

C^* -SIMPLICITY AND THE UNIQUE TRACE PROPERTY

LÉO RACLOT, TISTOU NORET

ABSTRACT. The purpose of this paper is to study, for discrete countable groups, the links between the reduced C^* -algebra, the Furstenberg boundary, and the amenable radical. First, following a recent proof by Uffe Haagerup, we show that such a group has the unique trace property, that is, its reduced $C_\lambda^*(G)$ has a unique trace state if and only if its amenable radical is trivial. Then, we characterize C^* -simplicity in terms of topological actions on boundaries, and especially the Furstenberg boundary. To prove this result, we draw on a paper by Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. Finally, we use this criterion to show that the free group on two generators is C^* -simple.

1. INTRODUCTION

Amenability is well known to be a fundamental notion in the study of a countable discrete group G because it has various characterizations that make it useful in many fields related to group theory. On the one hand, because it involves, in particular, the Hermitian structure of $l^2(G)$, one may naturally wonder about the links between amenability and unitary representations of G . In order to study those, the associated C^* -algebras and especially the reduced C^* -algebra are powerful tools. On the other hand, amenability is closely related to topological G -spaces (through their measure spaces), which brings into play G -boundaries and notably the Furstenberg boundary.

More precisely, we define the reduced C^* -algebra $C_\lambda^*(G)$ of G as the norm closure of the operator algebra in $l^2(G)$ generated by the left regular representation. If $C_\lambda^*(G)$ is simple, G is said to be C^* -simple, and if $C_\lambda^*(G)$ has a unique trace (given by $a \mapsto \langle a\delta_e, \delta_e \rangle$), G is said to have the unique trace property.

Powers [Pow75] proved in 1975 that the free group on two generators is both C^* -simple and has the unique trace property, which we will see in the fifth part (Section 5) as a consequence of general theorems. Since then, a lot of work has been done to characterize the groups with those properties and to determine whether they are equivalent. Studies in this direction have shown two important facts.

First, the unique trace property is completely characterized in terms of amenability, and more precisely, the amenable radical of a group G entirely determines whether G has the unique trace property or not.

Theorem A. *Let G be a discrete countable group. Then G has the unique trace property if and only if $\text{Rad}(G) = \{e\}$.*

This theorem is the main topic of the third part of this paper (Section 3), and was originally established by Kalantar and Kennedy [KK17]. However, we will prove it with more basic ideas, following Haagerup's paper [Haa17], which uses Furman's

characterization of the amenable radical of a group as consisting of those elements that act trivially under any boundary action.

Then, if X is a compact space, the simplicity of the reduced crossed product $C(X) \rtimes_{\lambda} G$ depends on the topological dynamics of the action of G on X . In fact, considering the Furstenberg boundary $\partial_F G$, one may find a space that completely determines whether $C_{\lambda}^*(G)$ is simple.

Theorem B. *Let G be a discrete group. The following are equivalent:*

- (1) G is C^* -simple;
- (2) G acts freely on its Furstenberg boundary $\partial_F G$, i.e., for any $g \in G \setminus \{e\}$ and $x \in \partial_F G$, one has $gx \neq x$;
- (3) There exists a topologically free G -boundary X , i.e., for any $g \in G \setminus \{e\}$, $\text{Fix}(g) := \{x \in X \mid gx = x\}$ has empty interior in X .

We will prove this theorem in the fourth part (Section 4), by refining some methods and ideas used to show Theorem A. We will closely follow the original proof of Breuillard, Kalantar, Kennedy, and Ozawa [BKKO17].

Noticing that if $C_{\lambda}^*(G)$ is simple, then the amenable radical of G must be trivial (which is a consequence of Proposition 2.27 and Lemma 4.5), one may observe that this last theorem proves C^* -simplicity implies the unique trace property. Let us mention for cultural purposes that the study of the link between these two properties was finally completed recently by Le Boudec, who gave examples of groups that have the unique trace property but are not C^* -simple.

2. PRELIMINARIES

We begin by introducing some notions of measure theory on groups, unitary representations, and C^* -algebras. All the groups that we consider will be countable and discrete.

2.1 Amenable groups

Definition 2.1. *Let X be a set. A mean on X is a function $m : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying $m(X) = 1$ and $m(A \sqcup B) = m(A) + m(B)$ for all disjoint subsets A and B . The set of means on X is denoted by $\mathcal{M}(X)$.*

Observe that if a group G acts by homeomorphisms on a space X , there is a natural action on $[0, 1]^{\mathcal{P}(X)}$ (given by $g \cdot f = f \circ g^{-1}$ for all $f \in [0, 1]^{\mathcal{P}(X)}$) for which $\mathcal{M}(X)$ is G -invariant. In particular, one can consider $X = G$ and the action of $g \in G$ by left-translation: $(g, h) \rightarrow g \cdot h$.

Definition 2.2. *A group G is **amenable** if it admits a left-invariant mean.*

This is the original definition of amenability by von Neumann, but there are many other characterizations (see [Tes25], sections 1.1, 1.6-1.9). Weak containment will be presented in the fifth part of this paper.

Proposition 2.3. *Let G be a group. The following are equivalent*

- (1) G is amenable
- (2) (Følner's criterion) For all $\varepsilon > 0$ and all finite subset $Q \subset G$ there exists $F \subset G$ finite such that $\frac{|Fg\Delta F|}{|F|} \leq \varepsilon$.
- (3) (Reiter's property) For all $\varepsilon > 0$ and all finite $Q \subset G$, there exists $f \in l^1(G)$ such that $\|f\|_1 = 1$ and $\|g \cdot f - f\| \leq \varepsilon$, for all $g \in Q$.

- (4) For all $\varepsilon > 0$ and all finite subsets $Q \subset G$, there exist $f \in l^2(G)$ such that $\|f\|_2 = 1$ and $\|f(\cdot g^{-1} - f)\|_2 \leq \varepsilon$, for all $g \in Q$.

Notice that if G is countable, all these properties can be easily formulated in sequential terms. Although (2) is geometric and can be seen on the Cayley graph of G , (3) makes the link with (4) which will be very useful to us as it shows the link with the Hilbertian structure of C^* -algebra.

But do amenable groups really exist? The answer is yes and, in fact, every abelian group is amenable (it can be proved using the fixed-point characterization of amenability, see [Tes25], Section 1.9). For \mathbb{Z} , "Følner sets" are given by sufficiently large integer intervals, and an invariant mean is the density.

Using the different characterizations, one can prove the following stability result, which is very strong.

Proposition 2.4. *Amenability is stable under sums, directed unions, subgroups, and quotients.*

We end this part by introducing another object whose existence follows from stability results, especially under group extension, that is, if G and H are amenable and there is an exact short sequence of groups

$$0 \longrightarrow G \longrightarrow Q \longrightarrow H \longrightarrow 0$$

then Q is amenable.

Proposition 2.5. *Let G be a group. There exists a unique maximal normal amenable subgroup of G called its **amenable radical**, denoted by $\text{Rad}(G)$.*

2.2 Topological boundaries Let G be a group. A G -space X is a compact separable metrizable space together with an action of G by homeomorphisms. We denote by $\text{Prob}(X)$ the set of Radon probability measures on X .

