Mémoire Math L3

Exact controllability of Euler Equations on a Smooth Domain in $\mathbb{R}^2$

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29.05.2025

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1 Introduction

The goal of this mémoire is to give a more thorough and self-contained version of the argument in *"Controlabilite exacte frontiere de l'equation d'Euler des fluides parfaits incompressibles bidimensionnels"* published in 1993 by Jean-Michel Coron [2]. This paper proves that the Euler equations are exactly controllable on a simply connected and smooth domain. We will give a short introduction to the underlying discipline of control theory to motivate main ideas. Then, we state the problem and present its proof as shown by J.M.Coron. Doing so, we expand on some statements used in the proof, which were not directly proven in the paper, as well as elaborate the omitted constructions.

1.1 Introduciton to control theory

The main question of control theory could be stated like this: can one steer a dynamical system $\dot{x}(t) = f\big(x(t), u(t)\big)$ from a state $x(T_1) = Y_1$ to a state $x(T_2) = Y_2$ ($T_2 > T_1$) by choosing a suitable control $u(t)$. If the control system is finite-dimensional and linear, there is a necessary and sufficient condition for controllability – the Kalman rank condition.

For a non-linear $f(x, u)$, no simple uniform criterion for controllability is known. One approach that yields a sufficient condition for controllability is the return method. We look for a non-constant trajectory, that starts and ends at equilibrium $f(x, u) = 0$. If the linearized control system around this trajectory is controllable, the implicit function theorem yields that one can find a trajectory from every state close to the equilibrium to every other state close to the equilibrium.(The exact theory behind the mentioned methods can be found in [1] Chapters 1,3 and 6.)

The reviewed paper makes use of the return method to prove the case of exact controllability of Euler equations.

1.2 The Euler equations on a smooth domain

The focus of the paper are the (incompressible) Euler equations, which can be stated as follows:

$$\partial_t \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = 0, \tag{1.1}$$

$$\operatorname{div} \mathbf{y} = 0 \tag{1.2}$$

The Euler equations are a system of non-linear partial differential equations and arise in fluid dynamics as a special case of the Navier-Stokes equations for a non-compressible and perfect (friction/viscosity-free) liquid. The vector field $\mathbf{y}(x, t)$ corresponds to the flow of liquid at a certain point in space. Equation (1.1) describes how the change in flow velocity at a point is determined by the pressure gradient ∇p acting as an accelerating force and by the influx of faster flowing liquid from neighbouring points, represented by $(\mathbf{y} \cdot \nabla)\mathbf{y}$. Equation (1.2) corresponds to the condition of incompressibility.

In our case, $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary $\partial\Omega$. The boundary condition is

$$\mathbf{y} \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega \tag{1.3}$$

where $\mathbf{n}$ denotes the unit normal on $\partial\Omega$ in outward direction. This is known as the free-slip condition. The physical interpretation is that, while no liquid can pass through the boundary, it can still glide along the wall without constraints, since we assume no friction.

1.3 Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be a nonempty bounded open set of class C^∞, which we assume to be simply connected. Let $\Gamma_0 \subset \partial\Omega = \Gamma$ be a subset with nonempty interior in Γ. Denote by $\mathbf{n}$ the unit outward normal on Γ. Let $T > 0$ and $\mathbf{y}_0, \mathbf{y}_1 \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfy

$$\operatorname{div} \mathbf{y}_0 = 0 \quad \text{in } \Omega, \tag{1.4}$$

$$\operatorname{div} \mathbf{y}_1 = 0 \quad \text{in } \Omega, \tag{1.5}$$

$$\mathbf{y}_0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \setminus \Gamma_0, \tag{1.6}$$

$$\mathbf{y}_1 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \setminus \Gamma_0, \tag{1.7}$$

The question is whether there exists such $(\mathbf{y}, p) \in C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2 \times \mathbb{R})$ that

$$\partial_t \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = 0, \qquad \text{in } \Omega \times (0,T), \tag{1.8}$$

$$\operatorname{div} \mathbf{y} = 0, \qquad \text{in } \Omega \times (0,T), \tag{1.9}$$

$$\mathbf{y} \cdot \mathbf{n} = 0, \qquad \text{on } (\Gamma \setminus \Gamma_0) \times (0,T), \tag{1.10}$$

$$\mathbf{y}(\cdot, 0) = \mathbf{y}_0, \qquad \text{in } \Omega, \tag{1.11}$$

$$\mathbf{y}(\cdot, T) = \mathbf{y}_1, \qquad \text{in } \Omega \tag{1.12}$$

If such $(\mathbf{y}, p)$ exists for any choice of $T > 0$ and $\mathbf{y}_0, \mathbf{y}_1 \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$, we say that the Euler equations are *exactly controllable* for (Ω, Γ_0).

The main result of the paper is the following:

Theorem 1. *The Euler equations are exactly controllable for* (Ω, Γ_0).

1.4 Outline of the proof

The proof by J. M. Coron uses the idea of the return method to reduce the problem of exact controllability to the existence of the trajectory going smoothly to the zero state.

The first main step is the extension of the domain and the construction of a vector field θ, the flow of which will enter and leave the extended domain.

This θ, in turn, allows the construction of the trajectory $(\bar{\mathbf{y}}, p)$ transitioning smoothly to the zero state with a controllable linearized system. This is the main motivation for the proof of controllability along the trajectory itself.

The last crucial step is the formulation of the system of equations for the evolution of curl for with fixed normal component of the flow velocity on $\partial\Omega$ and initial value of curl. The solution uses the idea of first solving for velocity flow from the given curl and later solving solving for curl from the velocity flow, which constructs a map to which we will apply the Schauder's fixed point theorem.

The obtained solution will verify that any sufficiently small perturbation of it smoothly goes into the zero state, which will prove the main result.

2 Proof of the statement

2.1 Standard definitions

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $k \in \mathbb{N}_0$. The* Sobolev space

$$H^k(\Omega) \ := \ \Big\{ f \in L^2(\Omega) \ \Big| \ D^\beta f \in L^2(\Omega) \text{ for every multi-index } \beta \text{ with } |\beta| \leq k \Big\}$$

is endowed with the norm

$$\|f\|_{H^k(\Omega)} \ := \ \Big(\sum_{|\beta| \leq k} \|D^\beta f\|^2_{L^2(\Omega)} \Big)^{1/2}.$$

Definition 2.2. *For natural m and $0 < \alpha \leq 1$ define*

$$\|f\|_{C^m(\overline{\Omega})} = \sum_{|\beta| \leq m} \|D^\beta f\|_{C^0(\overline{\Omega})} \ \text{ with } \ \|.\|_{C^0(\overline{\Omega})} \ \text{ denoting the supremum norm.}$$

$$C^\alpha(\overline{\Omega}) \ := \ \Big\{ f \in C^0(\overline{\Omega}) \ \Big| \ [f]_{C^\alpha(\overline{\Omega})} < \infty \Big\}, \qquad [f]_{C^\alpha(\overline{\Omega})} \ := \ \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

The norm is $\|f\|_{C^\alpha(\overline{\Omega})} := \|f\|_{C^0(\overline{\Omega})} + [f]_{C^\alpha(\overline{\Omega})}$. For integers $m \geq 0$ one sets $C^{m+\alpha}(\overline{\Omega}) := \{f \in C^m(\overline{\Omega}) \mid D^\beta f \in C^\alpha(\overline{\Omega}) \ \forall |\beta| = m\}$ and $\|f\|_{C^{m+\alpha}(\overline{\Omega})} = \|f\|_{C^m(\overline{\Omega})} + \sum_{|\beta|=m} \|D^\beta f\|_{C^\alpha(\overline{\Omega})}$.

Definition 2.3. *Let $\partial\Omega$ carry the outward unit normal vector $\mathbf{n}$.*
(Inhomogeneous) Dirichlet condition:

$$u = g \quad \text{on } \partial\Omega.$$

For $g \equiv 0$ one speaks of the homogeneous Dirichlet condition*.*
(Inhomogeneous) Neumann condition:

$$\frac{\partial u}{\partial \mathbf{n}} := \nabla u \cdot \mathbf{n} \ = \ g \quad \text{on } \partial\Omega.$$

Setting $g \equiv 0$ yields the homogeneous Neumann (van Neumann) condition*.*

2.2 Reduction of the problem from global to local controllability

Firstly, by the following scaling argument, we are able to reduce the problem of exact global controllability to one of local controllability, which means that we are able to obtain Theorem 1 as a corollary of the following theorem:

Theorem 2. *There exists a real number $\nu > 0$, such that for all $\mathbf{y}_0 \in C^\infty(\overline{\Omega};\mathbb{R}^2)$ verifying* (1.4), (1.6) *and* $\|\mathbf{y}_0\|_{C^1} < \nu$, *there exist* $\mathbf{y} \in C^\infty(\overline{\Omega}\times[0,1];\mathbb{R}^2)$ *and* $p \in C^\infty(\overline{\Omega}\times[0,1];\mathbb{R})$, *which satisfy* (1.8) *to* (1.12)*with* $(T,\mathbf{y}_1) = (1,0)$ *and satisfying:*

$$\frac{\partial^i \mathbf{y}}{\partial t^i}(\cdot,1) = 0 \quad \frac{\partial^i p}{\partial t^i}(\cdot,1) = 0 \quad \text{in } \overline{\Omega} \quad \forall i \in \mathbb{N} \tag{2.1}$$

