Rapport de stage de M1

Baptiste Serraille, encadré par Lev Buhovsky

August 6, 2022

Contents

1 Déroulement du stage

moins longtemps que prévu à cause du délai d'obtention du visa israélien et du fait que mon encadrant devait partir sur la fin Juin. Le stage a été prolongé pendant le reste de l'été, en particulier pour clôturer la pré-publication et discuter des différentes possibilités de généralisation des résultats.

Un peu avant le début du stage j'avais demandé à mon encadrant ce que je pouvais lire pour préparer le stage au mieux, il m'a ainsi conseillé de lire quelques chapitres de [9] ainsi que les sections des articles [6] et [14] consacrées à la propriété de C^0-fragmentation des difféomorphismes hamiltoniens. C'est ainsi que les préoccupations administratives et ces références ont remplis le mois précédent mon départ. Le livre [9] m'a par ailleurs accompagné à différents moments du stage pour compléter des lacunes éventuelles dans certains domaines.

En arrivant j'ai presque immédiatement essayé de répondre à la première question de mon encadrant et j'ai répondu par l'affirmative dans le Theorem 1 qui est un amélioration des propriété de C^0-fragmentation précédente. J'ai ensuite passé un peu de temps à rédiger une première version de la preuve avec l'aide de Maksim Stokic un thésard de Lev. Après un peu de travail je me rends compte qu'il y a un passage de la preuve emprunté à [6] que je ne comprend pas, Lev non plus et il me donne alors une autre preuve qui, après quelques ajustements techniques, finira dans l'article (Section 4). Dans un second temps Lev me pose une seconde question : est-ce que ce que l'on vient de démontrer peut aussi se démontrer dans le cas des homéomorphismes? La piste de recherche qu'il me donne est alors d'essayer d'approcher les homéomorphismes par limite uniforme de difféomorphismes. Après avoir réfléchi pendant une semaine sur cette approche il me semble qu'elle se heurte à un problème qui semble extrêmement technique voir impossible à résoudre. Après discussion avec Lev il semble d'accord avec moi mais je continue de chercher dans cette direction avant de me rendre compte que l'on peut plutôt adapter la preuve du théorème précédent avec des méthodes pour les homéomorphismes seulement. Je réponds alors par l'affirmative à la deuxième question avec le Theorem 2.

La vie à Tel-Aviv était très agréable mlagré les prix très élevés, j'ai rencontré beaucoup de gens très sympathiques dans le laboratoire d'étudiants du bâtiment du département de mathématiques de l'université dans une ambiance très accueillante et chaleureuse. C'est d'ailleurs avec quelques uns d'entre eux que j'ai visité Jérusalem et Tel-Aviv. Je tiens d'ailleurs à remercier tous les étudiants du labo qui ont rendu mon séjour agréable, plus que ce que je pensais possible. Je voudrais remercier plus particulièrement Sahar, Ilay, Michal, Nofar et Omer qui sont devenus des très bons amis.

Mon séjour a également été l'occasion de prendre des cours de combinatoire qui existent beaucoup moins en France et au DMA. J'ai ainsi participé à un cours de Michael Krivelevich sur les coloriages de graphes et d'hypergraphes. J'ai également participé au séminaire de combinatoire en plus du séminaire de géométrie symplectique.
Je voudrais remercier l'université de Tel-Aviv et Menashe Dvir-D'Ancona qui m'ont permis de trouver un logement dans l'université et m'ont donné un bureau. Je tiens également à remercier vivement Lev Buhovsky ainsi que mon tuteur François Charles qui m'a mis en contact avec lui.

2 Introduction à la topologie symplectique

On présente ici une introduction à la topologie symplectique, on voudrait faire en sorte que le lecteur qui a suivi un cours de géométrie différentielle puisse comprendre l'article qui suit après la lecture de l'introduction.

2.1 Définition d'une variété symplectique

2.1.1 Géométrie symplectique linéaire

On commence dans un premier temps par définir les formes symplectiques linéaires ainsi que quelques théorèmes de réduction. On suit en partie [9].

Définition 1. *Une* ***forme symplectique linéaire*** $\omega(\cdot,\cdot)$ *sur* E *un espace vectoriel de dimension fini est une forme bilinéaire antisymétrique non-dégénérée. On notera* (E,ω) *un espace vectoriel munie d'une forme symplectique.*

On peut maintenant définir la notion de symplectomorphisme linéaire.

Définition 2. *Soit* (V, ω) *un espace vectoriel munie d'une forme symplectique* ω*, et soit* $\Psi : V \to V$ *un isomorphisme d'espace vectoriel qui préserve la forme symplectique* ω*, c'est-à-dire que l'on a l'égalité suivante*

$$\omega = \Psi^*\omega := \omega(\Psi\cdot, \Psi\cdot),$$

on dira que Ψ *est un* ***symplectomorphisme linéaire****. On note* $Sp(V, \omega)$ *le groupe des symplectomorphismes linéaires.*

De manière analogue à la notion de sous-espace orthogonal à sous-espace vectoriel on peut définir des sous-espaces vectoriels complémentaires symplectiques.

Définition 3. *Soit* (V, ω) *un espace vectoriel symplectique et* $W^\omega \subset V$ *un sous-espace vectoriel, on note*

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \quad \forall w \in W\},$$

le ***complémentaire symplectique*** *de* W*.*

La non-dégénerescence de la forme symplectique ω montre alors le lemme suivant.

Lemme 1. *On a les deux identités suivantes*

$$\dim(W) + \dim(W^\omega) = \dim(V), \quad W^{\omega\omega} = W.$$

Si de plus, $W \cap W^\omega = \emptyset$ *on a que* $W \oplus W^\omega = V$ *et on peut alors former deux espaces vectoriels symplectiques* $(W, \omega \mid_W)$ *et* $(W^\omega, \omega \mid_{W^\omega})$*. On dira d'ailleurs que* W *est un* ***sous-espace vectoriel symplectique****.*

Théorème 1. *Soit* (V, ω) *espace vectoriel symplectique, alors* V *est un espace vectoriel de dimension paire et admet une* ***base symplectique****, c'est-à-dire une base* $u_1, \ldots, u_n, v_1, \ldots, v_n$ *de* V *telle que*

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \quad \omega(u_j, v_k) = \delta_{j,k}.$$

Proof. On va procéder par récurrence sur la dimension de V.

Tout d'abord les espaces vectoriels symplectiques de dimension 1 n'existent pas. En effet, supposons par l'absurde l'existence de (E, ω) un espace vectoriel symplectique de dimension 1, alors pour $x \neq 0$, $\omega(x, x) = -\omega(x, x)$ donc $\omega(x, x) = 0$ et ω est dégénérée. Cela est la contradiction recherchée.