Definition 2.6. *A G -space X is said to be*

- (i) **minimal** if it has no non-empty G -invariant closed strict subset;
- (ii) **strongly proximal** if for all μ in $\text{Prob}(X)$, there exists $x \in X$ such that $\delta_x \in \overline{G \cdot \mu}$;
- (iii) a **G -boundary** if it is both minimal and strongly proximal.

Notice that if X is a G -boundary, by minimality every orbit contains all Dirac masses in its closure. We will now focus on a special boundary called the Furstenberg boundary.

Definition 2.7. *A G -space X is called a **Furstenberg boundary** if for all G -boundary Y there exists a unique continuous surjective G -equivariant map $\pi : X \rightarrow Y$*

The existence of such a boundary is proved in [Tes25] (see 3.3 and 3.4).

Proposition 2.8. *Every group G admits a Furstenberg boundary, which is unique up to a G -equivariant homeomorphism. It is denoted by $\partial_F G$.*

The following lemma will be useful. Its first point is, in fact, an equivalence, but we do not need the reciprocity in the proofs.

Lemma 2.9. *Let X be a G -boundary. Then:*

- (1) X is the only minimal G -subspace of $\text{Prob}(X)$

- (2) If Y is a G -space, then for every G -equivariant map $f : Y \rightarrow \text{Prob}(X)$, $f(Y)$ contains X , which means all the dirac masses. If Y is minimal, this map is unique.

Proof. (1) If W is another minimal G -subspace of $\text{Prob}(X)$, then by strong proximality $X \subset W$ and $X = W$

- (2) We can assume that Y is minimal (if not we can take a minimal G -subspace of Y that exists by compactity). Then $f(Y)$ is minimal, closed, and invariant. By the previous point $f(Y) = X$. If g is another such map, $(f + g)/2$ is also and its image is X if and only if $f = g$. \square

A fact that will be important for our topic is the existence of **boundary maps**, that we define in the following proposition:

Proposition 2.10. *For every G -space X , there exists a G -equivariant map $\partial_F G \rightarrow \text{Prob}(X)$. Such maps are called boundary maps.*

It is a consequence of the following theorem, which directly implies the previous proposition.

Theorem 2.11. *Let K be a convex G -space. Then $\overline{\text{Ext}(K)}$ is a G -boundary.*

For a proof, see Proposition 3.27 in [Tes25]. We end this subsection by stating the following theorem, which will be crucial in the proofs of theorems A and B below.

Theorem 2.12 (Furman's theorem [Fur03]). *The kernel of the Furstenberg boundary action $\rho : G \curvearrowright \partial_F G$ is the amenable radical of G , that is, $\text{Ker}(\rho) := \{g \in G \mid gx = x \text{ for any } x \in \partial_F G\}$ satisfies $\text{Ker}(\rho) = \text{Rad}(G)$.*

Proof. We consider the set of means $M(G)$ on G as defined in Section 2.1, and let K be a minimal compact subset of $M(G)$. By Theorem 2.11, $X := \overline{\text{Ext}(K)}$ is a G -boundary. Let $m \in X$. Each $g \in \text{Stab}(m)$ stabilizes the mean m . If we write $G = \bigsqcup g_i \text{Stab}(m)$, then we can pose the mean on $\text{Stab}(m)$ defined by $m' : A \mapsto m(\bigsqcup g_i A)$. It is $\text{Stab}(m)$ -invariant so $\text{Stab}(m)$ is amenable. By the universal property of $\partial_F G$, we have a surjective G -equivariant map $\pi : \partial_F G \rightarrow X$. Then for $y \in \partial_F G$, $\text{Stab}(x) \subset \text{Stab}(\pi(x))$ and the stabilizer is amenable. Hence, $\text{Ker}(\rho) \subset \text{Rad}(G)$.

For the other inclusion, let $x \in \partial_F G$ and $f : \text{Rad}(G) \rightarrow \partial_F G$, $g \mapsto g \cdot x$. It is $\text{Rad}(G)$ -equivariant, Since $\text{Rad}(G)$ is amenable, it admits an invariant mean m , and the image of m by f is then $\text{Rad}(G)$ -invariant. Thus, the set I of $\text{Rad}(G)$ -invariant means on $\partial_F G$ is not empty and is $\text{Rad}(G)$ -invariant. But if $g \in G$ and $m \in I$ then for $n \in \text{Rad}(G)$ we have

$$n \cdot (g \cdot m) = g \cdot (n' \cdot m) = g \cdot m$$

where $n' \in \text{Rad}(G)$ because the amenable radical is normal. Hence, I is a non-empty closed G -invariant subset of $\text{Prob}(\partial_F G)$; according to Lemma 2.9, it must contain all dirac masses, so $\text{Rad}(G)$ acts trivially on $\partial_F G$. \square

2.3 C^* -algebras

Definition 2.13. *A C^* -algebra is a complex Banach algebra A together with an involution $*$: $A \rightarrow A$ such that*

- (1) $\forall x, y \in A, (x + y)^* = x^* + y^*$,
- (2) $\forall \lambda \in \mathbb{C}, \forall x \in A, (\lambda x)^* = \bar{\lambda}x^*$,
- (3) $\forall x, y \in A, (xy)^* = y^*x^*$,
- (4) $\forall x \in A, \|x\| = \|x^*\|$,
- (5) $\forall x \in A, \|x^*x\| = \|x\|^2$.

Here, all algebras are unital. We recall a few definitions or results about C^* -algebras.

Definition 2.14. *Let A be a C^* -algebra and $a \in A$. Then a is said to be positive if $a = a^*$ and $\text{Sp}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda \text{ is not invertible}\} \subset \mathbb{R}_+$ or, equivalently, if there exist $x \in A$ such that $a = xx^*$.*

Definition 2.15. *Let A, B be two C^* -algebras and $\phi : A \rightarrow B$ be a bounded linear map. We say that ϕ is*

- (1) **positive** if for all $a \in A$ positive, $\phi(a)$ is positive.
- (2) a **state** if it is positive, $B = \mathbb{C}$, and $\phi(1) = 1$.
- (3) **completely positive** if for all $n \in \mathbb{N}$, $\phi_n : M_n(A) \rightarrow M_n(B)$ (that is, $\text{id} \otimes \phi : M_n(\mathbb{C}) \otimes A \rightarrow M_n(\mathbb{C}) \otimes B$) is positive.
- (4) a **UCP map** if it is unital, i.e., $\phi(1) = 1$, and completely positive.

An example of a positive map that is not completely positive is the transposition on matrices. In particular, any state is completely positive. If π is a $*$ -representation of a C^* -algebra A in a Hilbert space \mathcal{H} , the map $\phi_{\pi, v} : a \mapsto \langle \pi(a)v, v \rangle$ is a positive linear form. These functions are called **functions of positive type**. The GNS construction below shows that every state is of this form.

Theorem 2.16 (GNS construction). *Let ϕ be a state in A . There exist a $*$ -representation π and a unit vector $v \in \mathcal{H}_\pi$ such that $\phi = \phi_{\pi, v}$*

In fact, this theorem can be generalized to completely positive maps, but we only present this version here.

