

# Self-Similar solution of the Navier-Stokes Equation

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## Abstract

We study the classical Cauchy problem for the incompressible 3d Navier-Stokes equations with  $(-1)$ -homogeneous initial data and show that it has a global scale-invariant solution which is smooth for positive times.

We are grateful to our supervisor Julien Guillod, who proposed and motivated us to work on this problem which lead us to study the theory of PDE and especially the Navier-Stokes equation.

## 1 Introduction

We consider the classical Cauchy problem for the incompressible Navier-Stokes equation in  $\mathbb{R}^3 \times (0, \infty)$

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p - \Delta u = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } \mathbb{R}^3 \times (0, \infty), \quad (1.1)$$

$$u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3. \quad (1.2)$$

We recall that the problem is invariant under the scaling

$$\begin{aligned} u(x, t) &\rightarrow u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \\ p(x, t) &\rightarrow p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t), \\ u_0(x) &\rightarrow u_{0\lambda}(x) = \lambda u_0(\lambda x), \end{aligned} \quad (1.3)$$

where  $\lambda > 0$ . We say that a solution  $u$  is *scale-invariant* if  $u_\lambda = u$  and  $p_\lambda = p$  for all  $\lambda > 0$ . Similarly, we say that an initial condition  $u_0$  is scale-invariant, if  $u_{0\lambda} = u_0$  for all  $\lambda > 0$ . This is of course the same as requiring that  $u_0$  be  $(-1)$ -homogeneous.

The report will discuss the following result.

**Theorem 1.1.** *Assume that  $u_0$  is scale-invariant and smooth in  $\mathbb{R}^3 \setminus \{0\}$  with  $\operatorname{div} u_0 = 0$ . Then the Cauchy problem (1.1), (1.2) has at least one scale-invariant Leray solution  $u$  which is smooth in  $\mathbb{R}^3 \times (0, \infty)$ .*

This theorem was first proved by Jia and Šverák in [4]. Their method is finding the solution in the form  $u(x, t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$  in order to satisfy the scaling invariance. The Navier-Stokes equation for  $u$  gives a Stokes equation for  $U$  in  $\mathbb{R}^3$

$$-\Delta U - \frac{1}{2} U - \frac{1}{2} x \cdot \nabla U + U \cdot \nabla U + \nabla P = 0 \quad (1.4)$$

with the asymptotics

$$|U(x) - u_0(x)| = o\left(\frac{1}{|x|}\right), \quad x \rightarrow \infty \quad (1.5)$$

The problem (1.4), (1.5) is reminiscent of the Leray's problem of finding steady-states solution of the Navier-Stokes equation in a bounded domain with given conditions of  $U$ . Jia and Šverák showed this problem by using Leray-Schauder Theorem for a good functional space and establish a-priori estimates, and we will sketch their proof in the last part of section 3.

Our report is organized as follows. In section 2 we discuss some results and proof of [4]. The first part of this section is mostly about criteria for solutions of (1.1) to be Hölder continuous in some domain of  $\mathbb{R}^3 \times (0, \infty)$ . The second one is about estimates of Leray-solutions to the Navier-Stokes equation near initial time. The last part of section 2 discusses about estimates of self-similar solutions. In the section 3, we then give an alternative proof of Theorem 1.1 by following the techniques in [3] but with a weaker conclusion.

The strategy of our proof is the following one: we first use the same idea as Jia and Šverák by trying to find a solution  $u$  of the form  $u(x, t) = \frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right)$ . We seek the solution  $U$  in the form

$$U(x) = U_0(x) + V(x) \quad \text{where } U_0 = e^\Delta u_0 \quad \text{and } V \in H_{0,\sigma}^1(\mathbb{R}^3)$$

then show that  $u$  satisfies almost the conditions of a Leray solution, except the locally square-integrable at  $t = 0$ . To do this, we try to find a weak solution  $V \in H_{0,\sigma}^1(\Omega)$  of a Stokes system when  $\Omega$  is a bounded domain of  $\mathbb{R}^3$  using the Leray-Schauder theorem, then extend the result to  $\mathbb{R}^3$  in the first subsection. Then in the second subsection, we show that  $u$  satisfies almost the conditions of Leray solutions. We note that our proof differs from the original one in that it requires a weaker decay  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$ , compared to the decay condition that Jia and Šverák gave in [4].

**Notation** We use the same notation as [4]. For instance,  $B_R(x_0)$  denotes a ball centered at  $x_0$  with radius  $R$  in  $\mathbb{R}^3$ ,  $B_R := B_R(0)$ ; for  $z_0 = (x_0, t_0)$ ,  $Q(R, z_0) := B_R(x_0) \times (t_0 - R^2, t_0]$ ,  $Q_R := Q(R, (0, 0))$ ; for any  $f$  in  $\mathcal{O}$ ,  $\int_{\mathcal{O}} f := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} f$ . We also use the following standard notations in the literature: for vectors  $a$  and  $v$ ,  $a \otimes v$  is a matrix with  $(a \otimes v)_{ij} = a_i v_j$ ; for two matrices  $a, b$ ,  $(a : b) = a_{ij} b_{ij}$  where we assume the usual Einstein summation convention; for a matrix valued function  $f = (f_{ij})$ ,  $\operatorname{div} f$  is a vector with  $(\operatorname{div} f)_i = \sum_j \partial_j f_{ij}$ . We use  $C$  to denote an absolute and often large positive number,  $c$  a positive small absolute number,  $\epsilon$  the positive small numbers,  $C(\alpha, \beta, \dots)$  when the number depends on the parameters  $\alpha, \beta, \dots$ .  $C_{\text{par}}^\alpha(\mathcal{O})$  denotes the Hölder space with respect to the parabolic distance when  $\mathcal{O}$  is a space-time domain.

## 2 Main Results from Jia and Šverák on Navier-Stokes Local Estimates

### 2.1 $\epsilon$ -regularity criteria

Our goal in this subsection is to explain an  $\epsilon$ -regularity criteria for a generalized Navier-Stokes equation. It shows that if a scaled local  $L^3$ -norm of a suitable weak solution is sufficiently small around the origin, it is regular in the parabolic Hölder sense.

First, we recall the definition of a suitable weak solution of Navier Stokes equation.

**Definition 2.1.** *Let  $\mathcal{O}$  be an open subset of  $\mathbb{R}_x^3 \times \mathbb{R}_t$ ,  $a \in L_{\text{loc}}^m(\mathcal{O})$  with  $m > 5$  (not necessarily an integer),  $\operatorname{div} a = 0$ . We call a pair of functions  $(u, p)$  suitable weak solution to*

$$\begin{cases} \partial_t u - \Delta u + a \cdot \nabla u + \operatorname{div}(a \otimes u) + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \quad (x, t) \in \mathcal{O}, \quad (2.1)$$

if  $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(\mathcal{O}')$  for any open subset  $\mathcal{O}' \subseteq \overline{\mathcal{O}'} \subset \mathcal{O}$ ,  $p \in L_{loc}^{3/2}(\mathcal{O})$ , such that  $(u, p)$  satisfies equations (2.1) in the sense of distributions in  $\mathcal{O}$ , and

$$\partial_t \frac{|u|^2}{2} - \Delta \frac{|u|^2}{2} + |\nabla u|^2 + \nabla \cdot \left( \frac{|u|^2}{2} (u + a) \right) + u \cdot \operatorname{div}(a \otimes u) + \operatorname{div}(up) \leq 0, \quad (2.2)$$

in the sense of distributions. Recall that a distribution  $v$  in  $\mathcal{O}$  is called nonnegative if  $(v, \phi) \geq 0$  for any  $\phi \in C_0^\infty(\mathcal{O})$  with  $\phi \geq 0$ ;  $u \cdot \operatorname{div}(a \otimes u)$  is a distribution with

$$(u \cdot \operatorname{div}(a \otimes u), \phi) = - \int_{\mathcal{O}} a_i u_j \partial_j u_i \phi(x, t) dx dt = \int_{\mathcal{O}} a_i u_j u_i \partial_j \phi(x, t) dx dt.$$

The terms in (2.2) make sense due to the regularity assumptions and  $u \in L_{loc}^{10/3}(\mathcal{O})$  by the Sobolev embeddings.

The main theorem in this section can be stated as the following:

**Theorem 2.2.** ( *$\epsilon$ -regularity criterion*) Let  $(u, p)$  be a suitable weak solution to (2.1) in  $Q_1$  with  $a \in L^m(Q_1)$ ,  $m > 5$ ,  $\operatorname{div} a = 0$ . Then there exists  $\epsilon_0 = \epsilon_0(m) > 0$  with the following property: if

$$\left( \int_{Q_1} |u|^3 dx dt \right)^{1/3} + \left( \int_{Q_1} |p|^{3/2} dx dt \right)^{2/3} + \left( \int_{Q_1} |a|^m dx dt \right)^{1/m} \leq \epsilon_0,$$

then  $u$  is Hölder continuous in  $Q_{1/2}$  with exponent  $\alpha = \alpha(m) > 0$  and

$$\|u\|_{C_{par}^\alpha(Q_{1/2})} \leq C(m, \epsilon_0).$$

By setting:  $a(x, t) = \frac{1}{R_0} b\left(\frac{x-x_0}{R_0}, \frac{t-t_0}{R_0}\right)$  where  $0 < R_0 < \frac{1}{2}$  is a small positive number and  $z_0 = (x_0, t_0) \in Q_{1/2}$ , then we can show by applying the preceding theorem that we can get rid of the smallness condition for  $a$ :

**Theorem 2.3.** (*Improved  $\epsilon$ -regularity criteria*) Let  $(u, p)$  be a suitable weak solution to (2.1) in  $Q_1$ , with  $a \in L^m(Q_1)$ ,  $\operatorname{div} a = 0$ ,  $\|a\|_{L^m(Q_1)} \leq M$ , for some  $M > 0$  and  $m > 5$ . Then there exists  $\epsilon_1 = \epsilon_1(m, M) > 0$  with the following properties: if

$$\left( \int_{Q_1} |u|^3 dx dt \right)^{1/3} + \left( \int_{Q_1} |p|^{3/2} dx dt \right)^{2/3} \leq \epsilon_1,$$

then  $u$  is Hölder continuous in  $Q_{1/2}$  with exponent  $\alpha = \alpha(m) > 0$  and

$$\|u\|_{C_{par}^\alpha(Q_{1/2})} \leq C(m, \epsilon_1, M).$$

These two theorems are useful when we want to show that we can gain some regularity on the solution just by proving some estimates.