Proposition 2.4. *Theorem 2* $\implies$ *Theorem 1*

Proof. We assume $T > 0$ and $\mathbf{y}_0, \mathbf{y}_1 \in C^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying (1.4) to (1.7). For an $\epsilon \in (0, T/2)$ small enough, we have $\|\epsilon\ \mathbf{y}_0\|_{C^1} < \nu$ and $\|-\epsilon\ \mathbf{y}_1\|_{C^1} < \nu$ and thus by Theorem 2 there are $\mathbf{y}'$ and $\mathbf{y}''$ in $C^\infty(\overline{\Omega}\times[0,1];\mathbb{R}^2)$ satisfying $\mathbf{y}'(\cdot,0) = \epsilon\ \mathbf{y}_0$ and $\mathbf{y}''(\cdot,0) = -\epsilon\ \mathbf{y}_0$, p' and p'' in $C^\infty(\overline{\Omega}\times[0,1];\mathbb{R})$ such that (1.8) to (1.10) and (2.1) are satisfied when $T = 1$ and $(y',p') = (y,p) = (y'',p'')$. We then choose an ϵ and define $\mathbf{y} : \overline{\Omega}\times[0,T] \mapsto \mathbb{R}^2; p : \overline{\Omega}\times[0,T] \mapsto \mathbb{R}$ by:

$$\mathbf{y}(x,t) = \epsilon^{-1}\mathbf{y}'(x,\epsilon^{-1}t), \quad p(x,t) = \epsilon^{-2}p'(x,\epsilon^{-1}t), \qquad \text{if } (x,t) \in \overline{\Omega}\times[0,\epsilon], \tag{2.2}$$

$$\mathbf{y}(x,t) = 0, \quad p(x,t) = 0, \qquad \text{if } (x,t) \in \overline{\Omega}\times(\epsilon, T-\epsilon), \tag{2.3}$$

$$\mathbf{y}(x,t) = \epsilon^{-1}\mathbf{y}''(x,\epsilon^{-1}(T-t)), \qquad \text{if } (x,t) \in \overline{\Omega}\times[T-\epsilon, T] \tag{2.4}$$

$$p(x,t) = \epsilon^{-2}p''(x,\epsilon^{-1}(T-t)), \qquad \text{if } (x,t) \in \overline{\Omega}\times[T-\epsilon, T] \tag{2.5}$$

From there we can see that $\mathbf{y}$ and p satisfy (1.8) to (1.12) and are of class C^∞:

Equation (1.8) holds on $\overline{\Omega}\times[\epsilon, T-\epsilon]$ trivially and for $\overline{\Omega}\times[0,\epsilon]$ and $\overline{\Omega}\times[\epsilon, T-\epsilon]$, which follows from $\mathbf{y}', \mathbf{y}''$ fulfilling (1.8) and by transforming in time like: $t \mapsto \epsilon t \implies \partial_t \mapsto \epsilon^{-1}\partial_t$ and $t \mapsto (T - \epsilon t) \implies \partial_t \mapsto -\epsilon^{-1}\partial_t$ respectively.
Equations (1.9) and (1.10) hold for $\mathbf{y}$ since they hold for $\mathbf{y}', \mathbf{y}''$ and obviously for $\mathbf{y} = 0$. Since $\forall x \in \overline{\Omega}$ we have $\mathbf{y}(x,0) = \epsilon^{-1}\mathbf{y}'(x,\epsilon^{-1}0) = \epsilon^{-1}\epsilon\ \mathbf{y}_0 = \mathbf{y}_0$ and $\mathbf{y}(x,1) = \epsilon^{-1}\mathbf{y}''(x,\epsilon^{-1}(1-1)) = \epsilon^{-1}\epsilon\ \mathbf{y}_1 = \mathbf{y}_1$, (1.11) and (1.12) are also fulfilled.
Lastly, for a fixed ϵ and a $\delta \le \epsilon$, when looking at $\mathbf{y}(\cdot,\delta)$ and $\mathbf{y}(\cdot, T-\delta)$ for $\delta \to \epsilon$ we get $\mathbf{y}(\cdot,T-\delta) \to \mathbf{y}'(\cdot,1)$ and $\mathbf{y}(\cdot,T-\delta) \to \mathbf{y}''(\cdot,1)$ and thus by (2.1), that we have a smooth transition to $\mathbf{y} = 0$ from the left and right because $\mathbf{y}'$ and $\mathbf{y}''$ are of class C^∞. Hence, $\mathbf{y}$ is of class C^∞. □

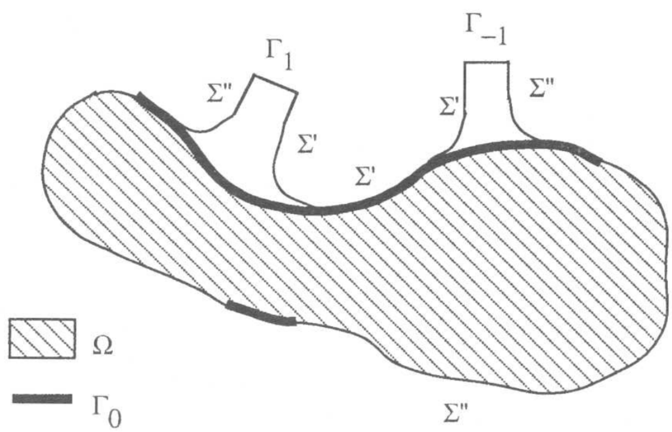
In the following, we prove Theorem 2 using the return method. In the context of our problem, this means we will construct a $\bar{\mathbf{y}}$ in $C^\infty(\bar{\Omega}\times[0,T];\mathbb{R}^2)$ and a $\bar{p}$ in $C^\infty(\bar{\Omega}\times[0,T];\mathbb{R})$ such that, if $\mathbf{y} = \bar{\mathbf{y}}$, $p = \bar{p}$, $\mathbf{y}_0 = 0$, $\mathbf{y}_1 = 0$ and $T = 1$, the equations (1.8) to (1.12) are satisfied and the linearized control system around $(\bar{\mathbf{y}}, \bar{p})$ is locally controllable. In fact, the fact that the linearized system is exactly controllable is not used in the argument. Nevertheless, the fact that this trajectory has an exactly controllable linearized system is the primary reason why such a construction works for our purpose.

2.3 Construction of the extended domain

We take Ω_1 to be an open bounded simply connected subset of $\mathbb{R}^2$ with its boundary being Lipschitz and composed of two disjoint straight segments Γ_{-1} and Γ_1 and of two closed and disjoint curves Σ' and Σ'' of class C^∞, connected at their boundary, such that $\partial\Sigma' \cup \partial\Sigma'' = \partial\Gamma_{-1} \cup \partial\Gamma_1$. We also assume the existence of a neighborhood of Γ_{-1} (resp. Γ_1) such that the intersection of Ω_1 with this neighborhood is the intersection of this neighborhood and the open half-plane limited by Γ_{-1} (resp. Γ_1) and two straight lines orthogonal to Γ_{-1} (resp. Γ_1) and passing through $\partial\Gamma_{-1}$ (resp. $\partial\Gamma_1$).

Lemma 2.5. *For Ω and Γ_0 one can to construct a Ω_1, with the following additional conditions:*

$$\Omega \subset \Omega_1 \tag{2.6}$$
$$(\partial\Omega/\Gamma_0) \subset \Sigma := \Sigma' \cup \Sigma'' \tag{2.7}$$



Proof. We start by taking a point $a \in \Gamma_0$. We can take a rectangular neighborhood V of a such that if we take the coordinate axes parallel to the sides of V, $\partial\Omega \cap V$ is a graph of some function $f : I \to J$ ($V = I \times J$) in these coordinates and $\Omega \cap V$ is the region below the graph of f. We can now draw a straight segment inside V parallel to the x-axis, which will be a graph of a constant function c on some subinterval I' of I, and which will be strictly above the graph of f. We can now stitch f and c smoothly by taking

$$g(x) = f(x) + \rho(x)(c - f(x)),$$

where $\rho : I \to [0, 1]$ is a smooth function, which is zero outside I' and a constant one on a subinterval I'' of I'.

The function g obtained is constant on I'', so we can replace its graph on I'' by two curves in the region of V above the graph of g on I'', each of these curves smoothly transitioning from a horizontal line into a vertical line, then right angle with a horizontal line, then another right angle with a vertical line and transitioning smoothly into a horizontal line again. This yields us a curve with two rectangular regions bounding an Ω_1 that contains Ω. □

2.4 Construction of a nonstationary trajectory

We now construct a $\bar{\mathbf{y}}$ with a locally controllable linearized system. For this, we will need to construct a flow that passes through our domain Ω_1. We reduce the problem of the construction of this θ to two elliptic problem in a smooth domains.

