O-minimality of restricted analytic field with exponentiation

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Abstract

Following van den Dries, Macintyre, and Marker's paper [10], we give a proof that the theory $(\mathbb{R}_{\text{an}}, \exp)$ of restricted analytic field with (unrestricted) exponentiation is o-minimal via quantifier elimination and Hardy field methods.

Contents

1 Introduction

For an integer $n > 0$, denote $\mathbb{R}\{X_1, \dots, X_n\}$ as the set of real power series in $X_1, \dots, X_n$ that converge in a neighborhood of I^n, where $\mathrm{I} = [-1, 1]$. Given $f \in \mathbb{R}\{X_1, \dots, X_n\}$, define the function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ as below:

$$\tilde{f}(x) = f(x) \cdot \mathbf{1}_{\mathrm{I}^n},$$

where $\mathbf{1}_{\mathrm{I}^n}$ is the characteristic function of I^n. The $\tilde{f}$s defined above are called the *restricted analytic functions.*

Let L_{an} be the language of ordered rings $(<, 0, 1, +, -, \cdot)$ added with a new function symbol for every such $\tilde{f}$, as n varies through all the positive integers. Let $\mathbb{R}_{\text{an}}$ be the natural L_{an}-structure on the real numbers $\mathbb{R}$, and T_{an} the theory of $\mathbb{R}_{\text{an}}$. Finally, add the function symbol for exponentiation to obtain the language $L_{\text{an}}(\exp)$, as well as the model $(\mathbb{R}_{\text{an}}, \exp)$, whose theory we denote as $\text{Th}(\mathbb{R}_{\text{an}}, \exp)$.

Definition 1.1 (O-minimality)**.** An L-structure $(M,<,\ldots)$, such that $<$ is a linear order, is called o-minimal, if every definable subset of M (with parameters taken from M) can be written as a finite union of points and intervals. A theory T is called o-minimal if every model of T is o-minimal.

Example 1.2. Since the theory T_0 of dense linear order without endpoints admits quantifier elimination (in the language with $<$ as the only symbol), T_0 is o-minimal. In particular, the model $(\mathbb{R},<)$ is o-minimal. Furthermore, by the Tarski–Seidenberg theorem, which states that the theory of $\mathcal{R} := (\mathbb{R},<,+,\cdot,0,1)$ admits quantifier elimination, $\mathcal{R}$ is o-minimal.

Remark 1.3. In fact, [4, Theorem 0.2] states that the complete theory T of an o-minimal structure is an o-minimal theory.

Before we dive into the proof of o-minimality, we give below a few insights as to why one would possibly care about the o-minimality of certain models, and its connections with other fields of mathematics.

In o-minimal structures, definable subsets have tame behavior, just like semi-algebraic subsets, and this motivates applications to arithmetic geometry like the Pila-Wilkie Theorem [6, Theorem 1.8], which gives an upper bound on the number of rational points of a set definable in an o-minimal expansion $\mathcal{R}$ of the real field. Using this theorem, a proof of the André-Oort conjecture can be seen in [8], where one uses the fact that the j-function is, in some sense, definable in the o-minimal structure $(\mathbb{R}_{\mathrm{an}},\exp)$. Furthermore, some links between Hodge theory and o-minimality are discussed in [3].

As a side note, due to the nature of this short survey, not all mathematical statements are proved. For those without proofs, proper references are provided.

1.1 Outline of the article

In §2, we introduce the standard real valuation of ordered fields, which will be used throughout the whole article. Then we give an axiomatization of the theory T_{an} and prove related results concerning quantifier elimination in §3. We go on to examine the value groups of models of T_{an} in §4. Using the results in §4, we establish the fact that $\mathrm{Th}(\mathbb{R}_{\mathrm{an}},\exp)$ admits quantifier elimination and present a universal axiomatization of it in §5. Finally, in §6, we prove that $(\mathbb{R}_{\mathrm{an}},\exp)$ is o-minimal via Hardy field methods.

2 Valuation of ordered fields

Definition 2.1 (Valuation)**.** Given a totally ordered abelian group Γ, a (Γ-valued) valuation on a field K is a function $v\colon K\to\Gamma\cup\{+\infty\}$ satisfying the following conditions:
(1) the restriction $v|_{K^\times}: K^\times\to\Gamma$ is a surjective group homomorphism;
(2) $v(0)=+\infty$;
(3) for all $x,y\in K$, $v(x+y)\geq\min(v(x),v(y))$, with the convention that $+\infty>a$ for all $a\in\Gamma$.
In this case, (K,v) is called a valued field.

For a valuation $v\colon K\to\Gamma$, the group Γ is called the *value group*. Notice that the set

$$\mathcal{O}_v := \{x\in K \mid v(x)\geq 0\}$$

is a subring of K, called the *valuation ring*. Moreover, the set

$$\mathcal{M}_v \coloneqq \{x \in K \mid v(x) > 0\}$$

is an ideal of $\mathcal{O}_v$, known as the *valuation ideal*. For any element $x \in \mathcal{O}_v \setminus \mathcal{M}_v$, we have $v(x) = 0$, so $v(x^{-1}) = -v(x) = 0$, and thus $x^{-1} \in \mathcal{O}_v$. Therefore, $\mathcal{O}_v$ is a local ring with maximal ideal $\mathcal{M}_v$. The *residue class field* of the valuation v is defined as the quotient ring $\mathcal{O}_v/\mathcal{M}_v$.

Definition 2.2 (Equivalent valuations)**.** Let $v\colon K \to \Gamma_v \cup \{+\infty\}$ and $w\colon K \to \Gamma_w \cup \{+\infty\}$ be valuations on a field K. We say that v and w are *equivalent* if there exists an order-preserving group isomorphism $\varphi\colon \Gamma_v \to \Gamma_w$ such that for all $x \in K^\times$, we have

$$w(x) = \varphi(v(x)).$$

Remark 2.3. Since φ is order-preserving, we have that $v(x) \geq 0 \Leftrightarrow w(x) \geq 0$, for every $x \in K$, and thus $\mathcal{O}_v = \mathcal{O}_w$.

Definition 2.4 (Henselian valuation)**.** A valued field (K, v) called Henselian if its valuation ring $\mathcal{O}_v$ is Henselian, i.e., if P is a monic polynomial in $\mathcal{O}_v[x]$, then any factorization of its image $\bar{P}$ in $(\mathcal{O}_v/\mathcal{M}_v)[x]$ into a product of coprime monic polynomials can be lifted to a factorization in $\mathcal{O}_v[x]$.

We now define the *standard real valuation* of an ordered field. Let $(F, <)$ be an ordered field, clearly $\mathrm{char}(F) = 0$ and thus F contains the rational numbers $\mathbb{Q}$. Define the set of bounded elements

$$V(F) := \{a \in F \,|\, \exists r \in \mathbb{Q}, \text{such that} - r < a < r\},$$

and the set of infinitesimals

$$\mu(F) := \{a \in F \,|\, \forall r \in \mathbb{Q}_+, -r < a < r\}.$$

Note that $\mu(F)$ is the maximal ideal of the local ring $V(F)$. The set $U(F) := V(F) \backslash \mu(F)$ is a multiplicative subgroup of $F^\times$, and the factor group $\Gamma := F^\times/U(F)$ is totally ordered by $aU(F) \geq bU(F)$ if and only if $a/b \in V(F)$.