## 2.2 Local in space near intial time smoothness of Leray solution

In this subsection, we explain how Jia and Šverák used the ' $\epsilon$ -regularity' theorem given in the last subsection to study the local in space near initial time smoothness of the so-called *Leray solutions*. We recall the definition of Leray solutions. The setting is as follows.

Let  $u_0 \in L_{loc}^2(\mathbb{R}^3)$  with  $\operatorname{div} u_0 = 0$  and  $\sup_{x_0 \in \mathbb{R}^3} \int_{B_1(x_0)} |u_0|^2 dx < \infty$ . Suppose first that  $u \in C^\infty(\mathbb{R}^3 \times (0, \infty)) \cap L_t^2 H_x^1(\mathbb{R}^3 \times (0, \infty))$  is a solution to (1.1). Then it is also a solution in the sense of distribution, it verifies some local energy inequality by taking the dot product in (1.1) and multiplying by a positive, compactly supported smooth function  $\phi$ . So a stronger condition for a solution in the sense of distribution is the following one:

**Definition 2.4.** (Leray solution) A vector field  $u \in L^2_{loc}(\mathbb{R}^3 \times [0, \infty))$  is called a Leray solution to Navier-Stokes equations with initial data  $u_0$  if it satisfies:

(i) Decay condition

$$\operatorname{ess\,sup}_{0 \leq t \leq R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x, t) \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u|^2 \, dx dt < \infty,$$

$$\lim_{|x_0| \rightarrow \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2(x, t) \, dx dt = 0$$

(ii) For some distribution  $p$  in  $\mathbb{R}^3 \times (0, \infty)$ , the pair  $(u, p)$  verifies the Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \quad (\text{NSE})$$

in the sense of distributions, and for any compact set  $K \subset \mathbb{R}^3$ ,

$$\lim_{t \rightarrow 0^+} \|u(\cdot, t) - u_0\|_{L^2(K)} = 0.$$

(iii)  $u$  is suitable in the sense of Caffarelli-Kohn-Nirenberg, more precisely, the following local energy inequality holds:

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \phi(x, t) \, dx dt \leq \int_0^\infty \int_{\mathbb{R}^3} \left( \frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \frac{|u|^2}{2} u \cdot \nabla \phi + p u \cdot \nabla \phi \right) \, dx dt$$

for any smooth  $\phi \geq 0$  with  $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, \infty))$ .

The set of all Leray solutions starting from  $u_0$  will be denoted as  $\mathcal{N}(u_0)$ .

**Remark 2.5.** Note that we impose a decay condition on  $u$  in (i). This condition allows us to calculate  $p$  in the following way: For all  $(x, t) \in B_r(x_0) \times (0, t_*) \subseteq \mathbb{R}^3 \times (0, \infty)$ , take a smooth cutoff function  $\phi$  with  $\phi|_{B_{2r}(x_0)} = 1$ , then there exists a function  $p(t)$  depending only on  $x_0, r, t, \phi$  (we suppress the dependence on  $x_0, r, \phi$  in our notation) such that for  $(x, t) \in B_r(x_0) \times (0, t_*)$

$$\begin{aligned} p(x, t) &= -\Delta^{-1} \operatorname{div} \operatorname{div}(u \otimes u \phi) \\ &\quad - \int_{\mathbb{R}^3} (k(x-y) - k(x_0-y)) u \otimes u(y, t) (1 - \phi(y)) \, dy + p(t) \end{aligned} \quad (3.3)$$

where  $k(x)$  is the kernel of  $\Delta^{-1} \operatorname{div} \operatorname{div}$ .

The right-hand side is well defined since  $u$  satisfies the estimates in (i) and

$$|k(x-y) - k(x_0-y)| = O\left(\frac{1}{|x_0-y|^4}\right) \quad \text{as } |y| \rightarrow \infty. \quad (3.4)$$

For Leray solution  $u \in \mathcal{N}(u_0)$ , we have the following a priori estimates, first proved in [5]. These estimates have played an important role in the proof of Theorem 2.9 for a priori estimate for forward self similar solutions, which will be explained in the next section.

**Lemma 2.6.** (A priori estimate for Leray solutions) Let  $\alpha = \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u_0|^2}{2}(x) \, dx < \infty$  for some  $R > 0$ , and let  $u$  be a Leray solution with initial data  $u_0$ . Then there exists some small absolute number  $c > 0$  such that for  $\lambda$  satisfying  $0 < \lambda < c \min\{\alpha^{-2} R^2, 1\}$ , we have

$$\operatorname{ess\,sup}_{0 \leq t \leq \lambda R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x, t) \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |\nabla u|^2(x, t) \, dx dt \leq C\alpha. \quad (3.5)$$

**Remark 2.7.** Note that from the formula (3.3) and the a priori estimate of  $u$ , we get the following estimate for  $p$  which will be useful:

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |p - p(t)|^{3/2} dx dt \leq C\alpha^{3/2} R^{1/2}. \quad (3.6)$$

Now we can explain the proof of first important result. It shows that modulo the usual (and quite mild) non-local influences of the pressure, local regularity of the initial data propagates for at least a short time.

**Theorem 2.8.** Let  $u_0 \in L^2_{loc}(\mathbb{R}^3)$  with  $\sup_{x_0 \in \mathbb{R}^3} \int_{B_1(x_0)} |u_0|^2(x) dx \leq \alpha < \infty$  and  $\operatorname{div} u_0 = 0$ . Suppose  $u_0$  is in  $L^m(B_2)$  with  $\|u_0\|_{L^m(B_2(0))} \leq M < \infty$  and  $m > 3$ . Let us decompose  $u_0 = u_0^1 + u_0^2$  with  $\operatorname{div} u_0^1 = 0$ ,  $u_0^1|_{B_{4/3}} = u_0$ ,  $\operatorname{supp} u_0^1 \subset B_2(0)$  and  $\|u_0^1\|_{L^m(\mathbb{R}^3)} \leq C(M, m)$ .

Let  $a$  be the locally in time defined mild solution to Navier–Stokes equations with initial data  $u_0^1$ . Then there exists a positive  $T = T(\alpha, m, M) > 0$ , such that any Leray solution  $u \in \mathcal{N}(u_0)$  satisfies:

$$u - a \in C_{par}^\gamma(B_{1/2} \times [0, T]), \quad \text{and} \quad \|u - a\|_{C_{par}^\gamma(B_{1/2} \times [0, T])} \leq C(M, m, \alpha),$$

for some  $\gamma = \gamma(m) \in (0, 1)$ .

*Proof.* First, we admit the existence local-in-time of the mild solution with the initial data in  $L^q$  for  $q > 3$ , see more details in chapter 5 of [6].

Now, let us decompose  $u_0 = u_0^1 + u_0^2$  with  $\operatorname{div} u_0^1 = 0$ ,  $u_0^1|_{B_{4/3}} = u_0$ ,  $\operatorname{supp} u_0^1 \subset B_2$  and  $\|u_0^1\|_{L^m(\mathbb{R}^3)} \leq C(M, m)$ . Indeed, let  $\chi$  be the cut-off function such that  $\chi|_{B_{4/3}} = 1$  and  $\operatorname{supp}(\chi) \subseteq B_2$ .

We write

$$u_0^1 = u_0^{1,1} + u_0^{1,2}, \quad \text{where } u_0^{1,2} = \chi u_0.$$

Since

$$0 = \operatorname{div}(u_0^1) = \operatorname{div}(u_0^{1,1} + u_0^{1,2}) = \nabla \chi \cdot u_0 + \chi \operatorname{div} u_0 + \operatorname{div}(u_0^{1,1}),$$

we obtain the equation

$$\operatorname{div}(u_0^{1,1}) = -\nabla \chi \cdot u_0.$$

Solve this equation; we obtain  $u_0^{1,1}$ , then we have  $u_0^1$  and our decomposition.

Now, by assumption  $a$  solves the Cauchy problem for Navier–Stokes equations with initial data  $u_0^1$  in  $\mathbb{R}^3 \times [0, T_1]$ , where  $T_1 = T_1(M, m)$ , namely:

$$\begin{cases} \partial_t a - \Delta a + a \cdot \nabla a + \nabla \tilde{p} = 0 \\ \operatorname{div} a = 0 & \text{in } \mathbb{R}^3 \times (0, T_1), \\ a(\cdot, 0) = u_0^1. \end{cases} \quad (3.7)$$

Since  $u_0^1 \in L^m$  with  $m > 3$ , we have the existence of the mild solution.

We can follow the arguments in the Appendix of [1], and obtain the regularity of the mild solution

$$a \in L^{\frac{5m}{3}}(\mathbb{R}^3 \times (0, T_1)) \quad \text{with} \quad \|a\|_{L^{\frac{5m}{3}}(\mathbb{R}^3 \times (0, T_1))} \leq CM.$$

Note that  $\frac{5m}{3} > 5$  since  $m > 3$ . Moreover, by the estimates on  $a$  and by choosing  $T_1$  small enough such that the nonlinear term is small, we can recover a local energy estimate for  $a$ :

$$\operatorname{ess\,sup}_{0 < t < T_1} \int_{B_1(x_0)} \frac{|a|^2}{2}(x, t) dx + \int_0^{T_1} \int_{B_1(x_0)} |\nabla a|^2(x, t) dx dt \leq C(M, m),$$

for any  $x_0 \in \mathbb{R}^3$ .

