# Bootstrap percolation

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### Abstract

This paper investigates the dynamics and threshold behaviour of bootstrap percolation in the lattice $\mathbb{Z}^d$, with particular emphasis on the strong bootstrap percolation model.

We rigorously prove that for all positive initial densities, the origin becomes occupied with probability one, and the convergence occurs exponentially fast. The core of the argument relies on combinatorial constructions and dimensional induction.

We also explore the model under finite-volume constraints, quantify convergence behaviour and metastability in two-dimensional settings. Central to the analysis is the use of an auxiliary function to estimate the spanning probabilities and derive the bounds above and below. Finally, we show that the probability of complete occupation in finite boxes becomes approximately constant over a broad parameter range, providing theoretical justification for the observed simulation results.

## Contents

# 1 Introduction

In this paper, we will analyse an important model called bootstrap percolation, which has a wide application in math, physics and population evolution. In order to describe this model, we first give some definitions.

**Definition 1.** *We will use $\eta$ to be a map from $\mathbb{Z}^d$ to $\{0,1\}$. In the following, we will call it a configuration and we will always use $\eta$ as a configuration. For any two configurations $\eta_1, \eta_2$, we call $\eta_1 \leqslant \eta_2$ if $\eta_1(x) \leqslant \eta_2(x)$ for any $x \in \mathbb{Z}^d$.*

For any configuration $\eta$, we will call the elements in $\mathbb{Z}^d$ "sites" and if $\eta$ takes the value 1 for some site, then we will call this site "occupied".
Let $S$ be a subset of $P(\mathbb{Z}^d\backslash\{0\})$, $P(\mathbb{Z}^d\backslash\{0\})$ is the set of all subsets of the set $\mathbb{Z}^d\backslash\{0\}$; here we think of $\mathbb{Z}^d\backslash\{0\}$ as a set of vectors, but not points.

**Definition 2.** *We will use $(\eta, S)$ as a model and define the evolution $\{\eta_t\}, \eta_\infty$ as follows: Let $\eta_0 = \eta$, and for $t \geqslant 1$ let*

$$\eta_t(x) = \begin{cases} 1 & \textit{if } \eta_{t-1}(x+A) = 1 \textit{ for some } A \in S \textit{ or } \eta_{t-1}(x) = 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$\eta_\infty(x) = \begin{cases} 1 & \textit{if } \eta_t(x) = 1 \textit{ for some } t \geqslant 0, \\ 0 & \textit{otherwise.} \end{cases}$$

**Definition 3.** *For any model $(\eta, S)$ and a subset $\Gamma \subseteq \mathbb{Z}^d$. We define a sub-evolution $\{\eta_t^\Gamma\}$, $\eta_\infty^\Gamma$ of $\eta$ as follows.*

$$\eta_0^\Gamma(x) = \begin{cases} \eta(x) & \textit{if } x \in \Gamma, \\ 0 & \textit{otherwise.} \end{cases}$$

*For $t \geqslant 1$ if $x \notin \Gamma$,*

$$\eta_t^\Gamma(x) = 0.$$

*If $x \in \Gamma$,*

$$\eta_t^\Gamma(x) = \begin{cases} 1 & \textit{if } \eta_{t-1}^\Gamma(x+A) = 1 \textit{ for some } A \in S \textit{ or } \eta_{t-1}^\Gamma(x) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\eta_\infty^\Gamma(x) = \begin{cases} 1 & \textit{if } \eta_t^\Gamma(x) = 1 \textit{ for some } t \geqslant 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Here we denote by $x + A$ the set $\{x + a \mid a \in A\}$, and $\eta_t(x + A) = 1$ means $\eta_t(x + a) = 1$ for any $a \in A$.
In the following, when we talk about two or more models, we will specify the set S (e. g. we will use $\eta_{S,t}$, instead of $\eta_t$) and we will omit it if there is no ambiguity.
Let $\Omega(d)$ be the set of all configurations, the $\sigma(d)$-algebra on $\Omega(d)$ is generated by all sets $C_{X,f}$ which $X \subseteq \mathbb{Z}^d$, $f$ is a map from $X$ to $\{0,1\}$.

$$C_{X,f} = \{\eta \in \Omega(d) \mid \forall x \in X : \eta(x) = f(x)\}$$

Fix $p \in [0,1]$, endow the measure of $C_{X,f}$ as $p^{|f^{-1}(1)|} \cdot (1-p)^{|f^{-1}(0)|}$, then we get a probability measure space $(\Omega(d), \sigma(d), p)$.

**Example 4.** *For an integer $l \in [0, 2d]$, let $U$ be the set of all unit vectors in $\mathbb{Z}^d$, define*

$$S_{bp} = \{A \mid A \subseteq U, |A| \geqslant l\}$$

*Then we name the model $(\eta, S_{bp})$ "bootstrap percolation".*

**Example 5.** *Let $x_i^0, x_i^1$ be the two unit vectors of the $i$-th direction $i = 1, 2, \ldots, d$, define*

$$S_{sbp} = \{A \mid A \subseteq U, x_i^0 \in A \text{ or } x_i^1 \in A\}$$

*Then we name the model $(\eta, S_{sbp})$ "strong bootstrap percolation".*

**Definition 6.** *For any model $S$ and $p \in [0, 1]$ we define*

$$\begin{aligned} \rho(p) &= \text{the measure of the set } \{\eta \in \Omega(d) \mid \eta_\infty(0) = 1\} \\ \gamma_d(p) &= \sup\{\gamma_d \geqslant 0 \mid \exists C : \forall t \geqslant 0 : P_p(\{\eta \in \Omega(d) : \eta_t(0) = 1\}) \leqslant Ce^{-\gamma_d t}\} \end{aligned}$$

*For a subset $X$ of $\Omega(d)$, let $P_p(X)$ denote the measure of the set in measure space $(\Omega(d), \sigma(d), p)$*

**Definition 7.** *For any model $(\eta, S)$, we now define two constants, the asymptotic density $P_S$ and the phase point $\pi_{d,S}$, by*

$$\begin{aligned} P_S &= \inf\{p \in [0, 1] \mid \rho(p) = 1\}, \\ \pi_{d,S} &= \inf\{p \in [0, 1] \mid \gamma_d(p) > 0\}. \end{aligned}$$

*Obviously, $\pi_{d,S} \geqslant P_S$ for any $S$.*

# 2 Main result for the bootstrap percolation

## 2.1 Introduction

In this section, we will analyse the bootstrap percolation and the strong bootstrap percolation. Following [Sch92], we will show that the value of the asymptotic density $p_c$ and the phase point $\pi_c$ are zero for the strong bootstrap percolation.