The *standard real valuation* is defined as the natural projection $v\colon F^\times \to \Gamma$; besides, $v(0)$ is set to be $+\infty$. One checks directly that it satisfies all the conditions for a valuation. For $F \subseteq K$ an inclusion of ordered fields, we can naturally identify $v(F^\times)$ as a subgroup of $v(K^\times)$.

Let us introduce the notion of *power series fields*, as they provide examples of valued fields. Given a field k and a ordered abelian group Γ, let t be a symbol and define the power series field $k((t^\Gamma))$ as the set below

$$\{x = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \,|\, a_\gamma \in k, \{\gamma \in \Gamma \,|\, a_\gamma \neq 0\} \text{ is a well-ordered subset of } \Gamma\}.$$

Addition is defined componentwise, and multiplication follows the usual rules for power series. With these operations, multiplication is well-defined, and $k((t^\Gamma))$ forms a field; see [2, pp. 83–84]. There is a natural valuation $\mathrm{ord}\colon k((t^\Gamma)) \to \Gamma$ defined as $\mathrm{ord}(x) = \min(\mathrm{supp}(x))$. In fact, ord is a Henselian valuation and $k((t^\Gamma))$ is maximal with value group Γ and residue class field k, that is, every extension of $k((t^\Gamma))$ must enlarge either the value group or the residue class field.

The field $k((t^\Gamma))$ can be ordered if k is: for $x = \sum a_\gamma t^\gamma$ and $y = \sum b_\gamma t^\gamma$, define $x < y$ if $a_{\gamma_0} < b_{\gamma_0}$, where γ_0 is the smallest γ such that $a_\gamma \neq b_\gamma$. When $k = \mathbb{R}$, the standard real valuation of the ordered field $k((t^\Gamma))$ is equivalent to the valuation ord defined in the previous paragraph.

By [7, Theorem 8.6], a Henselian valued field with real closed residue class field and divisible value group is real closed; therefore, if k is real closed and Γ is divisible, then $k((t^\Gamma))$ is real closed. For more on valued fields, we refer the reader to the book by Engler and Prestel [2].

3 Axiomatization of T_{an}

Let us consider the following sentences in the language L_{an}.

A1) For any $f, g \in \mathbb{R}\{X_1, \ldots, X_m\}, m \in \mathbb{N}$,

$$\begin{aligned} \widetilde{f+g}(\bar{x}) &= \tilde{f}(\bar{x}) + \tilde{g}(\bar{x}); \\ \widetilde{fg}(\bar{x}) &= \tilde{f}(\bar{x}) \cdot \tilde{g}(\bar{x}); \\ \bigwedge_{i=1}^{m} |x_i| \le 1 &\to \tilde{0}(\bar{x}) = 0 \wedge \tilde{1}(\bar{x}) = 1; \\ \bigvee_{i=1}^{m} |x_i| > 1 &\to \tilde{0}(\bar{x}) = \tilde{1}(\bar{x}) = 0, \end{aligned}$$

where $\bar{x} := (x_1, \ldots, x_m)$ and $\tilde{1}, \tilde{0}$ are the function symbols corresponding to the constant functions 1 and 0.

A2)

$$\begin{aligned} \bigwedge_{i=1}^{m} |x_i| \le 1 &\to \widetilde{X_j}(\bar{x}) = x_j, \\ \bigvee_{i=1}^{m} |x_i| > 1 &\to \widetilde{X_j}(\bar{x}) = 0, \end{aligned}$$

where X_j is the function symbol corresponding to taking the j-th coordinate.

A3) For any $f \in \mathbb{R}\{X_1, \ldots, X_n\}$ and polynomials $g_1, \ldots, g_n \in \mathbb{R}[X_1, \ldots, X_m]$ such that $g := (g_1, \ldots, g_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ satisfies $g(\mathrm{I}^m) \subseteq \mathrm{I}^n$, the polynomials g_i have no constant terms, and that $f(g_1, \ldots, g_n) \in \mathbb{R}\{X_1, \ldots, X_m\}$:

$$\bigwedge_{i=1}^{m} |x_i| \le 1 \to \widetilde{f(g_1, \ldots, g_n)}(\bar{x}) = \tilde{f}(g_1(\bar{x}), \ldots, g_n(\bar{x})).$$

A4) For any $f, g \in \mathbb{R}\{X_1, \ldots, X_m\}, \epsilon \in \mathbb{R}_{>0}, a := (a_1, \ldots, a_m) \in \mathrm{I}^m$, such that $g = f_a(\epsilon X_1, \ldots, \epsilon X_m)$, where $f_a := \sum_i \frac{1}{i!} \frac{\partial^{|i|} f}{\partial X^i}(a) X^i$ is the Taylor series of f at a:

$$(\bigwedge_{i=1}^{m} |x_i| \le 1) \wedge (\bigwedge_{i=1}^{m} |a_i + \epsilon x_i| \le 1) \to \tilde{f}(\tilde{a}_1 + \tilde{\epsilon} x_1, \ldots, \tilde{a}_m + \tilde{\epsilon} x_m) = \tilde{g}(\bar{x}).$$

Lemma 3.1. *Let K be an L_{an}-structure which is a field and satisfies* A1) − A3)*, then K is Henselian.*

Lemma 3.1 is proved by verifying the following classical criterion: if $g(x) \in V(K)[x]$ is such that the reduction $\bar{g}(x) \in V(K)/\mu(K)[x]$ has a simple root e, then $g(x)$ itself has a simple root $f \in K$ satisfying $\bar{f} = e \in V(K)/\mu(K)$. For details, consult [10, Lemmas 2.4 and 2.5].

Corollary 3.2. *Keeping the notation in Lemma 3.1, suppose furthermore that each positive element of K has an n-th root, for all $n \in \mathbb{N}_{>0}$. Then K is real closed.*

Proof. Since every positive element of K has an n-th root, the value group of K is divisible. Moreover, as K is Henselian by Lemma 3.1 and has residue class field $\mathbb{R}$, we deduce that K is real closed by [7, Theorem 8.6], which states that a Henselian valued field with real closed residue class field and divisible value group is real closed. □

We define a unary function symbol $(\cdot)^{-1}$ which corresponds to taking the multiplicative inverse. By following essentially the same ideas as in [1, Theorem 4.6], one could obtain the quantifier-elimination result below.

Proposition 3.3. *Let K be an L_{an}-structure which is a real closed field, and moreover satisfies* A1) − A4)*. Suppose $\varphi(x_1, \ldots, x_n)$ is an $L_{\mathrm{an}}(^{-1})$-formula, then there exists a quantifier-free formula $\phi(x_1, \ldots, x_n)$ that is independent of K, such that $K \models \varphi \leftrightarrow \phi$ (with the natural $L_{\mathrm{an}}(^{-1})$-structure on K).*

Recall that T_{an} denotes the theory of $\mathbb{R}_{\mathrm{an}}$. We give an axiomatization of T_{an} below.