Write  $u = a + v$ , we can verify that  $v$  satisfies:

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + a \cdot \nabla v + \operatorname{div}(a \otimes v) + \nabla q = 0 \\ \operatorname{div} v = 0 \end{cases}, \quad (3.9)$$

in the sense of distributions in  $\mathbb{R}^3 \times (0, T_1)$ , here  $q = p - \tilde{p}$  with  $p$  being the associated pressure for  $u$ ; and the local energy inequality

$$\partial_t \frac{|v|^2}{2} - \Delta \frac{|v|^2}{2} + |\nabla v|^2 + \operatorname{div} \left( \frac{|v|^2}{2} (v + a) \right) + v \cdot \operatorname{div}(a \otimes v) + \operatorname{div}(vq) \leq 0,$$

in the sense of distributions in  $\mathbb{R}^3 \times (0, T_1)$ ;

$$\lim_{t \rightarrow 0^+} \|v(\cdot, t) - u_0^2\|_{L^2(B_1(x_0))} = 0, \quad \text{for any } x_0 \in \mathbb{R}^3.$$

Note also that  $u_0^2|_{B_{4/3}} \equiv 0$ . Since  $(u, p)$  satisfies the a priori estimates in Lemma 2.6 (and the remarks below it),  $(a, \tilde{p})$  is regular, we obtain the following estimates for  $(v, q)$  in  $B_2(0) \times [0, T_2]$ ,  $T_2 = T_2(\alpha, M, m)$ :

$$\operatorname{ess\,sup}_{0 < t < T_2} \frac{1}{2} \int_{B_2} |v|^2(x, t) dx + \int_0^{T_2} \int_{B_2} |\nabla v|^2(x, s) dx ds \leq C(\alpha, m, M).$$

From the local energy inequality for  $v$ , and  $\lim_{t \rightarrow 0^+} \|v(\cdot, t)\|_{L^2(B_{4/3})} = 0$ , we obtain

$$\begin{aligned} & \frac{1}{2} \int_{B_{4/3}} |v|^2(x, t) \phi(x) dx + \int_0^t \int_{B_{4/3}} |\nabla v|^2(x, s) \phi(x) dx ds \\ & \leq \int_0^t \int_{B_{4/3}} \frac{|v|^2}{2} \Delta \phi dx ds + \int_0^t \int_{B_{4/3}} \frac{|v|^2}{2} (v + a) \cdot \nabla \phi dx ds \\ & \quad + \int_0^t \int_{B_{4/3}} [a \otimes v : (\nabla v \phi + v \otimes \nabla \phi)] + qv \cdot \nabla \phi dx ds, \end{aligned}$$

where  $\phi \in C_c^\infty(B_{4/3})$ ,  $\phi|_{B_1} \equiv 1$ ,  $\phi \geq 0$ .

By the Sobolev embedding, we know

$$\left( \int_0^{T_2} \int_{B_2} |v|^{10/3} dx dt \right)^{3/10} \leq C(\alpha, m, M).$$

From  $\Delta q = -\operatorname{div} \operatorname{div}(v \otimes v + a \otimes v + v \otimes a)$ , we can see  $q \in L_{\operatorname{loc}}^{5/3}$ . Thus, by Cauchy-Schwartz inequality,

$$\left( \int_0^t \int_{B_1} |q|^{3/2} dx ds \right)^{2/3} \leq C(\alpha, m, M) t^{1/15}.$$

The importance of these estimates lies in the fact that they provide crucial ‘‘quantitative’’ information on the decay in time as  $t \rightarrow 0^+$ . Now for  $t_0$  fixed, whose precise value is to be determined later, extend  $v, q$  to  $B_1 \times (-1 + t_0, t_0]$  by setting  $v = 0$ ,  $q = 0$  for  $(x, t) \in B_1 \times (-1 + t_0, 0]$ . Extend  $a$  to  $B_1 \times (-1 + t_0, t_0]$  by setting  $a(t, x) = 0$  for  $t < 0$ . The extended

function  $(v, q)$  is a suitable weak solution to the generalized Navier–Stokes equations with the extended  $a$  in  $B_1 \times [-1 + t_0, t_0]$ . Note here that

$$\lim_{t \rightarrow 0^+} \|v(\cdot, t)\|_{L^2(B_1)} = 0$$

plays a crucial role: it guarantees that  $\frac{|v|^2}{2}$  will not cause any problem across  $t = 0$ . Then clearly if we choose  $t_0 = t_0(\alpha, m, M)$  sufficiently small, we can apply Theorem 2.3 and conclude  $v$  is Hölder continuous in  $B_{1/2} \times [0, t_0]$ , with

$$\|v\|_{C_{\text{par}}^\gamma(B_{1/2} \times [0, t_0])} \leq C(\alpha, m, M),$$

for some  $\gamma = \gamma(m)$ . The theorem is proved.  $\square$

### 2.3 Estimates of forward self-similar solutions

In this subsection, we begin by studying self-similar solutions to the Navier–Stokes equations and explain a result about a priori estimate for forward self-similar solutions  $u \in \mathcal{N}(u_0)$  when the initial data  $u_0$  is scale-invariant and it restricted in  $\partial B_1$  is smooth. The setting is as follows.

Let  $u$  be a Leray solution with initial data  $u_0$ . Suppose  $\lambda u_0(\lambda x) = u_0(x)$ ,  $\lambda u(\lambda x, \lambda^2 t) = u(x, t)$  for any  $\lambda > 0$ . We also assume  $u_0|_{\partial B_1(0)} \in C^\infty(\partial B_1)$ . Then the self-similar condition and the smoothness on  $\partial B_1$  imply that  $u_0$  is smooth on  $\mathbb{R}^3 \setminus \{0\}$ . We also get estimates on the derivative:

$$|\nabla^\alpha u_0(x)| \leq \frac{C(\alpha, u_0)}{|x|^{1+|\alpha|}}, \quad \forall |\alpha| \geq 0.$$

Indeed: we take  $\lambda = \frac{1}{|x|}$  for  $x \neq 0$ , then we have  $\frac{1}{|x|} u_0\left(\frac{x}{|x|}\right) = u_0(x)$ . Since  $u_0\left(\frac{x}{|x|}\right) \in C^\infty(\partial B_1)$ , we have the above inequality for  $\alpha = 0$ . Similarly, for all  $|\alpha| \geq 0$  and  $\lambda > 0$ , we have  $\nabla^\alpha u_0(x) = \lambda^{1+|\alpha|} (\nabla^\alpha u_0)(\lambda x)$ , by taking  $\lambda = \frac{1}{|x|}$  for  $x \neq 0$ , we thus have

$$|\nabla^\alpha u_0(x)| \leq \frac{C(\alpha, u_0)}{|x|^{1+|\alpha|}}, \quad \forall |\alpha| \geq 0.$$

The main result in this subsection is the following theorem, which gives a priori estimate of the forward self-similar solution  $u \in \mathcal{N}(u_0)$  by comparing it with the solution profile of the heat equation at time  $t = 1$ .

**Theorem 2.9.** (*A-priori estimate for forward self-similar solutions*) *Let  $u_0$  be a scale-invariant divergence-free initial data,  $u \in \mathcal{N}(u_0)$  be scale-invariant. Then  $U(\cdot) := u(\cdot, 1)$ , the solution profile at time  $t = 1$ , belongs to  $C^\infty(\mathbb{R}^3)$  and*

$$|\partial^\alpha (U(x) - e^\Delta u_0(x))| \leq \frac{C(\alpha, u_0)}{(1 + |x|)^{3+|\alpha|}}, \quad \forall |\alpha| \geq 0.$$

*Proof.* By applying Lemma 2.6 with  $R = 1$  and setting  $M := \|u_0\|_{C(\partial B_1)} = \sup_{x \in B_1} |u_0(x)|$ , this gives us:

$$\sup_{0 < t < T_1} \frac{1}{2} \int_{B_1(0)} |u(x, t)|^2 dx + \int_0^{T_1} \int_{B_1(0)} |\nabla u(x, t)|^2 dx dt \leq C(M), \quad T_1 = T_1(M). \quad (4.2)$$

Since  $u(x, t) = \frac{1}{\sqrt{t}}u\left(\frac{x}{\sqrt{t}}, 1\right) = \frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right)$ , we have for a fixed  $t_* < T_1$  which is to be determined later

$$\begin{aligned} C(M) &\geq \frac{1}{2} \int_{B_1(0)} |u(x, t_*)|^2 dx + \int_{t_*/2}^{t_*} \int_{B_1(0)} |\nabla u(x, t)|^2 dx dt \\ &\geq \frac{\sqrt{t_*}}{2} \int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |u(x, 1)|^2 dx + \frac{\sqrt{t_*}}{8} \int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |\nabla u(x, 1)|^2 dx. \end{aligned} \quad (4.3)$$

$$\geq \frac{\sqrt{t_*}}{2} \int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |U(x)|^2 dx + \frac{\sqrt{t_*}}{8} \int_{B_{\frac{1}{\sqrt{t_*}}}(0)} |\nabla U(x)|^2 dx. \quad (4.4)$$

On the other hand, for all  $x_0$ ,  $|x_0| = 8$ , since  $u_0 \in C^\infty(B_4(x_0))$  and since the Hölder continuity is weaker than the continuous derivative, we can apply Theorem 2.8 and some bootstrapping arguments the same as how Jia and Šverák worked in Theorem 3.2 in [4] to show the following: for any  $u \in \mathcal{N}(u_0)$ , there exists  $T_2 = T_2(M) > 0$  such that for all  $\alpha$ ,

$$\|\partial_t \partial_x^\alpha u\|_{L^\infty(B_{1/8}(x_0) \times [0, T_2])} \leq C(\alpha, u_0),$$

In particular  $u$  is a smooth function.