**Theorem 8.** *For strong bootstrap percolation, $P_{S_{sbp}} = \pi_{S_{sbp}} = 0$.*

In other words, we will prove that if the initial density $p$ is positive, the site 0 in $\mathbb{Z}^d$ will become occupied with probability 1. In addition, for any initial density $p > 0$, we will have two positive constants $\gamma$ and $C$ such that

$$P_p(\{\eta \in \Omega(d) \mid \eta_t(0) = 1\}) \leqslant Ce^{-\gamma t}.$$

Following from Theorem 8, we get the following corollary.

**Corollary 9.** *For any $l \leqslant d$, the bootstrap percolation will have $P_{S_{bp}} = \pi_{S_{bp}} = 0$.*

*Proof.* For any configuration $\eta \in \Omega(d)$, we have $\eta_{S_{bp},t} \geqslant \eta_{S_{sbp},t}$, $\eta_{S_{bp},\infty} \geqslant \eta_{S_{sbp},\infty}$, and therefore $P_{S_{bp}} \leqslant P_{S_{sbp}}$, $\pi_{S_{bp}} \leqslant \pi_{S_{sbp}}$. The claim follows from Theorem 8. $\square$

In the following subsections, we will restrict ourselves to the strong bootstrap percolation model.

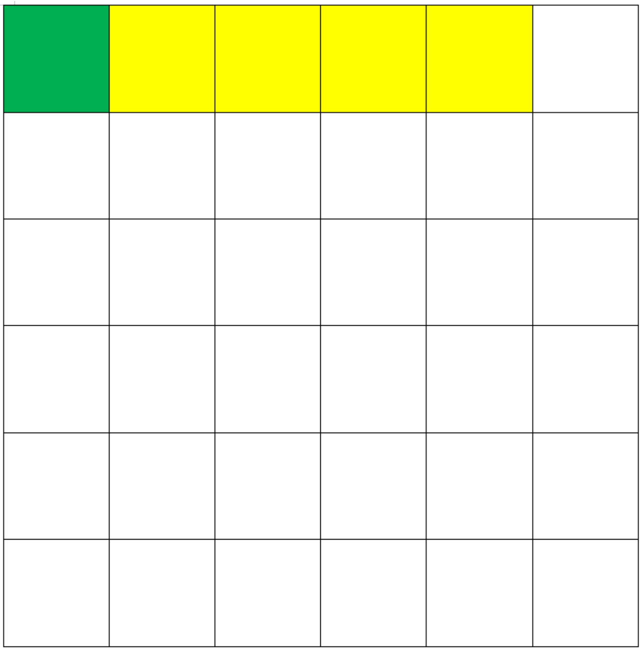


Figure 1: The yellow set is a 1-face and the green set is a 0-face

## 2.2 Proof of Theorem 8

In this subsection, we will give some important definitions that we will use later in the proof.

**Definition 10.** *For any $k \geqslant 0$, define a subset of $\mathbb{Z}^d$, called the $k$-*block *of $\mathbb{Z}^d$, by*

$$B_k = \{x \in \mathbb{Z}^d \mid \forall i \in \{1, \dots, d\} : |x_i| \leqslant k\}.$$

**Definition 11.** *For any $k \geqslant 0$, $r \in \{0, 1, \dots, d-1\}$, $1 \leqslant i_1 < i_2 < \cdots < i_{d-r} \leqslant d$ and $a_i \in \{+1, -1\}$ with $1 \leqslant i \leqslant d-r$, we define a subset of a $k$-block called $r$-face by*

$$F^d_{(i),(a)}(k) = \left\{x \in \mathbb{Z}^d \mid x_{i_s} = a_s k \text{ for } s = 1, 2, \dots, d-r \text{ and } |x_{i_s}| < k \text{ for } s \notin \{i_1, i_2, \dots, i_{d-r}\}\right\}$$

$$Z_{(i),(a)} = \left\{x \in \mathbb{Z}^d \mid x_{i_s} = a_s * k \text{ for } s = 1, 2, \dots, d-r\right\}$$

*here $(i)$ is the assay $(i_1, i_2, \dots, i_{d-r})$ and $(a)$ is the assay $(a_1, a_2, \dots, a_{d-r})$*

*Then $Z_{(i),(a)} \cong \mathbb{Z}^r$*

*In the following paragraph, we will denote by $F_r(k)$ the set of $r$-faces and*

$$F(k) = \bigcup_{r=0}^{d-1} F_r(k).$$

Following the above definitions, it follows that

$$B_k = B_{k-1} \cup \left(\bigcup_{I \in F} F^d_I(k)\right)$$

**Definition 12.** *For any $\Gamma \subseteq \mathbb{Z}^d$ and $\eta \in \Omega(d)$, we will say that $\Gamma$ is internally spanned by $\eta$ if $\eta^\Gamma_\infty(x) = 1$ for any $x \in \Gamma$. Define*

$$R^d(N, p) = P_p(\{\eta \in \Omega(d) \mid B_k \text{ is internally spanned by } \eta\})$$

**Definition 13.** *For any* $I \in F_r(k), \eta \in \Omega(d)$*, we restrict* $\eta$ *in* $Z_{(i),(a)}$*, we get a new configuration called* $\eta_I$*, we call* $F_I^d(k)$ *in dimension* $r$ *internally spanned by* $\eta$ *if* $\eta_{I,\infty}^{F_I^d(k)}(x) = 1$ *for* $x \in F_I^d(k)$

**Lemma 14.** *For any* $k \geqslant 1$ *If* $B_{k-1}$ *is completed occupied in configuration* $\eta \in \Omega(d)$ *and for each* $r \in \{1, 2, \ldots, d-1\}$ *and* $I \in F_r(k)$*,* $F_I^d(k)$ *in dimension* $r$ *is internally spanned by* $\eta$ *then* $B_k$ *internally spanned by* $\eta$*.*

*Proof.* We will prove the lemma by induction to the dimension $d$, for any $I$, $|I| = d-1$, because $B_{k-1}$ is completed occupied in the configuration $\eta \in \Omega(d)$ so for any $I \in F_{d-1}$, $\eta_{I,t}^{F_I^d(k)} \leqslant \eta_t^{B_k}$ and then by the definition $\eta_{I,\infty}^{F_I^d(k)} \leqslant \eta_\infty^{B_k}$, therefore $F_I^d(k)$ is completed occupied in $\eta_\infty^{F_d}$.
Now we would restrict the above proof to the $\mathbb{Z}^{d-1}$ occupied by $F_I^d(k)$ therefore we can finish the proof by induction of the dimension d. ☐

**Lemma 15.** *For the strong bootstrap percolation model,* $\gamma_d(p) \leqslant \gamma_{d-1}(p)$*,* $\pi_S(d) \geqslant \pi_S(d-1)$*.*

