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)\tau+(s(\mathbf{L}_{\mathrm{ss}})-\epsilon_0)s}\end{aligned}$$

Thus we could simply choose $\epsilon_0=\frac{1}{2}s(\mathbf{L}_{\mathrm{ss}})$, and we could choose T sufficiently small such that

$$\int_{-\infty}^{T}\frac{e^{\frac{1}{2}s(\mathbf{L}_{\mathrm{ss}})s}}{(T-s)^{\frac{1}{2}}}\mathrm{d}s\ll 1$$

then we have $\mathcal{T}$ maps the unit ball of X to itself.

Similarly, for the contraction, assume that $U,V\in X$, we have the following estimate:

$$\begin{aligned}\|e^{(\tau-s)\mathbf{L}_{\mathrm{ss}}}\mathbb{P}[(U(s)\cdot\nabla)U(s)-(V(s)\cdot\nabla)V(s)]\|_{H^N}&\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau+s)}(\|U\|_X+\|V\|_X)\|U-V\|_X\\ &\lesssim(\tau-s)^{-\frac{1}{2}}e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau+s)}\|U-V\|_X\end{aligned}$$

and,

$$\begin{aligned}&\|e^{(\tau-s)\mathbf{L}_{\mathrm{ss}}}\mathbb{P}[(U^{\mathrm{lin}}(s)\cdot\nabla)(U-V)(s)+((U-V)(s)\cdot\nabla)U^{\mathrm{lin}}(s)]\|_{H^N}\\ &\qquad\qquad\lesssim\frac{e^{(s(\mathbf{L}_{\mathrm{ss}})+\epsilon_0)(\tau)+s(\mathbf{L}_{\mathrm{ss}})s}}{(\tau-s)^{\frac{1}{2}}}\|U-V\|_X\end{aligned}$$

Then, we could choose T sufficiently small such that $\mathcal{T}$ is a contraction. □

Apply fix point theorem, Proposition (6.3) is a direct corollary of the contraction property.

Indeed, repeat the norm estimate, we have:

$$\sup_{\tau\in(-\infty,T]} e^{-2s(\mathbf{L}_{\mathrm{ss}})\tau}\|\mathcal{T}U(\tau)\|_{H^N} < \infty$$

Since U^{per} is a fixed point of $\mathcal{T}$, it satisfies the same estimate, thus the decay rate of $\|U^{\mathrm{per}}\|_{H^N}$ is actually $O(e^{2s(\mathbf{L}_{\mathrm{ss}})\tau})$.

Combine all the conclusions we have obtained, the proof of Theorem (3.1) is done.

A Appendix: Preliminaries in Spectral Analysis

We recall some properties in spectral analysis for unbounded operators, for detailed proof, please refer to T.Kato's book [Kat66].

Here are the useful facts of the spectrum:

Proposition A.1. *For all linear operator* $\mathbf{L}$*,* $\sigma(\mathbf{L})$ *is closed, and the following operator-valued function is holomorphic.*

$$\begin{array}{rcl} R_{\mathbf{T}} : \mathrm{res}(\mathbf{T}) & \to & \mathcal{L}(\mathcal{H}) \\ z & \mapsto & (\mathbf{T}-z)^{-1} \end{array}$$

Proposition A.2. *For a bounded linear operator* $\mathbf{L} : \mathcal{H} \to \mathcal{H}$*, we have:*

$$\sup\{|z| | z \in \sigma(\mathbf{L})\} = \lim_{n\to\infty} \|\mathbf{L}^n\|_{\mathcal{H}\to\mathcal{H}}^{\frac{1}{n}}$$

Proposition A.3. *For a closed normal operator* $\mathbf{L} : \mathcal{H} \to \mathcal{H}$*, i.e.* $\mathbf{L}\mathbf{L}^* = \mathbf{L}^*\mathbf{L}$*, for* $\lambda \in \mathrm{res}(\mathbf{L})$ *we have:*

$$\|(\lambda - \mathbf{L})^{-1}\|_{\mathcal{H}\to\mathcal{H}} = d(\lambda, \sigma(\mathbf{L}))^{-1}$$