Since for all  $\lambda > 0$ ,  $\lambda u(\lambda x, \lambda^2 t)$  is also a Leray solution with initial data  $u_0$ , we obtain by integrating in time and using the preceding inequality:

$$|\lambda|^{|\alpha|+1} |\partial^\alpha u(\lambda x_0, \lambda^2 t) - \partial^\alpha u_0(x_0)| \leq C(\alpha, u_0)t,$$

for any  $\lambda > 0$ ,  $|\alpha| \geq 0$ ,  $t \leq T_2(u_0)$ .

Taking  $\lambda = \frac{1}{\sqrt{t}}$ , we obtain

$$\left(\frac{1}{\sqrt{t}}\right)^{|\alpha|+1} \left| \partial^\alpha u\left(\frac{x_0}{\sqrt{t}}, 1\right) - \partial^\alpha u_0(x_0) \right| \leq C(\alpha, u_0)t.$$

Setting  $y = \frac{x_0}{\sqrt{t}}$ , and using the homogeneity of  $\partial^\alpha u$ , we get

$$|\partial^\alpha U(y) - \partial^\alpha u_0(y)| \leq \frac{C(\alpha, u_0)}{|y|^{|\alpha|+3}}, \quad \forall |y| > \frac{8}{\sqrt{T_2}}. \quad (4.5)$$

Now choosing  $t_*$  sufficiently small,  $t_* = t_*(M)$ , we see from inequality (4.3):

$$\int_{B_{\frac{16}{\sqrt{T_2}}}} (|U(y)|^2 + |\nabla U(y)|^2) dy \leq C(M).$$

Since  $u(x, t)$  satisfies Navier-Stokes equations, it is easy to verify  $U$  satisfies

$$\begin{cases} -\Delta U - \frac{x}{2} \cdot \nabla U - \frac{U}{2} + U \cdot \nabla U + \nabla P = 0 \\ \operatorname{div} U = 0 \end{cases} \quad \text{in } \mathbb{R}^3. \quad (4.6)$$

Thus by estimates for solution of elliptic equations (see section 6.3 of [2]) combined with general Sobolev inequalities (see section 5.6.3 of [2]) we get:

$$\|U\|_{C^k(B_{\frac{9}{\sqrt{T_2}}})} \leq C(k, M).$$

These estimates, combined with the properties of the heat equation finish the proof (see [2] section 2.3.3 for estimates of derivatives of solution to the heat equation).  $\square$

### 3 A New Proof to the existence of forward self-similar solution for large initial data

In this section we give a new proof for the existence of the forward self-similar solutions as stated in Theorem 3.1, with weaker conclusion compared with Theorem 1.1. Let us briefly describe the general strategy and compare it with Jia and Šverák's result in [4]. We are trying to find a solution  $u$  of the form  $u(x, t) = \frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}}U_0\left(\frac{x}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}}V\left(\frac{x}{\sqrt{t}}\right)$  where  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$ , then proving that  $u$  satisfies almost the conditions of a Leray solution, except the locally square-integrable at  $t = 0$ . To do this, we try to find a weak solution  $V \in H_{0,\sigma}^1(\Omega)$  of a Stokes system when  $\Omega$  is a bounded domain of  $\mathbb{R}^3$  using the Leray-Schauder theorem, then extend the result to  $\mathbb{R}^3$  in the first subsection. In the second subsection, we show that  $u$  satisfies some regular conditions the same as Leray solutions. Comparing with us, in [4], Jia and Šverák proved this theorem by finding the above  $V$  with a better decay such that the condition  $|U(x) - U_0(x)| = o\left(\frac{1}{|x|}\right)$  is verified, by using a decay for the linear singularly forced Stokes systems, we will discuss more in the final subsection.

First, we state the theorem, and then follow the above strategy to prove it.

**Theorem 3.1.** *Let  $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  satisfy  $\lambda u_0(\lambda x) = u_0(x)$  for all  $\lambda > 0$ ,  $\operatorname{div} u_0 = 0$ . Then there exists  $u \in H_{loc}^1(\mathbb{R}^3 \times (0, \infty))$  with  $\lambda u(\lambda x, \lambda^2 t) = u(x, t)$  for all  $\lambda > 0$  such that  $u$  satisfies:*

(i) *Decay condition*

$$\operatorname{ess\,sup}_{0 \leq t \leq R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2}(x, t) \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u|^2 \, dx dt < \infty,$$

$$\lim_{|x_0| \rightarrow \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2(x, t) \, dx dt = 0$$

(ii) *For some distribution  $p$  in  $\mathbb{R}^3 \times (0, \infty)$ , the pair  $(u, p)$  verifies the Navier-Stokes equations*

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \quad (\text{NSE})$$

*in the sense of distributions, and for any compact set  $K \subset \mathbb{R}^3$ ,*

$$\lim_{t \rightarrow 0^+} \|u(\cdot, t) - u_0\|_{L^2(K)} = 0.$$

(iii)  *$u$  is suitable in the sense of Caffarelli-Kohn-Nirenberg, more precisely, the following local energy inequality holds:*

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \phi(x, t) \, dx dt \leq \int_0^\infty \int_{\mathbb{R}^3} \left( \frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \frac{|u|^2}{2} u \cdot \nabla \phi + p u \cdot \nabla \phi \right) \, dx dt$$

*for any smooth  $\phi \geq 0$  with  $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, \infty))$ .*

**Remark 3.2.** *The solution  $u$  given in Theorem 3.1 is not a Leray solution in the sense of 2.4, since  $u \notin L_{loc}^2(\mathbb{R}^3 \times [0, \infty))$ , more specifically, it is not locally square-integrable at  $t = 0$ . Then we can not apply Theorem 2.9 to obtain the smoothness of  $u$ , since in the proof of Theorem 2.9, they use the regularity of  $u$  at  $t = 0$ .*

*Proof.* First we set  $U_0 = e^\Delta u_0$ . Due to the scaling invariance of  $u(x, t)$ , we are essentially seeking the profile function  $U(x) = u(x, 1)$ : since  $u(x, t) = \frac{1}{\sqrt{t}}u(\frac{x}{\sqrt{t}}, 1)$ .

Our goal is to obtain  $U(x)$  satisfying

$$\begin{cases} -\Delta U + U \cdot \nabla U - \frac{U}{2} - \frac{x}{2} \cdot \nabla U + \nabla P = 0, \\ \operatorname{div} U = 0, \end{cases} \quad \text{in } \mathbb{R}^3, \quad (5.5)$$

in the sense of distributions, and  $U - U_0 \in H_{0,\sigma}^1(\mathbb{R}^3)$ .

A weak solution to (5.5) is defined as:

$$(\nabla U, \nabla \varphi) + (U \cdot \nabla U - \frac{U}{2} - \frac{x}{2} \cdot \nabla U, \varphi) = 0 \quad (5.6)$$

for all  $\varphi \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ .

We will seek  $U$  in the form

$$U = U_0 + V, \quad \text{where } V \in H_{0,\sigma}^1(\mathbb{R}^3).$$

Thus we have reduced the problem to finding  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$  satisfying

$$(\nabla V, \nabla \varphi) + (V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 - \frac{V}{2} - \frac{x}{2} \cdot \nabla V, \varphi) = (-U_0 \cdot \nabla U_0, \varphi) \quad (5.7)$$

for all  $\varphi \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ . We rewrite the above as:

$$(\nabla V, \nabla \varphi) - (\frac{V}{2} + \frac{x}{2} \cdot \nabla V, \varphi) = (-V \cdot \nabla V - U_0 \cdot \nabla V - V \cdot \nabla U_0 - U_0 \cdot \nabla U_0, \varphi) \quad (5.8)$$

for all  $\varphi \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ .

To find  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$  satisfies 5.8, we will use Lemma 3.7 in the below subsection, which work with the linear problem of the Stokes system. Then, we finish the proof by using Lemma 3.8 in order to check the regular conditions of the solution  $u$ .  $\square$

The following two subsections are parts of the proof of 3.1

### 3.1 Linear problem of the Stokes system

In this subsection, we try to find a weak solution  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$  of (5.8). We will treat this problem first in a bounded smooth domain  $\Omega$  by using Leray-Schauder Theorem, then extend the result to  $\mathbb{R}^3$ . First, we will recall Leray-Schauder Theorem without the proof. See chapter 9 of [2] for a proof.

**Theorem 3.3.** (*Leray Schauder Theorem*) *Suppose  $A : X \rightarrow X$  is a continuous and compact mapping. Assume further that the set  $\{u \in X \mid u = \lambda A(u) \text{ for some } 0 \leq \lambda \leq 1\}$  is bounded. Then  $A$  has a fixed point.*

Now, we recall the definition of a weak solution of a Stokes equation.

**Definition 3.4.** *Let  $\Omega$  be a smooth domain. We consider the Stokes system*

$$\begin{cases} -\Delta V + \nabla P - \frac{x}{2} \cdot \nabla V - \frac{V}{2} = f \\ \operatorname{div} V = 0, \quad V|_{\partial\Omega} = 0 \end{cases} \quad (\text{SSE})$$

where  $f \in (H_{0,\sigma}^1(\Omega))'$ .

A vector field  $V \in H_{0,\sigma}^1(\Omega)$  is called a weak solution of the equation if  $V$  satisfies

$$(\nabla V, \nabla \varphi) - \left(\frac{x}{2} \cdot \nabla V + \frac{V}{2}, \varphi\right) = f(\varphi) \quad (5.9)$$

for all  $\varphi \in C_{0,\sigma}^\infty(\Omega)$ .