Theorem 3.4. *The theory T_{an} is axiomatized by*
(1) *the universal axioms for ordered fields;*
(2) *the universal axioms* A1) − A4)*;*
(3) *the axiom stating that each positive element has an n-th root, for all $n \in \mathbb{N}_{>0}$.*

Proof. Clearly, $\mathbb{R}_{\mathrm{an}}$ satisfies these axioms. On the other hand, let M be an L_{an}-model of the axioms $(1), (2)$, and (3). If $c \in \mathbb{R}$ and $\tilde{c}$ is the constant symbol corresponding to c, then $c \mapsto \tilde{c}_M$ defines an ordered field embedding from $\mathbb{R}$ to M by A1). Due to (1) and (2), M can then be viewed as an $L_{\mathrm{an}}(^{-1})$-extension of $(\mathbb{R}_{\mathrm{an}}, ^{-1})$. By Corollary 3.2 and Proposition 3.3, $(\mathbb{R}_{\mathrm{an}}, ^{-1}) \preceq M$, and thus M is a model of T_{an}. □

Remark 3.5. A substructure M of a model N of T_{an} that is closed under taking n-th roots and inverses is therefore also a model of T_{an}. Using Proposition 3.3, T_{an} added with the defining axiom for $(\cdot)^{-1}$, which we denote by $T_{\mathrm{an}}(^{-1})$, admits quantifier elimination in $L_{\mathrm{an}}(^{-1})$, and thus $M \preceq N$.

From now on, the theory T_{an} can be seen interchangeably with its axiomatization in Theorem 3.4. Let us add the symbols $\sqrt[n]{\cdot}$ that correspond to taking n-th roots to $L_{\mathrm{an}}(^{-1})$ to obtain the language $L_{\mathrm{an}}(^{-1}, \sqrt[n]{\cdot})$. Define the theory $T_{\mathrm{an}}(^{-1}, \sqrt[n]{\cdot})$ to be the theory obtained by adding the following two universal axioms to T_{an}:

1) $0^{-1} = 0 \wedge \forall x \neq 0 \ (x \cdot x^{-1} = 1)$;

2_n) $\forall x \geq 0 \ (\sqrt[n]{x} \geq 0 \wedge (\sqrt[n]{x})^n = x \wedge \sqrt[n]{-x} = 0)$, $n \in \mathbb{N}_{>0}$.

Proposition 3.6. *The theory* $T_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$ *admits quantifier elimination in the language* $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$*, and it is a universal axiomatization of* $\mathrm{Th}(\mathbb{R}_{\text{an}}, {}^{-1}, \sqrt[n]{\cdot})$*.*

Proof. Notice that the function symbol $\sqrt[n]{\cdot}$ is definable in $L_{\text{an}}({}^{-1})$:

$$\sqrt[n]{x} = y \leftrightarrow (y \geq 0 \wedge y^n = x) \vee (y = 0 \wedge x < 0).$$

Therefore, quantifier elimination of $T_{\text{an}}({}^{-1})$ (Proposition 3.3) implies quantifier elimination of $T_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$.

We now prove the statement concerning universal axiomatization. Let $M \models T_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$, then M can be viewed as an $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$-extension of $(\mathbb{R}_{\text{an}}, {}^{-1}, \sqrt[n]{\cdot})$. By quantifier elimination, $(\mathbb{R}_{\text{an}}, {}^{-1}, \sqrt[n]{\cdot}) \preceq M$, and thus $M \models \mathrm{Th}(\mathbb{R}_{\text{an}}, {}^{-1}, \sqrt[n]{\cdot})$. Therefore, $T_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$ indeed gives a universal axiomatization of $\mathrm{Th}(\mathbb{R}_{\text{an}}, {}^{-1}, \sqrt[n]{\cdot})$. □

Keep Proposition 3.6 in mind as it will be used multiple times in upcoming sections. We now show that every definable function in $\mathbb{R}_{\text{an}}$ is piecewise given by terms of the language $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$.

Corollary 3.7. *For a function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *definable in* $\mathbb{R}_{\text{an}}$*, there exist* $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$*-terms* $t_1(x_1, \ldots, x_n), \ldots t_k(x_1, \ldots, x_n)$ *such that for all* $a \in \mathbb{R}^n$*, there is an integer* j *satisfying* $f(a) = t_j(a)$*.*

Proof. Suppose the result does not hold and let $\phi(\bar{x}, y)$ be the L_{an}-formula defining "$f(\bar{x}) = y$". Then the set of formulas

$$\mathcal{P}(\bar{x}) := T_{\text{an}}({}^{-1}, \sqrt[n]{\cdot}) \cup \{\neg\phi(\bar{x}, t(\bar{x})) \,|\, t \text{ is a term in } L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})\}$$

is finitely consistent (realized by $\mathbb{R}_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$), and thus consistent by the compactness theorem. Let M be an $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$-structure containing a realization $\bar{x}_0$ of $\mathcal{P}(\bar{x})$, and let $N \subseteq M$ be the $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$-substructure generated by $\bar{x}_0$. By Remark 3.5, $N \models T_{\text{an}}$ and that $N \preceq M$ as $L_{\text{an}}({}^{-1})$-structures. However, by the definition of $\bar{x}_0$, $N \models \forall y \, \neg\phi(\bar{x}_0, y)$ while $M \models \forall \bar{x} \exists y \, \phi(\bar{x}, y)$. □

4 Valuation of models of T_{an}

4.1 Types and definable closures

Let us introduce two notations in model theory used in this section.

Definition 4.1 (Type)**.** Let M be an L-structure, and $B \subseteq M$ a subset. An n-type of M over B is defined as a set $\mathcal{P}$ of formulas $p(x_1, \ldots, x_n)$ in n free variables $x_1, \ldots, x_n$ with parameters from B, such that for every finite subset $\mathcal{P}_0 \subseteq \mathcal{P}$, there exists $c_1, \ldots, c_n \in M$ with $M \models \phi(c_1, \ldots, c_n)$ for all $\phi \in \mathcal{P}_0$.

An n-type $\mathcal{P}$ is said to be realized in M if there exists $\bar{c} \in M^n$ such that $M \models \phi(\bar{c})$, for all $\phi \in \mathcal{P}$. The type of an element $a \in M$ over B is defined as the set of formulas $\{\phi(x, \bar{b}) \,|\, \bar{b} \in B^{|\bar{b}|}, M \models \phi(a, \bar{b})\}$.

Example 4.2. Consider $M := (\mathbb{R}, <)$ and the 1-type $\mathcal{P} := \{0 < x < 1/n \,|\, n \in \mathbb{N}\}$ of M over $\mathbb{Q}$. Note that for every finite subset of formulas in $\mathcal{P}$, there always exists an element of M that realizes them. However, no element of M realizes the entire type, while such an element does exist in some elementary extension of M.

Definition 4.3 (Definable closure)**.** Given S a subset of an L-structure M, the definable closure of S in M is defined to be the set below:

$$\mathrm{dcl}(S) := \{a \in M \mid \{a\} \subseteq M \text{ is a definable subset over } S\}.$$

From the definition, $S \subseteq \mathrm{dcl}(S)$ and $\mathrm{dcl}(S)$ is an L-substructure of M.

Example 4.4. (1) In the language of rings, the definable closure of $\mathbb{F}_p(t^p)$ in the field $\mathbb{F}_p(t)$ is the entire $\mathbb{F}_p(t)$ (note that $\{t\} = \{x \in \mathbb{F}_p(t) \mid x^p = t^p\}$).

(2) Meanwhile, the definable closure of $\mathbb{R}$ in $\mathbb{C}$ is $\mathbb{R}$ itself: complex conjugation is an automorphism of $\mathbb{C}$ which fixes $\mathbb{R}$, hence the type of $z \in \mathbb{C} \setminus \mathbb{R}$ (over $\mathbb{R}$) is the same as that of $\bar{z} \in \mathbb{C}$, therefore $\{z\}$ is not definable over $\mathbb{R}$ (while $\{z, \bar{z}\}$ is).