Proof sketch. Firstly, we recall that, for a normal operator $\mathbf{L}$,

$$\lim_{n\to\infty} \|\mathbf{L}^n\|_{\mathcal{H}\to\mathcal{H}}^{\frac{1}{n}} = \|\mathbf{L}\|_{\mathcal{H}\to\mathcal{H}}$$

For $\lambda \in \mathrm{res}(\mathbf{L})$, we have $(\lambda - \mathbf{L})^{-1}$ is a bounded and normal operator, and observe that for $z \in \mathbb{Z}$,

$$z - (\lambda - \mathbf{L})^{-1} = (\lambda - \mathbf{L})^{-1}(z\lambda - 1 - z\mathbf{L})$$

thus, we have:

$$\sigma((\lambda - \mathbf{L})^{-1}) = \{\frac{1}{\lambda - z} | z \in \sigma(\mathbf{L})\} \cup \{0\}$$

Apply the proposition (A.2), and we obtain the conclusion. □

The following spectrum structure theorem for compact operators is useful:

Proposition A.4. *Let* $\mathcal{H}$ *be a infinite-dimensional Hilbert space,* $\mathbf{K} : \mathcal{H} \to \mathcal{H}$ *is a compact operator, then we have:*

1

$$0 \in \sigma(\mathbf{K})$$

2

$$\sigma(\mathbf{K}) \backslash \{0\} = \sigma_p(\mathbf{K})$$

and zero is the only cluster point of $\sigma_p(\mathbf{K})$*.*

3 $\sigma_p(\mathbf{K})$ *is consisted of countable isolated eigenvalues with finite algebraic multiplicity.*

From the structure theorem above, we could easily obtain the following corollary by projecting onto eigenspaces:

Corollary A.1. *Every compact operator is a limit in operator norm of a sequence of finite-ranked operators.*

[Kat66, Chapter 4] has provided us with some powerful tools:

Proposition A.5. *Given a closed linear operator* $\mathbf{L}$ *on a Hilbert space* $\mathcal{H}$ *and* $\mathbf{T}$ *a compact operator, then:*

1 $\sigma_{\text{ess}}(\mathbf{L}) \subset \mathbb{C}$ *is closed, in each connected component* $z \in A$ *of* $\mathbb{C} \backslash \sigma_{\text{ess}}(\mathbf{L})$*, either* $\mathbf{L} - z$ *is not invertible everywhere in* A*, or* $\mathbf{L} - z$ *is invertible except isolated points.*

2

$$\sigma_{\text{ess}}(\mathbf{L} + \mathbf{T}) = \sigma_{\text{ess}}(\mathbf{L})$$

Remark A.1. *Thanks to the proposition above, we are going to use frequently the following reasoning:*

Consider a closed linear operator $\mathbf{L}$ *define on* $\mathcal{H}$*, and assume that:*

$$\sigma_{\text{ess}}(\mathbf{L}) \subset \{x \in \mathbb{C} | \Re(x) \leq r\}$$

Then, we perturb $\mathbf{L}$ *with a relative compact operator* $\mathbf{T}$*, and a priori, we have that* $\mathbf{L} + \mathbf{T} - z$ *is invertible for large* $z \in \mathbb{R}$*, which remains true when* L *is a bounded perturbation of a asymmetrical operator. Then,*

$$\sigma_{\text{ess}}(\mathbf{L} + \mathbf{T}) \subset \{x \in \mathbb{C} | \Re(x) \leq r\}$$

and the remaining spectrum of $\mathbf{L} + \mathbf{T}$ *consists of isolated points of finite order in* $\{x \in \mathbb{C} | \Re(x) > r\}$*.*

B Appendix: Technical Tools

B.1 2D Radial Fourier Series

Since in polar coordinate, $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R} \backslash 2\pi\mathbb{Z}$, we could apply the Fourier series to the angular variable, to obtain a decomposition of $L^2(\mathbb{R}^2)$ with rotational symmetry:

Theorem B.1. *For $f \in L^2(\mathbb{R}^2)$, writing in polar coordinate $(r\theta)$, we have:*

$$f(r, \theta) = \sum_{k \in Z} f_k(r) e^{-ik\theta} \quad \text{in } L^2(\mathbb{R}^2)$$

where,

$$f_k(r) := \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) e^{ik\theta} \mathrm{d}\theta \ \in L^2(\mathbb{R}_+, r\mathrm{d}r)$$

And further satisfies the Plancherel identity:

$$\|f\|^2_{L^2(\mathbb{R}^2)} = 2\pi \sum_{k \in \mathbb{Z}} \|f_k\|^2_{L^2(\mathbb{R}_+, r\mathrm{d}r)}$$

Proof. For $\varphi \in L^2(\mathbb{T}, \mathbb{C})$ where $(\mathbb{T} = \mathbb{R}/\mathbb{Z})$, we define

$$\varphi_k = \int_0^1 \varphi(t) e^{-2i\pi kt} \mathrm{d}t \ , \ k \in \mathbb{Z}$$

φ_k is called the $k-$th coefficient of the Fourier series and $\sum_{k \in \mathbb{Z}} \varphi_k e^{2i\pi kt}$ the Fourier series.

If $\varphi \in C^0(\mathbb{T}, \mathbb{C})$, and if $\sum_{k \in \mathbb{Z}} |\varphi_k| < \infty$, then the function series $\sum_{k \in \mathbb{Z}} \varphi_k e^{2i\pi kt}$ converges uniformly to φ on $\mathbb{T}$.

Moreover,

$$\varphi \in C^2(\mathbb{T}, \mathbb{C}) \implies \varphi''_k = -4k^2\pi^2\varphi_k \implies \varphi_k \in O(\frac{1}{k^2}) \implies \sum_{k \in \mathbb{Z}} |\varphi_k| < \infty$$

Thus, the restriction of the linear map $(\varphi \in L^2(\mathbb{T}) \mapsto \sum_{k \in \mathbb{Z}} \varphi_k e^{2i\pi kt} \in L^2(\mathbb{T}))$ on $C^2(\mathbb{T})$ is the identity.

Because of the uniqueness of a continuous extension

$$\overline{L^2(\mathbb{T}) \bigcap C^2(\mathbb{T})} = L^2(\mathbb{T})$$

we have

$$\varphi(t) = \sum_{k \in \mathbb{Z}} \varphi_k e^{2i\pi kt}, \forall \varphi \in L^2(\mathbb{T})$$

Finally, for $f \in L^2(\mathbb{R})$, we have $f(r, 2\pi\cdot) \in L^2(\mathbb{T})$ for almost all r, so,

$$f(r,\theta) = \sum_{k\in Z} f_k(r) e^{-ik\theta} \in L^2(\mathbb{R})$$

. $\square$

B.2 Generalized Young's Inequality

In this subsection, we want to consider the behavior of convolution product of the form $|x|^{-\theta} * f$ in dimension d with $\theta > 0$, which the normal Young's inequality could be applied to, since $x^{-\theta}$ doesn't belong to any L^p space.

But we could see that $|x|^{-\theta}$ satisfies the Markov inequality, thus it is "almost" in the space $L^{\frac{N}{\theta}}$. We consider the weak L^p_w space defined with this intuition:

$$L^p_w(\mathbb{R}^d) := \left\{ f \in \mathcal{M}(\mathbb{R}^d, \mathcal{B}) \middle| \exists C \geq 0,\ \forall \lambda,\ m(\{|f| \geq \lambda\}) \leq \frac{C}{\lambda^p} \right\} \tag{B.1}$$

where $\mathcal{M}(\mathbb{R}^d, \mathcal{B}, m)$ denotes the set of measurable functions with respect to Borel σ-algebra, and m is the Lebesgue measure. Define the norm:

$$\|f\|_{L^p_w(\mathbb{R}^d)} := \inf \left\{ C \geq 0 \middle| \forall \lambda,\ m(\{|f| \geq \lambda\}) \leq \frac{C}{\lambda^p} \right\}$$

We have the following generalized Young's inequality for weak L^p norms:

Proposition B.1. *Let $p, q, r \in (1, \infty)$, satisfying $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, then there exists $C > 0$, for $f \in L^p(\mathbb{R}^d)$, $g \in L^q_w(\mathbb{R}^d)$, we have:*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q_w(\mathbb{R}^d)}$$

For detailed proof, please refer to [Gra14][Theorem 1.4.25].