These integrations are well defined, since  $V \in H_{0,\sigma}^1(\Omega)$  and  $\varphi \in C_{0,\sigma}^\infty(\Omega)$ . By the density of  $C_{0,\sigma}^\infty(\Omega)$  in  $H_{0,\sigma}^1(\Omega)$ , we can obtain (5.9) for all  $\varphi \in H_{0,\sigma}^1(\Omega)$ . Moreover, by integration by parts, (5.9) becomes

$$\int_{\Omega} \nabla V : \nabla \varphi + \left(\frac{x}{2} \cdot \nabla \varphi\right) \cdot V + V \cdot \varphi = f(\varphi) \quad (5.10)$$

for all  $\varphi \in H_{0,\sigma}^1(\Omega)$ .

We have a lemma for the existence and unicity of weak solutions for a general Stokes systems in a bounded smooth domain.

**Lemma 3.5.** *For all  $f \in (H_{0,\sigma}^1(\Omega))'$ , where  $\Omega \subset \mathbb{R}^3$  is a bounded smooth domain, the Stokes system (SSE) has a unique weak solution  $T(f) \in H_{0,\sigma}^1(\Omega)$ . Moreover:  $T : (H_{0,\sigma}^1(\Omega))' \rightarrow H_{0,\sigma}^1(\Omega)$ , is a linear bounded operator and:  $\|T(f)\|_{H_{0,\sigma}^1(\Omega)} \leq 4\|f\|_{(H_{0,\sigma}^1(\Omega))'}$ .*

*Proof.* We will prove the lemma by using Lax-Milgram theorem.

Consider the bilinear form

$$a : H_{0,\sigma}^1(\Omega) \times H_{0,\sigma}^1(\Omega) \rightarrow \mathbb{R}, \quad a(V_1, V_2) = \int_{\Omega} \nabla V_1 : \nabla V_2 + \left(\frac{x}{2} \cdot \nabla V_2\right) \cdot V_1 + V_1 \cdot V_2.$$

We first need to need to prove that  $a$  is a well-defined, continuous, bilinear form:

$$\int_{\Omega} |\nabla V_1 : \nabla V_2 + \left(\frac{x}{2} \cdot \nabla V_2\right) \cdot V_1 + V_1 \cdot V_2| \leq C(\Omega) \|V_1\|_{H_{0,\sigma}^1(\Omega)} \|V_2\|_{H_{0,\sigma}^1(\Omega)}$$

where we used here the fact that  $\Omega$  is bounded to bound the term  $\frac{x}{2}$ , so  $C(\Omega)$  is a constant depending on the bounded domain  $\Omega$ . It is clear that  $a$  is a bilinear form, and then we proved that is continous bilinear form.

To apply Lax-Milgram theorem we also need to show that this form is coercive. By definition:

$$a(V, V) = \int_{\Omega} \nabla V : \nabla V + \left(\frac{x}{2} \cdot \nabla V\right) \cdot \nabla V + V \cdot V = \|V\|_{L^2(\Omega)}^2 + \|\nabla V\|_{L^2(\Omega)}^2 + \int_{\Omega} \left(\frac{x}{2} \cdot \nabla V\right) \cdot \nabla V$$

Now we need to compute the last term. By integration by parts, we get:

$$\int_{\Omega} \left(\frac{x}{2} \cdot \nabla V\right) \cdot \nabla V = -\frac{3}{2} \int_{\Omega} V \cdot V - \int_{\Omega} \left(\frac{x}{2} \cdot \nabla V\right) \cdot V$$

this imply that:  $\int_{\Omega} \left(\frac{x}{2} \cdot \nabla V\right) \cdot \nabla V = -\frac{3}{4} \|V\|_{L^2(\Omega)}^2$ . So we finally get:

$$a(V, V) = \frac{1}{4} \|V\|_{L^2(\Omega)}^2 + \|\nabla V\|_{L^2(\Omega)}^2 \geq \frac{1}{4} \|V\|_{H_{0,\sigma}^1(\Omega)}^2$$

Thus,  $a$  is a continuous and coercive bilinear form on the Hilbert space  $H_{0,\sigma}^1(\Omega)$ . Then by Lax-Milgram theorem, there is a unique weak solution to the system (SSE). We prove now the second statement of the lemma. The linearity of  $T$  comes from the bilinearity of  $a$  and

the uniqueness of the solution. Now we prove that  $T$  is bounded: by using the inequality we get when proving  $a$  is coercive we have:

$$\|T(f)\|_{H_{0,\sigma}^1(\Omega)}^2 \leq 4a(T(f), T(f)) = 4f(T(f)) \leq 4\|f\|_{(H_{0,\sigma}^1(\Omega))'} \|T(f)\|_{H_{0,\sigma}^1(\Omega)}$$

so:  $\|T(f)\|_{H_{0,\sigma}^1(\Omega)} \leq 4\|f\|_{(H_{0,\sigma}^1(\Omega))'}$ . The lemma is proved.  $\square$

Now, we state and prove the existence of the weak solution  $V$  of the equation (5.8) by using Leray-Schauder theorem.

**Lemma 3.6.** *Let  $\Omega$  be a smooth bounded domain. Let  $u_0 \in C^\infty(\Omega \cap \mathbb{R}^3 \setminus \{0\})$  satisfy  $\lambda u_0(\lambda x) = u_0(x)$  for all  $\lambda > 0$ ,  $\operatorname{div} u_0 = 0$  and  $U_0 = e^\Delta u_0$ . Then the equation (5.8) has a weak solution  $V \in H_{0,\sigma}^1(\Omega)$  with the boundary condition  $V|_{\partial\Omega} = 0$ .*

*Proof.* For  $V \in H_{0,\sigma}^1(\Omega)$ , consider

$$N_{U_0}(V) = -U_0 \cdot \nabla V - V \cdot \nabla U_0 - V \cdot \nabla V - U_0 \cdot \nabla U_0$$

If we fix  $V \in H_{0,\sigma}^1(\Omega)$ , we can see  $N_{U_0}(V) : H_{0,\sigma}^1(\Omega) \rightarrow \mathbb{R}$ ,  $\varphi \mapsto \int_{\Omega} N_{U_0}(V) \cdot \varphi$  as a continuous linear form, since for all  $\varphi \in H_{0,\sigma}^1(\Omega)$ :

$$\left| \int_{\Omega} (U_0 \cdot \nabla V) \cdot \varphi \right| \leq \|U_0\|_{L^\infty(\Omega)} \|V\|_{H_{0,\sigma}^1(\Omega)} \|\varphi\|_{H_{0,\sigma}^1(\Omega)}$$

by Holder inequality and using the fact  $U_0$  is bounded on  $\Omega$  since it is a bounded domain. We can do the same for the term  $V \cdot \nabla U_0$  and  $U_0 \cdot \nabla U_0$ . We also have:

$$\left| \int_{\Omega} (V \cdot \nabla V) \cdot \varphi \right| = \left| \int_{\Omega} (V \cdot \nabla \varphi) \cdot V \right| \leq \|V\|_{L^4(\Omega)}^2 \|\nabla \varphi\|_{L^2(\Omega)} \leq \|V\|_{L^4(\Omega)}^2 \|\varphi\|_{H_{0,\sigma}^1(\Omega)}.$$

So we define:  $N_{U_0} : H_{0,\sigma}^1(\Omega) \rightarrow (H_{0,\sigma}^1(\Omega))'$ , which is a nonlinear function.

Then: we can rewrite the system as

$$V = T(N_{U_0}(V)) =: K(V).$$

we can forget the index  $U_0$  since it is fixed for the rest of our problem.

We reformulated the equation to be the solution of a fixed point problem, we now want to apply the Leray-Schauder theorem. To apply Leray-Schauder theorem to conclude that there is a weak solution of (5.8) when  $\Omega$  is bounded, we will show that  $K$  verifies the condition of the theorem:  $K$  is a continuous and compact operator, and the set of solutions of the equation  $V = \lambda K(V)$  for some  $\lambda \in [0, 1]$  is uniformly bounded in  $H_{0,\sigma}^1(\Omega)$ .

First, since  $T : (H_{0,\sigma}^1(\Omega))' \rightarrow H_{0,\sigma}^1(\Omega)$  is a bounded operator by Lemma 3.5, we want to have some estimates of  $\|N_{U_0}(V_1) - N_{U_0}(V_2)\|_{(H_{0,\sigma}^1(\Omega))'}$  to get the estimates of  $K$ .

We first have,

$$\begin{aligned} N_{U_0}(V_1) - N_{U_0}(V_2) &= (-U_0 \cdot \nabla V_1 - V_1 \cdot \nabla U_0 - V_1 \cdot \nabla V_1 - U_0 \cdot \nabla U_0) \\ &\quad - (-U_0 \cdot \nabla V_2 - V_2 \cdot \nabla U_0 - V_2 \cdot \nabla V_2 - U_0 \cdot \nabla U_0) \\ &= -(U_0 + V_1) \cdot \nabla W - W \cdot \nabla (U_0 + V_2) \end{aligned}$$

where  $W = V_1 - V_2$ .

By integration by parts, we finally get,

$$\int_{\Omega} [N_{U_0}(V_1) - N_{U_0}(V_2)] \cdot \varphi = \int_{\Omega} [(U_0 + V_1) \cdot W + W \cdot (U_0 + V_2)] \cdot \nabla \varphi,$$

Thus we have:

$$\begin{aligned} \|N_{U_0}(V_1) - N_{U_0}(V_2)\|_{(H_{0,\sigma}^1(\Omega))'} &\leq \|(U_0 + V_1) \cdot W + W \cdot (U_0 + V_2)\|_{L^2(\Omega)} \\ &\leq (\|U_0 + V_1\|_{L^4} + \|U_0 + V_2\|_{L^4}) \|V_1 - V_2\|_{L^4(\Omega)}. \end{aligned}$$

Now, by using the fact that  $T$  is a bounded linear operator, we thus have an estimate of  $K$ :

$$\|K(V_1) - K(V_2)\|_{H_{0,\sigma}^1(\Omega)} \leq 4 (\|U_0 + V_1\|_{L^4(\Omega)} + \|U_0 + V_2\|_{L^4(\Omega)}) \|V_1 - V_2\|_{L^4(\Omega)}.$$

By the same argument as above with Sobolev embedding:  $H^1(\Omega) \subseteq L^4(\Omega)$  so we get:

$$\|K(V_1) - K(V_2)\|_{H_{0,\sigma}^1(\Omega)} \leq c \left( \|U_0 + V_1\|_{H_{0,\sigma}^1(\Omega)} + \|U_0 + V_2\|_{H_{0,\sigma}^1(\Omega)} \right) \|V_1 - V_2\|_{H_{0,\sigma}^1(\Omega)}.$$

for some constant  $c$ . This shows the continuity of  $K$ .