*Proof.* For $p \in [0, 1]$, consider a map $F$ from $(\Omega(d), \sigma(d), p)$ to $(\Omega(d-1), \sigma(d-1), p)$.

$$F(\eta \in \Omega(d)) = \eta \text{ on } \{0\} \times \mathbb{Z}^{d-1}$$

Then we have $P_{p,d-1}(A) = P_{p,d}(F^{-1}(A))$, for any $A \in \sigma(d-1)$. Obviously, $\eta_t(0) \leqslant F(\eta)_t(0)$, so we have $\gamma_d(p) \leqslant \gamma_{d-1}(p)$ and therefore $\pi_S(d) \geqslant \pi_S(d-1)$. ☐

**Lemma 16.** *For the strong bootstrap percolation model,*

$$1 - R^d(N, p) \leqslant C \cdot e^{-\gamma_d(p) \cdot N/4^d} \text{ where } C \text{ depends on } p \text{ and the dimension } d.$$

*Proof.* We prove the lemma by induction to the dimension. The lemma is easily verified for $d = 1$. We now suppose that it holds in dimension $1, 2, \ldots, d-1$ and will show that it holds in dimension $d$. If $\gamma_d(p) = 0$, there is nothing to prove, so we also suppose that $\gamma_d(p) > 0$. From Lemma 14, then

$$\gamma_1(p) \geqslant \gamma_2(p) \geqslant \cdots \geqslant \gamma_d(p) > 0$$

Set $M = [N/4]$ and let $F_N$ be the event that for every $k > M$, $r = 1, 2, \ldots, d-1$ and $I \in F_r(k)$, $F_I^d(k)$ in dimension $r$ is internally spanned by $\eta$. Let $F_I$ be the event that $F_I^d(k)$ in dimension $r$ is internally spanned by $\eta$. Because $\{F_I\}_{I \in F}$ are independent, by induction,

$$P((F_N)^c) = 1 - P\left(\bigcap_{r=1}^{d-1} \bigcap_{k=M+1}^{N} \bigcap_{I \in F_r} \{F_I\}\right)$$

$$\leqslant 1 - \sum_{r=1}^{d-1} \sum_{k=M+1}^{N} \sum_{I \in F_r} (R^r(k, p))$$

$$\leqslant 1 - \sum_{r=1}^{d-1} \sum_{k=M+1}^{N} \sum_{I \in F_r} (1 - \exp(-\gamma_r(p) \cdot k/4^r)$$

$$\leqslant \sum_{r=1}^{d-1} \sum_{k=M+1}^{N} \sum_{I \in F_r} C_r(p) \exp(-\gamma_r(p) \cdot k/4^r)$$

Let $G_N$ be the event that $\eta_\infty^{B_N}(x) = 1$ for any $x \in B_M$. Now we use the fact that the interaction is among nearest neighbours. So if we define

$$r(x) = \min\{n \mid x \in B_n\}$$

Then if $\eta_\infty^{B_N}(x) = 0$ we get $\eta_{N-r(x)}(x) = 0$ for any $x \in B_M$, therefore

$$\begin{aligned} P((G_N)^c) &\leqslant P_p\left(\bigcup_{x \in B_M} \{\eta_\infty^{B_N}(x) = 0\}\right) \\ &\leqslant P_p\left(\bigcup_{x \in B_M} \{\eta_{N-r(x)}(x) = 0\}\right) \\ &\leqslant P_p\left(\bigcup_{x \in B_M} \{\eta_{N-M}(x) = 0\}\right) \\ &\leqslant |B_M| \cdot P_p(\{\eta_{N-r(x)}(x) = 0\}) \\ &\leqslant |B_M| \cdot C_\varepsilon(p) \cdot \exp(-(\gamma_d(p) - \varepsilon) \cdot (N - M)) \\ &\text{here } \varepsilon \in (0, \gamma(p)/2) \end{aligned}$$

Now, from Lemma 14 and the induction, it follows that

$$\begin{aligned} 1 - R^d(N,p) &\leqslant P((F_N)^c) + P((G_N)^c) \\ &\leqslant C \cdot \exp(\gamma_d(p) \cdot N/4^d) \end{aligned}$$

□

**Lemma 17.** *For strong bootstrap percolation, if* $\pi_{d-1,sbp} = 0$ *then for any* $p > 0$*,*

$$\lim_{N \to \infty} R^d(N,p) = 1$$

*Proof.* From the last lemma, we only need to prove that

$$\lim_{N \to \infty} P((F_N)^c) = \lim_{N \to \infty} P((G_N)^c) = 0.$$

By the estimate in the last lemma for $P((F_N)^c)$,

$$P((F_N)^c) \leqslant \sum_{r=1}^{d-1} \sum_{k=M+1}^{N} \sum_{I \in F_r} C_r(p) exp(-\gamma_r(p) \cdot k/4^r)$$

Because $\pi_{d-1,sbp} > 0$, so $\gamma_1(p) \geqslant \gamma_2(p) \geqslant \cdots \geqslant \gamma_d(p) > 0$, so

$$\lim_{N \to \infty} P((F_N)^c) = 0$$

Define event $G$ be the event $\{\eta \in \Omega \mid \eta(0) = 1, F_I^d(k)$ in dimension r is internally spanned by $\eta$ for $k = 1, 2, \ldots,\ \ r = 1, 2, \ldots, d-1$ *and* $I \in F_r\}$, then

$$P(G) = p \prod_{k=1}^{\infty} \prod_{r=1}^{d-1} R^r(k,p)^{|F_r|}$$

By the proof of last lemma, $P(G) > 0$. For any $x \in \mathbb{Z}^d$, define $\theta_{-x}(\eta)(y) = \eta(y+x)$ for any $\eta \in \Omega$ is a function from $\Omega$ to $\Omega$. Define event $G_\theta$ as event $\{\eta \in \Omega \mid$ exists $x \in \mathbb{Z}^d$ such that $\theta_{-x}(\eta) \in G$, because $G_\theta$ is invariant by translations, from ergodicity of translations, $P(G_\theta) = 1$ so

$$\lim_{N\to\infty} P(G_N) = 1$$

This completes the proof. ☐

**Lemma 18.** $\pi_{d,sbp} > 1 - 1/2d$

*Proof.* For $p, 2d \cdot (1-p) < 1$, define

$$C_{t,\eta}(0) = \{x \in \mathbb{Z}^d \mid \text{there are } 0{=}x_0, x_1, \ldots, x_n = x \text{ such that } |x_i - x_{i+1}| = 1, i = 0, 1, \ldots, n-1 \\ \text{and } \eta_t(x_j) = 0 \text{ for } j = 1, 2, \ldots, n\}$$
$$R_{t,\eta}(0) = \sup\nolimits_{x \in C_{t,\eta}(0)} |x|, sup\varnothing = -1$$