On peut donc supposer que $\dim(V) \geq 2$, on choisit alors u_1 et v_1 deux vecteurs de telle sorte que $\omega(u_1, v_1) = 1$.

On note $W :=< u_1, v_1 >$ le sous-espace engendré par u_1 et v_1. Dans ce cas ni u_1, ni u_2 ne sont dans W^ω et ainsi d'après le Lemme 1, $W \cap W^\omega = \emptyset$ et W^ω est un sous-espace vectoriel munie de la forme symplectique $\omega \mid_{W^\omega}$. On peut alors appliquer l'hypothèse de récurrence sur W^ω, alors W^ω est un espace vectoriel de dimension paire et il existe $u_2, \ldots, u_n, v_2, \ldots, v_n$ une base symplectique de W^ω. Dans ce cas la base $u_1, \ldots, u_n, v_1, \ldots, v_n$ est une base symplectique de V ce qui achève la récurrence. □

Le Théorème 1 montre alors que tous les espaces vectoriels n'admettent pas une forme symplectique, en effet les espaces vectoriels de dimension impaire ne peuvent pas admettre de forme symplectique. On donne une construction pour tous les espaces de dimension paire.

Remarque 1. *On munit l'espace vectoriel* $\mathbb{R}^{2n}$ *de la forme symplectique*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

où $x_1, \ldots, x_n, y_1, \ldots, y_n$ *est une base quelconque de* $\mathbb{R}^{2n}$*. Le Théorème 1 nous montre qu'il s'agit du seul type de forme symplectique sur une espace vectoriel.*

La géométrie des formes symplectiques est reliée à la géométrie des formes volumes de la manière suivante.

Corollaire 1. *Soit* (V, ω) *un espace vectoriel symplectique de dimension* $2n$*, alors* $\omega^n = \omega \wedge \ldots \wedge \omega$ *est une forme volume sur* V*.*

Notre travail sur les espaces vectoriels symplectiques nous permet maintenant de définir les variétés symplectiques.

2.1.2 Les variétés symplectiques

Définition 4. *Soit* M *une variété différentielle, une* ***forme symplectique*** *est une 2-forme différentielle* $\omega \in \Omega^2(M)$ *fermée et non-dégénérée, c'est-à-dire que* $d\omega = 0$ *et pour tout point* $q \in M$*,* $(T_qM, \omega \mid_{T_qM})$ *est un espace vectoriel symplectique. On dira que* (M, ω) *est une* ***variété symplectique****.*

Le corollaire suivant découle encore du Théorème 1.

Corollaire 2. *Si* (M, ω) *est une variété symplectique, alors la dimension de* M *est paire.*

Corollaire 3. *Soit* (M, ω) *une variété symplectique de dimension* $2n$*, alors* $\Omega = \omega^n$ *est une forme volume sur* M*. Il suit que* M *dispose d'une orientation.*

L'exemple de le Remarque 1 est un premier exemple de variété symplectique. De manière générale, si l'on voit $\mathbb{R}^{2n}$ comme $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$ (ou autrement dit que $(y_i)_i$ est la base duale des $(x_i)_i$), on peut de manière analogue définir une forme symplectique sur la fibré cotangent T^*M d'une variété différentielle M.

2.1.3 Le groupe des symplectomorphismes

Définition 5. *Soit* (M, ω) *une variété symplectique et soit* $f \in Diff(M)$*, on dit que* f *est un* ***symplectomorphisme*** *lorsque* f *préserve la forme symplectique, c'est-à-dire que*

$$\omega = f^*\omega := \omega(df\cdot, df\cdot).$$

On note $Symp(M, \omega)$ *le* ***groupe des symplectomophismes*** *de* (M, ω)*.*

Comprendre les isotopies symplectiques. Supposons que l'on dispose d'une isotopie symplectique Ψ_t, c'est-à-dire que Ψ_t est une famille lisse de symplectomorphismes. On voudrait comprendre la famille Ψ_t vu comme un flot. En effet, ces deux points de vue sont équivalents de la manière suivante. Pour Ψ_t une isotopie on peut définir un champs de vecteur dépendant du temps X_t par l'identité qui suit.

$$\partial_t \Psi_t = X_t \circ \Psi_t.$$

L'isotopie Ψ_t est alors le flot du champs de vecteur (dépendant du temps) X_t. On voudrait trouver une condition nécessaire et suffisante sur le champ de vecteur X_t pour que Ψ_t soit e, effet une isotopie symplectique. Comme il faut forcément que Ψ_0 soit lui-même un difféormorphismes symplectique on supposera cette condition vérifiée (en regardant $\Psi_0^{-1} \circ \Psi_t$ ou bien $\Psi_t \circ \Psi_0^{-1}$on remarque que l'on peut même supposer que $\Psi_0 = Id$). On effectue alors le calcul suivant en dérivant la relation $\Psi_t^*\omega = \omega$.

$$\begin{aligned} 0 &= \partial_t \left(\Psi_t^*\omega\right) \\ &= \Psi_t^* \left(\mathcal{L}_{X_t}\omega\right) \\ &= \Psi_t^* \left(d(\iota_{X_t}\omega) + \iota_{X_t} d\omega\right). \end{aligned}$$

Ce qui montre l'identité suivante

$$d(\iota_{X_t}\omega) = 0$$

De manière équivalent si $d(\iota_{X_t}\omega) = 0$ et que Ψ_0 est un symplectomorphisme, alors Ψ_t est une istotopie symplectique. Cela motive la définition suivante.

Définition 6. *Soit X_t un champs de vecteur, on dira que X_t est un **champs de vecteur symplectique** si $d(\iota_{X_t}\omega) = 0$. Si de plus, $\iota_{X_t}\omega$ est exacte on dira que X_t est un **champs de vecteur hamiltonien** et une famille de fonction $H_t : M \to \mathbb{R}$ telle que $\iota_{X_t}\omega = dH_t$ sera dite **hamiltonienne**.*

Remarque 2. *Le calcul que l'on a déroulé précédemment peut être affiné pour une famille de forme symplectique ω_t, on appelle cela le "trick de Moser", un énoncé précis du resultat peut-être trouvé dans la Proposition 5.2.*

2.2 Quelques résultats utilisant des propriétés de fragmentation

Les propriétés de fragmentation des difféomorphismes et symplectomorphismes ont servi à démontrer plusieurs résultats importants sur les structures des groupes de difféomorphismes. On donne ici les énoncés de certains résultats (on réfère à [1] pour plus de détails). On rappelle dans un premier temps quelques définitions sur les groupes et leurs commutateurs.

Définition 7. *Soit G un groupe, on dira que G est **parfait** si il est égal à son groupe des commutateurs ($H_1(G) := G/[G,G]$ est trivial).*

*On dira que G est **simple** si il ne possède pas de sous-groupe distingué autre que lui-même et le sous-groupe trivial.*

Tout d'abord dans le cas des $\mathcal{C}^k$-difféomorphismes isotopiques à l'identité.