We next show that  $K$  is compact. Let  $(V_n)$  be a bounded sequence in  $H_{0,\sigma}^1(\Omega)$ , since the embedding  $H^1(\Omega) \rightarrow L^4(\Omega)$  is compact by the Rellich-Kondrachov theorem, we can extract a subsequence (also called  $V_n$ ) such that  $V_n \rightarrow V$  strongly in  $L^4(\Omega)$ , and then

$$\|K(V_n) - K(V)\|_{H_{0,\sigma}^1(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This shows the compactness of  $K$ .

Finally, in order to apply the Leray-Schauder theorem, we prove that the set of solutions  $V = \lambda K(V)$ , with  $\lambda \in [0, 1]$  is uniformly bounded in  $H_{0,\sigma}^1(\Omega)$ .

The equation becomes:

$$(\nabla V, \nabla \varphi) - \left( \frac{V}{2} + \frac{x}{2} \cdot \nabla V, \varphi \right) = \lambda (-V \cdot \nabla V - U_0 \cdot \nabla V - V \cdot \nabla U_0 - U_0 \cdot \nabla U_0, \varphi)$$

for all  $\varphi \in H_{0,\sigma}^1(\Omega)$ .

Choosing  $\varphi = V$ , we have

$$\begin{aligned} \|\nabla V\|_{L^2(\Omega)}^2 + \frac{1}{4} \|V\|_{L^2(\Omega)}^2 &= -\lambda \int_{\Omega} (\nabla U_0 \cdot U_0) \cdot V \\ &\leq \lambda \|U_0 \cdot \nabla U_0\|_{L^2(\Omega)} \|V\|_{L^2(\Omega)} \leq C \|V\|_{H_{0,\sigma}^1(\Omega)} \end{aligned}$$

Then,  $\|V\|_{H_{0,\sigma}^1(\Omega)} \leq C'$ . By Leray-Schauder Theorem, we have a weak solution  $V \in H_{0,\sigma}^1(\Omega)$  of (5.8) by choosing  $\lambda = 1$ . The lemma is proved.  $\square$

Now, we extend the result to  $\mathbb{R}^3$  in the below lemma.

**Lemma 3.7.** *Let  $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  satisfy  $\lambda u_0(\lambda x) = u_0(x)$  for all  $\lambda > 0$ ,  $\operatorname{div} u_0 = 0$  and  $U_0 = e^\Delta u_0$ . Then the equation (5.8) has a weak solution  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$ .*

*Proof.* We consider  $\Omega_n = B_n$ . By the lemma 3.6, we know there exists a weak solution  $V_n \in H^1(\Omega_n)$  of (5.8) in  $\Omega_n$ . Since  $V|_{\partial\Omega_n} = 0$ , we can extend  $V_n$  to  $\mathbb{R}^3$  by setting  $V_n = 0$  on  $\mathbb{R}^3 \setminus \Omega_n$ . From the lemma 3.6, we have:

$$\|\nabla V_n\|_{L^2(\Omega_n)}^2 + \frac{1}{4} \|V_n\|_{L^2(\Omega_n)}^2 = - \int_{\Omega_n} (U_0 \cdot \nabla U_0) \cdot V_n$$

We will bound  $V_n$  by a constant  $C$  depending on only  $U_0$ . To do that, we will show that  $U_0 \in L^4(\mathbb{R}^3)$  and  $\nabla U_0 \in L^2(\mathbb{R}^3)$ , then use the Hölder inequality to have the above boundedness.

Indeed, for a convex function  $\beta$  of class  $C^2$  such that these integrations below are well defined, we have:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \beta(e^{t\Delta} u_0(x)) dx &= \int_{\mathbb{R}^3} \beta'(e^{t\Delta} u_0(x)) \Delta(e^{t\Delta} u_0(x)) dx \\ &= - \int_{\mathbb{R}^3} \beta''(e^{t\Delta} u_0(x)) |\nabla e^{t\Delta} u_0(x)|^2 dx \leq 0 \end{aligned}$$

Then

$$\int_{\mathbb{R}^3} \beta(e^{t\Delta} u_0(x)) dx \leq \int_{\mathbb{R}^3} \beta(u_0(x)) dx$$

By the scaling invariance of  $u_0$ , we have  $u_0(x) = O_{|x| \rightarrow \infty}(\frac{1}{|x|})$ , then  $u_0 \in L^4(\mathbb{R}^3)$ . Then  $U_0 = e^{\Delta} u_0 \in L^4(\mathbb{R}^3)$ . Similarly, since we have  $\nabla e^{t\Delta} u_0 = e^{t\Delta}(\nabla u_0)$ , and  $\nabla u_0 = O_{|x| \rightarrow \infty}(\frac{1}{|x|^2})$ ,  $\nabla u_0 \in L^2(\mathbb{R}^3)$  then  $\nabla U_0 \in L^2(\mathbb{R}^3)$ . Thus we have the boundedness of  $V_n$  in  $H_{0,\sigma}^1(\mathbb{R}^3)$ :

$$\begin{aligned} \|\nabla V_n\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{4} \|V_n\|_{L^2(\mathbb{R}^3)}^2 &= \|\nabla V_n\|_{L^2(\Omega_n)}^2 + \frac{1}{4} \|V_n\|_{L^2(\Omega_n)}^2 = - \int_{\Omega_n} (U_0 \cdot \nabla U_0) \cdot V_n \\ &\leq \|U_0\|_{L^4(\Omega_n)} \|\nabla U_0\|_{L^2(\Omega_n)} \|V\|_{L^4(\Omega_n)} \leq C \|U_0\|_{L^4(\mathbb{R}^3)} \|\nabla U_0\|_{L^2(\mathbb{R}^3)} \cdot \|V\|_{H_{0,\sigma}^1(\mathbb{R}^3)} \end{aligned}$$

by the Sobolev embedding  $H_{0,\sigma}^1(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)$ . This shows that  $(V_n)_n$  is bounded in  $H_{0,\sigma}^1(\mathbb{R}^3)$ .

Then, up to a subsequence,  $V_n \rightharpoonup V$  in  $H_{0,\sigma}^1(\mathbb{R}^3)$  weakly. This subsequence exists since the closed unit ball of  $H_{0,\sigma}^1(\mathbb{R}^3)$  is compact for the weak topology by Kakutani's theorem for reflexive space.

By the definition of weak convergence, for all  $\varphi \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ :

$$\lim_{n \rightarrow \infty} (\nabla V_n, \nabla \varphi) = (\nabla V, \nabla \varphi), \quad \lim_{n \rightarrow \infty} (V_n, \varphi) = (V, \varphi)$$

We will prove that  $V$  is a weak solution of (5.8) on  $\mathbb{R}^3$ . Fix  $\varphi \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ ; there exists  $m \in \mathbb{N}$  such that  $\text{supp } \varphi \subset \Omega_m$ . Then for all  $n \geq m$ , we have

$$(\nabla V_n, \nabla \varphi) + (\nabla P - \frac{x}{2} \cdot \nabla V_n - \frac{V_n}{2}, \varphi) = (-U_0 \cdot \nabla V_n - V_n \cdot \nabla U_0 - U_0 \cdot \nabla U_0 - V_n \cdot \nabla V_n, \varphi)$$

and it only remains to show the equation is valid in the limit  $n \rightarrow \infty$ .

Since  $\text{supp } \varphi \subset \Omega_m$  for all  $n \geq m$ , we have

$$\begin{aligned} &| (U_0 \cdot \nabla V_n + V_n \cdot \nabla U_0 + V_n \cdot \nabla V_n - U_0 \cdot \nabla V - U_0 \cdot \nabla U_0 - V_n \cdot \nabla V_n, \varphi) | \\ &\leq |((U_0 + V) \cdot \nabla (V_n - V), \varphi)| + |((V_n - V) \cdot \nabla (U_0 + V_n), \varphi)| \\ &= |((U_0 + V) \cdot \nabla \varphi, V_n - V)| + |((V_n - V) \cdot \nabla \varphi, U_0 + V_n)| \\ &\leq (\|U_0 + V\|_{L^4(\Omega_m)} \cdot \|\nabla \varphi\|_{L^2(\Omega_m)} + \|\nabla \varphi\|_{L^2(\Omega_m)} \cdot \|U_0 + V_n\|_{L^4(\Omega_m)}) \cdot \|V_n - V\|_{L^4(\Omega_m)} \end{aligned}$$

Since  $(V_n)$  is bounded in  $H_{0,\sigma}^1(\Omega_m)$  and the embedding  $H^1(\Omega_m) \hookrightarrow L^4(\Omega_m)$  is compact, there exists a subsequence (also denoted by  $V_n$ ) that converges strongly to  $V$  in  $L^4(\Omega_m)$ . Then:

$$\lim_{n \rightarrow \infty} (U_0 \cdot \nabla V_n + V_n \cdot \nabla U_0 + V_n \cdot \nabla V_n, \varphi) = (U_0 \cdot \nabla V + U_0 \cdot \nabla U_0 + V \cdot \nabla V, \varphi)$$

Thus  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$  is a weak solution of (5.8) in  $\mathbb{R}^3$ . The lemma is proved.  $\square$

### 3.2 Verification of the Leray conditions

In this subsection, we will finish the proof in the theorem 3.1 by checking  $u(x, t) = \frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}}U_0\left(\frac{x}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}}V\left(\frac{x}{\sqrt{t}}\right)$  satisfies almost the conditions of Leray solutions, except the locally square-integrable in spacetime condition, where  $V$  is given in Lemma 3.7 in the below Lemma.