It is easy to see that $R_{t+1}(0) \leqslant R_t(0) - 1$, so

$$\begin{aligned}
P_p(\{\eta \in \Omega \mid \eta_t(0) = 0\}) &= P_p(\{\eta \in \Omega \mid R_{t,\eta}(0) \geqslant 0\}) \\
&\leqslant P_p(\{\eta \in \Omega \mid R_{0,\eta}(0) \geqslant t\}) \\
&= P_p\left(\bigcup_{l=t}^{\infty} \cup_{|path|=t} \{\eta \in \Omega \mid \eta(x) = 1, x \ in \ the \ path\}\right) \\
&\leqslant \sum_{l=t}^{\infty} (1-p)^{l+1}(2d)^l \\
&= \frac{1-p}{1-2d\cdot(1-p)} \cdot e^{-ln(\frac{1}{2d\cdot(1-p)})}.
\end{aligned}$$

So $\pi_{d,sbp} > 1 - 1/2d$. ☐

**Lemma 19.** *Define* $\zeta_d(p) = \inf\{N \mid R^d(N,p) > 1 - 1/(4d)\}$, *then*

$$\gamma_d(p) \geqslant C/\zeta_d(p),$$

*where* $C$ *depends only on the dimension* $d$.

*Proof.* We can assume $\zeta_d(p) < \infty$, for $N > 0$, $k \in \mathbb{Z}^d$. Define

$$B_{N,k} = \{x \in \mathbb{Z}^d \mid x - (2N+1)\cdot k \in B_N\}$$

For any $\eta \in (\Omega, \sigma, p)$, define $\theta(\eta)(k) = 1$ if and only if $B_{N,k}$ is internally spanned by $\eta$, then $\theta$ is a map from $\eta \in (\Omega, \sigma, p)$ to $\eta \in (\Omega, \sigma, R^d(N,p))$ and keep the measure. In addition, for any $\tau \geqslant 0$, let $t = (2N+1)^d + (2dN+1)\tau$ then if $\theta(\eta)_t(k) = 1$, $\eta_\tau(x) = 1$ for any $x \in B_{N,k}$. Take $N = \zeta_d(p)$

Following the above argument and using the last lemma, we have

$$\begin{aligned}
&P_p(\eta \in \Omega \mid \eta_t(0) = 0) \\
&\leqslant P_{R^d(N,p)}(\eta \in \Omega \mid \theta(\eta)_\tau(0) = 0) \\
&\leqslant \frac{1 - R^d(N,p)}{1 - 2d\cdot(1 - R^d(N,p))} \cdot \exp\left(-\ln\left(\frac{1}{2d\cdot(1-R^d(N,p))}\right)\cdot\tau\right) \\
&\leqslant \frac{1 - R^d(N,p)}{1 - 2d\cdot(1 - R^d(N,p))} \cdot \exp\left(-\ln\left(\frac{1}{2d\cdot(1-R^d(N,p))}\right)\cdot\left(\frac{t-(2N+1)^d}{(2dN+1)}\right)\right).
\end{aligned}$$

Therefore,

$$\gamma_d(p) \geqslant \ln\left(\frac{1}{2d\cdot(1-R^d(N,p))}\right)/(2dN+1) \geqslant \frac{\ln 2/4}{\zeta_d(p)}.$$

This completes the proof. ☐

**Corollary 20.** *Proof of Theorem 8.*

*Proof.* By induction, if $\pi_{d-1,sbp}=0$, by Lemma 17, for any $p>0$,

$$\lim_{N\to\infty} R^d(N,p)=1.$$

So $\zeta_d(p)<\infty$ and $\gamma_d(p)>0$ for any $p>0$, therefore $\pi_{d,sbp}=0$. ☐

# 3 Simulating bootstrap percolation

## 3.1 Introduction

Following [AL88], considering only the case of $d=2$ dimensions, we now want to analyse the behaviour of the bootstrap bercolation on a finite square. As proved in the last section, we have $\pi_c=p_c=0$, that is, for all positive density $p$, the plane is filled exponentially fast with probability 1. However, this result is not supported by practical simulations on a finite $n\times n$ square at first sight: The square is only filled if $n$ is "large enough compared to $p$".
In order to simplify the calculations, we will let $1-e^{-p}$ be the probability of a site being occupied.

**Observation.** *The percolation dynamics of the bootstrap restricted to $\Lambda_L$ will eventually end. More precisely, a configuration is stable if and only if the filled sites form pairwise disjoint rectangles with a (Manhattan) distance of at least 2.*

We consider the quadratic region $\Lambda_n=(-\frac{n}{2},\frac{n}{2})^2$, so $\Lambda_n$ is a square of side length $n-2$ if $n$ is even and a square of side length $n-1$ if $n$ is odd. In particular, $R(2m-1,p)=R(2m,p)$ for all $m$, so without loss of generality, it suffices to consider only even $n$.

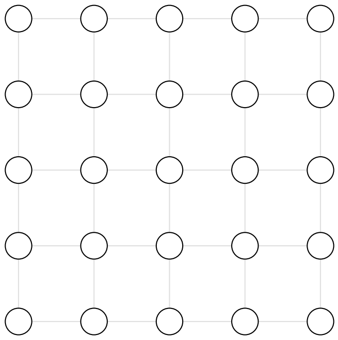


Figure 2: $\Lambda_5=\Lambda_6$

As in [AL88], we will quantify the above introductory ideas by the following main result:

**Theorem 21.** *For small $p$, $R(e^{\lambda/p},p)$ is essentially a step function in $\lambda$. More explicitly, there exists a constant $\lambda_c$ bounded by two constants, $c_1\leqslant\lambda_c\leqslant c_2$ which are independent of $p$, such that*

$$R(e^{\lambda/p},p)=e^{-(\lambda_c-\lambda+o(1))/p}\quad \textit{for } o(1)<\lambda<\lambda_c$$

*and*

$$R(e^{\lambda/p},p)\geqslant 1-\exp(-c\cdot e^{\lambda/p})\quad \textit{for } \lambda>\lambda_c.$$

## 3.2 The plan

Our first big goal is Lemma 38: It states essentially that for fixed $p$, our spanning probability $R(n,p)$ is approximately constant for medium values of $n$, providing a concrete bound on its value and deviation. For this sake, we will need to attain four smaller goals:

- Bound $\frac{R(n,p)}{R(k,p)}$ from above and below, for $k \leqslant n$. We need to prove that the quotient is approximately 1 in order for $R(n,p)$ to be approximately constant.
- Bound $R(n,p)$ from above and below (for a fixed value of $n$). As $R(n,p)$ is approximately constant, this will yield a bound on $R(n,p)$ for medium-sized $n$.