2.4.1 Construction of θ

Proposition 2.6. *There exists $\theta \in C^\infty\big(\overline{\Omega_1}\big)$ such that*

$$\Delta\theta = 0 \quad \text{in } \Omega_1, \qquad \theta = \pm 1 \text{ on } \Gamma_{\pm 1} \quad \text{and} \quad \partial_n\theta = 0 \text{ on } \Sigma \setminus \partial\Sigma. \tag{2.8}$$

We prove the solution exists in two steps: first, that it exists in a weak sense, and second, that it is smooth on $\overline{\Omega_1}$.
Take R_1 and R_{-1} to be rectangles in $\overline{\Omega_1}$ containing Γ_1 and Γ_{-1} respectively, each having exactly one side in the interior of Ω_1 and the other three on the boundary.
Take Γ_1 as the x-axis and define

$$\psi\colon R_1 \longrightarrow [0,1]$$

as $\psi(x,y) = \rho(y)$ with $\rho = 1$ in some neighborhood of 0 and supported in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Similarly define ψ on R_{-1}.
Extend this ψ smoothly to $\overline{\Omega_1}$ as 0 outside $R_1 \cup R_{-1}$.

Thus, by considering $\varphi := \theta - \psi$, the problem (2.8) is reduced to finding such a $\varphi \in C^\infty(\overline{\Omega_1})$ that

$$\Delta\varphi = f \ \text{ in } \Omega_1, \quad \varphi = 0 \ \text{ on } \Gamma_1 \cup \Gamma_{-1}, \quad \partial_n\varphi = 0 \ \text{ on } \Sigma \setminus \partial\Sigma, \tag{2.9}$$

with $f := -\Delta\psi$.

Lemma 2.7. *For a $\varphi \in H^2\big(\Omega_1\big)$, (2.9) is equivalent to*

$$\int_{\Omega_1} \nabla\varphi \cdot \nabla v = -\int_{\Omega_1} fv \quad \forall v \in H^1(\Omega_1) \text{ such that } v\,|_{\Gamma_1 \cup \Gamma_{-1}} = 0. \tag{2.10}$$

Proof. "$\Rightarrow$": Take $v \in H^1(\Omega_1)$ with $v\,|_{\Gamma_1 \cup \Gamma_{-1}} = 0$.
Then

$$\int_{\Omega_1} \nabla\varphi \cdot \nabla v = -\int_{\Omega_1} \Delta\varphi v + \int_{\partial\Omega_1} v\partial_n\varphi = -\int_{\Omega_1} fv \tag{2.11}$$

"$\Leftarrow$": For any test function $\psi \in C_c^\infty(\Omega_1)$,

$$\int_{\Omega_1} f\psi = -\int_{\Omega_1} \nabla\varphi \cdot \nabla\psi = \int_{\Omega_1} \psi\Delta\varphi,$$

hence $\Delta\varphi = f$. Using (2.11),

$$\int_{\Sigma} v\partial_n\varphi = 0 \quad \forall v \in H^1(\Omega_1) \text{ such that } v\,|_{\Gamma_1 \cup \Gamma_{-1}} = 0,$$

so $\partial_n\varphi = 0$ on Σ. □ □

Definition 2.8. *A $\varphi \in H^1(\Omega_1)$ is called a weak solution of (2.9) if (2.10) holds for φ.*

Lemma 2.9. *For any $f \in L^2(\Omega_1)$, (2.9) has a weak solution $\varphi \in H^1(\Omega_1)$.*

Proof. We first define

$$V = \left\{ v \in H^1(\Omega_1) \mid v \,|_{\Gamma_1 \cup \Gamma_{-1}} = 0 \right\}$$

and prove that it is strongly and weakly closed in $H^1(\Omega_1)$. Strong closedness follows from the bound

$$\|v\|_{L^2(\Gamma_1 \cup \Gamma_{-1})} \le \|v\|_{L^2(\partial \Omega_1)} \le C \|v\|_{H^1(\Omega_1)},$$

which follows from the usual trace theory, since Ω_1 has Lipschitz boundary. Since V is convex, it is also weakly closed.

The main part of the proof is *Poincaré's inequality* for V that we state below:

$$\exists C > 0 \text{ such that } \|v\|_{L^2} \le C \|\nabla v\|_{L^2} \quad \forall v \in V.$$

We prove it by contradiction: assuming the contrary, there exists a sequence $\{v_n\}$ such that $\|v_n\|_{L^2} = 1$, $\|\nabla v_n\|_{L^2} \to 0$ and $v_n \in V$. By Banach–Alaoglu, we can extract a subsequence $\{v_{n_k}\}$ which converges weakly in $H^1(\Omega_1)$ and strongly in $L^2(\Omega_1)$ by Rellich's theorem. The two limits correspond to the same $u \in H^1(\Omega_1)$, which satisfies

$$\|\nabla u\|_{L^2} = 0, \quad \|u\|_{L^2} = 1, \quad u \,|_{\Gamma_1 \cup \Gamma_{-1}} = 0$$

by weak and strong closedness of V. $\|\nabla u\|_{L^2} = 0$ implies u is constant, and $u = 0$ on $\Gamma_1 \cup \Gamma_{-1}$ implies u has to be identically zero, contradicting $\|u\|_{L^2} = 1$. We can now prove the statement of the lemma.

$$B(v_1, v_2) := \int_{\Omega_1} \nabla v_1 \cdot \nabla v_2$$

defines a bounded and coercive (by Poincaré's inequality) bilinear form and

$$L(v) = -\int f v$$

is a bounded linear functional on the Hilbert space V. By the Lax–Milgram theorem we get that there exists a $\varphi \in V$ such that

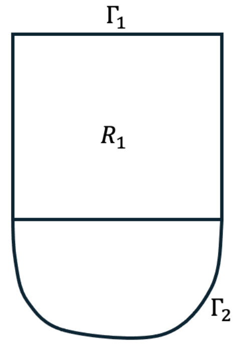
$$B(\varphi, v) = L(v) \quad \forall v \in V,$$

meaning φ is the wanted weak solution. □

Having obtained a weak solution to our system by Lemma 2.9, we want to show that it must be smooth. For this, we set up a bootstrap argument to obtain elliptic regularity for more familiar settings. The same ψ we constructed above to be identically one in a small neighborhood of $\Gamma_1 \cup \Gamma_{-1}$ and identically zero outside $R_1 \cup R_{-1}$ works to write φ as a sum of functions $\psi\varphi$ and $(1-\psi)\varphi$, one with support in small rectangular neighborhoods of Γ_1 and Γ_{-1} and the other with support outside some small neighborhoods of Γ_1 and Γ_{-1}.

Lemma 2.10. *If φ is a weak solution of* (2.9) *with smooth f, then $\varphi \in H^k(\Omega_1)$ implies $\psi\varphi \in H^{k+1}(\Omega_1)$.*

Figure 1: Construction of Γ_2



Proof. We will turn the problem for $\psi\varphi$ into a Poisson equation on a smooth open set with Neumann boundary condition.

For this, first restrict $\psi\varphi$ to a function φ_1 on R_1 and smoothly join two sides of R_1 to a smooth curve $\Gamma_2 \subset \Omega_1$ as in Figure 1 below.

Denote the interior bounded by Γ_1, Γ_2 and two sides of R_1 as Ω'. We extend φ_1 to $\overline{\Omega' \setminus R_1}$ by 0. Denote reflection across Γ_1 as r. Extend φ_1 to $r(\overline{\Omega'})$ by setting $\varphi_1(r(x)) = -\varphi_1(x)$ for all $x \in \Omega'$.

Now we can see that $\varphi_1 \in H^1(\Omega'')$, where Ω'' is the interior of $\overline{\Omega'} \cup \Gamma_1 \cup r(\overline{\Omega'})$. Indeed, for any i, we can express the integral

$$\int_{\Omega''} \varphi_1 \partial_i u \quad \text{for } u \in C_c^\infty(\Omega'')$$

as the sum of integrals over Ω' and $r(\Omega')$ and apply Green's theorem to each of them, using $\varphi_1 = 0$ on Γ_1, since we obviously have $\varphi_1 \in H^1(\Omega')$ and $\varphi_1 \in H^1(r(\Omega'))$. This yields $\partial_i \varphi_1 \in L^2(\Omega'')$.

Now, for any $u \in C^\infty(\overline{\Omega''})$,

$$\int_{\Omega''} \nabla\varphi_1 \cdot \nabla u = \frac{1}{2}\int_{\Omega''} \nabla\varphi_1 \cdot \nabla(u - \hat{u}) = \int_{\Omega'} \nabla\varphi_1 \cdot \nabla(u - \hat{u}) = -\int_{\Omega'} g(u - \hat{u}) = -\int_{\Omega''} gu, \quad (2.12)$$

where $\hat{u}(x) := u(r(x)) \quad \forall x \in \Omega''$ and g is defined as $\varphi\Delta\psi + 2\nabla\varphi \cdot \nabla\psi + \psi f$ on R_1 and extended to $\overline{\Omega''}$ the same way as φ_1. Here, we used that $\varphi_1, u - \hat{u}$ and g are odd across Γ_1, and the third equality follows by writing $\varphi_1 = \psi\varphi$, applying (2.10) and Green's theorem in Ω_1 repeatedly, knowing that $\partial_n \psi = 0$ on $\partial\Omega''$ and $\varphi = 0$ on Γ_1.