Proof. We will try to define a scalar product on A . We consider the sesquilinear positive form on A defined by $\forall a, b \in A, \langle a, b \rangle := \phi(b^*a)$. Then the set I of elements $a \in A$ such that $\phi(a^*a)$ is a left-ideal because if $a, b \in I$, by Cauchy-Schwarz inequality

$$\begin{aligned} |\phi((a+b)^*(a+b))| &\leq |\phi(a^*b)| + |\phi(b^*a)| \\ &\leq 2|\phi(a^*a)||\phi(b^*b)| \\ &= 0 \end{aligned}$$

and if $a \in I, c \in A$ then

$$\begin{aligned} |\phi((ca)^*ca)| &= |\phi(a^*c^*ca)| \\ &\leq |\phi((a^*c^*c)^*(a^*c^*c))||\phi(a^*a)| \\ &= 0, \end{aligned}$$

again with Cauchy-Schwarz. Then we can take the quotient of A by I and the induced hermitian product that makes it a pre-Hilbertian space with a $*$ -representation of A by left-multiplication.

Now, for $b \in A, \|b\| \leq 1$, the map $f : x \mapsto \phi(b^*xb)$ is positive. So for $x \in A$ unit,

since $1 - x^*x$ is positive by the spectral radius equality (see [Ska19]), $f(1) \geq f(x^*x)$. By Cauchy-Schwarz's inequality,

$$|f(x)|^2 \leq |f(x^*x)||f(1)| \leq \phi(b^*b)^2.$$

Applying this to a^*a where $a \in A$ we obtain $\|\pi(a)\| \leq \|a\|$ and we can extend the representation to the completion of A/I , and we only have to take $v = 1$. \square

Let us finish with two important theorems for our topic. Proposition 2.18 will be used in the proofs of the indirect way of Theorem A and of (2) \Rightarrow (1) in Theorem B. Theorem 2.23 will be useful in the proof of (2) \Rightarrow (1) too.

Definition 2.17. Let $\phi : A \rightarrow B$ be a positive map. Its multiplicative domain is the subspace $D_\phi := \{a \in A : \forall b \in A, \phi(ab) = \phi(a)\phi(b) \text{ and } \phi(ba) = \phi(b)\phi(a)\}$.

Proposition 2.18. Let $\phi : A \rightarrow B$ be a positive map. Then $D_\phi := \{a \in A : \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a^*)\}$. In particular, D_ϕ is a C^* -subalgebra.

We show this proposition in the special case where ψ is a state of a unital C^* -algebra (see [BO08], Section 1.5 for a complete proof).

Proof. Using Theorem 2.16, one may write $\psi(a) = \langle \pi(a)v, v \rangle$ where π is a $*$ -representation of A and v a unit vector. For $a \in A$ such that $\psi(a^*a) = \psi(aa^*) = \overline{\psi(a)}\psi(a)$, let us observe the equality case of the following inequality.

$$\begin{aligned} \psi(a)\psi(a^*) &= |\psi(a)|^2 \\ &= |\langle \pi(a)v, v \rangle|^2 \\ &\leq \|\pi(a)v\|_2^2 \\ &= |\langle \pi(a^*a)v, v \rangle|^2 \\ &= \psi(a^*a) \end{aligned}$$

It happens if and only if $\pi(a)v$ and v are colinear, that is $\pi(a)v = \psi(a)v$. Hence, for all $b \in A$, $\psi(ba) = \langle \pi(ba)v, v \rangle = \psi(a)\langle \pi(b)v, v \rangle = \psi(a)\psi(b)$. Since a^* satisfies the same properties as a , we also have $\psi(ab) = \psi(b^*a^*) = \psi(a^*)\psi(b^*) = \psi(a)\psi(b)$. We conclude that $a \in D_\psi$. \square

Theorem 2.19 (Arveson's extension theorem). Let B be a C^* -algebra, A be a C^* -subalgebra of B , \mathcal{H} be a Hilbert space and $\Phi : A \rightarrow B(\mathcal{H})$ be a completely positive map. Then we can extend Φ to B .

For a proof, see [BO08].

We can now announce another important result. A C^* -algebra is said to be a G - C^* -algebra if it is equipped with an action of G by automorphisms. Given two G - C^* -algebras A and B , we denote by $\text{UCP}_G(A, B)$ the set of G -equivariant UCP maps from A to B . A proof of the following theorem can be found in [Ham85].

Theorem 2.20. Let G be a discrete countable group. Then

- $C(\partial_F G)$ is G -**injective**: for all G - C^* -algebras $A \subset B$, every UCP map $\Phi \in \text{UCP}_G(A, C(\partial_F G))$ can be extended to a G -UCP map $\Phi : B \rightarrow C(\partial_F G)$.
- $C(\partial_F G)$ is G -**essential**: for all G - C^* -algebra A , any $\Phi \in \text{UCP}_G(C(\partial_F G), A)$ is completely isometric, or equivalently, $\Phi^{-1} : \Phi(C(\partial_F G)) \rightarrow C(\partial_F G)$ is also a UCP map.

Since a linear map ϕ is positive if and only $\phi(1) = \|\phi\|$, the denomination of a "completely isometric map" takes all its meaning.

2.4 Group C*-algebras Let G be a discrete countable group. Its algebra $\mathbb{C}[G]$ is naturally equipped with an involution that gives it a C*-algebra structure:

$$\left(\sum_{g \in G} a_g g\right)^* = \sum_{g \in G} \overline{a_g} g^{-1} = \sum_{g \in G} \overline{a_{g^{-1}}} g$$

If π is a unitary representation of G on a Hilbert space \mathcal{H} , we can extend π by linearity to a *-homomorphism $\pi : \mathbb{C}[G] \rightarrow \mathcal{B}(\mathcal{H})$. The closure of its image is then a C*-algebra denoted by $C_\pi^*(G)$, which admits a natural G -action given by $\sigma_g(T) = \pi(g)T\pi(g)^*$. Let us denote by λ the left regular representation of G on $l^2(G)$ that is unitary.

Definition 2.21. *The C*-algebra $C_\lambda^*(G)$ is called the reduced C*-algebra of G .*

Each unitary representation of G induces a norm on $\mathbb{C}[G]$ given by $\|a\|_\pi := \|\pi(a)\|$. We can then consider the supremum of all these norms and consider

$$\|a\|_u := \sup\{\|a\|_\pi \mid \pi \text{ is a unitary representation of } G\},$$

which is well defined because this supremum is finite (for the reason of unitary representations). We then obtain the following algebra:

Definition 2.22. *The completion of $\mathbb{C}[G]$ for $\|\cdot\|_u$ is called the full C*-algebra of G and is denoted by $C^*(G)$.*

This C*-algebra satisfies the following universal property:

Proposition 2.23. *For every unitary representation π of G , there exists a unique *-homomorphism $\Phi : C^*(G) \rightarrow C_\pi^*(G)$ with $\Phi(\pi_u(g)) = \pi(g)$.*

Proof. The representation π extends by linearity to a *-representation of $\mathbb{C}[G]$, which is contractive by the definition of $\|\cdot\|_u$ and then we can extend it to $C^*(G)$. \square

The following theorem is a well-known characterization of amenability from the approach of C*-algebra. We will prove it after introducing weak containment tool in subsection 2.5.