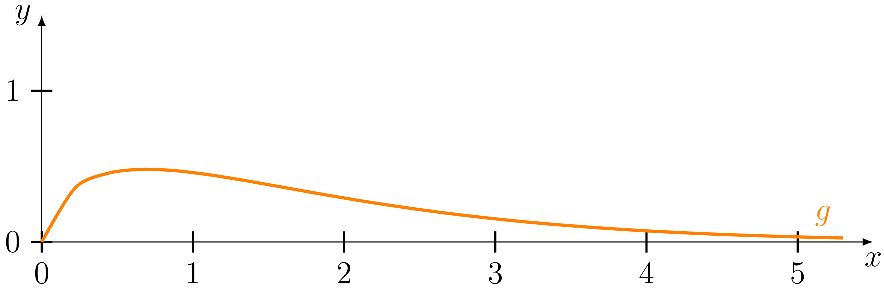
The bounds will consist of combinatorial ideas, combined with some analytic work in order to obtain an understanding of their orders of magnitude. In the analytic part, we will deal every time with similar terms; for this sake, we will first define and analyse an auxiliary function $g$.

## 3.3 An auxiliary function

Throughout our argumentation, the following function will appear quite naturally. We will first state and prove some of its basic analytic properties which we will exploit later.

**Definition 22.** *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be defined by*

$$g(x) = x \ln\left(\frac{1}{1-e^{-x}}\right).$$



**Lemma 23.** *The function $g$ has the following properties:*

*(1)* $g(x) \geqslant 0$ *for all* $x \geqslant 0$.

*(2)* $\lim_{x\to 0} g(x) = \lim_{x\to+\infty} g(x) = 0$.

*(3)* $g$ *has exactly one local maximum* $M$*, is increasing on* $(0,M)$ *and decreasing on* $(0,+\infty)$.

*(4) For* $x \geqslant 1$*, we have* $g(x) \leqslant e^{-x/2}$.

*(5) For* $t \geqslant 1$*, we have* $\int_t^\infty \frac{g(z)}{z} dz \leqslant e^{\frac{-t}{2}}$.

*Proof.* The first three properties are straightforward to verify. For the last two properties, note that

$$\ln\left(\frac{1}{1-e^{-x}}\right) = \ln\left(1 + \frac{e^{-x}}{1-e^{-x}}\right) \leqslant \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x - 1}$$

for all $x \geqslant 1$. As $e^x - 1 \geqslant \frac{1}{2}e^x > x \cdot e^{x/2}$, this implies that $g(x) \leqslant \frac{x}{e^x-1} < \frac{1}{e^{x/2}}$, proving the third property. Furthermore, it implies that

$$\int_t^\infty \frac{g(z)}{z} dz \leqslant \int_t^\infty \frac{1}{e^{x/2}} dz = \left[-\frac{1}{2}e^{-x/2}\right]_t^\infty = \frac{1}{2}e^{-t/2} < e^{-t/2}.$$

□

## 3.4 Bounding $R(n,p)$ and $\frac{R(n,p)}{R(k,p)}$ from below

### 3.4.1 The combinatorial idea: Rings

In order to bound $R(n,p)$ from below, we provide a sufficient condition for $\Lambda_n$ to be internally spanned. For this, we dissect $\Lambda_n \backslash \{(0,0)\}$ into rings $L_1, \ldots, L_{\frac{n}{2}-1}$, where $L_i$ is the set of sites on the border of $[-i,i]^2$.

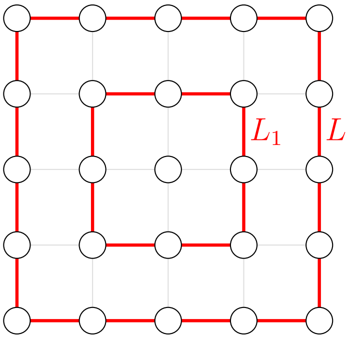
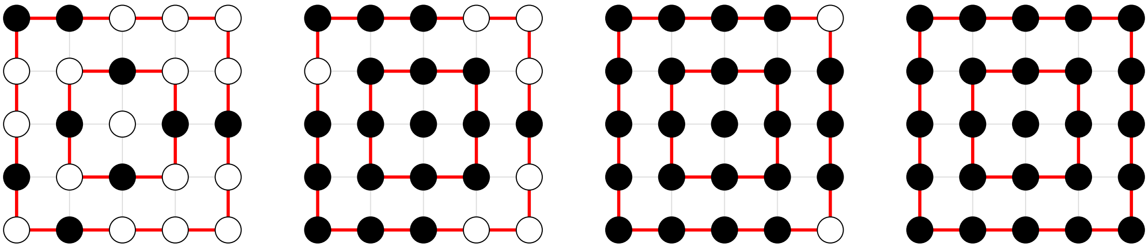


Figure 3: $\Lambda_6$ dissected into two rings

Now, for $\Lambda_n$ to be internally spanned it is sufficient that the following condition holds:

- Each ring $L_i$ contains an infected site on each of its four sides (without the corners). For a fixed ring $L_i$, this happens with probability $\left(1 - e^{-(2i-1)p}\right)^4$, as each of the four sides of $L_i$ contains $2i-1$ sites.

Indeed, for $L_1$ this implies that the neighbours of the origin are occupied, such that the origin will be occupied in the second step. The following pictures illustrate the continuing process filling up the entire region $\Lambda_n$:



This yields the following bound:

**Corollary 24.** *We have*

$$R(n,p) \geqslant \prod_{i=1}^{\frac{n}{2}-1} \left(1 - e^{-(2i-1)p}\right)^4 .$$

The same idea can be used for comparing $R(n,p)$ and $R(k,p)$; this will be crucial in Lemma 38 to prove that these two values are almost equal under certain conditions. In fact, for $\Lambda_n$ to be internally spanned the following two conditions are sufficient:

- $\Lambda_k$ is internally spanned; this happens with probability $R(k,p)$.
- Each ring $L_i$ with $i = \frac{k}{2}, \ldots, \frac{n}{2} - 1$ contains an infected site on each of its four sides.