Now (2.12) can be understood as the determining relation for a weak solution φ_1 of $\Delta\varphi_1 = g$ in Ω'' with Neumann boundary condition. Besides, we have $g \in H^{k-1}(\Omega'')$ by assumption on φ. To see this, it is enough to notice that g is identically zero in a neighborhood of Γ_1.

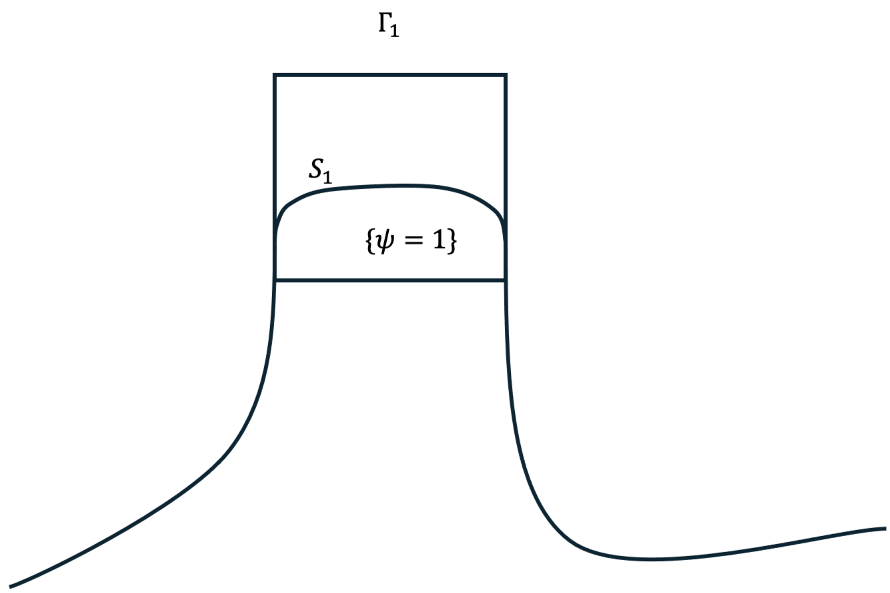
Hence, since $\partial\Omega''$ is smooth and $g \in H^{k-1}(\Omega'')$, elliptic regularity [[5]; p. 217, Theorem 4] ensues $\varphi_1 \in H^{k+1}(\Omega'')$ and hence $\psi\varphi \in H^{k+1}(R_1)$. Similarly, $\psi\varphi \in H^{k+1}(R_{-1})$ and $\psi\varphi \in H^{k+1}(\overline{\Omega_1})$. □

Lemma 2.11. *If φ is a weak solution of* (2.9) *with smooth f, then $\varphi \in H^k(\Omega_1)$ implies* $(1-\psi)\varphi \in H^{k+1}(\Omega_1)$.

Proof. We again turn the problem for $\psi\varphi$ into a Poisson equation on a smooth open set with Neumann boundary condition.

$\psi = 1$ in some neighborhood of Γ_1, hence we can smoothly join a part of Σ to a smooth curve S_1 in the region where $\psi = 1$ as in Figure 2 below.

Figure 2: Construction of S_1



Similarly, we join a portion of the resulting curve to S_{-1} near Γ_{-1}, and call the open set bounded by this resulting curve Ω'''. We now obtain that $\varphi_2 := (1-\psi)\varphi$ is a weak solution of $\Delta\psi_2 = h$ on Ω''' with Neumann boundary condition, in the sense that

$$\int_{\Omega'''} \nabla\varphi_2 \cdot \nabla u = -\int_{\Omega'''} hu \quad \forall u \in C^\infty(\overline{\Omega'''}), \tag{2.13}$$

where $h := \Delta(1-\psi)\varphi + 2\nabla(1-\psi)\cdot\nabla\varphi - (1-\psi)\Delta\varphi$.

Relation (2.13) can be obtained as in the proof of Lemma 2.10 by using (2.10) and Green's theorem repeatedly, along with the fact that $\partial_n\psi = 0$ on $\partial\Omega'''$.

Since $h \in H^{k-1}(\Omega''')$ and $\partial\Omega'''$ is smooth, we have $\varphi_2 \in H^{k+1}(\Omega''')$ by elliptic regularity again [[5]; p. 217, Theorem 4]. Hence, $(1-\psi)\varphi \in H^{k+1}(\Omega_1)$. □

Proof of Proposition 2.6. By Lemma 2.9, we obtain a weak solution $\varphi \in H^1(\Omega_1)$ of (2.9) with $f := -\Delta\psi$. By Lemma 2.10 and Lemma 2.11, $\varphi \in H^k(\Omega_1)$ implies

$$\varphi = \psi\varphi + (1-\psi)\varphi \in H^{k+1}(\Omega_1).$$

Hence $\varphi \in H^k(\Omega_1)$ for all k and therefore,

$$\varphi \in C^\infty\big(\overline{\Omega_1} \setminus \partial\Sigma\big)$$

by Sobolev injection.

In fact, all derivatives of φ extend continuously to $\partial\Sigma$ by the same argument of extension by symmetry as in Lemma 2.10. Hence, $\varphi \in C^\infty(\overline{\Omega_1})$ and $\theta = \varphi + \psi \in C^\infty(\overline{\Omega_1})$ satisfies (2.8).

□

2.4.2 Construction of $\bar{\mathbf{y}}$

We now construct the good trajectory $\bar{\mathbf{y}}$.

Let $\gamma \in C^\infty\big([0,1];[0,+\infty[\big)$ be an application not identical to zero with a support included in $]0,\frac{1}{2}[$ and let M be a positive real number.

One defines $\bar{\mathbf{y}} \in C^\infty\big(\overline{\Omega} \times [0,1]; \mathbb{R}^2\big)$ by

$$\bar{\mathbf{y}}(x,t) = M\gamma(t)\nabla\theta(x) \tag{2.14}$$

and $\bar{p} \in C^\infty\big(\overline{\Omega} \times [0,1]; \mathbb{R}\big)$ by

$$\bar{p}(x,t) = -M\gamma'(t)\theta(x) - \frac{M^2}{2}\gamma(t)^2|\nabla\theta(x)|^2, \tag{2.15}$$

then (1.8) to (1.12) and (2.1) are fulfilled by $p = \bar{p}$, $y = \bar{\mathbf{y}}$, $T = 1$, $\mathbf{y}_0 = \mathbf{y}_1 = 0$.

Let Ω_3 be an open and bounded set of class C^∞, which contains $\overline{\Omega}_1$; we extend θ on Ω_3 as a function of class C^∞, with compact support in Ω_3 ; We call this extension θ as well. Let

$$\bar{\phi} \colon \overline{\Omega}_3 \times [0,1] \to \overline{\Omega}_3$$

be defined by

$$\frac{\partial\bar{\phi}}{\partial t} = M\gamma(t)\nabla\theta\big(\bar{\phi}\big), \qquad \bar{\phi}(x,0) = x, \quad \forall x \in \overline{\Omega}_3. \tag{2.16}$$

From this we can conclude that:

$$\frac{\partial(\theta\circ\bar{\phi})}{\partial t} = M\gamma(t)|\nabla\theta\big(\bar{\phi}\big)|^2 \geq 0, \tag{2.17}$$

The following lemma is crucial for the control of curl in Ω_1.

Lemma 2.12. *For θ constructed in Proposition 2.6, we have*

$$\nabla\theta(x) \neq 0 \quad \forall x \in \overline{\Omega}_1. \tag{2.18}$$

The proof of Lemma 2.12, which is not detailed here, mainly relies on applying Morse theory to count the number of zeros of $g(x) = \partial\theta/\partial x_1 - i\partial\theta/\partial x_2$ and uses the strong maximum principle to bound the normal derivative of θ on $\Gamma_{-1} \cup \Gamma_1$.

Lemma 2.13. *If M is large enough,*

$$\theta\big(\bar{\phi}(x,1/2)\big) > 1, \qquad \forall x \in \bar{\Omega}_1. \tag{2.19}$$

Proof. For a fixed $x \in \Omega$, (2.17) and Neumann condition for θ imply that $\bar{\phi}(x,t)$ exits the domain $\overline{\Omega}_1$ only by passing through Γ_1, hence we obtain for $\bar{\phi}\big(x, \frac{1}{2}\big)$ the following possibilities:

(i) there is a $t' \in [0, \frac{1}{2})$ such that $\bar{\phi}(x,t') \in \Gamma_1$,

(ii) $\bar{\phi}(x,t) \in \overline{\Omega}_1 \forall t \in [0, \frac{1}{2}]$.