Théorème 2. *(Mather [10] et Thurston [15]) Soit M une variété différentielle paracompacte, connexe et de dimension n, alors $Diff_0^k(M)$ où $k \neq n+1$ est un groupe simple.*

On s'est ensuite intéressé à des groupes de difféomorphismes plus particuliers où les difféomorphismes devaient de plus préserver un forme différentielle, dans un premier temps pour les formes volumes.

Théorème 3. *(Thurston [16]) Si Ω est une forme volume sur une variété différentielle close et connexe M de dimension n, alors si $\widetilde{Diff_0^\infty}(M,\Omega)$ est le revêtement universel de $Diff_0^\infty(M,\Omega)$, on a :*

(i) $H_1(\widetilde{Diff_0^\infty}(M,\Omega)) \cong H^{n-1}(M)$ et $H_1(Diff_0^\infty(M,\Omega)) \cong H^{n-1}(M)/\Gamma$ avec Γ un sous-groupe de $H^{n-1}(M)$;

(ii) le groupe des commutateurs $[\widetilde{Diff_0^\infty}(M,\Omega), \widetilde{Diff_0^\infty}(M,\Omega)]$ est parfait et $[Diff_0^\infty(M,\Omega), Diff_0^\infty(M,\Omega)]$ est simple.

Finalement, en raffinant les théorèmes de fragmentation utilisés, Banyaga démontre le théorème suivant.

Théorème 4. *(Banyaga [2]) Si ω est une forme symplectique sur une variété différentielle close et connexe M de dimension n, alors si $\widetilde{Diff_0^\infty}(M,\omega)$ est le revêtement universel de $Diff_0^\infty(M,\omega)$, on a :*

(i) $H_1(\widetilde{Diff_0^\infty}(M,\omega)) \cong H^{n-1}(M)$ et $H_1(Diff_0^\infty(M,\omega)) \cong H^{n-1}(M)/\Gamma$ avec Γ un sous-groupe de $H^{n-1}(M)$;

(ii) le groupe des commutateurs $[\widetilde{Diff_0^\infty}(M,\omega), \widetilde{Diff_0^\infty}(M,\omega)]$ est parfait et $[Diff_0^\infty(M,\omega), Diff_0^\infty(M,\omega)]$ est simple.

Des résultats plus récents. Dans [5] Cristofaro-Gardiner, Humilière et Seyfaddini ont démontré "the infinite twist conjecture" qui implique que certaines propriétés (propriété P_ρ définies par Le Roux dans [8]) de fragmentation des homéomorphismes ne sont pas vérifiées, on montre cependant une propriété plus faible de fragmentation des homéomorphismes dans la pré-publication.

Après cette introduction voici la pré-publication qui a été faite à l'issue du stage.

Abstract

In this paper, we present a C^0-fragmentation property for Hamiltonian. More precisely, it is known that for given an open covering $\mathcal{U}$ of a compact surface we can decompose each C^0-small enough Hamiltonian diffeomorphism with Hamiltonian diffeomorphism compactly supported inside the open sets of the covering $\mathcal{U}$. We show that such a decompostion can be done with a Lipschitz estimate on the C^0-norm of the fragments. We will also show the same property for the kernel of θ, the mass-flow homomorphism for homeomorphisms.

3 Introduction and main results

3.1 C^0-fragmentation for Ker(Flux)

Let (M,ω) be a connected symplectic manifold equipped with a symplectic form ω. If M is closed we define $\mathrm{Symp}(M,\omega)$ the set of smooth diffeomorphisms that preserve ω. We then define $\mathrm{Symp}_0(M,\omega)$ the connected component of Id in $\mathrm{Symp}(M,\omega)$, i.e $\phi \in \mathrm{Symp}_0(M,\omega)$ if and only if there exists a smooth family of symplectic diffeomorphism $(\phi_t)_{t\in[0,1]}$ such that $\phi_0 = Id$ and $\phi_1 = \phi$.

A smooth *Hamiltonian* H is a smooth function $H : [0,1]\times M \to \mathbb{R}$ compactly supported in $[0,1]\times \mathrm{Interior}(M)$. H induces a *Hamiltonian flow*

$$\phi_H^t : M \to M, \quad (0 \le t \le 1),$$

by integrating the unique time-dependant vector field X_H satisfying $\iota_{X_H}\omega = dH_t$. A *Hamiltonian diffeomorphism* is a diffeomorphism obtained as the time-1 map of a Hamiltonian flow, we will denote $\mathrm{Ham}(M,\omega)$ the set of such diffeomorphism. We will eliminate the area form ω from the above notation when no confusion is possible.

It is possible to give an other definition of $\mathrm{Ham}(M)$ using the flux we will need this definition later on so we recall some of the definition, for the full proof of what we are going to state we refer the reader to the Chapter 10 of [9]. We define first the $\widetilde{\mathrm{Flux}}$: $\widetilde{\mathrm{Symp}_0(M,\omega)} \to H^1(M,\mathbb{R})$. Let $\{\phi_t\}$ a symplectic isotopy from Id to ϕ_1. Let X_t the time-dependant vector field defined via the relation

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

Since $\{\phi_t\}$ is a symplectic isotopy the 1-form $\iota_{X_t}\omega$ is a closed form. We then define $\widetilde{\mathrm{Flux}}(\phi_t) \in H^1(M,\mathbb{R})$ by the formula :

$$\int_0^1 [\iota_{X_t}\omega]\, dt.$$

This 1-form does not depend on the choice of the homotopy class of the isotopy ϕ_t with fixed endpoint. Also one can see by the natural identification of $H^1(M,\mathbb{R})$ and $\mathrm{Hom}(\pi_1(M),\mathbb{R})$ that $\widetilde{\mathrm{Flux}}(\{\phi_t\})$ acts on $\pi_1(M)$ this action is describing how much "mass" is going through a loop γ in M during the isotopy. Then one can see that a Hamiltonian diffeomorphism ϕ is one such that there exists a symplectic isotopy $\{\phi_t\}$ such that $\phi_0 = Id$, $\phi_1 = \phi$ and $\widetilde{\text{Flux}}(\{\phi_t\}) = 0$. In order to define the flux we will take the quotient by the space of loops inside $\text{Symp}_c(M, \omega)$. If we denote by Γ the image under the $\widetilde{\text{Flux}}$ of the isotopies from Id to itself, the $\widetilde{\text{Flux}}$ descends to the Flux : $\text{Symp}_0(M) \to H^1(M, \mathbb{R})/\Gamma$.