This yields the following more general result:

**Corollary 25.** *We have*

$$R(n,p) \geqslant R(k,p) \cdot \prod_{i=\frac{k}{2}}^{\frac{n}{2}-1} \left(1 - e^{-(2i-1)p}\right)^4 .$$

### 3.4.2 Rewriting the bounds in terms of our auxiliary function $g$

In order to work with the products in Corollary 24 and Corollary 25, we aim to compare them to an integral. For this sake, we take their logarithms: For any integers $a, b \geqslant 1$, we have

$$\ln\left(\prod_{i=a}^{b} \left(1 - e^{-(2i-1)p}\right)^4\right) = 4\sum_{i=a}^{b} \ln\left(1 - e^{-(2i-1)p}\right).$$

Now let $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \ln\left(1 - e^{-xp}\right)$, so that the expression above becomes $4\sum_{i=a}^{b} f(2i-1)$. One can verify that $f$ is concave and integrable, which yields by Jensen's inequality that $2f(t) \geqslant \int_{t-1}^{t+1} f(x)dx$ for all $t \geqslant 1$. Therefore,

$$4\sum_{i=a}^{b} f(2i-1) \geqslant 2\sum_{i=a}^{b} \int_{2i-2}^{2i} f(x)dx = \int_{2a-2}^{2b} f(x)dx.$$

Here, we will make use of the function $g$ from Definition 22: Note that $f(x) = -\frac{g(px)}{px}$, which implies that

$$\int_{2a-2}^{2b} f(x)dx = -\frac{1}{p} \cdot \int_{(2a-2)p}^{2bp} \frac{g(x)}{x}dx.$$

For simplicity, we remove the upper limit of the integral and obtain, using the positivity of $g$, that

$$-\frac{1}{p}\int_{(2a-2)p}^{2bp} \frac{g(x)}{x}dx \geqslant -\frac{1}{p}\int_{(2a-2)p}^{\infty} \frac{g(x)}{x}dx.$$

Together with Corollary 24, where $a = 1$ and $b = \frac{n}{2} - 1$, this finally yields the following simpler bound on $R(n,p)$:

**Corollary 26.** *We have*

$$\ln R(n,p) \geqslant -\frac{1}{p}\int_{0}^{\infty} \frac{g(x)}{x}dx$$

Similarly, Corollary 25 with $a = \frac{k}{2}$ and $b = \frac{n}{2} - 1$ becomes

**Corollary 27.** *We have*

$$\ln R(n,p) - \ln R(k,p) \geqslant \int_{(k-2)p}^{\infty} \frac{g(x)}{x}dx.$$

## 3.5 Bounding $R(n,p)$ from above

### 3.5.1 The combinatorial idea: Slabs

In order to bound $R(n,p)$ from above, we dissect $\Lambda_n$ into $\frac{n}{2}$ columns of width 2. Each of these must contain at least one particle in the initial configuration. Otherwise, such a column would never contain an infected site, as every site in the column is adjacent to at most one site in another column. This yields

$$R(n,p) \leqslant \left(1 - e^{-2pn}\right)^{n/2}.$$

### 3.5.2 Rephrasing the bound in terms of $g$

Taking the logarithm in the above bound yields

$$\ln R(n,p) \leqslant \frac{n}{2}\ln\left(1-e^{-2pn}\right) = -\frac{n}{2}\cdot\ln\frac{1}{1-e^{-2pn}} = -\frac{1}{4p}\cdot g(2pn).$$

Therefore, we obtain the following:

**Corollary 28.** *For any $n$, we have*

$$\ln R(n,p) \leqslant -\frac{1}{4p}\cdot g(2pn).$$

## 3.6 Bounding $\frac{R(n,p)}{R(k,p)}$ from above

### 3.6.1 The combinatorial idea: Internally spanned rectangles of certain dimensions

For an upper bound on $R(n,p)$, we need to find a necessary condition for $\Lambda_n$ to be internally spanned. For this, we consider rectangles inside $\Lambda_n$ with nonnegative side lengths whose corners are lattice points and whose sides are parallel to the coordinate axes. Note that we denote by "side length" of a rectangle the number of sites on its edges (to match the notation in [AL88]). We will see in this paragraph that such rectangles are "succeedingly spanned" in an increasing order.

**Definition 29.** *For $k \geqslant 0$, let $\mathcal{R}_k$ be the set of rectangles whose longest side length is in the interval $[k, 2k+1]$.*

**Lemma 30.** *We have $|\mathcal{R}_k| \leqslant (2k+1)^2\cdot n^2$.*

*Proof.* For any $a, b \in \{1,\dots,n\}$, there are $(n-a+1)\cdot(n-b+1) \leqslant n^2$ rectangles of dimension $a \times b$. The side lengths of a rectangle in $\mathcal{R}_k$ are between 1 and $2k+1$, which corresponds to at most the choices of $(2k+1)^2$ for its dimension. This yields the claim. ☐

The crucial observation in this paragraph is the following:

**Lemma 31.** *If $\Lambda_n$ is internally spanned, then for all $k \in \{1,\dots,\frac{n}{2}\}$ there exists an internally spanned rectangle in $\mathcal{R}_k$.*

*Proof.* The essential idea is the following: If $S_1$ and $S_2$ are two internally spanned rectangles with a Manhattan distance of at most 2, the smallest rectangle $S$ containing $S_1$ and $S_2$ is also internally spanned. The largest side length of $S$ is at most double the largest side length of $S_1$ or $S_2$ plus two (for the Manhattan distance). We will prove in a certain sense that this unification process reflects the entire dynamics; this will yield the claim as a sequence of integers who double at most in every step hits every interval $[k, 2k+1]$.

We will proceed by induction, which resembles the evolution of the spanning process. For $k = 1$, the statement is clear, as at least one site needs to be occupied. Suppose that the statement holds for $k = 1,\dots,K$ with $1 \leqslant K \leqslant \frac{n}{2}-1$. Let $\mathcal{S}_{K-1} \subseteq \bigcup_{k=0}^{K-1}\mathcal{R}_k$ be the set of rectangles with both side lengths at most $2K-1$ which are internally spanned and inclusion-maximal with this property.

If there is a rectangle in $\mathcal{S}_{K-1}$ with a side length of at least $K$, this rectangle is in $\mathcal{R}_K$, proving the induction hypothesis. Otherwise, we assume that both sides lengths of all rectangles in $\mathcal{S}_{K-1}$ are at most $K-1$.