Theorem 2.24. *Let G be a discrete group. Then G is amenable if and only if the canonical quotient map from $C^*(G)$ to $C_\lambda^*(G)$ is an isomorphism.*

Now, given a C*-algebra A , a homomorphism $\alpha : G \rightarrow \text{Aut}(A) : g \mapsto \alpha_g$ and a faithful *-representation ρ of A on \mathcal{H}_1 we want to consider a C*-algebra containing the information of both $C_\lambda^*(G)$ and A . We will hence construct their reduced crossed product.

Firstly, we consider the twisted group ring $A_\alpha[G]$ whose elements are formal linear combinations of elements of G with coefficients in A . And the product is given by

$$\forall g \in G, \forall a \in A, ga = \alpha_g(a)g,$$

exactly as in the case of the crossed-product of two groups.

Then we can consider the Hilbert space $\mathcal{H}_1 \otimes l^2(G)$ and the representations $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ given by

$$\pi(a)(w \otimes \delta_g) = \rho(\alpha_{g^{-1}}(a))(w) \otimes \delta_g,$$

and $\tilde{\lambda} := \text{id} \otimes \lambda$ of G in \mathcal{H} .

One can easily verify that

$$(2.1) \quad \forall a \in A, \forall g \in G, \tilde{\lambda}(g)\pi(a)\tilde{\lambda}(g)^* = \pi(\alpha_g(a))$$

Then, we can construct a $*$ -representation of $A_\alpha[G]$ in \mathcal{H} extending both π and $\tilde{\lambda}$. The closure of its image in $\mathcal{B}(\mathcal{H})$ is called the **reduced crossed-product** of A and G and is denoted by $A \rtimes_r G$.

In particular, we can apply this construction to $A := C(X)$, where X is a G -space with $\alpha_g(f) = f(g^{-1}\cdot)$ and $\rho : C(X) \rightarrow B(L^2(X))$ with $\rho(f)u = fu$ for $f \in C(X)$ and $u \in L^2(X)$.

2.5 Weak containment In this last part, we prove several results on weak containment, which will be useful later in the proofs.

If A is a group C^* -algebra, we only need to consider the representation of the group. We will need it to recall the following definition of weak containment.

Definition 2.25. *Let π and σ be two unitary representations of a discrete group G in two Hilbert spaces \mathcal{H}_π and \mathcal{H}_σ . We say that π is **weakly contained** in σ and we note that $\pi \prec \sigma$ if every function of positive type associated with π can be approximated with finite sums of functions of positive type associated with σ . More explicitly, for all $v \in \mathcal{H}_\pi$, $\varepsilon > 0$, and $Q \subset G$ finite, there exist $n \geq 1$ and $w_1, \dots, w_n \in \mathcal{H}_\sigma$ such that*

$$\forall g \in Q, |\langle \pi(g)v, v \rangle - \sum_{i=1}^n \langle \sigma(g)w_i, w_i \rangle| < \varepsilon$$

Some remarks will be useful, especially on weak containment of the trivial representation.

Remark 2.26. *We have the following propositions regarding weak containment:*

- (1) *Weak containment passes to direct sum and tensor product of representations.*
- (2) *The relation \prec is reflexive and transitive.*
- (3) *$\pi \prec \sigma$ if and only if every function of positive type of π associated with a unit vector v can be approximated by convex combinations of functions of positive type of π associated with unit vectors.*

Proof. The “if” part follows from the definitions. For the “only if” part, let us take a unit vector $v \in \mathcal{H}_\pi$, $1 > \varepsilon > 0$ and $Q \subset G$ finite. We can assume that $e \in Q$. We take w_1, \dots, w_n given by the definition. Then $|1 - \sum_{i=1}^n \|w_i\|^2| < \varepsilon$ by evaluating in e and we can pose $p_i = \frac{\|w_i\|}{\sum_{i=1}^n \|w_i\|^2}$

and $w'_i = \frac{w_i}{\|w_i\|}$. Then for $g \in Q$,

$$\begin{aligned}
& |1 - \sum_{i=1}^n p_i \langle \pi(g)w'_i, w'_i \rangle| \\
& \leq |1 - \sum_{i=1}^n \langle \pi(g)w_i, w_i \rangle| + |\sum_{i=1}^n p_i \langle \pi(g)w'_i, w'_i \rangle - \sum_{i=1}^n \langle \pi(g)w_i, w_i \rangle| \\
& < \varepsilon + |1 - \frac{1}{\sum_{i=1}^n \|w_i\|^2}| \cdot |\sum_{i=1}^n \langle \pi(g)w_i, w_i \rangle| \\
& < \varepsilon + \frac{\varepsilon}{1 - \varepsilon}(1 + \varepsilon),
\end{aligned}$$

which is as small as we want. \square

- (4) We have $1_G \prec \pi$ if and only if π admits almost invariant vectors, that is $\forall \varepsilon > 0, \forall Q \subset G$ finite, $\exists \xi \in \mathcal{H}_\pi, \forall g \in Q, \|\pi(g)\xi - \xi\| < \varepsilon$.

Proof. This comes from the fact that if ξ is a unit vector,

$$\|\pi(g)\xi - \xi\|^2 = 2(1 - \Re(\langle \pi(g)\xi, \xi \rangle)) \leq 2|1 - \langle \pi(g)\xi, \xi \rangle|$$

and

$$|1 - \langle \pi(g)\xi, \xi \rangle| \leq \|\pi(g)\xi\| \cdot \|\pi(g)\xi - \xi\| \leq \|\pi(g)\xi - \xi\|$$

by Cauchy-Schwarz inequality. The second inequality proves the indirect way. If $|1 - \sum_{i=1}^n \langle \pi(g)w_i, w_i \rangle| < \varepsilon$, $\bigoplus_{i=1}^n w_i$ is (Q, ε') -invariant where $\varepsilon' = \frac{\varepsilon}{\sum_{i=1}^n \|w_i\|^2}$ can be taken as small as we want if $e \in Q$. Then $\pi^{\oplus \infty}$ has (Q, ε) -invariants vectors for all ε and so has π , and we conclude by the first inequality. \square

Now we can tackle the proof of an important proposition:

Proposition 2.27. *Let π, σ be two unitary representations of a discrete group G . The following conditions are equivalent:*

- (i) $\pi \prec \sigma$,
- (ii) $\|\cdot\|_\pi \leq \|\cdot\|_\sigma$,
- (iii) $C^*\text{Ker}(\sigma) \subset C^*\text{Ker}(\pi)$,

where the representations are seen through the full C^* -algebra. We will use Ker instead of $C^*\text{Ker}$ (the kernel by factorization).

We denote by $\mathcal{S}_\pi(G)$ the closure of the convex hull of the set of states on $C^*(G)$ of the form $\phi_{\pi, v}$, and $\mathcal{S}(G) = \mathcal{S}(C^*(G))$. Every state on a representation group C^* -algebra can be seen as a state on the full group C^* -algebra. We have the following lemma:

Lemma 2.28. *For every unitary representation π , $\mathcal{S}_\pi(G) = \mathcal{S}(C^*_\pi(G))$.*

Proof. The inclusion $\mathcal{S}_\pi(G) \subset \mathcal{S}(C^*_\pi(G))$ is true by definition.