On (M, ω) a symplectic manifold with distance d induced by a riemannian metric, the C^0-distance (or uniform distance) is defined by

$$d_{C^0}(\phi, \psi) := \max_x d(\psi(x), \phi(x)).$$

Similarly, for two symplectic isotopy $\{\phi_t\}$ and $\{\psi_t\}$ we define their C^0-distance by

$$d_{C^0}^{\text{path}}(\{\phi_t\}, \{\psi_t\}) := \max_{x,t} d(\psi_t(x), \phi_t(x)).$$

Those two distances induce what is called the C^0-topology.

We give now two definitions that will be used later on.

Definition 1. *For a surface (Σ, ω) we will call a disk in Σ the image of an area-preserving embedding of $D_r := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$ in Σ for some $r \in \mathbb{R}^+$.*

Definition 2. *We define the annuli $\mathbb{A}_y = S^1 \times [-y, y]$ for y a positive real number and we will equip it with the area form $dx \wedge dy$ where x is the angular coordinate and y the radius coordinate. We also define $\mathbb{A} := \mathbb{A}_1$.*

The fragmentation property has been introduced in [2], [3] and [16] by Thurston and Banyaga in order to study the symplecity of perfectness of the groups of diffeomorphisms preserving a symplectic form or preserving a volume form. Later, in order to study the structure of the groupe of homeomorphisms preserving a measure Fathi [7] proved a similar fragmentation property for measure-preserving homeomorphisms in dimension $n \geq 3$. More recently, Entov, Polterovich and Py and then Seyfaddini described sharper versions of the fragmentation property of C^0-small Hamiltonian diffeomorphisms preserving a volume form in the 2 dimensional case, see [6] and [14]. They gave a Hölder-type bound on the C^0-norm of the fragments. In the present paper we adapt their proof to show a Lipschitz bound on the C^0-norm of the fragments. This is the subject of our first theorem.

Theorem 1. *Let (Σ, ω) be a closed surface equipped with an area form ω and d a distance induced by a riemannian metric. Let $\mathcal{W} = (W_i)_{i=1}^m$ be an open covering of the surface. Then there exists a C^0-neighborhood N of the identity in Ham(Σ) such that for each $\phi \in \mathrm{N}$, we can decompose as :*

$$\phi = \phi_1 \phi_2 \cdots \phi_m,$$

and for every $1 \leq i \leq m$, ϕ_i belongs to Ham(W_i). Moreover, we have the following estimate for all $1 \leq i \leq m$,

$$d_{C^0}(Id, \phi_i) \leq C d_{C^0}(Id, \phi),$$

for some constant $C > 0$ independant of ϕ.

Remark 3.1. *The result proved by Seyfaddini in [14] (Proposition 3.1) is the estimate :*

$$d_{C^0}(Id, \phi_i) \leq C \left(d_{C^0}(Id, \phi)\right)^{2^{1-N}},$$

where N is an integer depending on the genus of the surface.

The Theorem 1 leads to the following corollary.

Corollary 1. *Let $\phi \in$ Ham(Σ), then there exists $\{\phi_t\}$ a Hamiltonian isotopy such that $\phi_0 = Id$, $\phi_1 = \phi$ and the following estimate is verified*

$$\|\phi_t\|_{C^0}^{path} \leq C \|\phi\|_{C_0},$$

for some $C > 0$ a constant independant of ϕ.

Proof. The proof follows immediatly from the fragmentation and the equivalent proposition for Σ a disk. This last result is proven, with an adaptation of Alexander's trick, in [14] (Lemma 3.2). $\square$

The fragmentation proposition will be proven using the Lemma 3.2, an improvment of the extension lemmas in [14].

Lemma 3.2 (Area-preserving extension lemma for the annulus)**.** *Let equipped $\mathbb{A}_3$ with an area form ω. Let ϕ be a smooth area-preserving embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$.*

We assume furthermore that, for some $y \in (-1,1)$, (and hence for all) $S^1 \times y$ and $\phi(S^1 \times y)$ are homotopic in $\mathbb{A}_2$ and that

$$\text{the area in } \mathbb{A}_2 \text{ bounded by } S^1 \times y \text{ and } \phi(S^1 \times y) \text{ is zero.} \tag{1}$$

Let $\delta := d_{C^0}(Id, \phi)$. Then if δ is sufficiently small, there exists D (independent of δ) and $\psi \in Ham(\mathbb{A}_3)$ such that $\psi\big|_{\mathbb{A}_{1-D\delta}} = \phi\big|_{\mathbb{A}_{1-D\delta}}$ and

$$d_{C^0}(Id, \psi) \leq C\delta,$$

for some constant $C > 0$.

Moreover, if for some arc $I \subset S^1$ we have that $\phi = Id$ outside a quadrilateral $I \times [-1,1]$ and $\phi(I \times [-1,1]) \subset I \times [-2,2]$, then ψ can be chosen to be the identity outside $I \times [-2,2]$.

3.2 C^0-fragmentation for Ker(θ)

In this section we fix (Σ, μ) a surface equipped with a measure μ. We define Homeo(Σ, μ) the set of homeomorphisms of Σ that preserve the measure μ and Homeo$_0(\Sigma, \mu)$ the identity component of Homeo(Σ, μ). We will describe first the mass flow homomorphism for homeomorphisms, it has been introduced by Schwartzman in [13]. We will give a definition quite geometric, if the reader is interested about the definition of the mass flow homomorphism on a general compact metric space we refer to [7]. Let $\widetilde{\text{Homeo}_0}(\Sigma, \mu)$ the set of isotopies starting at the identity in Homeo$_0(\Sigma, \mu)$. We want to define θ, in order to do that we will define first $\widetilde{\theta} : \widetilde{\text{Homeo}_0}(\Sigma, \mu) \to H^1(\Sigma, \mathbb{R})$.

Let γ a loop in Σ. And let $\{\phi_t\}$ a symplectic isotopy from the identity to ϕ. We define $\widetilde{\theta}(\{\phi_t\})(\gamma) = \int_\sigma \mu$, where σ is the 2-cell

$$\sigma : [0,1] \times [0,1] \to \Sigma, (s,t) \mapsto \phi_s(\gamma(t)).$$

One can show that this definition can descend to cohomology and then $\widetilde{\theta}(\phi_t)$ defines indeed a cohomology class, moreover we can check that $\widetilde{\theta}$ is a homomorphism. If we denote by Γ the image of the subset of loops based at Id in Homeo(Σ). Then $\widetilde{\theta}$ descends by quotient to θ : Homeo$(\Sigma) \to H^1(\Sigma, \mathbb{R})/\Gamma$. In what follows we will show a C^0-fragmentation property for Ker(θ) which is the analog of the one for Ham(Σ).