Case (i): Since $\dfrac{\partial(\theta \circ \phi)}{\partial t} > 0$, we get that

$$\theta\big(\bar{\phi}(x, \tfrac{1}{2})\big) > \theta\big(\bar{\phi}(x,t')\big) = 1 \implies \theta\big(\bar{\phi}(x, \tfrac{1}{2})\big) > 1.$$

Case (ii): We have from (2.17) and (2.18) that $|\nabla \theta|^2$ is strictly positive, thus $\overline{\Omega}_1$ being closed allows us to find an $\varepsilon > 0$ such that $|\nabla\theta|^2 > \varepsilon$ everywhere on $\overline{\Omega}_1$. We obtain that $\forall x \in \overline{\Omega}_1$,

$$\theta\big(\bar{\phi}(x, \tfrac{1}{2})\big) = \theta\big(\bar{\phi}(x,0)\big) + \int_0^{\frac{1}{2}} \frac{\partial(\theta \circ \phi)}{\partial t}(x,t)dt \tag{2.20}$$

$$= \theta(x) + \int_0^{\frac{1}{2}} M\gamma(t)|\nabla\theta\big(\bar{\phi}(x,t)\big)|^2 dt \tag{2.21}$$

$$\geq -1 + M\varepsilon^2 \int_0^{\frac{1}{2}} \gamma(t)dt. \tag{2.22}$$

Hence, for M large enough,

$$\theta\big(\bar{\phi}(x, \tfrac{1}{2})\big) > 1 \quad \forall x \in \overline{\Omega_1}.$$

□

We fix M big enough to ensure that (2.19) holds. Since $\theta(\bar{\Omega}_1) \subset [-1,1]$ it follows from (2.16) and (2.19)) that there is a open set Ω_2 containing $\bar{\Omega}_1$ of which the closure is included in Ω_3 and which verifies

$$\phi(x,t) \notin \bar{\Omega}_1, \qquad \forall (x,t) \in \bar{\Omega}_2 \times [1/2, 1]. \tag{2.23}$$

Definition 2.14. *For all f in $C^0(\bar{U}; \mathbb{R}^i)$ we set*

$$q(f) = |f|_0 + \sup\Big\{\frac{|f(x_1) - f(x_2)|}{X(|x_1 - x_2|)}; x_1 \in U, x_2 \in U, x_1 \neq x_2\Big\}, \tag{2.24}$$

where $X(s) = s + s\log(1/s)$ for s in (0,1) and $X(s) = s$ for $s \geq 1$.

2.4.3 Construction of an extension map

Proposition 2.15. *There exists a linear map*

$$\pi : C^0\big(\overline{\Omega_1}; \mathbb{R}^2\big) \to C^0\big(\overline{\Omega_3}; \mathbb{R}^2\big)$$

such that the following conditions hold:

$$\pi(f)(x) = f(x) \quad \forall x \in \overline{\Omega_1} \quad \forall f \in C^0\big(\overline{\Omega_1}; \mathbb{R}^2\big), \tag{2.25}$$

$$\pi(f) \text{ has support in } \Omega_2 \quad \forall f \in C^0\big(\overline{\Omega_1}; \mathbb{R}^2\big), \tag{2.26}$$

$$\pi \text{ sends } C^\lambda\big(\overline{\Omega_1}, \mathbb{R}^2\big) \text{ to } C^\lambda\big(\overline{\Omega_3}, \mathbb{R}^2\big) \text{ continuously } \ \forall \lambda \in [0, +\infty), \tag{2.27}$$

$$q\big(\pi(f)\big) \leq c q(f) \qquad \forall f \in C^0\big(\overline{\Omega_1}, \mathbb{R}^2\big) \quad \text{for some } c > 0. \tag{2.28}$$

Definition 2.16. *We define an extension* $\sigma : C^0(\mathbb{R}^2_+; \mathbb{R}^2) \to C^0(\mathbb{R}^2; \mathbb{R}^2)$ *as in [3]:*

$$\sigma(f)(x, -y) := \int_0^\infty \varphi(t) f(x, ty) dt \quad \forall y > 0, x \in \mathbb{R}, \tag{2.29}$$

$$\sigma(f)(x, y) = f(x, y) \quad \forall y \geq 0, x \in \mathbb{R}, \tag{2.30}$$

where

$$\varphi(t) = \frac{e^{2\sqrt{2}}}{\pi(1+t)} e^{-\left(t^{\frac{1}{4}} + t^{-\frac{1}{4}}\right)} \sin\big(t^{\frac{1}{4}} - t^{-\frac{1}{4}}\big)$$

and $\mathbb{R}^2_+$ *denotes the closed upper half-plane* $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$.

The function φ is chosen this way to satisfy

$$\int_0^\infty t^n \varphi(t) dt = (-1)^n \quad \forall n \in \{0, 1, \dots\}, \tag{2.31}$$

and this relation can be verified by writing φ as the real part of a holomorphic function and using contour integration. In turn, (2.31) ensures that the derivatives of $\sigma(f)$ for an f smooth on $\mathbb{R}^2_+$ coincide above and below on the x-axis.

Lemma 2.17. *We also have*

$$X(at) \leq (a+1) X(t) \qquad \forall a, t > 0.$$

Proof.

$$\frac{X(at)}{at} \leq \frac{X(t)}{t} \qquad \forall a \geq 1$$

and

$$X(at) \leq X(t) \quad \forall a < 1,$$

from what the statement follows. □

Lemma 2.18. *Let $\lambda \geq 0$. There is a constant $A > 0$ such that*

$$q\big(\sigma(f)\big) \leq Aq(f) \quad \forall f \in C^0\big(\mathbb{R}^2_+, \mathbb{R}^2\big)$$

and $A' > 0$ depending on λ such that

$$\|\sigma(f)\|_{C^\lambda} \leq A'\|f\|_{C^\lambda} \quad \forall f \in C^\lambda\big(\mathbb{R}^2_+, \mathbb{R}^2\big).$$

Proof. Let $x, y \in \mathbb{R}^2$ with $x_2, y_2 \leq 0$.

$$\begin{aligned}
|\sigma(f)(x) - \sigma(f)(y)| &= \Big|\int_0^\infty \varphi(t)\big(f(x_1, -tx_2) - f(y_1, -ty_2)\big)dt\Big| \\
&\leq \int_0^\infty |\varphi(t)|q(f)X\big(\sqrt{(x_1-y_1)^2 + t^2(x_2-y_2)^2}\big)dt \\
&\leq \int_0^\infty |\varphi(t)|q(f)(X\big((1+t)|x-y|\big)dt \\
&\leq \int_0^\infty |\varphi(t)|q(f)(2+t)X\big(|x-y|\big)dt \\
&= A_1 q(f) X\big(|x-y|\big)
\end{aligned}$$

by Lemma 2.17 and since $|\varphi(t)|$ decreases fast enough at infinity.

For $x, y \in \mathbb{R}^2$ with $x_2 \leq 0, y_2 \geq 0$, we take $z \in \mathbb{R}^2$ as the intersection of the segment xy with the x-axis. Then

$$\begin{aligned}
|\sigma(f)(x) - \sigma(f)(y)| &\leq |f(x) - f(z)| + |f(z) - f(y)| \\
&\leq A_1 q(f) X\big(|x-z|\big) + A_1 q(f) X\big(|z-y|\big) \\
&\leq 2A_1 q(f) X\big(|x-y|\big).
\end{aligned}$$

We also have

$$|\sigma(f)(x)| = \Big|\int_0^\infty \varphi(t) f(x_1, -tx_2)dt\Big| \leq \|f\|_{C^0}\int_0^\infty |\varphi(t)|dt = A_2\|f\|_{C^0} \quad \forall x \text{ such that } x_2 \leq 0.$$

Thus

$$q\big(\sigma(f)\big) \leq (2A_1 + A_2 + 1)q(f).$$

Writing $\lambda = m + \varepsilon$ with $m \in \mathbb{Z}$ and $\varepsilon \in [0, 1)$, we can easily check that for $f \in C^\lambda(\mathbb{R}^2_+; \mathbb{R}^2)$, $\sigma(f) \in C^m$ by bringing the derivative under the integral sign and checking that the upper and lower limits of derivatives match along the x-axis.