Let $M \subseteq N$ be models of T_{an}. By Proposition 3.3, $M \preceq N$. For an element $y \in N$, let $M\langle y\rangle$ denote the L_{an}-definable closure of $M \cup \{y\}$ in N. If $a \in M\langle y\rangle$ is defined by the formula $\phi(x, b_1, \ldots, b_m)$ with $b_1, \ldots, b_m \in N$, that is to say

$$\{a\} = \{c \in N \mid N \models \phi(c, b_1, \ldots, b_m)\}.$$

If $a > 0$, then

$$\{\sqrt[n]{a}\} = \{c \in N \mid N \models \phi(c^n, b_1, \ldots, b_m)\},$$

and thus $\sqrt[n]{a} \in M\langle y\rangle$. If $a \neq 0$, then

$$\{a^{-1}\} = \{c \in N \mid N \models \exists d\, (dy = 1) \wedge (\phi(d, b_1, \ldots, b_m))\},$$

and therefore $a^{-1} \in M\langle y\rangle$. Hence, $M\langle y\rangle$ is closed under taking n-th roots and inverses, by Remark 3.5, $M\langle y\rangle \preceq N$.

Notice that by Proposition 3.3, every $\mathbb{R}_{\mathrm{an}}$-formula is equivalent to a quantifier-free $L_{\mathrm{an}}(^{-1})$-formula. Combining this with the fact that nonzero analytic functions have isolated zeros implies the o-minimality of $\mathbb{R}_{\mathrm{an}}$ and hence T_{an} by Remark 1.3.

Lemma 4.5. *Let $M \models T_{\mathrm{an}}$, and let N, N' be elementary extensions of M. Suppose $y \in N$ and $y' \in N'$ make the same cut in M, then there exists an L_{an}-isomorphism $M\langle y\rangle \to M\langle y'\rangle$ sending y to y' and fixing M pointwise.*

Proof. It suffices to prove that the type of $y \in N$ over M depends only on the cut y makes in M. In fact, by o-minimality of T_{an}, any definable set $E := \{a \in N \mid N \models \phi(a, \bar{b}), \bar{b} \in M^{|\bar{b}|}\}$ is a finite union of intervals and points, hence whether $y \in E$ or not depends only on the cut y makes in M. □

4.2 Valuations

Let us now look at the value group $\Gamma := v(M^\times)$ of a model $M \models T_{\mathrm{an}}$. We claim that Γ has a natural $\mathbb{Q}$-vector space structure. For any $v(a) \in \Gamma$, we can suppose $a > 0$ as $v(a) = v(-a)$. Then a admits a unique n-th root $a_n \in M$ and $n \cdot v(a_n) = v(a)$. If there is another $b \in M$ such that $n \cdot v(b) = v(b^n) = v(a)$, then $\frac{(a_n)^n}{b^n} \in U(M)$, the set of bounded non-infinitesimal elements, i.e., elements with valuation zero. Hence, $\frac{a_n}{b} \in U(M)$, and $v(a_n) = v(b)$. Therefore, for every rational number $\frac{m}{n} \in \mathbb{Q}$, we can define $\frac{m}{n} \cdot v(a)$ as $v(a_n^m)$.

We call a group homomorphism f from Γ to $M^\times$ a *section* if $v(f(g)) = g$ for all $g \in \Gamma$. Take a $\mathbb{Q}$-basis $\{v_j\}_{j \in J}$ of Γ, and suppose $v(b_j) = v_j$ for $b_j \in M_{>0}$. Define a map

$s\colon \Gamma \to M^\times$ by setting $s(v_j) = b_j$ and extending $\mathbb{Q}$-linearly. Then we note that s is a section, and thus sections always exist.

We now introduce an L_{an}-structure on the power series field $\mathbb{R}((t^\Gamma))$, where Γ is a totally ordered abelian group. This structure plays an important role in the results to follow. Let $\mu(\mathbb{R}((t^\Gamma)))$ denote the maximal ideal of the valuation ring of $\mathbb{R}((t^\Gamma))$, i.e.,

$$\mu(\mathbb{R}((t^\Gamma))) := \{x \in \mathbb{R}((t^\Gamma)) \mid v(x) > 0\}.$$

For each power series $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{R}[[X_1,\dots,X_n]]$ and tuple $\bar{y} = (y_1,\dots,y_n) \in \mu(\mathbb{R}((t^\Gamma)))^n \subseteq \mathbb{R}((t^\Gamma))^n$, we define

$$f(\bar{y}) := \sum_{\alpha\in\mathbb{N}^n} c_\alpha \bar{y}^\alpha.$$

This sum is well-defined in $\mathbb{R}((t^\Gamma))$ because each y_i lies in the maximal ideal and hence has only positive powers of t, ensuring that the resulting series is well-defined and its support is well-ordered.

Now let $f \in \mathbb{R}\{X_1,\dots,X_n\}$, the ring of restricted analytic functions, and consider $x = (x_1,\dots,x_n) \in I(\mathbb{R}((t^\Gamma)))^n := \{x \in \mathbb{R}((t^\Gamma))^n \mid -1 \le x_i \le 1 \text{ for all } i\}$. We write each coordinate $x_i = a_i + y_i$, where $a_i \in [-1,1] \subseteq \mathbb{R}$ and $y_i \in \mu(\mathbb{R}((t^\Gamma)))$. That is, $x = a + y$ with $a \in [-1,1]^n$ and $y \in \mu(\mathbb{R}((t^\Gamma)))^n$.

Let $f_a(X) \in \mathbb{R}[[X_1,\dots,X_n]]$ denote the Taylor expansion of f at the point a, given by:

$$f_a(X) = \sum_{i\in\mathbb{N}^n} \frac{1}{i!}\frac{\partial^{|i|} f}{\partial X^i}(a) X^i.$$

We then define $f(x) := f_a(y)$.

For $x \notin I(\mathbb{R}((t^\Gamma)))^n$, we set $f(x) = 0$. This endows $\mathbb{R}((t^\Gamma))$ with an L_{an}-structure, which we denote by $\mathbb{R}((t^\Gamma))_{\mathrm{an}}$. It can be verified (with some effort) that this structure satisfies the axioms A1) $-$ A4) from the axiomatization of T_{an}. Since $\mathbb{R}((t^\Gamma))$ is real closed (see end of §2), it follows from Theorem 3.4 and Proposition 3.6 that $\mathbb{R}_{\mathrm{an}} \preceq \mathbb{R}((t^\Gamma))_{\mathrm{an}}$.

Lemma 4.6. *Let $M, N \models T_{\mathrm{an}}$ be such that M is an L_{an}-substructure of N, and let $s\colon v(M^\times) \to M^\times$ be a section. Suppose we have an L_{an}-embedding $\sigma\colon M \to \mathbb{R}((t^{v(M^\times)}))_{\mathrm{an}}$, with $\sigma(s(g)) = t^g$ for all $g \in v(M^\times)$. If $y \in N \setminus M$ satisfies $v(M\langle y\rangle^\times) = v(M^\times)$, then σ can be extended to an L_{an}-embedding from $M\langle y\rangle$ to $\mathbb{R}((t^{v(M^\times)}))_{\mathrm{an}}$.*

For the proof of Lemma 4.6, consult [10, Lemma 3.3]: the idea is to find an element of $\mathbb{R}((t^{v(M^\times)}))_{\mathrm{an}}$ that realizes the cut of y in M and apply Lemma 4.5. We next prove a small technical lemma used in the proof of Lemma 4.8.