Theorem 2. *Let (Σ, ω) be a closed surface equipped with an area form ω and d a distance induced by a riemannian metric. Let $\mathcal{W} = (W_i)_{i=1}^m$ an open covering of the surface. Then there exists a C^0-neighborhood N of the identity in Ker(θ) such that for each $\phi \in \mathrm{N}$, we can decompose as :*

$$\phi = \phi_1 \phi_2 \cdots \phi_m,$$

and for every $1 \leq i \leq m$,ϕ_i belongs to Homeo$_{0,c}(W_i, \omega)$. Moreover, we have the following estimate for all $1 \leq i \leq m$,

$$d_{C^0}(Id, \phi_i) \leq C d_{C^0}(Id, \phi),$$

for some constant $C > 0$ independant of ϕ.

We will prove Theorem 2 by using an analog of Lemma 3.2 for homeomorphisms.

Lemma 3.3. *Let equipped $\mathbb{A}_3$ with an area form ω. Let ϕ be a continuous area-preserving embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$.*

We assume furthermore that for some $y \in [-1,1]$ (and hence for all) $S^1 \times y$ and $\phi(S^1 \times y)$ are homotopic in $\mathbb{A}_2$ and that

$$\text{the area in } \mathbb{A}_2 \text{ bounded by } S^1 \times y \text{ and } \phi(S^1 \times y) \text{ is zero.} \tag{2}$$

Let $\delta := d_{C^0}(Id, \phi)$. Then if δ is sufficiently small, there exists D (independent of δ) and ψ is in the kernel of the mass-flow homomorphism on $\mathbb{A}_3$ such that $\psi\big|_{\mathbb{A}_{1-D\delta}} = \phi\big|_{\mathbb{A}_{1-D\delta}}$ and

$$d_{C^0}(Id, \psi) \leq C\delta,$$

for some constant $C > 0$.

Moreover, if for some arc $I \subset S^1$ we have that $\phi = Id$ outside a quadrilateral $I \times [-1,1]$ and $\phi(I \times [-1,1]) \subset I \times [-2,2]$, then ψ can be chosen to be the identity outside $I \times [-2,2]$.

4 Proof of the fragmentation properties

In this section we prove the Theorem 1 and the Theorem 2. The proof of both theorems works as follows, we define a triangulation T and an open covering associated to it in Section 4.2 and we will decompose the diffeomorphisms on small ball around the vertices, then around the edges and finally around the faces of the triangulation to finish the proof in Section 4.4, what will make this work is the two extension corollaries in Section 4.1 and the care of the two obstruction classes defined in Section 4.3.

4.1 Two corollaries of the area-preserving extension lemma for the annulus

We will prove two corollaries of Lemma 3.2. The corollaries are improved versions of Lemmas 6.2 and 6.3 in [6].

Corollary 2 (Area-preserving extension lemma for disks)**.** *Let $D_1 \subset D_2 \subset D \subset \mathbb{R}^2$ be closed disks such that $D_1 \subset Interior(D_2) \subset D_2 \subset Interior(D)$. Let $\phi : D_2 \to D$ be a smooth area-preserving embedding. If ϕ is sufficiently C^0-small, then there exists $\psi \in Ham(D)$ such that*

$$\psi\big|_{D_1} = \phi\big|_{D_1} \text{ and } d_{C^0}(Id, \psi) \leq C d_{C^0}(Id, \phi),$$

for some constant $C > 0$.

Remark 4.1. *The Corollary 2 transposes completely on the continuous setting if we ask to extend ϕ a continuous embedding by ψ an element of $Ker(\theta)$. We can then prove it by adapting the proof and using Lemma 3.3 instead of Lemma 3.2.*

Proof. We copy the proof in [6], adding only the estimate of Lemma 3.2.

Up to replacing D_2 by a slightly smaller disk, we can assume that ϕ is defined in a neighborhood of D_2. Identify some small neighborhood of ∂D_2 with $\mathbb{A} = S^1 \times [-3,3]$ so that ∂D_2 is identified with $S^1 \times 0 \subset \mathbb{A}_1 \subset \mathbb{A}_2 \subset \mathbb{A}_3$ and $\phi(\mathbb{A}_1) \subset \text{Interior}(\mathbb{A}_2) \subset \mathbb{A}_3 \subset \text{Interior}(D) \backslash \phi(D_1)$. Denote $\delta := d_{C^0}(Id, \phi)$.

Apply Lemma 3.2 and find $h \in \text{Ham}(\mathbb{A})$,

$$d_{C^0}(Id, h) \leq C d_{C^0}(Id, \phi),$$

for some constant $C > 0$ and so that $h\big|_{\mathbb{A}_{1-D\delta}} = \phi$. Set $\phi_1 := h^{-1} \circ \phi \in \text{Ham}(D)$. Note that $\phi_1\big|_{D_1} = \phi$ and ϕ_1 is the identity on $\mathbb{A}_{1-D\delta}$. Therefore we can extend $\phi_1|_{D_2 \cup \mathbb{A}_1}$ to D by the identity and get the required ψ. ☐

Corollary 3 (Area-preserving extension lemma for rectangles)**.** *Let $\Pi = [0,R] \times [-c,c]$ be a rectangle and let $\Pi_1 \subset \Pi_2 \subset \Pi$ be two smaller rectangles of the form $\Pi_i = [0,R] \times [-c_i, c_i], (i=1,2)$, $0 < c_1 < c_2 < c$. Let $\phi : \Pi_2 \to \Pi$ be a smooth area-preserving embedding such that*

- *ϕ is the identity near $0 \times [-c_2, c_2]$ and $R \times [-c_2, c_2]$,*
- *The area in Π bounded by the curve $[0,R] \times y$ and its image under ϕ is zero for some (and hence for all) $y \in [-c_2, c_2]$.*

If ϕ is sufficiently C^0-small, then there exists $\psi \in Ham(\Pi)$ such that

$$\psi\big|_{D_1} = \phi\big|_{D_1} \text{ and } d_{C^0}(Id, \psi) \le C d_{C^0}(Id, \phi),$$

for some constant $C > 0$.

Remark 4.2. *Again Corollary 3 transposes completely in the continuous setting by taking ϕ a continuous embedding and extending it by $\psi \in Ker(\theta)$.*

Proof. The proof relies on the last assumption of Lemma 3.2, indeed we can identify the rectangles $\Pi_1 \subset \Pi_2 \subset \Pi$ by a diffeomorphism with $I \times [-1,1] \subset I \times [-2,2] \subset I \times [-3,3]$ for some arc $I \subset S^1$. □

4.2 Covering associated to a triangulation

In this section we associate to a triangulation $T = (\Delta_i^k)_{i \in I_k}$, $k = 0,1,2$ three open coverings $\mathcal{V} = (V_i^k)_{i \in I_k}$, $\mathcal{V}' = (V_i^{'k})_{i \in I_k}$ and $\mathcal{U} = (U_i^k)_{i \in I_k}$, following the construction of Thurston and Banyaga [2].