For $\alpha \in \mathbb{N}^2, |\alpha| = n$ and x, y with $x_2, y_2 < 0$,

$$\begin{aligned}
\big|\partial_\alpha(\sigma(f))(x) - \partial_\alpha(\sigma(f))(y)\big| &= \Big|\int_0^\infty \varphi(t)(-t)^{\alpha_2}\big(\partial_\alpha f(x_1, -tx_2) - \partial_\alpha f(y_1, -ty_2)\big)dt\Big| \\
&\leq \int_0^\infty |\varphi(t)|t^{\alpha_2}\|\partial_\alpha f\|_{C^\varepsilon} \cdot (1 + t^\varepsilon)|x-y|^\varepsilon dt \\
&\leq A_3\|\partial_\alpha f\|_{C^\varepsilon}|x-y|^\varepsilon,
\end{aligned} \tag{2.32}$$

since $|\varphi(t)|$ is rapidly decreasing. Hence

$$\|\sigma(f)\|_{C^\lambda} \leq A_4\|f\|_{C^\lambda} \qquad \text{for some } A_4 > 0.$$

□

Lemma 2.19. *Let $\lambda > 0$. For a compactly supported smooth $\rho \in C^\infty(U)$ there is a $B > 0$ such that*

$$q\big(\rho(f)\big) \leq Bq(f) \quad \forall f \in C^0(U, \mathbb{R}^2)$$

and a $B' > 0$ depending on λ such that

$$\|\rho f\|_{C^\lambda} \leq B'\|f\|_{C^\lambda} \quad \forall f \in C^\lambda(U, \mathbb{R}^2). \tag{2.33}$$

Proof.

$$\begin{aligned} |\rho(x)f(x) - \rho(y)f(y)| &\leq |\rho(x)||f(x) - f(y)| + |\rho(x) - \rho(y)||f(y)| \\ &\leq \|\rho\|_{C^1}q(f)\big(X(|x-y|) + |x-y|\big) \\ &\leq 2\|\rho\|_{C^1}q(f)X(|x-y|). \end{aligned} \tag{2.34}$$

Hence $q(\rho f) \leq 3\|\rho\|_{C^1}q(f)$.

For the case of C^λ-norms, we can express a derivative of any order of ρf as a sum of products of the form $\partial_\beta\rho$ and $\partial_\gamma f$. The same reasoning as in (2.34) applied to $\partial_\beta\rho\partial_\gamma f$, using $|x-y| \leq c|x-y|^\varepsilon$ for any x, y in the support of ρ and a fixed $\varepsilon \in [0, 1)$, grants us

$$\|\rho f\|_{C^\lambda} \leq B_1\|\rho\|_{C^{\lambda+1}}\|f\|_{C^\lambda}.$$

□

Lemma 2.20. *Let $U, V \subset \mathbb{R}^2$ be open sets and let $g : U \to V$ be a smooth diffeomorphism. Let $\lambda \geq 0$.*

Then for any open U' with $\overline{U'} \subset U$ compact, there is a $C > 0$ such that

$$q\big(f \circ g\big) \leq Cq(f) \quad \forall f \in C^0\big(V, \mathbb{R}^2\big)$$

and a $C' > 0$ depending on λ such that

$$\|\sigma(f)\|_{C^\lambda} \leq C'\|f\|_{C^\lambda} \quad \forall f \in C^\lambda\big(V, \mathbb{R}^2\big).$$

Proof. Since $\overline{U'}$ is compact, there is an $M > 0$ such that

$$|g(x) - g(y)| \leq M|x-y| \quad \forall x, y \in \overline{U'}.$$

Therefore, using (2.17)

$$|f(g(x)) - f(g(y))| \leq q(f)X(|g(x) - g(y)|) \leq q(f)X\big(M|x-y|\big) \leq q(f)(M+1)X(|x-y|)$$

and

$$q(f \circ g) \leq (M+2)q(f) \qquad \forall f \in C^0\big(V, \mathbb{R}^2\big).$$

If $f \in C^\lambda\big(V, \mathbb{R}^2\big)$ with $\lambda = m + \varepsilon, m \in \mathbb{N}, \varepsilon \in [0,1)$, every $\partial_\alpha(f \circ g)$ with $|\alpha| \leq m$ can be expressed as a sum of products with terms $\big(\partial_\beta f\big) \circ g$ and $\big(\partial_\gamma g_i\big) \circ g$ with $\beta, \gamma \leq \alpha$. Since $\overline{U'}$ is compact, $\big(\partial_\gamma g_i\big) \circ g$ is bounded on $\overline{V'}$ for each γ.

The existence of $C' > 0$ such that

$$\|f \circ g\|_{C^\lambda} \leq C' \|f\|_{C^\lambda}$$

thus follows from Lemma 2.19. □

Proof of Proposition 2.15. We take a smooth diffeomorphism $g : \mathbb{S}^1 \times (-\delta, \delta) \longrightarrow U$, where U is a neighbourhood of $\partial\Omega$, which exists by the tubular neighbourhood theorem. Cover $\partial\Omega$ by open sets $V_1, V_2, \ldots, V_n$ such that each V_i satisfies $\overline{V_i} \subset U \cap \Omega_2$.

Take g_i as a smooth diffeomorphism from a neighbourhood of $g^{-1}\big(\overline{V_i}\big)$ to $\mathbb{R}^2$, which sends $(\mathbb{S}^1 \times \{0\}) \cap g^{-1}\big(\overline{V_i}\big)$ onto the x-axis for each i.

Set $h_i := g_i \circ g^{-1}$ and, without loss of generality, assume that h_i sends $\Omega \cap V_i$ onto $\mathbb{R}^2_+$ for each i. Next we take $\rho_0, \rho_1, \ldots, \rho_n$ a partition of unity subordinate to $\Omega, V_1, \ldots, V_n$. We can now define π. For a given $f \in C^0(\overline{\Omega}; \mathbb{R}^2)$ set

$$f_i := \sigma \circ \big(\rho_i f\big) \circ h_i$$

and extend it to $\Omega_3 \setminus V_i$ by 0 for each i. Then set f_0 as ρf in $\overline{\Omega}$ and extend it by 0 to $\overline{\Omega}_3 \setminus \overline{\Omega}$.

$$\pi(f) := \sum_{i=0}^{n} f_i.$$

Lemma 2.18, Lemma 2.19, and Lemma 2.20 now ensure that this π satisfies all the needed conditions. □

2.5 Construction of the control trajectory y

Continuing with the proof of Theorem 2, what is now only left show is that we can obtain a trajectory $\mathbf{y}$ using the tools, which we constructed up until now, such that we are able to go smoothly from small a $\mathbf{y}_0$ to $\mathbf{y}(x,T) = 0$ To do this, we look at how the curl of $\mathbf{y}$ is transported near our retruning trajectory. First we denote for f in $C^0(\bar{\Omega} \times [0,1]; \mathbb{R}^2)$ $\pi(f)$ the function in $C^0(\bar{\Omega}_3 \times [0,1]; \mathbb{R}^2)$ such that $\pi(f)(x,t) = \pi\big(f(\cdot,t)\big)(x)$ for all (x,t) in $\bar{\Omega}_3 \times [0,1]$.

Let $\mathbf{y}^* \in C^\infty(\bar{\Omega}_3 \times [0,1]; \mathbb{R}^2)$ be defined by $\mathbf{y}^*(x,t) = M\gamma(t)\nabla\theta(x)$; we denote with $\bar{\mathbf{y}}$ the restriction of $\mathbf{y}^*$ to $\bar{\Omega} \times [0,1]$.

Let $\mu \in C^\infty\big([0,1]; [0,1]\big)$ be a map equal to 1 on $[0,1/4]$ and to 0 on $[1/2,1]$. We try to find a $\mathbf{y} \colon \bar{\Omega} \times [0,1] \to \mathbb{R}^2$ and a $\omega \colon \bar{\Omega}_3 \times [0,1] \to \mathbb{R}$, which are solutions of:

$$\frac{\partial \omega}{\partial t} + \big((\mathbf{y}^* + \pi(\mathbf{y} - \bar{\mathbf{y}})) \cdot \nabla\big)\omega = 0 \qquad \text{in } \bar{\Omega}_3 \times [0,1], \tag{2.35}$$

$$\omega(\cdot, 0) = \operatorname{rot} \pi(\mathbf{y}_0) \qquad \text{on } \bar{\Omega}_3, \tag{2.36}$$

$$\operatorname{div} \mathbf{y}(\cdot,t) = 0, \qquad \operatorname{rot} \mathbf{y}(\cdot,t) = \omega(\cdot,t) \qquad \text{on } \bar{\Omega}, \qquad \forall t \in [0,1], \tag{2.37}$$

$$\mathbf{y}(x,t) \cdot \mathbf{n}(x) = \big(\bar{\mathbf{y}}(x,t) + \mu(t)\mathbf{y}_0(x)\big) \cdot \mathbf{n}(x) \qquad \forall x \in \partial\Omega,\ t \in [0,1]. \tag{2.38}$$

We will refer to this system of partial differential equations as (P).

2.5.1 Suitedness of the solution of the (P)

We will show now that the solutions of (P) will yield our suited $\mathbf{y}$ as a solution.