Lemma 4.7. *Let $K \subseteq L$ be real closed fields and $y \in L \setminus K$. If $v(K(y)^\times) \neq v(K^\times)$, then there exists $a \in K$ such that $v(y - a) \notin v(K^\times)$.*

Proof. There exists a monic K-polynomial $P(x)$ such that $v(P(y)) \notin v(K^\times)$. Since K is real closed, $P(x)$ can be factored into a product of linear and quadratic K-polynomials: $P(x) = \prod_i (x - c_i) \prod_j ((x - a_j)^2 + b_j^2)$. Suppose the result does not hold, then there exists j_0 such that $v((x - a_{j_0})^2 + b_{j_0}^2) \notin v(K^\times)$. Then $v((y - a_{j_0})^2) = v(b_{j_0}^2)$, and that $v((x - a_{j_0})^2 + b_{j_0}^2) > v(b_{j_0}^2)$. However, as $(x - a_{j_0})^2 + b_{j_0}^2 > b_{j_0}^2$, we have $v((x - a_{j_0})^2 + b_{j_0}^2) \le v(b_{j_0}^2)$. This is a contradiction. □

Lemma 4.8. *Keeping the notations in Lemma 4.6, let $y \in N \setminus M$ such that $v(M(y)^\times) \neq v(M^\times)$. Then σ can be extended to an L_{an}-embedding σ_1 from $M\langle y\rangle$ to $\mathbb{R}((t^{\Gamma_1}))_{\mathrm{an}}$, where Γ_1 is the divisible hull of $v(M(y)^\times)$ in $v(N^\times)$. Moreover, s can be extended to a section $s_1\colon \Gamma_1 \to M\langle y\rangle^\times$ such that $\sigma_1(s_1(h)) = t^h$, for all $h \in \Gamma_1$.*

Proof. Due to Lemma 4.7, there exists an $a \in M$ such that $v(y-a) \notin v(M^\times)$. Hence, by replacing y with $y-a$, we can assume that $v(y) \notin v(M^\times)$. Moreover, by replacing y with $-y$ if necessary, we assume that $y > 0$.

Let Γ_1 be the divisible subgroup generated by $v(M^\times) \cup \{v(y)\}$ in $v(N^\times)$. The section s can then be extended to $s_1\colon \Gamma_1 \to M\langle y\rangle^\times$ by defining $s_1(\gamma + qv(y)) = s(\gamma)y^q$ for all $\gamma \in \Gamma$ and $q \in \mathbb{Q}$. Notice that for a positive element $m \in M$, $m < y$ if and only if $v(m) > v(y)$ if and only if $m < t^{v(y)}$. Therefore, using Lemma 4.5, σ can be extended to an L_{an}-embedding σ_1 from $M\langle y\rangle$ to $\mathbb{R}((t^{\Gamma_1}))_{\mathrm{an}}$ sending y to $t^{v(y)}$, such that $\sigma_1(s_1(h)) = t^h$, for all $h \in \Gamma_1$. We see that $v(M\langle y\rangle^\times) = \Gamma_1$, and thus Γ_1 is indeed the divisible hull of $v(M(y)^\times)$ in $v(N^\times)$. □

Theorem 4.9. *Suppose $M \models T_{\mathrm{an}}$, and $v(M^\times) = \Gamma$, then there exists an L_{an}-embedding i of M into $\mathbb{R}((t^\Gamma))_{\mathrm{an}}$. Besides, given any section $s\colon \Gamma \to M^\times$, there is an embedding $i : M \hookrightarrow \mathbb{R}((t^\Gamma))_{\mathrm{an}}$ such that $i(s(g)) = t^g$, for all $g \in \Gamma$.*

Proof. The theorem follows by combining Zorn's lemma (begin with $M_0 = \mathbb{R}_{\mathrm{an}} \subseteq M, \Gamma_0 = \{0\}, s_0(0) = s(0) = 1$, and $i = \mathrm{id}_{\mathbb{R}_{\mathrm{an}}}$), Lemma 4.6, and Lemma 4.8. □

Corollary 4.10. *Suppose $M \hookrightarrow N$ is an embedding of T_{an}-models. Let $y \in N \setminus M$, then $v(M\langle y\rangle^\times)$ is the divisible hull of $v(M(y))$.*

Proof. By Theorem 4.9, we have an embedding σ from M into $\mathbb{R}_{\mathrm{an}}((t^\Gamma))$, where $\Gamma := v(M^\times)$, and a section s such that $\sigma(s(g)) = t^g$, for all $g \in \Gamma$.

If $v(M(y)^\times) = \Gamma$, then we can apply Lemma 4.6, and obtain an L_{an}-embedding $M\langle y\rangle \hookrightarrow \mathbb{R}((t^\Gamma))_{\mathrm{an}}$. Hence $v(M\langle y\rangle^\times) = \Gamma$.

If $v(M(y)^\times) \neq \Gamma$, then we can apply Lemma 4.8, to obtain an L_{an}-embedding $M\langle y\rangle \hookrightarrow \mathbb{R}((t^{\Gamma_1}))_{\mathrm{an}}$, where Γ_1 is the divisible hull of $v(M(y)^\times)$ in $v(N^\times)$. Hence, $v(M\langle y\rangle^\times) = \Gamma_1$. □

5 Quantifier elimination of $\mathrm{Th}(\mathbb{R}_{\mathrm{an}}, \exp)$

Recall that $L_{\mathrm{an}}(\exp)$ is the language L_{an} with a new function symbol "exp" corresponding to exponentiation. Let $T_{\mathrm{an}}(\exp)$ to be the theory obtained by adding the following five axioms to T_{an}:

E1) $\exp(x+y) = \exp(x)\exp(y)$;

E2) $x < y \to \exp(x) < \exp(y)$;

E3) $x > 0 \to \exists y \exp(y) = x$;

E4$_n$) $x > n^2 \to \exp(x) > x^n$, for each $n \in \mathbb{N}$;

E5) $-1 \leq x \leq 1 \to \exp(x) = E(x)$, where $E(x)$ is the function symbol of L_{an} corresponding to the power series $\sum_{i\geq 0} \frac{1}{i!}x^i$.

Let us define $L_{\mathrm{an}}(\exp, \log)$ to be the language obtained by adding a new unary function symbol corresponding to logarithm. Note that if $K \models T_{\mathrm{an}}(\exp)$, one can define "log" on K by setting $\exp(\log(x)) = x$ when $x > 0$, and $\log(x) = 0$ otherwise.

We write $F \subseteq_{\mathrm{an}} K$ to signify that F is an L_{an}-substructure of K, and we say F is *log-closed* if $\log(x) \in F$, for all $x \in F$. If moreover we have $L \models T_{\mathrm{an}}(\exp)$, we say that $\sigma : F \to L$ is a *log-preserving* embedding if it is an L_{an}-embedding and $\sigma(\log(x)) = \log(\sigma(x))$, for all $x \in F$. For $x \in K \setminus F$, we denote $F\langle x\rangle$ as the L_{an}-definable closure of $F \cup \{x\}$ in K.

To state the next theorem, we need to introduce the notion of saturated models.

Definition 5.1 (Saturated model)**.** For κ a cardinal number, and M a structure of a first-order language, M is called κ-saturated if for any subset $E \subseteq M$ of cardinality strictly less than κ, all types over E are realized in M.