Let $T = (\Delta_i^k)_{i \in I_k}$ of Σ. We build the covering by induction on the skeletons of the triangulation. The disks V_i^0 are balls containing Δ_i^0 and such that $V_i^0 \cap V_j^0 \neq \emptyset$ whenever $i \neq j$. We assume that we have already constructed the disks $(V_i^\ell)_{i \in I_\ell}$, $\ell = 0,1,\dots,k-1$ such that $\widetilde{\Delta}_i^k = \Delta_i^k - \bigcup_{\ell \le k-1} V^\ell$ is a shrinkage of Δ_i^1, where $V^\ell = \bigcup_{j \in I_0} V_j^\ell$. Let $\widehat{\Delta}_i^k$ a small thickening of $\widetilde{\Delta}_i^k$. Then V_i^k is defined as a C^∞-tubular neighborhood of $\widehat{\Delta}_i^k$. If this tubular neighborhood is small enough the open sets will verify $V_i^k \cap V_j^k = \emptyset$ for all $i \neq j$.

Now we define $\mathcal{V}' = (V_i^{'k})_{i \in I_k}$ and $\mathcal{U} = (U_i^k)_{i \in I_k}$ two other open coverings obtained by thickening the V_i^k. Such that for all $k \le 2, i \in I_k$, $\overline{V_i^k} \subset V_i^{'k} \subset \overline{V_i^{'k}} \subset U_i^k$ and $V_i^{'k} \cap V_j^{'k} = U_i^k \cap U_j^k = \emptyset$ whenever $i \neq j$.

In the rest of this section we will fix $T = (\Delta_i^k)_{i \in I_k}$, $\mathcal{V} = (V_i^k)_{i \in I_k}$, $\mathcal{V}' = (V_i^{'k})_{i \in I_k}$ and $\mathcal{U} = (U_i^k)_{i \in I_k}$ good openings associated to T.

4.3 Definition of two obstructions

In this section we fix (Σ, ω) a symplectic surface and T, $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}'$ a triangulation and 3 open converings associated to it as in Section 4.2. We will define first $\mathcal{O} : \mathrm{Symp}_{0,c}(\mathbb{A}_1) \to \mathbb{R}$ which represents the obstruction for a diffeomorphism $\phi \in \mathrm{Symp}_c(\mathbb{A}_1)$ to belong to $\mathrm{Ham}(\mathbb{A}_1)$. We also define $\mathcal{A}_{i_1,i_2}(\phi)$ for i_1 and i_2 in I_0 which will represent the obstruction of the extension of a certain embedding of V_j^1 in $V_j^{'1}$ if the two vertices $\Delta_{i_1}^0$ and $\Delta_{i_2}^0$ are joined by the edge Δ_j^1 (see Corollary 3).

Definition 3 (Definition of $\mathcal{O}$)**.** *Let $\phi \in Symp_{0,c}(\mathbb{A}_1)$, A and B two points on the boundary of $\mathbb{A}_1$ as in the figure below and $\gamma = [0,1] \to \mathbb{A}_1$, $t \mapsto \gamma(t)$ an arc with endpoints $\gamma(0) = A$ and $\gamma(0) = B$. Then let $h : [0,1] \times [0,1] \to \mathbb{A}_1$ an isotopy from γ to $\phi(\gamma)$ such that $h_{0,t} = \gamma(t)$ and $h_{1,t} = \phi(\gamma(t))$. We can then define :*

$$\mathcal{O}(\phi) := \int_0^1 \int_0^1 \omega(\partial_s h_{s,t}, \partial_t h_{s,t})\, ds\, dt.$$

Remark 4.3. *We can also define $\mathcal{O}$ in a more general setting by defining it as the area of a 2-cell defined by h. We can also replace ω by a Borel measure. This will be useful for Theorem 2.*

The next proposition is gives some properties of $\mathcal{O}$, those properties can also be proven in a continuous setting.

Proposition 4.4. *$\mathcal{O}$ is well-defined, i.e. it does not depend on the choice of h, γ nor on the choice of the points A and B.*

Moreover, $\mathcal{O}$ is exactly the obstruction for $\phi \in Symp_{0,c}(\mathbb{A}_1)$ to be in $Ham(\mathbb{A}_1)$, i.e $\phi \in Ham(\mathbb{A}_1)$ if and only if $\mathcal{O}(\phi) = 0$.

Proof. We show first that $\mathcal{O}(\phi)$ does not depend on the choice of the homotopy $h_{s,t}$. Let h' a second homotopy between γ and $\phi \circ \gamma$. Then define $g_{s,t} = h'_{1-2s,t} \circ h_{2s,t}$, g is then a homotopy from γ to itself. What we want to show is the identity :

$$\mathcal{O}(\phi) = \int_0^1 \int_0^1 h^*\omega = \int_0^1 \int_0^1 (h')^*\omega$$

or equivalently,

$$\int_0^1 \int_0^1 g^*\omega = 0.$$

This last identity is true since, if we denote $\sigma : [0,1] \times [0,1] \to \mathbb{A}$, then $\int_0^1 \int_0^1 g^*\omega = \int_\sigma \omega$ but $\pi_2(\mathbb{A}) = 0$ so the last integral is 0.

Let $\{\phi_s\}_s$ an isotopy such that $\phi_0 = Id$, $\phi_1 = \phi$ and for all $s \in [0,1]$, $\phi_s \in \mathrm{Symp}_c(\mathbb{A}_1)$. Then we let $h(s,t) = \phi_s(\gamma(t))$ and we now use the description of $\mathcal{O}(\phi)$ in terms of $h_{s,t}$. We will now link $\mathcal{O}$ to the Flux of the isotopy $\{\phi_s\}_s$.

The partial derivatives are now :

$$\partial_s h(s,t) = \partial_s \phi_s \circ \gamma(t) = X_s \circ \phi_s(\gamma(t)) = d\phi_s(X_s(\gamma(t))),$$

and also

$$\partial_t h(s,t) = \partial_t \phi_s(\gamma(t)) = d\phi_s(\dot{\gamma}(t)).$$

So we can compute

$$\begin{aligned}
\mathcal{O}(\phi) &= \int_0^1 \int_0^1 \omega(\partial_s h(s,t), \partial_t h(s,t))\, ds\, dt \\
&= \int_0^1 \int_0^1 \omega(d\phi_s(X_s(\gamma(t))), d\phi_s(\dot{\gamma}(t)))\, ds\, dt \\
&= \int_0^1 \int_0^1 \phi_s^*\omega(X_s(\gamma(t)), \dot{\gamma}(t))\, ds\, dt \\
&= \int_0^1 \int_0^1 \omega(X_s(\gamma(t)), \dot{\gamma}(t))\, ds\, dt = \int_\gamma \int_0^1 \iota_{X_s}\omega\, ds.
\end{aligned}$$

The fourth equality come from the fact that ϕ_s is a symplectomorphism for each s. The last identity is now $\widetilde{\mathrm{Flux}}(\{\phi_t\})$ applied to γ. We define now $\sigma_s := \iota_{X_s}\omega$ and we define also $H : [0,1] \times \mathbb{A}_1 \to \mathbb{R}$ a function such that $dH_s = \sigma$ and $H_s(A) = 0$ at any time s (H is like a Hamiltonian). Since ϕ_s is compactly supported H_s is constant near the boundary of $\mathbb{A}_1$. So $H_s \equiv 0$ near $S \times \{-1\}$ and $H_s \equiv c(s)$ near $S \times \{1\}$.