**Lemma 3.8.** *Let  $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  satisfy  $\lambda u_0(\lambda x) = u_0(x)$  for all  $\lambda > 0$ ,  $\operatorname{div} u_0 = 0$  and  $U_0 = e^\Delta u_0$ . Let  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$  be a weak solution of (5.8) given in Lemma 3.7. Then  $u(x, t) = \frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}}U_0\left(\frac{x}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}}V\left(\frac{x}{\sqrt{t}}\right)$  satisfies condition (i), (ii), (iii) in the sense of 2.4.*

*Proof.* We will check the conditions given in 2.4.

(i) First, we will prove that  $R > 0$ ; we have

$$\operatorname{ess\,sup}_{0 \leq t < R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2} dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u|^2 dx dt < \infty.$$

and

$$\lim_{|x_0| \rightarrow \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2 dx dt = 0.$$

Indeed, since  $U_0 = e^\Delta u_0$  and  $u_0$  is scale-invariant then by the formula of heat equation, we have  $\frac{1}{\sqrt{t}}U_0\left(\frac{x}{\sqrt{t}}\right) = e^{\Delta t}u_0 \lesssim u_0 \lesssim \frac{1}{|x|}$ . Then we have

$$\begin{aligned} \int_{B_R(x_0)} \frac{|u|^2}{2} dx &\leq \int_{B_R(x_0)} |e^{\Delta t}u_0|^2 dx + \int_{B_R(x_0)} \frac{1}{t} \left| V\left(\frac{x}{\sqrt{t}}\right) \right|^2 dx \\ &\leq C \int_{B_R(x_0)} \frac{1}{|x|^2} dx + \sqrt{t} \cdot \|V\|_{L^2(\mathbb{R}^3)}^2 \end{aligned}$$

Remark that by change of variables in spherical coordinate, we have  $\int_{B_R} \frac{1}{|x|^2} dx < \infty$  and  $1/x^2$  is smooth on  $\mathbb{R}^3 \setminus \{0\}$ , so  $\sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{1}{|x|^2} dx = \alpha < \infty$ . So we have

$$\operatorname{ess\,sup}_{0 \leq t < R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2} dx dt \leq \alpha + R \|V\|_{L^2(\mathbb{R}^3)}^2 < \infty$$

Similarly, since  $\nabla U_0 \in L^2(\mathbb{R}^3)$  and  $V \in H_{0,\sigma}^1(\mathbb{R}^3)$ , then we have

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u|^2 dx dt \leq 2 \frac{R^{3/2}}{3/2} \left( \|\nabla U_0\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 \right) < \infty$$

This shows that

$$\operatorname{ess\,sup}_{0 \leq t < R^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{|u|^2}{2} dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u|^2 dx dt < \infty.$$

To show that  $\lim_{|x_0| \rightarrow \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2 dx dt = 0$ , we have the below estimates:

For  $x_0$  large such that  $0 \notin B_R(x_0)$ , and use the change of variables, we have

$$\begin{aligned} \int_{B_R(x_0)} |u|^2 dx &\leq 2 \int_{B_R(x_0)} |e^{\Delta t}u_0(x)|^2 dx + 2 \int_{B_R(x_0)} \frac{1}{t} \left| V\left(\frac{x}{\sqrt{t}}\right) \right|^2 dx \\ &\leq 2 \int_{B_R(x_0)} |e^{\Delta t}u_0(x)|^2 dx + 2 \int_{B_R\left(\frac{x_0}{\sqrt{t}}\right)} \sqrt{t} |V(x)|^2 dx. \end{aligned}$$

Since  $V \in L^2(\mathbb{R}^3)$ , we have  $\lim_{|x_0| \rightarrow \infty} \int_{B_R(\frac{x_0}{\sqrt{t}})} \sqrt{t}|V(x)|^2 dx = 0$  for all  $t > 0$ .

Then

$$\lim_{|x_0| \rightarrow \infty} \int_0^{R^2} \int_{B_R(x_0)} \sqrt{t}|V(x)|^2 dx dt = 0.$$

For  $x \in B_R(x_0)$ , we have  $|x| \geq |x_0| - R$  and by the scaling invariance,  $e^{\Delta t} u_0 \lesssim u_0 \lesssim \frac{1}{|x|}$   
 $\Rightarrow |e^{\Delta t} u_0(x)|^2 \lesssim \frac{1}{|x|^2} \lesssim \frac{1}{(|x_0| - R)^2}$

Then

$$\begin{aligned} \int_{B_R(x_0)} |e^{\Delta t} u_0(x)|^2 dx &\lesssim \int_{B_R(x_0)} \frac{1}{(|x_0| - R)^2} dx \\ &= \frac{1}{(|x_0| - R)^2} \cdot \text{Vol}(B_R(x_0)) \rightarrow 0 \quad \text{when } |x_0| \rightarrow \infty. \end{aligned}$$

This shows that

$$\lim_{|x_0| \rightarrow \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2 dx dt = 0.$$

(ii)  $(u, p)$  verifies the Navier-Stokes equations (NSE) in the sense of distributions, as we explain at the beginning of the proof of Theorem 3.1.

We will show that for any compact set  $K \subset \mathbb{R}^3$ ,  $\lim_{t \rightarrow 0^+} \|u(t, \cdot) - u_0(\cdot)\|_{L^2(K)} = 0$ . It suffits to prove it only for balls  $B_R$  with  $R > 0$ . Fix  $R > 0$ , we have

$$\begin{aligned} \|u(t, \cdot) - u_0(x)\|_{L^2(B_R)}^2 &= \int_{B_R(0)} |u(x, t) - u_0(x)|^2 dx \\ &= \int_{B_R(0)} \left| \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{\sqrt{t}} u_0\left(\frac{x}{\sqrt{t}}\right) \right|^2 dx = \sqrt{t} \int_{B_{R/\sqrt{t}}(0)} |U(z) - u_0(z)|^2 dz. \end{aligned}$$

We will prove that  $U - u_0 \in L^2(\mathbb{R}^3)$ , then  $\|u(t, \cdot) - u_0(x)\|_{L^2(B_R)}^2 \leq \sqrt{t} \|U - u_0\|_{L^2(\mathbb{R}^3)}$ , so we have  $\lim_{t \rightarrow 0^+} \|u(t, \cdot) - u_0\|_{L^2(B_R)} = 0$ . To do that,

$$\begin{aligned} \|U - u_0\|_{L^2(\mathbb{R}^3)}^2 &= \int_{B_R} |U(x) - u_0(x)|^2 dx + \int_{\mathbb{R}^3 \setminus B_R} |U(x) - u_0(x)|^2 dx \\ &\leq 2\|U\|_{L^2(B_R)}^2 + 2\|u_0\|_{L^2(B_R)}^2 + 2 \int_{\mathbb{R}^3 \setminus B_R} (|V(x)|^2 + |e^{\Delta} u_0(x) - u_0(x)|^2) dx. \end{aligned}$$

We have:  $|u_0(x)| = |u_0(\frac{x}{|x|})| \cdot \frac{1}{|x|} \leq \frac{C}{|x|}$  by definition of being scale-invariant and the fact that  $u_0$  is bounded on the sphere  $\mathbb{S}^2$ . So we get:

$$\|u_0\|_{L^2(B_R)}^2 \leq C \left\| \frac{1}{|x|} \right\|_{L^2(B_R)}^2 = C \int_{B_R} \frac{1}{|x|^2} dx \leq C'$$

by the change of variables in spherical coordinates. Since  $V \in H_{0,\sigma}^1(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$ , we only need to prove that  $\|e^{\Delta} u_0 - u_0\|_{L^2(\mathbb{R}^3 \setminus B_R)}^2 < \infty$ .

To do that, we use the formula of the solution of the heat equation and the Taylor expansion for  $u_0$ . Indeed, let  $K_t(x) = K(t, x)$  be the heat kernel:

$$K(t, x) = \begin{cases} \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}, & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

By the formula of the solution of heat equation, we have:

$$e^\Delta u_0(x) - u_0(x) = \int_{\mathbb{R}^3} K_1(x-y) [u_0(y) - u_0(x)] dy$$

By Taylor expansion, for large  $|x|$  and fixed  $y$ :

$$u_0(y) - u_0(x) = \nabla u_0(x) \cdot (y-x) + \frac{1}{2} D^2 u_0(x)(y-x, y-x) + O(|y-x|^3)$$

Then

$$e^\Delta u_0(x) - u_0(x) \lesssim \int_{\mathbb{R}^3} K_1(x-y) (\nabla u_0(x) \cdot (y-x)) dy + \frac{1}{2} \int_{\mathbb{R}^3} K_1(y) D^2 u_0(x)(y-x, y-x) dy$$

We have  $\forall i, j \in \{1, 2, 3\}$ ,  $\int_{\mathbb{R}^3} K_1(z) z_j dz = 0$ ,  $\int_{\mathbb{R}^3} K_1(z) z_i z_j dz = 2\delta_{ij}$ . By the above estimates and the scaling invariance of  $u_0$ , we have

$$e^\Delta u_0 - u_0 \lesssim \Delta u_0 \lesssim \frac{1}{|x|^3}$$

Then  $\|e^\Delta u_0 - u_0\|_{L^2(\mathbb{R}^3 \setminus B_R)}^2 \lesssim \left\| \frac{1}{|x|^3} \right\|_{L^2(\mathbb{R}^3 \setminus B_R)}^2 < \infty$ .