Theorem 5.2. *Suppose $K \models T_{\text{an}}(\exp)$, $F_0 \subseteq_{\text{an}} K$ is log-closed and $F_0 \models T_{\text{an}}$. Let L be a $|K|^+$-saturated model of $T_{\text{an}}(\exp)$ and $\sigma_0 : F_0 \to L$ be a log-preserving embedding, then one can extend it to a log-preserving embedding from K to L.*

We approach Theorem 5.2 by proving three lemmas below (each one of which extends σ_0 by a little). Due to the length of this survey, we present only the proof of Lemma 5.3; the other two lemmas can be proved using roughly the same ideas, though in a slightly more sophisticated manner (see [10, Lemmas 4.3 and 4.4]).

Lemma 5.3. *Keeping the notations in Theorem 5.2, suppose $x \in K \setminus F_0$ and $v(F_0^\times) = v(F_0(x)^\times)$. Then $F := F_0\langle x\rangle$ is log-closed and σ_0 can be extended to a log-preserving embedding from F to L.*

Proof. By Corollary 4.10, $v(F^\times) = v(F_0(x)^\times) = v(F_0^\times)$. We first prove that F is log-closed. For any positive element $w \in F$, there exists $z \in F_0$ such that $v(z) = v(w)$; therefore, $v(w/z) = 0$, and there exists $r \in \mathbb{R}$ such that $\epsilon := w/z - r$ is an infinitesimal. Thus, $w = z(r + \epsilon) = rz(1 + \epsilon/r)$.

Then $\log(w) = \log(rz) + \log(1 + \epsilon/r)$. Note that $\log(rz) \in F_0$, since F_0 is log-closed. Moreover, as log is analytic at 1, there is an L_{an}-term t such that $\log(1 + \delta) = t(\delta)$ for all infinitesimals δ. Hence, $\log(1 + \epsilon/r) = t(\epsilon/r) \in F$, and we conclude that $\log(w) \in F$.

Since L is $|K|^+$-saturated, there exists $y \in L$ whose type over $\sigma_0(F_0)$ is the same as the type of x over F_0. Thus, σ_0 extends to an L_{an}-embedding $\sigma\colon F \to L$ sending x to y. We now show that σ is log-preserving.

For the same $w \in F$ as above, $\sigma(w) = \sigma(rz)(1 + \sigma(\epsilon/r))$. Since σ_0 is log-preserving, $\log\sigma(rz) = \sigma\log(rz)$. Furthermore,

$$\sigma(\log(1 + \epsilon/r)) = \sigma(t(\epsilon/r)) = t(\sigma(\epsilon/r)) = \log(1 + \sigma(\epsilon/r)).$$

Therefore, $\sigma(\log(w)) = \log\sigma(w)$. □

Lemma 5.4. *Keeping the notations in Theorem 5.2, suppose for any $y \in K \setminus F_0$, $v(F_0^\times) \neq v(F_0(y)^\times)$. Suppose $x \in F_0$ and $\exp x \notin F_0$. Then $F := F_0\langle\exp x\rangle$ is log-closed and σ_0 can be extended to a log-preserving embedding σ from F to L such that $\sigma(\exp x) = \exp\sigma(x)$.*

Lemma 5.5. *Keeping the notations in Theorem 5.2, suppose for any $y \in K \setminus F_0$, $v(F_0^\times) \neq v(F_0(y)^\times)$, and that F_0 is closed under* exp*. Let $x \in K \setminus F_0$, then there exists a log-closed $F \subseteq_{\text{an}} K$ containing $F_0(x)$ such that $F \models T_{\text{an}}$, and moreover, σ_0 can be extended to a log-preserving embedding from F to L.*

Proof of Theorem 5.2. Let us consider the set $\mathcal{P}$ of L_{an}-substructures of K that are log-closed and model T_{an}. Suppose we have an ascending chain $(F_i)_{i\in I}$ of elements in

$\mathcal{P}$, ordered by inclusion. Consider $\hat{F} := \bigcup_{i\in I} F_i$, which is clearly an L_{an}-substructure of K and log-closed.

To show that $\hat{F} \in \mathcal{P}$, it remains to prove $\hat{F} \models T_{\text{an}}$. Since $F_i \models T_{\text{an}}$, the structure F_i is in particular closed under taking n-th roots and inverses, hence $\hat{F}$ is as well. Thus, $\hat{F}$ can be seen as an $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$-substructure of K. As $K \models T_{\text{an}}$ and by Proposition 3.6 the theory T_{an} has a universal axiomatization in $L_{\text{an}}({}^{-1}, \sqrt[n]{\cdot})$, it follows that $\hat{F} \models T_{\text{an}}$. We conclude by applying Zorn's lemma, Lemma 5.3, Lemma 5.4, and Lemma 5.5. □

Proposition 5.6. *The theory $T_{\text{an}}(\exp)$ has quantifier elimination in $L_{\text{an}}(\exp, \log)$.*

Proof. We use the following criterion: for $M, N \models T$ such that N is $|M|^+$-saturated, $E \subseteq M$ an L-substructure, and $\sigma\colon E \to N$ an L-embedding, if there always exists an extension of σ to an L-embedding from M to N, then T admits quantifier elimination.

Now let M, N, E as in the criterion with $L = L_{\text{an}}(\exp, \log)$. Since E is an L-substructure, for a positive element $x \in E$, $\sqrt[n]{x} = \exp(\frac{\log(x)}{n}) \in E$, and $x^{-1} = \exp(-\log(x)) \in E$, while for a negative element $x \in E$, $x^{-1} = -\exp(-\log(-x)) \in E$. Hence, E is closed under taking n-th roots and inverses, by Remark 3.5, $E \models T_{\text{an}}$. We finish by applying Theorem 5.2 with M, N, E, σ in place of K, L, F_0, and σ_0. □

Corollary 5.7. *The theory $T_{\text{an}}(\exp)$ gives an axiomatization of* $\mathrm{Th}(\mathbb{R}_{\text{an}}, \exp)$.

Proof. Suppose $M \models T_{\text{an}}(\exp)$, then it can be viewed as an $L_{\text{an}}(\exp, \log)$-extension of $(\mathbb{R}_{\text{an}}, \exp, \log)$ (as said in the beginning of this section, log can be defined naturally on M). By Proposition 5.6, $(\mathbb{R}_{\text{an}}, \exp, \log) \preceq M$ as $L_{\text{an}}(\exp, \log)$-structures. Hence, $(\mathbb{R}_{\text{an}}, \exp) \equiv M$ as $L_{\text{an}}(\exp)$-structures and $T_{\text{an}}(\exp)$ is an axiomatization of $\mathrm{Th}(\mathbb{R}_{\text{an}}, \exp)$. □

Remark 5.8. Define $T_{\text{an}}(\exp, \log)$ to be the theory obtained by replacing the axiom E3) in $T_{\text{an}}(\exp)$ by

$$\forall x > 0 \ (\exp(\log x) = x).$$

Then by Proposition 5.6, $T_{\text{an}}(\exp, \log)$ admits quantifier elimination in $L_{\text{an}}(\exp, \log)$, and we have shown implicitly in the proof of Corollary 5.7 that $T_{\text{an}}(\exp, \log)$ is a universal axiomatization of $\mathrm{Th}(\mathbb{R}_{\text{an}}, \exp, \log)$.

By rewriting the proof of Corollary 3.7, we obtain the following parallel result.