If all pairs of rectangles in $\mathcal{S}_{K-1}$ had a Manhattan distance of at least 3, the configuration where all rectangles of $\mathcal{S}_{K-1}$ are fully occupied would be stable, so the dynamics would stop in this configuration, contradicting the fact that $\Lambda_n$ is internally spanned. Therefore, there are two rectangles $S_1, S_2 \in \mathcal{S}_{K-1}$ that have a distance of at most 2. This means that the smallest rectangle $S$ that contains $S_1$ and $S_2$ is also spanned internally.
As $\mathcal{S}_{K-1}$ consists by definition of rectangles with maximum inclusion of both sides with at most $2K-1$, on the one hand $S$ is strictly larger than $S_1$ and $S_2$, and on the other hand one of its side lengths is at least $2K$. However, since both side lengths of $S_1$ and $S_2$ are by our assumption at most $K-1$ and the Manhattan distance between $S_1$ and $S_2$ is at most 2, the side lengths of $S$ are at most $(K-1)+(K-1)+2 = 2K$. In conclusion, this yields the result the result that the longest side length of $S$ is exactly $2K$, which yields that $S \in \mathcal{R}_K$. □

Lemma 30 and Lemma 31 can be rephrased into an upper bound on $R(n,p)$ if we introduce the following notation:

**Definition 32.** *For $k \leqslant n$, let $R_1(k,p)$ be the maximum probability of a rectangle in $\mathcal{R}_k$ to be internally spanned, i. e. $R_1(k,p) = \max_{R\in\mathcal{R}_k} \mathbb{P}(R \text{ is internally spanned})$.*

(Note that $R_1(k,p)$ is independent of $n$, as long as $k \leqslant n$.)

**Corollary 33.** *For all $k \leqslant n$, we have $R(n,p) \leqslant (2k+1)^2 n^2 \cdot R_1(k,p)$.*

*Proof.* By Lemma 31, $R(n,p)$ is less than or equal to the probability for at least one rectangle in $\mathcal{R}_k$ to be internally spanned. By the union bound and Lemma 30, this probability is at most $(2k+1)^2 n^2 \cdot R_1(k,p)$. □

### 3.6.2 Comparing $R_1(k,p)$ and $R(k,p)$

As explained in the initial plan, it will later be crucial in Lemma 38 to compare $R(n,p)$ and $R(k,p)$ for $k \leqslant n$. We have already established a lower bound on $\frac{R(n,p)}{R(k,p)}$ in Corollary 25 and Corollary 27. In this section, we will deduce from Corollary 33 an upper bound on $\frac{R(n,p)}{R(k,p)}$.

**Lemma 34.** *We have $R(2k+1,p) \geqslant R_1(k,p) \cdot \left(1-e^{-pk}\right)^{2k+1}$.*

*Proof.* For $R(2k+1,p)$ to be internally spanned, it is sufficient that a rectangle in $\mathcal{R}_k$ with maximum spanning probability is internally spanned, and each of the surrounding "rings" contains at least an infected site on each edge. This yields the inequality as in Corollary 24. □

This yields our desired bound:

**Corollary 35.** *For all $k \leqslant n$, we have*

$$R(n,p) \leqslant R(k,p) \cdot \frac{k^2 n^2}{\left(1-e^{-p(k-1)/2}\right)^{2k}}.$$

Again, we will resort to our auxiliary function $g$ in order to simplify this bound. Taking the logarithms on both sides of Corollary 35, we get the following result:

**Corollary 36.** *We have*

$$\ln R(n,p) - \ln R(k,p) \leqslant 2\ln(kn) + \frac{2}{p} \cdot g\left(\frac{pk}{2}\right)$$

### 3.7 $R(n,p)$ is approximately constant in a certain range

We will now combine and rephrase Corollary 27 and Corollary 36. For the sake of readability, we introduce a new notation to work with the logarithm of $R(n,p)$.

**Definition 37.** *Let* $\sigma(n,p) = -p \ln R(n,p)$.

**Lemma 38.** *There exist positive constants* $A_0 \gg 1 \gg B_0$ *such that for every* $p$, *every* $A > \max\left(A_0, 3\ln\frac{1}{p}\right)$ *and every* $B < B_0$, *the following holds: The value of* $\sigma(k,p)$ *is approximately constant throughout the regime*

$$\frac{A}{p} \leqslant k \leqslant \exp\left(\frac{B}{p}\right).$$

*Concretely, for these values of* $n$ *we have*

$$|\sigma(k,p) - \hat{\sigma}| \leqslant 4B + 4\exp\left(-\frac{A}{3}\right),$$

*where* $\hat{\sigma}$ *is a constant bounded by*

$$C_1 := \sup_{x \geqslant 0} g(x) \leqslant \hat{\sigma} \leqslant 2\int_0^\infty \frac{g(x)}{x} dx =: C_2.$$

*Proof.* We will pick $A_0$ and $B_0$ throughout the proof. The strategy will be to set $n = \exp\left(\frac{B}{p}\right)$ to the maximum value of the considered regime and $\hat{\sigma} = \sigma(n,p)$ in order to bound $|\sigma(k,p) - \hat{\sigma}| = |\sigma(k,p) - \sigma(n,p)|$ by Corollary 27 and Corollary 36.
By Corollary 27, for any $n \geqslant k$ we have

$$\sigma(n,p) - \sigma(k,p) \leqslant p \int_{(k-2)p} \frac{g(x)}{x} dx.$$

As $g$ is positive, this implies for $(k-2)p \geqslant A$ (which is equivalent to $k \geqslant \frac{A}{p} + 2$) that

$$\sigma(n,p) - \sigma(k,p) \leqslant p \int_A^\infty \frac{g(x)}{x}.$$

By Lemma 23, we can pick $A_0$ sufficiently large such that for any $A > A_0$ we have

$$\int_A^\infty \frac{g(x)}{x} \leqslant e^{-A/2}.$$

On the other hand, Corollary 36 yields

$$\sigma(n,p) - \sigma(k,p) \geqslant -2p\ln(kn) - 2g\left(\frac{pk}{2}\right).$$

As $k \leqslant n$, we have $\ln(kn) \leqslant \ln\left(n^2\right) = 2\ln n$. Moreover, if we choose $A_0$ larger than the only local maximum of $g$, this implies by monotonicity of $g$ that for $kp \geqslant A > A_0$ we have

$$g\left(\frac{pk}{2}\right) \leqslant g\left(\frac{A}{2}\right) \leqslant e^{-A/3},$$

where we used a claim from Lemma 23 in the last inequality.

In conclusion, we have

$$|\sigma(n,p) - \sigma(k,p)| \leqslant \min\left(pe^{-A/2},\ 4p\ln n + 2e^{-A/3}\right).$$

If we put $B = p\ln n$ and $\hat{\sigma} = \sigma(n,p)$, this yields that for $\frac{A}{p} + 2 \leqslant k \leqslant n = e^{Bp}$ we have

$$|\sigma(k,p) - \hat{\sigma}| \leqslant \min\left(pe^{-A/2},\ 4B + 2e^{-A/3}\right),$$

proving that $\sigma(k,p)$ is approximately constant. Finally, the bounds on $\hat{\sigma}$ follow from Corollary 28 and Corollary 24. ☐

## 3.8 Analysing the behaviour of $R(n,p)$ for $n = \exp(\frac{\lambda}{p})$

We now continue with our final goal: We want to prove that the percolation process on an infinite grid can only be observed with an "exponential size" simulation. More concretely, if we consider $n = e^{\lambda/p}$ in terms of a parameter $\lambda$, we prove essentially that $R(n,p) \approx 0$ if and only if $\lambda$ is higher than some threshold $\lambda_c$.
For this, we will first compare $R(e^{\lambda/p}, p)$ to $R(\frac{1}{p^3}, p)$ in order to be able to apply Lemma 38. This will enable us to see that $R(e^{\lambda/p}, p)$ behaves like an exponential function of the form $\exp(-p^{-1}\cdot(\lambda_c(p) - \lambda + o(1)))$, which is approximately a step function for small $p$.