Thus, we have:  $\lim_{t \rightarrow 0^+} \|u(t, \cdot) - u_0\|_{L^2(B_R)} = 0$ . This shows that for any compact set  $K \subset \mathbb{R}^3$ ,  $\lim_{t \rightarrow 0^+} \|u(t, \cdot) - u_0(\cdot)\|_{L^2(K)} = 0$ .

(iii) We will prove the **local energy inequality**:

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \phi(x, t) dx dt \leq \int_0^\infty \int_{\mathbb{R}^3} \left[ \frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \frac{|u|^2}{2} u \cdot \nabla \phi + pu \cdot \nabla \phi \right] dx dt \quad (5.11)$$

for any  $\phi \geq 0$ ,  $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, \infty))$ . Indeed, since  $(u, p)$  satisfies NSE in the sense of distribution, we have  $\forall \varphi \in C_{0,\sigma}^\infty(\mathbb{R}^3 \times (0, \infty))$

$$(\partial_t u, \varphi) + (u \cdot \nabla u, \varphi) = -(\nabla u, \nabla \varphi) - (\nabla p, \varphi)$$

Fix  $\phi \in C_{0,\sigma}^\infty(\mathbb{R}^3 \times (0, \infty))$ ,  $\phi \geq 0$ . Let  $\Omega$  be the support of  $\phi$ . Since  $u \in H_{loc}^1(\mathbb{R}^3 \times (0, \infty))$ , we have  $u\phi \in H^1(\Omega)$ .

By the density of  $C_{0,\sigma}^\infty(\Omega)$  in  $H_\sigma^1(\Omega)$ , we can choose the test function  $\varphi = u\phi$  then (5.11) becomes

$$(\partial_t u, u\phi) + (u \cdot \nabla u, u\phi) = -(\nabla u, \nabla(u\phi)) - (\nabla p, u\phi)$$

Since  $u, \phi$  are divergence free and  $\phi \geq 0$ , using integration by parts, we have

$$\int_\Omega |\nabla u|^2 \phi(x, t) dx dt \leq \int_\Omega \left[ \frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \frac{|u|^2}{2} u \cdot \nabla \phi + pu \cdot \nabla \phi \right] dx dt$$

Since  $\Omega$  is the support compact of  $\phi$ , then we obtain the local energy inequality:

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \phi(x, t) dx dt \leq \int_0^\infty \int_{\mathbb{R}^3} \left[ \frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \frac{|u|^2}{2} u \cdot \nabla \phi + pu \cdot \nabla \phi \right] dx dt$$

□

### 3.3 Comparing with the results of Jia and Šverák

In this subsection, we sketch the proof by Jia and Šverák for Theorem 1.1, which is a stronger conclusion for forward self-similar solutions. An improvement of this result, compared with ours, is that they find a better decay:  $V(x) = o\left(\frac{1}{|x|}\right)$  when  $x \rightarrow \infty$ , which we will not have for general function in  $H_{0,\sigma}^1(\mathbb{R}^3)$ . This decay allows us to show that the solution  $u$  is locally square-integrable at  $t = 0$ , so by Theorem 2.9,  $u$  is smooth and has a good estimate. For convenience, we first restate the theorem and then sketch the proof.

**Theorem 3.9.** *Let  $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  satisfy  $\lambda u_0(\lambda x) = u_0(x)$  for all  $\lambda > 0$ ,  $\operatorname{div} u_0 = 0$ . Then there exists  $u \in C^\infty(\mathbb{R}^3 \times (0, \infty))$ , with  $\lambda u(\lambda x, \lambda^2 t) = u(x, t)$  for all  $\lambda > 0$ , and  $u \in \mathcal{N}(u_0)$ , that is,  $u$  satisfies*

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } \mathbb{R}^3 \times (0, \infty) \quad \text{for some distribution } p.$$

Moreover, let  $U(x) = u(x, 1)$ , then

$$|\partial^\alpha (U(x) - e^\Delta u_0(x))| \leq \frac{C(\alpha, u_0)}{(1 + |x|)^{3+|\alpha|}} \quad \forall |\alpha| \geq 0.$$

*Proof.* By Theorem 2.9, it suffices to show there exists  $u \in \mathcal{N}(u_0)$  with the scaling  $\lambda u(\lambda x, \lambda^2 t) = u(x, t)$  for all  $\lambda > 0$  since the hypothesis are verified.

Jia and Šverák defined a good function space

$$X = \left\{ V \in C^1(\mathbb{R}^3) : \operatorname{div} V = 0, \quad \sup_{x \in \mathbb{R}^3} ((1 + |x|)^2 |V(x)| + (1 + |x|)^3 |\nabla V(x)|) < \infty \right\}$$

and any  $V \in X$ , they defined a natural norm

$$\|V\|_X = \sup_{x \in \mathbb{R}^3} ((1 + |x|)^2 |V(x)| + (1 + |x|)^3 |\nabla V(x)|).$$

Then their strategy is as follows. Set  $U_0 = e^\Delta u_0$  and a parameter  $\mu \in [0, 1]$ , set  $U_{0\mu} = \mu U_0$ , in order to use the general Leray-Schauder degree theorem to prove the existence of  $u_\mu \in \mathcal{N}(\mu u_0)$  with

$$\lambda u_\mu(\lambda x, \lambda^2 t) = u_\mu(x, t) \quad \text{for all } \lambda > 0 \text{ and } \mu \in [0, 1].$$

Similarly with our proof, their goal was seeking the profile function  $U_\mu(x) = u_\mu(x, 1)$ : since  $u(x, t) = \frac{1}{\sqrt{t}} u\left(\frac{1}{\sqrt{t}} x, 1\right)$  and obtaining  $U_\mu(x)$  satisfying 5.5 and  $|U_\mu(x) - U_{0\mu}(x)| = o\left(\frac{1}{|x|}\right)$  as  $|x| \rightarrow \infty$ . To do that, they sought  $U_\mu$  in the form  $U = U_{0\mu} + V$  where  $V \in X$ , so they could reduce the problem to find  $V \in X$  satisfies 5.7. Then  $V \in X$  satisfies 5.7 if and only if  $v(x, t) := \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right)$  satisfies the linear singularly forced Stokes system:

$$\begin{cases} \partial_t v - \Delta v + \nabla p = t^{-3/2} F\left(\frac{x}{\sqrt{t}}\right), \\ \operatorname{div} v = 0, \\ v(\cdot, 0) = 0, \end{cases} \quad (5.12)$$

where  $F = -V \cdot \nabla V - U_{0\mu} \cdot \nabla V - V \cdot \nabla U_{0\mu} - U_{0\mu} \cdot \nabla U_{0\mu}$ . Then they used the below Lemma to show that 5.12 has a unique solution  $V \in X$  for such  $F$  and denoted the solution profile at time  $t = 1$  as  $\mathcal{G}(F) \in X$ . We will state the lemma first, then continue to Jia and Šverák's proof.

**Lemma 3.10.** (Decay for the linear singularly forced Stokes system) Let  $f \in C(\mathbb{R}^3)$ , suppose  $v \in L_t^\infty L_x^\gamma(\mathbb{R}^3 \times (0, T))$  for any  $T < \infty$ , and some  $\gamma > 1$ , suppose  $v$  satisfies

$$\begin{cases} \partial_t v - \Delta v + \nabla p = t^{-3/2} f\left(\frac{x}{\sqrt{t}}\right) \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$

for some distribution  $p$ , and  $\lim_{t \rightarrow 0^+} \|v(\cdot, t)\|_{L^\gamma(\mathbb{R}^3)} = 0$ . Then

- (i) If  $\tilde{v}$  also satisfies the above conditions, then  $v = \tilde{v}$
- (ii) If  $f$  satisfies  $M := \sup_{x \in \mathbb{R}^3} (1 + |x|)^3 |f(x)| < \infty$ , then

$$v(\cdot, t) = \int_0^t e^{(t-s)\Delta} P \left( \frac{1}{s^{3/2}} f\left(\frac{\cdot}{\sqrt{s}}\right) \right) ds$$

where  $P$  is the Helmholtz projection operator.

Let  $V(x) = v(x, 1)$ , then  $\|V\|_{C^{1,\alpha}(B_R)} \leq C(\alpha, R)M$  for  $\alpha \in (0, 1)$  and

$$\sup_{x \in \mathbb{R}^3} \left( (1 + |x|)^2 |V(x)| + (1 + |x|)^3 |\nabla V(x)| \right) \leq CM.$$

To continue to Jia and Šverák's proof, the unicity of the solution  $V \in X$  of 5.12 allows them to consider the following equivalent formulation: find  $V \in X$  with

$$V = \mathcal{G}(-V \cdot \nabla V - U_{0\mu} \cdot \nabla V - V \cdot \nabla U_{0\mu} - U_{0\mu} \cdot \nabla U_{0\mu}). \quad (5.13)$$

Then they defined an operator  $K : X \times [0, 1] \rightarrow X$  as

$$\forall V \in X, \quad \mu \in [0, 1], \quad K(V, \mu) := \mathcal{G}(U_{0\mu} \cdot \nabla U_{0\mu}) + \mathcal{G}(U_{0\mu} \cdot \nabla V + V \cdot \nabla U_{0\mu} + V \cdot \nabla V).$$

and solved the problem by applying Leray-Schauder degree theorem.

$$V \in X, \quad \text{such that } V + K(V, \mu) = 0, \quad \text{for some } \mu \in [0, 1] \quad (5.14)$$

by applying Leray-Schauder degree theorem. Note that the decay properties in Lemma 3.10 helped them to obtain the conditions of Leray-Schauder degree theorem. More specifically, they obtained that  $K : X \times [0, 1] \rightarrow X$  is a continuous compact operator, 5.14 is solvable for  $\mu$  small and the set of solutions of 5.14 is uniformly bounded in  $X$ . You can see more details calculations in [4].  $\square$

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