Corollary 5.9. *For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ definable in $\mathbb{R}_{\text{an}}(\exp)$, there are $L_{\text{an}}(\exp, \log)$-terms $t_1(x_1, \ldots, x_n), \ldots t_k(x_1, \ldots, x_n)$ such that for all $a \in \mathbb{R}^n$, there is an integer j satisfying $f(a) = t_j(a)$.*

6 Hardy field methods

Let $L := \{0, 1, <, +, -, \cdot, \ldots\}$ be an expansion of the language of ordered rings where no relation symbols are added. Denote $\mathcal{R}$ as the L-structure on the real numbers $\mathbb{R}$, and T its theory. We also define $\mathbb{R}$-terms as L-terms with parameters taken from $\mathbb{R}$. Let us first give a criterion for the o-minimality of T (consult [10, Lemma 5.2] for a proof).

Lemma 6.1. *Suppose the theory T admits quantifier elimination, then T is o-minimal if and only if for each $\mathbb{R}$-term $t(x)$, there exists $n \in \mathbb{R}$ such that either $t(x) > 0$ when $x > n$, or $t(x) = 0$ when $x > n$, or $t(x) < 0$ when $x > n$.*

We define $\mathcal{G}$ to be the ring of functions (not necessarily continuous) $f\colon \mathbb{R} \to \mathbb{R}$ modulo the ideal

$$I := \{h\colon \mathbb{R} \to \mathbb{R} \mid \text{there exists } n \in \mathbb{N} \text{ such that } h(x) = 0 \text{ for all } x > n\}.$$

Essentially, only the behavior of a function near $+\infty$ is taken into account. Elements of $\mathcal{G}$ are called germs at $+\infty$ of functions. There is a natural partial order on $\mathcal{G}$ defined by $f < g$ if and only if $(f-g)(x) < 0$ for all sufficiently large x.

A subring A of $\mathcal{G}$ is called a $\mathcal{G}$-domain if for any element $f \in A \setminus \{0\}$, when x is sufficiently large, either $f(x)$ is always positive, or $f(x)$ is always negative. Note that a $\mathcal{G}$-domain is certainly an integral domain and that the partial order of $\mathcal{G}$ restricted on it is a total order.

For a term $t(x_1, \ldots, x_n)$ of the language L, define $t_{\mathcal{G}}\colon \mathcal{G}^n \to \mathcal{G}$ by setting $t_{\mathcal{G}}(f_1, \ldots, f_n)$ to be the function that sends x to $t(f_1(x), \ldots, f_n(x))$. A $\mathcal{G}$-domain B is called an $\mathcal{R}$-domain if it is closed under $t_{\mathcal{G}}$ for all terms t, it is further called an $\mathcal{R}$-field if B itself is a field. Note that we have a natural L-structure on an $\mathcal{R}$-domain B: for an n-nary function symbol F of L, define $F_B : B^n \to B$ as $F_{\mathcal{G}}|_{B^n}$.

Example 6.2. Consider $\mathcal{R} = \{\mathbb{R}, <, 0, 1, +, -, \cdot, ^{-1}\}$, the ring of polynomials $\mathbb{R}[x]$ is a $\mathcal{G}$-domain and the field of rational functions $\mathbb{R}(x)$ is an $\mathcal{R}$-field.

The next result follows immediately from Lemma 6.1.

Proposition 6.3. *If T admits quantifier elimination, and there exists an $\mathcal{R}$-field containing $\mathbb{R}(x)$, then $\mathcal{R}$ is o-minimal.*

From now on, we assume that the theory T admits quantifier elimination and a universal axiomatization.

Example 6.4. Both $T_{\text{an}}(^{-1}, \sqrt[n]{\cdot})$ and $T_{\text{an}}(\exp, \log)$ satisfy these assumptions (see Proposition 3.6 and Remark 5.8).

Lemma 6.5. *Let K be an $\mathcal{R}$-domain, then $K \models T$.*

Proof. For each function $f\colon \mathbb{R} \to \mathbb{R}$, add a unary function symbol f to the language L to obtain a new language L'. By Löwenheim–Skolem theorem, there exists a proper elementary extension M of $\mathscr{R} := (\mathcal{R}, (f)_{f\colon \mathbb{R}\to\mathbb{R}})$. Pick a positive infinite element $a \in M$, and define a map $i_a\colon \mathcal{G} \to M$ by sending f to $f_M(a)$. The map i_a is well-defined: if there exists $n \in \mathbb{N}$ such that $\mathscr{R} \models \forall x > n\,(f(x) = g(x))$, then $M \models \forall x > n\,(f(x) = g(x))$, and thus $f_M(a) = g_M(a)$.

Note that for any functions $f, g\colon \mathbb{R} \to \mathbb{R}$, we have

$$\mathscr{R} \models \forall x\, \big((f+g)(x) = f(x) + g(x)\big) \wedge \big((f \cdot g)(x) = f(x) \cdot g(x)\big),$$

and

$$\mathscr{R} \models \forall x\, \big((F(f_1, \ldots, f_n))(x) = F(f_1(x), \ldots, f_n(x))\big)$$

for any n-ary function symbol F of L, and functions $f_1, \ldots, f_n\colon \mathbb{R} \to \mathbb{R}$. Thus, M models these formulas as well, and i_a is a ring morphism satisfying

$$i_a(F_{\mathcal{G}}(f_1, \ldots, f_n)) = F_M(i_a(f_1), \ldots, i_a(f_n)),$$

for any n-ary function symbol F of L, and functions $f_1, \ldots, f_n \in \mathcal{G}$.

For two distinct elements $f, g \in K$, we have $\mathscr{R} \models \forall x\,(f(x) \neq g(x))$; hence, $M \models \forall x\,(f(x) \neq g(x))$, and so $i_a(f) \neq i_a(g)$. We conclude that $i_a|_K\colon K \to M$ is an L-embedding. Since T has a universal axiomatization, it follows that $K \models T$. $\square$

Remark 6.6. Note that $T \models \forall x \neq 0\, \exists y\, (xy = 1)$. Thus, $K \models \forall x \neq 0\, \exists y\, (xy = 1)$. Hence, any $\mathcal{R}$-domain K is in fact an $\mathcal{R}$-field.

Using the model M constructed in Lemma 6.5 and some reasoning concerning elementary extensions, one can prove the following result (see [10, Lemma 5.9]).

Proposition 6.7. *Let K be an $\mathcal{R}$-field, and suppose $g \in \mathcal{G}$ is comparable to every element of K under the partial order $<$ on $\mathcal{G}$. Assuming T is o-minimal, then*

$$K\langle g\rangle := \{t_{\mathcal{G}}(f_1, \dots, f_n, g) \mid t(x_1, \dots, x_{n+1}) \text{ is a term of } L \text{ and } f_1, \dots, f_n \in K\}$$

is an $\mathcal{R}$-field.

Remark 6.8. The field $K\langle g\rangle$ is clearly the smallest $\mathcal{R}$-field containing $K \cup \{g\}$.

For an element $f \in \mathcal{G}$, we say that f is a $\mathcal{C}^1$-germ (at $+\infty$) if $f(x)$ is $\mathcal{C}^1$ for x sufficiently large. In this case, we can define $f' \in \mathcal{G}$ as the germ at $+\infty$ of $f'(x)\,(x \gg 1)$.

Definition 6.9 (Hardy field)**.** A field $K \subseteq \mathcal{G}$ is called a Hardy field if it is a $\mathcal{G}$-domain and that for any $f \in K$, f is a $\mathcal{C}^1$-germ, and $f' \in K$. It is furthermore called an $\mathcal{R}$-Hardy field if K is an $\mathcal{R}$-field.

We have below a small lemma concerning comparability (consult [10, Lemma 5.11] for a quick proof).