Let $a \in C^*(G)$. Then $\pi(a)$ is positive if and only if $\forall v \in \mathcal{H}_\pi, \phi_{\pi, v}(a) \geq 0$. This remains true for any element of $C^*_\pi(G)$. Suppose by contradiction that there is $\varphi \in \mathcal{S}(C^*_\pi(G)) \setminus \mathcal{S}_\pi(G)$. Then, by geometric Hahn-Banach applied to the dual

equipped with weak-* topology, there exists a linear form, which identifies to an element $x \in C_\pi^*(G)$ such that $\varphi(x) < 0$ and $\forall \psi \in \mathcal{S}_\pi(G), \psi(x) \geq 0$, which is contradictory by the above fact. \square

Proof of Proposition 2.27. From Remark 2.26, we deduce that $\pi \prec \sigma$ if and only if $\mathcal{S}_\sigma(G) \supset \mathcal{S}_\pi(G)$.

(i) \Rightarrow (iii): Assume that (i) holds and take $x \in \text{Ker}(\sigma)$. For all $\phi \in \mathcal{S}_\sigma(G)$, $\phi(x) = 0$. So, it stays true for ϕ in $\mathcal{S}_\pi(G)$, which means that $\pi(x) = 0$.

(ii) \iff (iii): Since (ii) clearly implies (iii), we prove the other way. (iii) means that π factorizes by σ through a *-morphism $C_\sigma^*(G) \rightarrow C_\pi^*(G)$. We can conclude because *-morphisms are contractive (see [Ska19]).

(ii) \Rightarrow (i): A state on $C^*(G)$ can pass to $C_\pi^*(G)$ if and only if it factorizes through the projection, that is, if and only if it is continuous for $\|\cdot\|_\pi$. By (ii), $\mathcal{S}(C_\pi^*(G)) \subset \mathcal{S}(C_\sigma^*(G))$ and we conclude with the lemma. \square

These general properties allow us to state two important results.

Theorem 2.29. *A discrete group G is C^* -simple, that is, $C_\lambda^*(G)$ is simple, if and only if every unitary representation of G that is weakly contained in the regular representation of G is weakly equivalent to it.*

Proof. First, assume that $C_\lambda^*(G)$ is simple and let π be a unitary representation weakly contained into λ . From the previous lemma we deduce $\text{Ker}(\lambda) \subset \text{Ker}(\pi)$. Thus, π induces a morphism $C_\lambda^*(G) \rightarrow C_\pi^*(G)$, which, by simplicity, must be injective. Hence, $\text{Ker} \lambda = \text{Ker} \pi$ and, therefore, using again the lemma, λ is weakly contained in π .

For the opposite direction, let us assume that any unitary representation weakly contained in λ is weakly equivalent to it. Then, we consider a *-representation $C_\lambda^*(G) \rightarrow \mathcal{B}(H)$, that is, a unitary representation π of G which factors through $C_\lambda^*(G)$. That means $\text{Ker} \lambda \subset \text{Ker} \pi$ or equivalently that π is weakly contained in λ . Therefore, the hypothesis implies that λ is weakly equivalent to π and thus $\text{Ker} \lambda = \text{Ker} \pi$. We deduce that the initial representation was faithful and finally that $C_\lambda^*(G)$ is simple. \square

Theorem 2.30. *Let G be a discrete amenable group. Then λ_G induces an isomorphism $C^*(G) \rightarrow C_\lambda^*(G)$*

Proof. From Proposition 2.3, we deduce that a group is amenable if and only if $1_G \prec \lambda_G$. Since $1_G \prec \lambda_G$, for every unitary representation π of G , $1_G \otimes \pi \prec \lambda_G \otimes \pi$. But $\lambda_G \otimes \pi$ is a multiple of λ_G . Indeed, we can see it by writing everything in \mathcal{H} -blocks. More precisely, we introduce $T : l^2(G) \otimes \mathcal{H} \rightarrow l^2(G) \otimes \mathcal{H}$, $\delta_x \otimes v \mapsto \delta_x \otimes \pi(x)v$. We can calculate $T^*(\lambda_G \otimes \pi)(g)T = \lambda_G \otimes \text{id}$. Then $\pi \prec \lambda_G^{\oplus n}$, so $\pi \prec \lambda_G$ and $\|\cdot\|_\pi \leq \|\cdot\|_{\lambda_G}$. So $C^*(G)$ and $C_\lambda^*(G)$ are isometric and therefore isomorphic with λ_G . \square

3. THE UNIQUE TRACE PROPERTY

In this section, we tackle the first theorem of this paper, Theorem A.

3.1 Traces. Given a group G a C^* -algebra A and an action $\sigma : G \curvearrowright A$, we naturally have an G -action on the set of states of A given by $\forall \varphi \in \mathcal{S}(A)$, $g\varphi = \varphi \circ \sigma_{g^{-1}}$.

Definition 3.1. A *trace* on a C^* -algebra A is a state on A such that

$$\forall a, b \in A, \varphi(ab) = \varphi(ba).$$

A group G is said to have the **unique trace property** if the only trace on $C_\lambda^*(G)$ is $u \mapsto \langle u(\delta_e), \delta_e \rangle$ (the coefficient of the neutral element in the decomposition of an element of the group ring).

We now have everything we need to establish Theorem A.

Proof of Theorem A. We start by proving the if direction. Assume that $N := \text{Rad}(G)$ is trivial and let τ be a trace on $C_\lambda^*(G)$. We just need to show that if $g \neq e$ then $\tau(g) = 0$. Let g be non-trivial. By Hahn-Banach extension theorem, we can extend τ to a state φ on $C(\partial_F G) \rtimes_\tau G$.

By assumption $g \notin \text{Rad}(G)$, and by Furman's theorem (Theorem 2.12), there exists $x \in \partial_F G$ such that $g \cdot x \neq x$. The restriction of φ to $C(\partial_F G)$ is a positive continuous linear form, so by Riesz's theorem it can be assimilated to a probability measure on $\partial_F G$ since $\partial_F G$ is compact. Hence, by the definition of $\partial_F G$ (we here use more than just strong proximality), there is a sequence (g_i) of elements of G such that the restriction of $g_i \cdot \varphi$ on $C(\partial_F G)$ weakly converges to δ_x . But $(g_i \cdot \varphi)$ is bounded, so by Banach-Alaoglu-Bourbaki's theorem we can extract a subsequence converging to a state ψ on the reduced crossed-product. Because τ is a trace, ψ still remains equal to τ on $C_\lambda^*(G)$.

Moreover, since ψ coincides with δ_x on $C(\partial_F G)$ by Proposition 2.18, it ensures that $C(\partial_F G)$ is included in the multiplicative domain of ψ ($\psi(f\bar{f}) = (f\bar{f})(x) = f(x)\bar{f}(x)$ for all f), and then by Proposition 2.18 that for all $f \in C(\partial_F G)$,

$$\psi(\tilde{\lambda}(g))f(x) = \psi(\tilde{\lambda}(g)f) = \psi(f(g^{-1}\cdot)\tilde{\lambda}(g)) = f(g^{-1}x)\psi(\tilde{\lambda}(g)).$$

Since $\partial_F G$ is Hausdorff compact and in particular normal, and $x \neq g^{-1}x$, we can take $0 \leq f \leq 1$ with $f(x) = 1$ and $f(g^{-1} \cdot x) = 0$. This implies that $\psi(\tilde{\lambda}(g)) = 0$ and so $\tau(\lambda(g)) = 0$.