B.3 Bogovskii's Operator

We want to solve the equation

$$\nabla \cdot f = g$$

Bogovskii has developed an operator in [Bog80] to treat this problem under certain constrains, now we give a brief state for the general N-dimensional case:

Firstly, we fix $\rho \in C_0^\infty(\mathbb{R}^N)_+$ to be non-negative smooth function with compact support, such that

$$\int_{\mathbb{R}^N} \rho(x) \mathrm{d}x = 1$$

It will serve as a local convolution kernel.

For $f \in C_0^\infty(\mathbb{R}^N)$, we define the following operator:

$$\mathrm{B}f(x) := \int_{\mathbb{R}^N} f(y) \frac{x-y}{|x-y|^N} \int_0^\infty (|x-y| + r)^{N-1} \rho\left(x + r\frac{x-y}{|x-y|}\right) \mathrm{d}r \mathrm{d}y$$

We have the following proposition:

Proposition B.2. *The operator above defines a linear mapping:*

$$\begin{array}{rcl} \mathrm{B} : C_0^\infty(\mathbb{R}^N) & \to & C_0^\infty(\mathbb{R}^N, \mathbb{R}^N) \\ f & \mapsto & \mathrm{B}f \end{array}$$

and satisfying the following identity:

$$\nabla \cdot \mathrm{B}f = f - \rho \int_{\mathbb{R}^N} f(x)\mathrm{d}x$$

Furthermore, for all $R > 0$ we could choose proper $\rho \in C_0^\infty(\mathbb{R}^N)$, such that:

$$f \in C_0^\infty(B_0(R)) \quad \Longrightarrow \quad \mathrm{B}f \in C_0^\infty(B_0(R))$$

Proof. Observe that the integral is not well-defined directly, since the function $\frac{x-y}{|x-y|^N}$ has a singularity at point $x = y$.

For x fixed, since the integral in r only take place in a line of finite length, thus there exists a constant $M > 0$, such that:

$$\int_0^\infty (|x-y|+r)^{N-1} \rho\left(x + r\frac{x-y}{|x-y|}\right) \mathrm{d}r \leq M, \quad \forall y \in \mathrm{supp}(f)\backslash\{x\}$$

as a consequence,

$$\frac{x-y}{|x-y|^N} \int_0^\infty (|x-y|+r)^{N-1} \rho\left(x + r\frac{x-y}{|x-y|}\right) \mathrm{d}r \in L^1_{loc}(\mathbb{R}^N)$$

since $f \in C_0^\infty(\mathbb{R}^N)$, the integral is absolutely integrable.

We then prove that $\mathrm{B}f$ is compactly supported. By change of variables, we could write $\mathrm{B}f$ in the following form:

$$\mathrm{B}f(x) = \int_{\mathbb{R}^N} (x-y)f(y) \int_1^\infty \rho(y + r(x-y))r^{n-1}\mathrm{d}r\mathrm{d}y$$

Note that the integrate of variable r take place on a half-line $I_{x,y}$ starting at x with direction $x - y$ in $\mathbb{R}^N$. Thus, for $|x|$ sufficiently large, we have:

$$y \in \mathrm{supp}(f) \implies I_{x,y} \cap \mathrm{supp}(\rho) = \emptyset$$

concluding that Bf is compactly supported.

Furthermore, we could choose that $\mathrm{supp}(\rho) \in B_0(\frac{R}{2})$, apply the same argument, we could also obtain that $\mathrm{supp}(\mathrm{B}f) \in B_0(R)$.