Lemma 6.10. *Let K be a Hardy field, and $f \in K$. Then $e^f \in \mathcal{G}$ is comparable to K, and moreover if $f > 0$, then $\log f$ is comparable to K.*

Proposition 6.11. *Suppose T is o-minimal and K is an $\mathcal{R}$-Hardy field. Let $f \in K$, then $K\langle e^f\rangle$ (defined in Proposition 6.7) is an $\mathcal{R}$-Hardy field. Besides, if $f > 0$, then $K\langle \log f\rangle$ is an $\mathcal{R}$-Hardy field.*

Proof. Let $g \in K\langle e^f\rangle$ be arbitrary. Then there exists an L-term $t(x_1, \dots, x_{n+1})$ and elements $f_1, \dots, f_n \in K$ such that $g = t(f_1, \dots, f_n, e^f)$. Our objective is to show that g represents the germ of a $\mathcal{C}^1$-function, and that its derivative also belongs to $K\langle e^f\rangle$.

Since T is o-minimal, the term $t(x_1, \dots, x_{n+1})$ in L admits a finite decomposition: there exist L-formulas $\phi_1(\bar{x}), \dots, \phi_k(\bar{x})$ such that for any model $\mathcal{A}$ of T, $\mathcal{A}^{n+1}$ is partitioned into finitely many definable sets $C_1, \dots, C_k$, such that each $C_j\, (1 \leq j \leq k)$ is defined by $\phi_j(\bar{x})$ and on which the term t coincides with a definable function h_j (independent of the choice of $\mathcal{A}$) that is $\mathcal{C}^1$ on a open neighborhood of C_j when interpreted in $\mathcal{R}$. This follows from the cell decomposition theorem for o-minimal structures; consult [9, Theorem 3.2.6].

By Lemma 6.5, $K \models T$. Using Lemma 6.10, the element e^f is comparable to K and thus it makes a cut in K. Let h_{i_0} be the C^1-function corresponding to the unique cell C_{i_0} such that the formula $\psi(y) := \phi_{i_0}(f_1, \dots, f_n, y)$ lies in the type of e^f over K. As T is o-minimal, there exists $k_1, k_2 \in K \cup \{\pm\infty\}$ such that $k_1 < e^f < k_2$ and $(f_1, \dots, f_n, z) \in C_{i_0}$, for all $z \in K \cap (k_1, k_2)$.

We denote M as the proper elementary extension of $\mathscr{R} := (\mathcal{R}, (f)_{f\colon \mathbb{R}\to\mathbb{R}})$ constructed in Lemma 6.5. As in Lemma 6.5, we have an L-embedding $i_a|_K\colon K \to M$ for every positive infinite element $a \in M$. Since T admits quantifier elimination and universal axiomatization, $K \preceq M$. Notice that

$$K \models \forall x\, (k_1 < x < k_2 \to \phi_{i_0}(f_1, \dots, f_n, x)),$$

therefore, $M \models \forall x\,(i_a(k_1) < x < i_a(k_2) \to \phi_{i_0}(i_a(f_1),\ldots,i_a(f_n),x))$, and in particular, taking $x = i_a(e^f)$ gives

$$M \models \phi_{i_0}(f_1(a),\ldots,f_n(a),e^{f(a)}).$$

Since this does not depend on a, we conclude that

$$M \models \exists x \forall y\,(y > x \to \phi_{i_0}(f_1(y),\ldots,f_n(y),e^{f(y)})),$$

as M is an elementary extension of $(\mathcal{R},(f)_{f\colon \mathbb{R}\to\mathbb{R}})$, therefore,

$$\mathcal{R} \models \exists x \forall y\,(y > x \to \phi_{i_0}(f_1(y),\ldots,f_n(y),e^{f(y)})).$$

Thus, $g(x) = h_{i_0}(f_1(x),\ldots,f_n(x),e^{f(x)})$ for $x \gg 1$. Because K is a Hardy field, the functions $f_1,\ldots,f_n$ are ultimately differentiable, and the function e^f is also differentiable, with derivative $f'e^f \in K\langle e^f\rangle$, as $f' \in K$. It follows that the composition

$$x \mapsto h_{i_0}(f_1(x),\ldots,f_n(x),e^{f(x)})$$

is $\mathcal{C}^1$ for all sufficiently large x, and its derivative can be expressed using the chain rule below:

$$\begin{aligned} g'(x) = {} & \sum_{i=1}^{n} \frac{\partial h_{i_0}}{\partial x_i}(f_1(x),\ldots,f_n(x),e^{f(x)}) \cdot f_i'(x) \\ & + \frac{\partial h_{i_0}}{\partial x_{n+1}}(f_1(x),\ldots,f_n(x),e^{f(x)}) \cdot f'(x)e^{f(x)}. \end{aligned}$$

Since T is o-minimal, [5, Lemma 3] states that the partial derivatives of a definable differentiable function are themselves definable. Therefore, $g' \in K\langle e^f\rangle$, establishing that $K\langle e^f\rangle$ is a Hardy field.

For the case of logarithms, suppose $f > 0$. Then $\log(f)$ is defined and differentiable for sufficiently large x, with $(\log f)' = \frac{f'}{f} \in K$, since both f and f' are in K. Arguing as above, one sees that $K\langle \log(f)\rangle$ is a Hardy field. □

Remark 6.12. The proof of Proposition 6.11 actually shows the following result: for T an o-minimal theory, K an $\mathcal{R}$-Hardy field, h a $\mathcal{C}^1$-germ comparable to K satisfying $h' \in K\langle h\rangle$, we have that $K\langle h\rangle$ is an $\mathcal{R}$-Hardy field.

By combining Zorn's lemma and Proposition 6.11, we obtain the result below.

Corollary 6.13. *Suppose the theory T is o-minimal, then every $\mathcal{R}$-Hardy field can be extended to an $\mathcal{R}$-Hardy field that is closed under exponentiation and logarithmization (of positive elements).*

We can now prove the o-minimality of $(\mathbb{R}_{\text{an}},\exp)$.

Theorem 6.14. *The structure $(\mathbb{R}_{\text{an}},\exp)$ is o-minimal.*

Proof. Let $\mathcal{R}$ denote $(\mathbb{R}_{\text{an}},{}^{-1},\sqrt[n]{\cdot})$, an expansion of the real field. We have seen in Proposition 3.6 that $T_{\text{an}}({}^{-1},\sqrt[n]{\cdot})$ is a universal axiomatization of $\text{Th}(\mathbb{R}_{\text{an}},{}^{-1},\sqrt[n]{\cdot})$ and that $T_{\text{an}}({}^{-1},\sqrt[n]{\cdot})$ admits quantifier elimination in $L_{\text{an}}({}^{-1},\sqrt[n]{\cdot})$. Therefore, $\mathcal{R}$ is o-minimal (the main idea is that analytic functions have isolated zeros), and thus $T_{\text{an}}({}^{-1},\sqrt[n]{\cdot})$ is as well by Remark 1.3.

Consider the $\mathcal{R}$-Hardy field of $\mathcal{R}$-definable functions, using Corollary 6.13, we can extend it to an $\mathcal{R}$-Hardy field H closed under exponentiation and logarithmization. We can view H as an $(\mathcal{R},\exp,\log)$-Hardy field. Since $T_{\text{an}}(\exp,\log)$ admits quantifier elimination (Remark 5.8), we see that the structure $(\mathcal{R},\exp,\log)$ is o-minimal by using Proposition 6.3. Therefore, $(\mathbb{R}_{\text{an}},\exp)$ is o-minimal. □

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