Now a little more computation shows that :

$$\mathcal{O}(\phi) = \int_\gamma \int_0^1 \iota_{X_s}\omega \, ds = \int_0^1 \int_\gamma dH_s \, ds \tag{3}$$

$$= \int_0^1 H_s(B) - H_s(A) \, ds = \int_0^1 c(s) \, ds. \tag{4}$$

So $\mathcal{O}$ does not depend on the choice of γ, A and B. This complete the well definition of this obstruction.

We now state the Lemma 4.5, it proves the surjectivity of this obstruction. We also give a bound on the norm of a particular antecedant of a real number (it will be useful in the proof of Lemma 4.8 for exemple). We will prove the Lemma 4.5 later.

Lemma 4.5 (Surjectivity of $\mathcal{O}$ and an estimate on the norm of an antecedant)**.** *The obstruction $\mathcal{O} : Symp_c(\mathbb{A}) \to \mathbb{R}$, $\psi \mapsto \mathcal{O}(\psi)$ is surjective.*

Moreover, there exists a constant $C > 0$ such that for all $\varepsilon \in \mathbb{R}$, there exists $\psi_\varepsilon \in Symp_c(\mathbb{A}_1)$, $\mathcal{O}(\psi_\varepsilon) = \varepsilon$ and

$$\|\psi_\varepsilon\|_{C^0} \leq C|\varepsilon|.$$

We can use the surjectivity of $\mathcal{O}$ in order to show that $\mathcal{O}$ is exactly the obstruction for being Hamiltonian. First of all, if ϕ is a Hamiltonian diffeomorphism then we obtain immediatly that $\mathcal{O}(\phi) = 0$.

Now if $\mathcal{O}(\phi) = 0$ we denote $\{\phi_t\}$ an isotopy from Id to ϕ. Let

$$\mathcal{O}_{t_0}(\{\phi_t\}) := \int_0^{t_0} c(s) \, ds$$

where $c(s)$ is defined as previously. Then

$$\Psi_{t_0} := \psi_{-\mathcal{O}_{t_0}(\{\phi_t\})} \circ \phi_{t_0}$$

is an isotopy from Id to ϕ. Moreover, $\mathcal{O}_{t_0}(\{\Psi_t\}) = -\mathcal{O}_{t_0}(\{\phi_t\}) + \mathcal{O}_{t_0}(\{\phi_t\}) = 0$ for all $t_0 \in [0, 1]$ which shows that $\{\Psi_t\}$ is an isotopy in $\mathrm{Ham}(\mathbb{A})$. It follows that $\phi \in \mathrm{Ham}(\mathbb{A})$. □

To complete the proof of Proposition 4.4 we now give a proof of Lemma 4.5.

Proof. (of Lemma 4.5) Let $\varepsilon \in \mathbb{R}$ and define $H^\varepsilon_s(x, y) = \chi(y)\varepsilon$ a Hamiltonian-like function, where χ is a smooth function supported in $[-1, 1]$ (and χ' is compactly supported in $[-1, 1]$), satisfying $\chi(-1) = 0$ and $\chi(1) = 1$. We let $C = \|\chi\|_\infty$. We then define ϕ_s as the flow generated by this Hamiltonian-like function. That is, define the vector field X_s by the equation

$$\iota_{X_s}\omega = dH^\varepsilon_s = \chi'(y)\varepsilon \, dy.$$

So $X_s = \chi'(y)\varepsilon \frac{\partial}{\partial x}$ and $\phi_s(x, y) = (x + \varepsilon\chi'(y)s, y)$, where x is in $\mathbb{R}/\mathbb{Z}$ and $x + \varepsilon\chi'(y)s$ is taken in $\mathbb{R}/\mathbb{Z}$. Then $\|\phi\|_{C^0} \leq C|\varepsilon|$ and $\mathcal{O}(\phi) = \int_0^1 H_s(1) \, ds = \varepsilon$ as wanted. □

The Proposition 4.4 transposes in the continuous setting, we just have to change the definition of $\mathcal{O}$ using integration on 2-cell instead.

Definition 4 (Definition of $\mathcal{A}$)**.** *Let T, $\mathcal{U}$ and $\mathcal{V}$ given as in Section 4.2. Let $\phi \in Symp_0(\Sigma)$ such that for all $i \in I_0$, $\phi\,|_{V^0_i} = Id$. Given $\Delta^0_{i_1}$ and $\Delta^0_{i_2}$ two vertices in T linked by an edge Δ^1_j parametrized by an arc γ with $\gamma(0) = \Delta^0_{i_1}$ and $\gamma(1) = \Delta^0_{i_2}$, we define :*

$$\mathcal{A}_{i_1,i_2}(\phi) = \int_0^1 \int_0^1 \omega(\partial_s h_{s,t}, \partial_t h_{s,t}),$$

where $h : [0, 1] \times [0, 1] \to U^1_j$ is an isotopy with fixed endpoints from γ to $\phi(\gamma)$.

Remark 4.6. *We can also define $\mathcal{A}$ via integration on 2-cells instead. Again it allows us to define $\mathcal{A}$ on the continuous case.*

We prove now several properties of this obstruction $\mathcal{A}$.

Proposition 4.7. *Let T, $\mathfrak{U}$ and $\mathcal{V}$ given as in Section 4.2. We assume, as before, that i_1 and i_2 are two indices in I_0 such that $\Delta^0_{i_1}$ and $\Delta^0_{i_2}$ are linked in T by an edge Δ^1_j, then if ϕ is C^0-small enough and verifies that, for all $i \in I_0$, $\phi|_{V^0_i} = Id$. We have the following four properties :*

(i) $\mathcal{A}_{i_1,i_2}(\phi)$ is well defined, i.e. it does not depends on the choice made in its definition.

(ii) The following identity holds :

$$\mathcal{A}_{i_1,i_2}(\phi) = -\mathcal{A}_{i_2,i_1}(\phi).$$

(iii) There exists $C > 0$ such that,

$$|\mathcal{A}_{i_1,i_2}(\phi)| \leq C\|\phi\|_{C^0}.$$

(iv) If $\phi \in Ham(\Sigma)$, then for a set of indices $i_1, i_2, \dots, i_m$ such that i_p and i_{p+1} are linked by an edge in T (with the convention that $i_{m+1} = i_1$), then

$$\sum_{p=1}^{m} \mathcal{A}_{i_p,i_{p+1}}(\phi) = 0.$$

Proof. We are going to prove the claims in the order they appear.