### 3.8.1 Three almost equivalent events

In order to compare $R(e^{\lambda/p}, p)$ to $R(\frac{1}{p^3}, p)$, we will consider an event that is "almost equivalent" to $\Lambda_n$ being internally spanned.
Let $G$ denote the event that each linear segment of length $\frac{1}{2p^3}$, parallel to a principal axis, has at least one occupied neighbouring site. Furthermore, we introduce the following events:

$A_1$ : $\Lambda_n$ is internally spanned,
$A_2$ : $\Lambda_n$ contains a translate of $\Lambda_{1/p^3}$ which is internally spanned,
$A_3$ : $\Lambda_n$ contains an internally spanned translate of $\Lambda_{1/p^3}$ centred at a point in $\frac{1}{2p^3}\mathbb{Z}^d$.

**Lemma 39.** *We have* $\mathbb{P}(G^c) \leqslant 4n^2\exp(-\frac{1}{p^2})$.

*Proof.* In a fixed row, there are $n \cdot 2p^3 \leqslant 2n$ horizontal segments of length $\frac{1}{2p^3}$; so over the total of $n$ rows, there are at most $2n^2$ horizontal such segments. The same number of vertical segments is added, yielding at most $4n^2$ segments of length $\frac{1}{2p^3}$ in $\Lambda_n$.
As such a segment has more than $2\cdot\frac{1}{2p^3} = \frac{1}{p^3}$ neighbours, the probability that none of its neighbours are infected is at most $(e^{-p})^{1/p^3} = e^{-1/p^2}$. The claim follows by the union bound. ☐

**Lemma 40.** *Given $G$, the events $A_1$, $A_2$ and $A_3$ are equivalent.*

*Proof.* It is clear that $A_3$ implies $A_2$.
One can see that $A_2$ implies $A_1$ by the same argument as in Corollary 25: By $G$, every "ring" around a translate of $\Lambda_{1/p^3}$ contains at least one occupied site on each of its edges.
It remains to prove that $A_1$ implies $A_3$. For this, observe that by Lemma 31, there exists an internally spanned rectangle $R$ whose longest side length is in the interval $[\frac{1}{4p^3} - 1, \frac{1}{2p^3}]$. Therefore, $R$ is contained in a translate of $\Lambda_{1/p^3}$ with its midpoint in $\frac{1}{2p^3}\mathbb{Z}^d$, and we can conclude as before: By $G$, each rectangular "ring" around $R$ contains at least an infected site on each of its edges. ☐

By these two lemmas, we can bound the symmetric differences $A_i \Delta A_j := (A_i \backslash A_j) \cup (A_j \backslash A_i)$:

**Corollary 41.** *For all $i, j \in \{1, 2, 3\}$, we have $\mathbb{P}(A_i \Delta A_j) \leqslant \exp(-\frac{C}{p^2})$, where $C < \infty$ is a constant independent from $p$.*

*Proof.* We have

$$\mathbb{P}(A_i \Delta A_j) = \underbrace{\mathbb{P}(A_i \Delta A_j \mid G)}_{=0} + \underbrace{\mathbb{P}(A_i \Delta A_j \mid G^c)}_{\leqslant \mathbb{P}(G^c)} \leqslant \mathbb{P}(G^c) \leqslant 8n^2 \exp\left(-\frac{2}{p^2}\right),$$

from which the claim follows. ☐

### 3.8.2 Concluding via a comparison of $R(e^{\lambda/p}, p)$ to $R(\frac{1}{p^2}, p)$

By Corollary 41, it suffices to bound $\mathbb{P}(A_3)$ in order to obtain a bound on $\mathbb{P}(A_1)$. As the number of considered translates of $\Lambda_{1/p^3}$ in $A_3$ is $(2p^3 n)^2$, we have for $n = e^{\lambda/p}$ that

$$R(e^{\lambda/p}, p) \approx \mathbb{P}(A_3) \leqslant (2p^3 e^{\lambda/p})^2 \cdot R\left(\frac{1}{p^3}, p\right) = e^{\ln 4 + 6 \ln p + \lambda/p - \sigma(n,p)/p}.$$

(Recall that $R(n, p) = e^{-\sigma(n,p)/p}$.) As for sufficiently small $p$, we have $\frac{A}{p} < \frac{1}{p^3} < \exp(\frac{B}{p})$, we obtain by Lemma 38 that $\sigma(n, p) = \hat{\sigma} + o(1)$, so that we have

$$R(e^{\lambda/p}, p) \leqslant e^{(\lambda - \hat{\sigma} + o(1))/p}.$$

This implies that for $\lambda < \hat{\sigma}$, $R(e^{\lambda/p}, p)$ converges rapidly to 0. Note that the exact value of $\hat{\sigma}$ may depend on $p$; however, Lemma 38 yields bounds $C_1 \leqslant \hat{\sigma} \leqslant C_2$ that are independent of $p$. This implies the first part of Theorem 21.
For the second part, observe that

$$\mathbb{P}(A_3) \geqslant 1 - \left(1 - R\left(\frac{1}{p^3}, p\right)\right)^{(2p^3 \exp(\lambda/p))^2},$$

as the right-hand side is the probability that none of $(2p^3 \exp(\lambda/p))^2$ independent translates of $\Lambda_{1/p^3}$ are internally spanned. Using $1 - x \leqslant e^{-x}$, it follows that

$$\mathbb{P}(A_3) \geqslant 1 - \exp\left(R\left(\frac{1}{p^3}, p\right) \cdot (2p^3 e^{\lambda/p})^2\right) \approx 1 - \exp\left(4\hat{\sigma} \cdot (p^3 e^{\lambda/p})^2\right).$$

This proves the second part of Theorem 21.

# 4 Conclusion

Bootstrap percolation is an important model to describe processes in physics. However, practical observations differ strongly from mathematical theory: While the probability that the origin becomes infected in an infinite grid is 1, this can only be observed in simulations of "exponential" size. We proved both observations by similar ideas, exploiting necessary and sufficient conditions for the dynamics to span squares or the plane.

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