Lemma 2.21. *If (P) possesses a solution* $(\mathbf{y}, \omega)$ *of class* C^∞*, such that* $\mathbf{y}$ *is zero on* $\bar{\Omega} \times [1/2, 1]$*, then, there exists a* $p \in C^\infty(\bar{\Omega} \times [0,1]; \mathbb{R})$ *such that* (1.8) *to* (1.11) *and* (2.1) *are fulfilled.*

Proof. That (1.9) is fulfilled follows trivially from $\operatorname{div} \mathbf{y} = 0$, similarly that y suffices (1.10) is not hard to obtain from $\mathbf{y}(x,t) \cdot \mathbf{n}(x) = \big(\bar{\mathbf{y}}(x,t) + \mu(t)\mathbf{y}_0(x)\big) \cdot \mathbf{n}(x) \forall x \in \partial\Omega, \quad \forall t \in [0,1]$ by (1.4) and the fact on $\overline{\Omega} \times [0,1]$ $\bar{\mathbf{y}} \cdot \mathbf{n}(x) = \mathbf{y}^* \cdot \mathbf{n}(x) = M\gamma(t)\nabla\theta(x) \cdot \mathbf{n}(x) = M\gamma(t)\partial_{\mathbf{n}}\theta(x)$, which is zero on $(\Gamma \setminus \Gamma_0) \times (0,T) \subset \Sigma$ due to the boundary conditions on θ.
We can see that (1.11) holds by Lemma 2.23, which is shown in the next subsection.
To show (1.8), we write:

$$\begin{aligned} \operatorname{rot}((\mathbf{y} \cdot \nabla)\mathbf{y}) &= \partial_1((y_1\partial_1 + y_2\partial_2)y_2) - \partial_2((y_1\partial_1 + y_2\partial_2)y_1) \\ &= \partial_1(y_1)\partial_1(y_2) + y_1\partial_1\partial_1(y_2) + \partial_1(y_2)\partial_2(y_2) + y_2\partial_1\partial_2(y_2) \\ &- \partial_2(y_1)\partial_1(y_1) - y_1\partial_1\partial_2(y_1) - \partial_2(y_2)\partial_2(y_1) - y_2\partial_2\partial_2(y_1) \\ &= \operatorname{rot}(\mathbf{y}) \cdot \operatorname{div}(\mathbf{y}) + \mathbf{y}\nabla(\operatorname{rot}(\mathbf{y})) = 0 + \mathbf{y} \cdot \nabla(\operatorname{rot}(\mathbf{y})) \\ &= \mathbf{y} \cdot \nabla(\operatorname{rot}(\mathbf{y})) \end{aligned} \tag{2.39}$$

With this we are able to obtain from (2.35) and the fact that $(\mathbf{y}^* + \pi(\mathbf{y} - \bar{\mathbf{y}}) = \mathbf{y}$ on $\overline{\Omega} \times [0,1]$ the following:

$$0 = \partial_t\omega + \big((\mathbf{y}^* + \pi(\mathbf{y} - \bar{\mathbf{y}})) \cdot \nabla\big)\omega = \partial_t \operatorname{rot}(\mathbf{y}) + \big(\mathbf{y} \cdot \nabla\big) \operatorname{rot}(\mathbf{y}) = \operatorname{rot}(\partial_t\mathbf{y} + \big(\mathbf{y} \cdot \nabla\big)\mathbf{y}) = 0$$

From where we get since rot is gauge invariant under addition of gradient-fields that there exists a $p \in C^\infty(\bar{\Omega} \times [0,1]; \mathbb{R})$ such that (1.8) is fulfilled. Finally it can easily be concluded that $\mathbf{y}$ fulfills (2.1) by our assumption of $\mathbf{y}$ being zero on $\overline{\Omega} \times [1/2, 1]$ and we can choose p such that it fulfills (2.1) by choosing it to be zero on $\overline{\Omega} \times [1/2, 1]$ since our constraint (1.8) allows us to choose p as constant on $\overline{\Omega} \times [1/2, 1]$ due to $\mathbf{y}$ being zero on $\overline{\Omega} \times [1/2, 1]$. □

2.5.2 Existence of the solution in C^∞ of (P)

Proposition 2.22. *The system (P) has a solution* $(\mathbf{y}, \omega)$ *with* $\mathbf{y} \in C^0\big([0,1]; C^{\delta+1}(\overline{\Omega}; \mathbb{R}^2)\big)$ *and* $\omega \in C^0\big([0,1]; C^\delta(\overline{\Omega_3})\big)$ *for some* $\delta > 0$.

We approach the problem in the following way – first, by solving (1.9) and (2.38) to obtain $\mathbf{y}$ as a function of ω, and second, by solving (2.35) and (1.4) to obtain ω' from this $\mathbf{y}$ constructed before. We will show that such a construction of ω' is a continuous function $F : S \to S$, where S is a compact convex subset of a Banach space and finish by applying the Schauder fixed point theorem.

Lemma 2.23. *For a given $a \in C^\infty(\overline{\Omega})$, The solution to the system*

$$\begin{cases} \operatorname{div} \mathbf{y} = 0 & \text{in } \overline{\Omega}, \\ \operatorname{rot} \mathbf{y} = a & \text{in } \overline{\Omega}, \\ \mathbf{y} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

is given by

$$\mathbf{y} = -\nabla^\perp Ga, \qquad \text{where} \quad \nabla^\perp f := \left(-\tfrac{\partial f}{\partial x_2}, \tfrac{\partial f}{\partial x_1}\right)^T,$$

and G is the Green function for Ω, which is the fundamental solution for the operator $-\Delta$ with Dirichlet boundary condition.

Proof. $\operatorname{div} \mathbf{y} = 0$ implies $\mathbf{y} = \nabla^\perp \psi$ for some $\psi \in C^\infty(\overline{\Omega})$. This is easy to see by checking that any contour integral of $J\mathbf{y}$, where J is the clockwise rotation by 90 degrees, is 0 by divergence theorem, hence $Jy = \nabla\psi$. The condition $\nabla^\perp \psi \cdot \mathbf{n} = 0$ implies $\nabla\psi \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$, meaning $\psi = \textit{const}.$ on $\partial\Omega$, where $\boldsymbol{\tau}$ is the unit tangential vector on $\partial\Omega$. We may therefore take $\psi = 0$ on $\partial\Omega$. Then $\Delta\psi = \operatorname{rot}\big(\nabla^\perp\psi\big) = a$, and we get

$$\psi = Ga \quad \text{and} \quad \mathbf{y} = -\nabla^\perp Ga.$$

□

Lemma 2.24. *Let $v \in C^0\big([0,1]; C^1(\overline{\Omega_3}; \mathbb{R}^2)\big)$, and let $b \in C^\infty(\overline{\Omega_3})$ with $v(\cdot, t)$ supported in Ω_2 for every t.*
The system

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + v \cdot \nabla \xi = 0 & \text{in } \overline{\Omega}_3 \times [0,1], \\ \xi(\cdot, 0) = b \end{cases}$$

has a unique solution $\xi \in C^1\big(\overline{\Omega_3} \times [0,1]\big)$ given by

$$\xi(x,t) = b\big(U_{0,t}(x)\big), \tag{2.40}$$

where $U_{s,t}$ are the stream lines of the flow v, with $U_{t,s}(x)$ defined as the value at time t of the solution to

$$\frac{du}{dt} = v(u,t) \qquad \text{with initial condition } u(s) = x.$$

Proof. Since $v(\cdot, t) = 0$ on $\overline{\Omega_3} \setminus \Omega_2$ for all $t \in [0,1]$, $U_{t,s}(x)$ is defined for all $s, t \in [0,1]$ and $x \in \overline{\Omega_3}$. Considering $g(x,t) := \xi\big(U_{t,0}(x), t\big)$ it is easy to see, that $\dfrac{\partial g}{\partial t} = 0$; hence

$$\xi(x,t) = \xi\big(U_{t,0}(x), t\big) = \xi\big(U_{0,t}(x), 0\big) = b\big(U_{0,t}(x)\big),$$

as claimed. □

Proof of Proposition 2.22. First, we solve the system of equations (2.37) and (2.38). It is well-known that G is a continuous operator from C^δ to $C^{2+\delta}$ ([6]; p. 335). By Lemma 2.23, for a given $\omega \in C^0\big([0,1], C^\delta(\overline{\Omega})\big)$, equations (2.37) and (2.38) give a unique solution $\mathbf{y} \in C^0\big([0,1], C^{\delta+1}(\overline{\Omega}; \mathbb{R}^2)\big)$ given by

$$\mathbf{y}(x,t) = \bar{\mathbf{y}}(x,t) + \mu(t)\mathbf{y}_0(x) - \nabla^\perp G\Big(\omega(.,t) - \operatorname{rot}\big(\bar{\mathbf{y}}(.,t) + \mu(t)\mathbf{y}_0\big)\Big)(x). \tag{2.41}$$

Second, we solve the system of equations (2.35) and (2.36) for the $\mathbf{y}$ obtained from ω. Applying Lemma 2.24 with $v = \mathbf{y}^* + \pi\big(\mathbf{y} - \bar{\mathbf{y}}\big)$ and $b = \operatorname{rot}\pi(\mathbf{y}_0)$, we obtain a solution ω' to (2.35) and (2.36).

Hence, we have constructed ω' as a function of ω, and we want to find a fixed point of this map. The only part of the argument left is finding a compact convex set that this function will map continuously into itself. For this, we need the following bound.