For the other direction, we introduce the notion of “conditional expectation” $E : \mathbb{C}[G] \rightarrow \mathbb{C}[N]$ defined as the linear map that is the identity on N and vanishes on $G \setminus N$. Equivalently, using the left regular representation of $\mathbb{C}[G]$ on $l^2(G)$, $\forall a \in \mathbb{C}[G], E(a) = PaP$ where $P : l^2(G) \rightarrow l^2(N)$ is the orthogonal projection. Therefore, E naturally extends to a map from $C_\lambda^*(G)$ to $C_\lambda^*(N)$ (its range is contained in the closure of $\mathbb{C}[N]$) which is the identity while restricted on $C_\lambda^*(N)$ since it is the identity on $\mathbb{C}[N]$.

Now, let us assume that N is non-trivial. As it is amenable, using Proposition 2.23 and the universal property of $C^*(N) \simeq C_\lambda^*(N)$, we can extend the trivial representation of G to a C^* -homomorphism $\psi : C_\lambda^*(N) \rightarrow \mathbb{C}$. Then $\phi := \psi \circ E$ is a trace. Indeed, clearly it is a state (E is positive) and for all $g, k \in G$, $\phi(k) = \mathcal{K}_N(k) = \phi(gkg^{-1})$ since N is a normal subgroup. We conclude by observing that because N is non-trivial, ϕ is different from the canonical trace. \square

4. TWO CHARACTERIZATIONS OF C^* -SIMPLICITY

Let us introduce the definition of C^* -simplicity.

Definition 4.1. Let A be a C^* -algebra. A **quotient** of A is a C^* -algebra B that admits a surjective $*$ -homomorphism from A to B . A is said to be **simple** if its only (two-sided closed) ideals are $\{0\}$ and A , or equivalently, if any of its quotient maps is an isomorphism. A group G is called **C^* -simple** if its reduced C^* -algebra is simple.

In this section, we focus on Theorem B, which is the following:

Theorem. Let G be a discrete group. The following are equivalent:

- (1) G is C^* -simple;
- (2) G acts freely on its Furstenberg boundary $\partial_F G$, i.e., for any $g \in G \setminus \{e\}$ and $x \in \partial_F G$, one has $gx \neq x$;
- (3) There exists a topologically free G -boundary X , i.e., for any $g \in G \setminus \{e\}$, $\text{Fix}(g) := \{x \in X \mid gx = x\}$ has empty interior in X .

4.1 Proof of (2) \iff (3). Since (2) \Rightarrow (3) occurs by definition, we only need to show the inverse. Recall that a topological space is said to be *extremally disconnected* if the closure of any open set is open. Firstly, we need to show the following lemma:

Lemma 4.2. The Furstenberg boundary $\partial_F G$ is extremally disconnected.

Proof. To do so, we need the following facts about G -boundaries:

Let U be an open subset of $\partial_F G$. We want to show that \overline{U} is open. Let $x \in \partial_F G$. We consider $Y := (\{0\} \times U^c) \cup (\{1\} \times \overline{U})$, we call π the projection onto the second variable and we introduce the function $\Psi : G \rightarrow Y$ defined by

$$\Psi(g) = \begin{cases} (1, g \cdot x) & \text{if } g \cdot x \in U \\ (0, g \cdot x) & \text{otherwise} \end{cases}$$

Then Ψ is G -equivariant and extends to a continuous map $\beta G \rightarrow Y$ by definition of the Stone-Cech compactification βG of G , that is still equivariant. By Proposition 2.10, there is a G -equivariant map $f : \partial_F G \rightarrow \text{Prob}(\beta G)$. Let $x \in \partial_F G$, and $\mu_x = \phi \cdot f(x)$. Then $x \mapsto \pi \cdot \mu_x$ is a G -equivariant map from X to $\text{Prob}(Y)$. By Lemma 2.9, $\pi \cdot \mu_x = \delta_x$ so $\mu_x = (1 - t_x)\delta_{(0,x)} + t_x\delta_{(1,x)}$ for $t_x \in [0, 1]$. The map $g : x \mapsto \mu_x(\{0\} \times U^c)$ is lower semi-continuous, and so is $x \mapsto \mu_x(\{1\} \times \overline{U})$. So these maps are continuous. But if $x \in U$, then $g(x) = 0$, so $g(\overline{U}) = \{0\}$. Furthermore, $g(\overline{U}^c) = \{1\}$. So, by continuity, \overline{U} is open. \square

Now we return to the proof of (3) \Rightarrow (2) and first prove that G acts topologically freely on the Furstenberg boundary. By contradiction, let us suppose that there exists $g \in G \setminus \{e\}$ that has fixed points in $\partial_F G$ and assume that G acts topologically freely on a G -boundary X . By the universal property of the Furstenberg boundary, there is a continuous surjective G -equivariant map $\pi : \partial_F G \rightarrow X$. If $\text{Fix}_{\partial_F G}(g)$ has non-empty interior, by compactness and minimality, $\partial_F G$ is covered by finitely many translates of $\text{Fix}_X(g)$, so by the properties of π , X is covered by finitely many translates of $\pi(\text{Fix}_{\partial_F G}(g)) \subset \text{Fix}_X(g)$. But this is impossible since $\text{Fix}_X(g)$ has an empty interior. Hence G acts topologically freely on the Furstenberg boundary.

We can then conclude with the following fact and Lemma 4.2 that $\text{Fix}_{\partial_F G}(g) = \emptyset$.

Lemma 4.3. Let X be a compact extremally disconnected space, and $h : X \rightarrow X$ be an homeomorphism. Then $\text{Fix}(h)$ is open.

Proof. We follow a proof by [Pit12]. Let S be the partially ordered set of open h -simple subsets W of X , that is, $h(W) \cap W = \emptyset$. We endow the set of chains of S a structure of inductive set with the relation of inclusion. By Zorn's lemma, it admits a maximal \mathcal{C} element which is a maximal chain of elements of S . The union C of the elements of \mathcal{C} is a maximal h -simple, open set and its closure is also h -simple and closed-open by Lemma 4.2. So, by maximality, C is closed.

Then, let us pose $A = C \cup h(C) \cup h^{-1}(C)$. Hence, $A \cap \text{Fix}(h) = \emptyset$ and A are closed. We will show that $A^c = \text{Fix}(h)$. If not, there is $x \in A^c$ such that $h(x) \neq x$. We now take a closed-open neighborhood V of x such that $h(V) \cap V = \emptyset$ and $V \cap A = \emptyset$. This point means that $C \cup V$ is still h -simple, which is contradictory due to the maximality of C . Then $\text{Fix}(h) = A^c$ is open. \square

4.2 Proof of (2) \Rightarrow (1). Every separable unital C^* -algebra, in particular, any quotient of $C_\lambda^*(G)$, admits a faithful representation in a Hilbert space because it admits states, and we can apply the GNS construction 2.16. Hence, let $\pi : C_\lambda^*(G) \rightarrow B(\mathcal{H})$ be a non-trivial $*$ -representation. We need to show that π is injective. The $*$ -homomorphism π is completely positive, so by Arveson's extension theorem we can extend it to a map Φ from the reduced crossed-product $C(\partial_F G) \rtimes_r G$. By Proposition 2.18, $C_\lambda^*(G)$ is contained in the multiplicative domain of Φ , which implies that Φ is G -equivariant by the definition of the crossed-product, where the G -action on $B(\mathcal{H})$ is given by $\text{Ad}(\pi(\lambda_g))$ ($g \in G$).