The smoothness of $\mathrm{B}f$ comes from the following facts: by change of variables $y \mapsto x - y$,

$$\mathrm{B}f(x) = \int_{\mathbb{R}^N} f(x-y)\frac{y}{|y|^N} \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y$$

since $f, \rho \in C_0^\infty(\mathbb{R}^N)$, by the interchangeability of integration and differentiation, $\mathrm{B}f$ is smooth.

It remains to prove the property of $\nabla \cdot \mathrm{B}f$. By direct computation, we have:

$$\begin{aligned}\nabla \cdot \mathrm{B}f(x) = &\int_{\mathbb{R}^N} y \cdot \nabla f(x-y)|y|^{-N} \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y \\ &+ \int_{\mathbb{R}^N} f(x-y)|y|^{-N} \int_0^\infty (|y|+r)^{N-1} y \cdot \nabla\rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y\end{aligned}$$

observe that the integral above has a singularity at $y = 0$, we cut away a ball $B_0(\epsilon)$ centered at $y = 0$ with radius ϵ, and finally we would let $\epsilon \to 0$.

Firstly, for the remainders, since the integral above is well-defined,

$$\lim_{\epsilon\to 0} \int_{B_0(\epsilon)} y \cdot \nabla f(x-y)|y|^{-N} \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y = 0$$

$$\lim_{\epsilon\to 0} \int_{B_0(\epsilon)} f(x-y)|y|^{-N} \int_0^\infty (|y|+r)^{N-1} y \cdot \nabla\rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y = 0$$

Apply divergence theorem to the first term outside the ball:

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_0(\epsilon)} y \cdot \nabla f(x-y)|y|^{-N} \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y \\ &= \int_{\partial B_0(\epsilon)} \epsilon^{-N+1} f(x-y) \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y \\ &\quad + (N-1) \int_{\mathbb{R}^N\setminus B_0(\epsilon)} f(x-y)|y|^{-N+1} \int_0^\infty (|y|+r)^{N-2} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y\end{aligned}$$

and integration by parts to the second term:

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_0(\epsilon)} f(x-y)|y|^{-N} \int_0^\infty (|y|+r)^{N-1} y \cdot \nabla\rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y \\ &= \int_{\mathbb{R}^N\setminus B_0(\epsilon)} f(x-y)|y|^{-N+1} \int_0^\infty (|y|+r)^{N-1} \partial_r \left[\rho\left(x + r\frac{y}{|y|}\right)\right] \mathrm{d}r\mathrm{d}y \\ &= -(N-1) \int_{\mathbb{R}^N\setminus B_0(\epsilon)} f(x-y)|y|^{-N+1} \int_0^\infty (|y|+r)^{N-2} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y \\ &\quad - \rho(x) \int_{\mathbb{R}^N\setminus B_0(\epsilon)} f(x-y)\mathrm{d}y\end{aligned}$$

For the behavior of the terms, we have:

$$\lim_{\epsilon\to 0} \int_{\partial B_0(\epsilon)} \epsilon^{-N+1} (f(x-y) - f(x) \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y = 0$$

$$\begin{aligned}&\int_{\partial B_0(\epsilon)} \epsilon^{-N+1} \int_0^\infty (|y|+r)^{N-1} \rho\left(x + r\frac{y}{|y|}\right) \mathrm{d}r\mathrm{d}y \\ &\qquad = \int_{B_0(1)} \int_0^\infty (\epsilon+r)^{N-1} \rho(x+yr)\mathrm{d}r\mathrm{d}y \underset{\epsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^N} \rho(y)\mathrm{d}y = 1\end{aligned}$$

$$\lim_{\epsilon \to 0} \rho(x) \int_{\mathbb{R}^N \setminus B_0(\epsilon)} f(x-y) \mathrm{d}y = \rho(x) \int_{\mathbb{R}^N} f(y) \mathrm{d}y$$

Combine all the computations, we obtain the desired result. $\square$

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