Define $M := \big\|\operatorname{rot}\pi(\mathbf{y}_0)\big\|_{C^0}$. Having defined $U_{t,s}$ as in Lemma 2.24 for $v = \mathbf{y}^* + \pi(\mathbf{y} - \bar{\mathbf{y}})$, we proceed exactly as in Lemma 2.6 of Kato [4] to obtain

$$\big|U_{t,s}(x) - U_{\bar{t},\bar{s}}(\bar{x})\big| \leq L\Big(|x-\bar{x}|^\delta + |t-\bar{t}|^\delta + |s-\bar{s}|^\delta\Big) \tag{2.42}$$

for $|x - \bar{x}| \leq 1$ and some $\delta \in (0,1)$, given that $\mathbf{y}$ is constructed by solving (2.37) and (2.38) for a $\omega \in C^0\big([0,1]; C^{\delta'}(\overline{\Omega})\big)$ with some $\delta' > 0$ and $\|\omega\|_{C^0} \leq M$. Here, the constants δ and L do not depend on $\omega \in C^0\big([0,1]; C^{\delta'}(\overline{\Omega})\big)$ provided that $\|\omega\|_{C^0} \leq M$.

Finally, we apply the fixed point argument. Define S as the set of $\omega \in C^0\big(\overline{\Omega} \times [0,1]\big)$ with

$$\|\omega\|_{C^0} \leq M \quad \text{and} \quad |\omega(x,t) - \omega(\bar{x},\bar{t})| \leq L\big(|x-\bar{x}|^\delta + |t-\bar{t}|^\delta\big)$$

for every $x, \bar{x} \in \overline{\Omega}$, $t, \bar{t} \in [0,1]$ such that $|x - \bar{x}| \leq 1$. We define $F : S \to S$ as

$$F(w) := w'|_{\overline{\Omega}\times[0,1]},$$

where w' is constructed from the given w as above. The fact that $w'|_{\overline{\Omega}\times[0,1]} \in S$ follows from (2.42) and the explicit formula for w' in Lemma 2.24.

It is easy to check that F is continuous in the $C^0\big(\overline{\Omega} \times [0,1]\big)$-topology. Hence, since S is a convex and compact (by Arzelà–Ascoli) subset of the Banach space $C^0\big(\overline{\Omega} \times [0,1]\big)$, by Schauder's fixed point theorem there exists $\omega \in S$ such that $F(\omega) = \omega$. ☐

Proposition 2.25. *The solution $(\mathbf{y}, \omega)$ obtained in Proposition 2.22 is smooth.*

Proof. We obtained the existence of a solution $(\mathbf{y}, \omega)$ of (P) such that, if $\delta > 0$ is small enough, $\mathbf{y} \in C^0\big([0,1]; C^{\delta+1}(\bar{\Omega}; \mathbb{R}^2)\big)$ and $\omega \in C^0\big([0,1]; C^\delta(\bar{\Omega}_3; \mathbb{R})\big)$. In the following discussion, (2.35) is understood in the sense of distributions by replacing $(z \cdot \nabla)\omega$ with $\operatorname{div}(\omega z) - \omega \operatorname{div} z$ with $z = \mathbf{y}^* + \pi(\mathbf{y} - \bar{\mathbf{y}})$.

Now, we apply a recursive argument. If $\omega \in L^\infty\big([0,1]; C^\lambda(\bar{\Omega}_3; \mathbb{R})\big)$ for some $\lambda \in (0, +\infty)\backslash\mathbb{N}$, Schauder estimates following from (2.41) give $\mathbf{y} \in L^\infty\big([0,1]; C^{\lambda+1}(\bar{\Omega}; \mathbb{R}^2)\big)$.

From Lemma 2.24, we can express ω through (2.40), and it suffices to see that the reverse flow map $U_{0,t}$ is uniformly bounded in the $C^{\lambda+1}$-norm to obtain $\omega \in L^\infty\big([0,1]; C^{\lambda+1}(\bar{\Omega}_3; \mathbb{R})\big)$.

We consider $U_{0,t}$ as the map from Ω_3 to Ω_3. The reverse flow map $U_{0,t}$ is the inverse of $U_{t,0}$, which is $C^{\lambda+1}$ since it has the same level of regularity as $\mathbf{y}$ by usual ODE theory.

We have $DU_{0,t}(x) = DU_{t,0}\big(U_{0,t}(x)\big)^{-1}$. By differentiating this equality m times (where $\lambda = m+\epsilon$), we get that $DU_{0,t}$ is uniformly bounded in the C^λ norm, having used that $U_{0,t}$ is differentiable hence Lipschitz and that $DU_{t,0}$ is uniformly bounded in C^λ norm.

By induction on $p \in \mathbb{N}$, for every $\lambda > 0$, we use (2.35) to obtain

$$\omega \in W^{p,\infty}\big([0,1]; C^\lambda(\bar\Omega_3;\mathbb{R})\big), \qquad \mathbf{y} \in W^{p,\infty}\big([0,1]; C^\lambda(\bar\Omega;\mathbb{R}^2)\big),$$

hence $\mathbf{y}$ and p are smooth. □

2.5.3 Proof that solution to (P) is our sought trajectory y

The only thing left to complete the proof of Theorem 2 is to show that the obtained $\mathbf{y}$ indeed goes to the equilibrium $\mathbf{y}(x) = 0$ for $t \in [1/2, 1]$ and is the returning trajectory.
Clearly, if $|\mathbf{y}_0|_{C^1} \to 0$, then $|\omega|_{C^0} \to 0$ and $|\mathbf{y} - \bar{\mathbf{y}}|_{C^0} \to 0$. From this, together with (2.23), we deduce that there exists $\nu > 0$ such that, if $|\mathbf{y}_0|_{C^1} < \nu$, we have $\phi(x,t) \notin \bar\Omega_1$ for all $(x,t) \in \bar\Omega_2 \times [1/2,1]$, where $\phi : \bar\Omega_3 \times [0,T] \to \bar\Omega_3$ is defined by

$$\frac{\partial \phi}{\partial t} = \big(\mathbf{y}^* + \pi(\mathbf{y} - \bar{\mathbf{y}})\big)(\phi, t), \qquad \phi(x,0) = x, \forall x \in \bar\Omega_3.$$

Lemma 2.26. *if $|\mathbf{y}_0|_{C^1} < \nu$, then ω vanishes on $\bar\Omega \times [1/2,1]$,*

Proof. we start by defining $\omega'(x,t) = \omega\big(\phi(x,t),t\big)$ and obtain:

$$\begin{aligned}\partial_t \omega'(x,t) &= \partial_t \omega\big(\phi(x,t),t\big) + \nabla\omega \cdot \partial_t \phi(x,t)\\ &= -\big((\mathbf{y}^* + \pi(\mathbf{y}-\bar{\mathbf{y}}))\cdot\nabla\big)\omega\big(\phi(x,t),t\big) + \Big(\big(\mathbf{y}^* + \pi(\mathbf{y}-\bar{\mathbf{y}})\big)\cdot\nabla\Big)\omega(\phi(x,t),t)\\ &= 0\end{aligned} \tag{2.43}$$

This shows, that $\omega(\phi(x,t),t) = const = \omega(\phi(x,0),0)$ meaning that the curl of $\mathbf{y}$ gets moved along the flow lines. From this we conclude, since $\phi(x,t) \notin \bar\Omega_1$ for all $(x,t) \in \bar\Omega_2 \times [1/2,1]$, that the whole curl, which is inside $\overline{\Omega}_2$ at $t=0$ gets transported outside $\overline{\Omega}_1$ by $t = 1/2$ and stays on this part of the domain for the rest of $t \in [1/2,1]$. But since ϕ is a dimorphism we can conclude that the flow which ends up in $\overline{\Omega}_1$ at time $t \in [1/2,1]$ has to come from $\overline{\Omega}_3 \setminus \overline{\Omega}_2$ meaning for $t \in [1/2,1]$ and $\phi(x,t) \in \overline{\Omega}_1$ we get $x \in \Omega_3 \setminus \overline{\Omega}_2$. By (2.36) we get, that the support of ω at $t=0$ lies in $\overline{\Omega}_2$ and therefore $\omega(x,0) = 0$.
Thus for a fixed $t \in [1/2,1]$ and any $x' \in \overline{\Omega}_1$ there is an $x \in \Omega_3 \setminus \overline{\Omega}_2$ such that $x' = \phi(x,t)$ and thus

$$\omega(x',t) = \omega(\phi(x,t),t) = \omega(\phi(x,0),0) = \omega(x,0) = 0$$

From this it follows by similar reasoning as in Lemma 2.21 that $\mathbf{y}$ is also zero on this same set, thanks to (2.37) and (2.38). □

This completes the proof of (2) and hence of Theorem 1.

References

[1] Jean-Michel Coron. *Control and Nonlinearity.* American Mathematical Society, 2007.

[2] Jean-Michel Coron. Controlabilite exacte frontiere de l'equation d'euler des fluides parfaits incompressibles bidimensionnels. *Comptes rendus de l'Académie des sciences*, 1:271–276, 1993.

[3] Richard S. Hamilton. The inverse function theorem of nash and moser. *Bulletin of the American Mathematical Society*, 7:65–222, 1982.

[4] Tosio Kato. On classical solutions of the two-dimensional non-stationary euler equation. *Archive for Rational Mechanics and Analysis*, 25:188–200, 1967.

[5] V. P. Mikhailov. *Partial Differential Equations.* Mir Publishers, 1978.

[6] D. Hilbert R. Courant. *Methods of Mathematical Physics Vol. II.* Interscience, 1962.