Then, by G -essentiality the restriction $\Phi|_{C(\partial_F G)}$ is completely isometric, so it is an isomorphism onto its image. After applying G -injectivity to extend to the whole $\text{Im}(\Phi)$ the inverse of the restriction of $\Phi|_{C(\partial_F G)}$ to its image, we get a G -equivariant completely positive map Ψ from $\text{Im}(\Phi)$ to $C(\partial_F G)$, and $\Psi \circ \Phi : C(\partial_F G) \rtimes_r G \rightarrow C(\partial_F G)$ is a G -equivariant completely positive map.

Let $g \in G \setminus \{e\}$. We want to show that $\Psi(\Phi(\lambda_g)) = 0$. Let $x \in \partial_F G$. By assumption, G acts freely on $\partial_F G$. So $g \cdot x \neq x$ and there exists $f \in C(\partial_F G)$ such that $f(x) = 0$ and $f(g^{-1} \cdot x) \neq 0$. The map $\Psi \circ \Phi$ is the identity on $C(\partial_F G)$, and Proposition 2.18 ensures that

$$\Psi(\Phi(\lambda_g f))(x) = \Psi(\Phi(\lambda_g))f(x) = 0.$$

But on the other side, by the definition of the crossed-product

$$\Psi(\Phi(\lambda_g f))(x) = \Psi(\Phi((g \cdot f)\lambda_g)) = f(g^{-1} \cdot x)\Psi(\Phi(\lambda_g))$$

So $\Psi(\Phi(\lambda_g)) = 0$ and we fully know $\Psi \circ \Phi$ which is the canonical expectation $E : \sum_{h \in G} f_h \lambda_h \mapsto f_e$ which is faithful in the sense that:

$$\forall a \in C(\partial_F G) \rtimes_r G, E(a^* a) = 0 \Rightarrow a = 0,$$

Then Φ is also faithful, and so is π . Recall that π is a representation and therefore if x belongs to the kernel of π , $\pi(x^* x) = \pi(x)^* \pi(x) = 0$ and so that $x^* x = 0$ and $x = 0$. Hence, π is injective.

4.3 Proof of (1) \Rightarrow (3). We recall the following theorem, whose proof has been done in the preliminaries:

Theorem 4.4. *A group is C^* -simple if and only if any unitary representation weakly contained in λ is actually equivalent to λ .*

Let X be a G -boundary and $s \in G$ such that the interior U of the set $\{x \in X \mid sx = x\}$ of fixed points is non-empty. We will show that for any $x \in X$, the left regular representation λ_G is not weakly contained in the quasi-regular representation $\lambda_{G/G_x} : G \rightarrow B(l^2(G/G_x))$. We will then conclude by combining the following result and Theorem 2.27.

Theorem 4.5. *Assume that H is an amenable subgroup of a group G . Then $\lambda_{G/H}$ is weakly contained in $\lambda := \lambda_G$.*

Proof. Let $f \in l^2(G/H)$ be a finitely supported unit vector. Since H is amenable, there exist $(v_n)_{n \in \mathbb{N}} \in (l^2(H))^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}, \|v_n\| = 1$ and $\|\lambda(h)v_n - v_n\| \rightarrow 0$, that is, $\langle \lambda(h)v_n, v_n \rangle \rightarrow 1$, for all $h \in H$. Let $X \subset G$ be a subset such that $G = \bigsqcup_{x \in X} xH$. We then define $\tilde{f}_n \in l^2(G)$ by $\tilde{f}_n(xh) = f(xH)v_n(h)$ for $x \in X$ and $h \in H$. Finally, for any $g_0 \in G$ and $x \in X$, we denote $h_{g_0, x} \in H$ the unique element such that $g_0^{-1}xh_{g_0, x} \in X$. Therefore,

$$\begin{aligned} \langle \lambda(g_0)\tilde{f}_n, \tilde{f}_n \rangle &= \sum_{f(xH) \neq 0} \sum_{h \in H} \tilde{f}_n(g_0^{-1}xh)\tilde{f}_n(xh) \\ &= \sum_{f(xH) \neq 0} \sum_{h \in H} f(g_0^{-1}xH)v_n(h_{g_0, x}^{-1}h)f(xH)v_n(h) \\ &= \sum_{f(xH) \neq 0} f(g_0^{-1}xH)f(xH)\langle \lambda(h_{g_0, x})v_n, v_n \rangle. \end{aligned}$$

Note that the sum is finite and converges to

$$\sum_{f(xH) \neq 0} f(g_0^{-1}xH)f(xH) = \langle \lambda_{G/H}(g_0)f, f \rangle.$$

Thus, we have pointwise convergence of $g \mapsto \langle \lambda(g)\tilde{f}_n, \tilde{f}_n \rangle$ to $g \mapsto \langle \lambda_{G/H}(g)f, f \rangle$, which implies in particular that the approximation condition required in the definition of weak containment is fulfilled for the matrix coefficient associated to f . We conclude by density of finitely supported functions in $l^2(G/H)$. \square

By the proof of Theorem 2.12, we know that $G_x < G$ is an amenable subgroup. Hence λ_{G/G_x} is weakly contained in λ by Theorem 4.5. Fixing $\epsilon = \frac{1}{3}$ and $\xi = \delta_e$ the Dirac mass at the unit of G . For any finite subset $F \subset U$, there exist unit vectors $\xi_1, \dots, \xi_m \in l^2(G/G_x)$ and weights $p_1, \dots, p_m \in [0, 1]$ such that $p_1 + \dots + p_m = 1$ and, for all $g \in F$,

$$(*) \quad \left| \langle \lambda_G(g)\xi, \xi \rangle - \sum_{j=1}^m p_j \langle \lambda_{G/G_x}(g)\xi_j, \xi_j \rangle \right| < \epsilon.$$

Meanwhile, the following holds.

Lemma 4.6. *Let G be a non-trivial discrete group and let X be a G -boundary. For every non-empty open subset $U \subset X$ and $\epsilon > 0$, there is a finite subset $E \subset G \setminus \{e\}$ such that for every probability measure μ on X , there is $t \in E$ satisfying $\mu(tU) > 1 - \epsilon$.*

Proof. We fix an arbitrary $x \in U$. For each probability measure μ on X , on one hand, there exists by strong proximality $y \in G$ such that $\delta_y \in G \cdot \mu$. On the other hand, by minimality, $x \in \overline{G \cdot y}$. Hence, there is t_μ such that $1 - \mu(t_\mu^{-1}U) = (\delta_x - t_\mu \mu)(U) < \epsilon$, and we can assume $t_\mu \in G \setminus \{e\}$. Working with weak topology,

there is an open neighborhood V_μ of μ such that $\nu(t_\mu U) > 1 - \epsilon$ for all $\nu \in V_\mu$ (because if a sequence (μ_n) weakly converges to μ , $\liminf \mu_n(t_\mu U) \geq \mu(t_\mu U)$). By compactness, we thus obtain a finite cover $V_{\mu_1}, \dots, V_{\mu_n}$ and $E = \{t_{\mu_1}, \dots, t_{\mu_n}\}$ satisfies the required properties. \square

Let us take $F = \{tst^{-1} \mid t \in E\}$. Note that since $e \notin E$, $\langle \lambda_G(g)\xi, \xi \rangle = 0$ for all $g \in F$.

Now, let us consider the probability measures on X defined by

$$\mu_j := \sum_{y \in G \cdot x} |\xi_j(y)|^2 \delta_y \text{ and } \mu := \sum_{j=1}^m p_j \mu_j,$$

where we identify $G \cdot x \subset X$ with G/G_x . By construction, there is $t \in E$ such that $\mu(t(X \setminus U)) < \epsilon$. Therefore,

$$\sum_{j=1}^m p_j \sum_{y \in (X \setminus U) \cap t^{-1}G \cdot x} |\xi_j(ty)|^2 = \sum_{j=1}^m p_j \mu_j(t(X \setminus U)) = \mu(t(X \setminus U)) < \epsilon.$$

Hence, denoting $v_j := \lambda_{G/G_x}(t^{-1})\xi_j$ and using the fact that U is s -invariant,

$$\begin{aligned} \langle \lambda_{G/G_x}(s)v_j, v_j \rangle &= \sum_{y \in G \cdot x} v_j(s^{-1}y) \overline{v_j}(y) \\ &= \sum_{y \in U \cap G \cdot x} |v_j(y)|^2 + \sum_{y \in (X \setminus U) \cap G \cdot x} v_j(s^{-1}y) \overline{v_j}(y). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} &|1 - \langle \lambda_{G/G_x}(tst^{-1})\xi_j, \xi_j \rangle| \\ &= |1 - \langle \lambda_{G/G_x}(s)v_j, v_j \rangle| \\ &\leq \sum_{y \in (X \setminus U) \cap G \cdot x} |v_j(y)|^2 + \sum_{y \in (X \setminus U) \cap G \cdot x} \frac{|v_j(s^{-1}y)|^2 + |\overline{v_j}(y)|^2}{2} \\ &= 2 \sum_{y \in (X \setminus U) \cap G \cdot x} |v_j(y)|^2 \\ &= 2\mu_j(t(X \setminus U)). \end{aligned}$$

Averaging over j , we get

$$|1 - \sum_{j=1}^m p_j \langle \lambda_{G/G_x}(g)\xi_j, \xi_j \rangle| \leq 2\mu(t(X \setminus U)) < \frac{2}{3}.$$

This is in contradiction with the fact that

$$|\sum_{j=1}^m p_j \langle \lambda_{G/G_x}(g)\xi_j, \xi_j \rangle| = |\langle \lambda_G(g)\xi, \xi \rangle - \sum_{j=1}^m p_j \langle \lambda_{G/G_x}(g)\xi_j, \xi_j \rangle| < \frac{1}{3},$$

which is obtained by applying (*) to $g = tst^{-1}$.

5. APPLICATION: $C_\lambda^*(F_2)$ IS SIMPLE

Theorem B is a powerful tool for showing that a group is C^* -simple. In this section, we focus on proving that the free group with 2 generators F_2 is C^* -simple by constructing a topologically free action of F_2 on a boundary. We denote by a and b the two generators of F_2 .

Recall that a reduced word on an alphabet is a finite sequence of letters such that a letter never succeeds its inverse. This definition, of course, generalizes to infinite sequences.

Definition 5.1 (Gromov boundary of $\text{Cay}(F_2)$). *The **Gromov boundary** ∂F_2 of the (Cayley graph of the) free group with two generators is the set of infinite reduced sequences of a, b, a^{-1} and b^{-1} .*

The relation consists in identifying two sequences which coincide at infinity. Intuitively, it represents the directions of the infinite geodesics of the Cayley graph of F_2 .

This is in fact a special case of a more general object, where we consider geodesics in graphs or trees, but here we can describe them properly with sequences of letters. For more information on this topic, you can consult [BO08] (Sections 5.2 and 5.3 for the general definitions).

The Gromov boundary is naturally equipped with an action of F_2 by left multiplication, which is continuous with the topology induced by the product topology on $\{a, b, a^{-1}, b^{-1}\}^{\mathbb{N}}$.

Proposition 5.2. *The Gromov boundary ∂F_2 is a F_2 -boundary*

Proof. We have three points to show.

Compactness: ∂F_2 is closed in the compact space $\{a, b, a^{-1}, b^{-1}\}^{\mathbb{N}}$ because we can control the convergence of pairs of letters.

Minimality: Let $u \in \partial F_2$. We want to show that its orbit is dense. Let $v \in \partial F_2$. By symmetry of the problem we can assume that v contains an infinity of a , at positions $i_0 < i_1 < \dots < i_k \dots$, and even if it means multiplying by b or b^{-1} , we suppose that u begins with one of these letters. For all $k \geq 0$, we pose $g_k = v_0 \dots v_{i_k}$. Then $g_k \cdot u$ converges to v .

Strong-proximality: Let $\mu \in \text{Prob}(\partial F_2)$, $u \in \partial F_2$. Denote by a^+, b^+, a^-, b^- the constant sequences. Then there exists $0 \leq c \leq 1$ such that $a^n \cdot \mu \rightarrow (1-c)\delta_{a^+} + c\delta_{a^-}$ (c is the probability of $\{a^-\}$). Indeed, it can be proved by taking an open subset O of the boundary and intersecting it with the set A_m of reduced words which do not have a^m as a prefix. When we multiply the measure by a^n we arrive at A_{n-m} and we can conclude by additivity of μ that

$$\liminf_n a^n \cdot \mu(O) \geq (1-c)\delta_{a^+}(O) + c\delta_{a^-}(O).$$

We can suppose that c is not 1) and thus $b^m a^n \cdot \mu \xrightarrow{n \rightarrow \infty} (1-c)\delta_{b^{-m}a^+} + c\delta_{b^{-m}a^-}$ which tends to δ_{b^-} when m goes to infinity. □

Corollary 5.3. *The group F_2 is C^* -simple.*

Proof. For a reduced word $g = g_0 \dots g_n \in F_2 \setminus \{e\}$ and $u = (u_k)_{k \geq 0} \in \partial F_2$ such that $g \cdot u = u$, we can take i the smallest integer such that $g_{i+1} \neq g_{n-(i+1)}$. Because the product $g \neq e$ is reduced, i exists and $i < (n+1)/2$. Then we can write $g = hg'h^{-1}$ where $h = g_0 \dots g_i$ and $g' \neq e$. Two cases can happen: if h is not a prefix of u , then we can write $h^{-1}u = g_{n-i}v$ as a reduced word and hence for $n \geq 0$, $g^n \cdot u = hg'^n g_{n-i+1}v \xrightarrow{n \rightarrow \infty} hg'^+$, so $u = hg'^+$. If $u = hv$, $u = g^n \cdot u = hg'^n v$ which tends to hg'^+ if and only if $v \neq g'^-$. Finally, $\text{Fix}(g) = \{hg'^+; hg'^-\}$ and has an empty interior. We can therefore conclude with the theorem. □

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ÉCOLE NORMALE SUPÉRIEURE, DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, 45 RUE
D’ULM, 75230 PARIS CEDEX 05, FRANCE

Email address: leo.raclot@ens.psl.eu, tistou.noret@ens.psl.eu