(i) The proof here is the same as for the Proposition 4.4.

(ii) If $h_{s,t}$ is an istopy from γ to $\phi(\gamma(t))$, then $h_{s,1-t}$ is an isotopy from $\gamma(1-t)$ to $\phi(\gamma(1-t))$, so after a change of variable in the integral we have the identity :

$$\mathcal{A}_{i_1,i_2}(\phi) = \int_0^1 \int_0^1 \omega(\partial_s h_{s,t}, \partial_t h_{s,t}) = -\mathcal{A}_{i_2,i_1}(\phi).$$

(iii) This claim is immediate since $\mathcal{A}_{i_1,i_2}(\phi)$ represents the area between Δ^1_j and $\phi(\Delta^1_j)$ in U^1_j and $\phi(\Delta^1_j)$ is stuck in the tubular neighborhood of Δ^1_j of radius $\|\phi\|_{C^0}$, hence we indeed have :

$$|\mathcal{A}_{i_1,i_2}(\phi)| \leq C\|\phi\|_{C^0},$$

for some constant $C > 0$.

(iv) Let γ the piecewiese smooth path going through $\Delta^1_{j_1}, \Delta^1_{j_2}, \dots \Delta^1_{j_m}$, where $\Delta^1_{j_p}$ links $\Delta^0_{i_p}$ to $\Delta^0_{i_{p+1}}$, the path shouldn't go twice through the same vertex. Let ϕ_t be a Hamiltonian isotopy from Id to ϕ, then $\widetilde{\text{Flux}}(\phi_t) = 0$, this means that the area of the cylinder $\phi_t(\gamma)$ is zero. However, nothing tells us that the area of the cylinder of $\phi_t(\gamma)$ is the same as the sum $\sum_{p=1}^m \mathcal{A}_{i_p,i_{p+1}}(\phi)$, obtained also as the area of some cylinder between γ and $\phi \circ \gamma$ but with support in $\bigcup U^1_{j_p}$. However, we can glue those two cylinder to obtain one closed 2-cycle σ_2 so

$$\int_{\sigma_2} \omega = \widetilde{\text{Flux}}(\phi_t) + \sum_{p=1}^{m} \mathcal{A}_{i_p,i_{p+1}}(\phi) = \sum_{p=1}^{m} \mathcal{A}_{i_p,i_{p+1}}(\phi)$$

has value in $\omega \cdot H_2(\Sigma, \mathbb{Z})$ a discrete subgroup of $\mathbb{R}$. If ϕ is C^0-small enough then by the third point of the Proposition, $\left|\sum_{p=1}^m \mathcal{A}_{i_p,i_{p+1}}(\phi)\right| \leq mC\|\phi\|_{C^0} \leq |I_k|\|\phi\|_{C^0}$ is in a discrete subgroup of $\mathbb{R}$ so must be 0. This finishes the proof of the Proposition 4.7.

∎

4.4 Proof of Theorem 1 and 2

We have now all the tools to prove the Theorem 1 and the Theorem 2.

Proof. (of Theorem 1)
Let T a triangulation such that the star of every vertices of T are included in one of the open sets of the subcovering of $\mathcal{W}$. We will consider the open coverings $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{U}$ associated to T. Then the three coverings $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{U}$ ar thiner than $\mathcal{W}$. We will prove the fragmentation theorem on $\mathcal{U}$ which will imply it for $\mathcal{W}$.

Lemma 4.8 (Fragmentation on the 0-skeleton)**.** *Let $T = (\Delta_i^k)_{i\in I_k}$ a good triangulation $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}'$ open coverings associated with T as described in Section 4.2. Let $\phi \in Ham(\Sigma)$ a C^0-small diffeomorphism. Then we can find the following C^0-fragmentation :*

$$\phi = \phi_1^{(0)} \circ \phi_2^{(0)} \cdots \circ \phi_\ell^{(0)} \circ \phi',$$

where $\ell := |I_0|$, for all $i \in I_0$, $\phi_i^{(0)} \in Ham(U_i^0)$, verifies $\phi_p^{(0)} |_{V_i'} = \phi$ and satisfy the estimate

$$\|\phi_p^{(0)}\|_{C^0} \leq C\|\phi\|_{C^0},$$

where $C > 0$ is a constant.
Moreover, $\mathcal{A}_{i,j}(\phi') = 0$ for all $i, j \in I_0$ linked by an edge in T.

The first step in order to prove Lemme 4.8 is to prove the following lemma, it is a fragmentation on the open sets on the vertices but we did not ask for the nullity of the obstruction $\mathcal{A}$.

Lemma 4.9. *Let $T = (\Delta_i^k)_{i\in I_k}$ a good triangulation $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}'$ open coverings associated with T as described in Section 4.2. Let $\phi \in Ham(\Sigma)$ a C^0-small diffeomorphism. Then we can find the following C^0-fragmentation*

$$\phi = \phi_1^{(-1)} \circ \phi_2^{(-1)} \cdots \circ \phi_\ell^{(-1)} \circ \widetilde{\phi},$$

where $\ell = |I_0|$, for all $i \in I_0$, $\phi_i^{(-1)} \in Ham(U_i^0)$, verifies $\phi_p^{(-1)} |_{V_i'} = \phi$ and satisfy the estimate

$$\|\phi_p^{(-1)}\|_{C^0} \leq C\|\phi\|_{C^0},$$

where $C > 0$ is a constant, $\widetilde{\phi}$ is then supported in $\Sigma\backslash V^0$ and its C^0-norm satisfy a Lipschitz bound in the C^0-norm of ϕ.

Proof. Let $i \in I_0$, by Corollary 2 and since for ϕ is a C^0-small diffeomorphism and an area-preserving embedding from V_i^0 to $V_i'^0$, there exists $\phi_i^{(-1)} \in \mathrm{Ham}(U_i^0)$ such that $\phi_i^{(-1)} |_{V_i} = \phi$. Moreover, there exists a constant $C > 0$ such that $\|\phi_i^{(-1)}\|_{C^0} \leq C\|\phi\|_{C^0}$. Then there exists a diffeomorphism $\widetilde{\phi}$ such that $\phi = \bigcirc_{i\in I_k} \phi_i \circ \widetilde{\phi}$ (there is no issue with the composition since the supports of the ϕ_i are disjoint). We then have that $\widetilde{\phi}$ is supported in $\Sigma\backslash V^0$ and is a Hamiltonian diffeomorphism of Σ. Also, $\|\widetilde{\phi}\|_{C^0} \leq \|\phi\|_{C^0} + \sum \|\phi_i^{(-1)}\|_{C^0} \leq C'\|\phi\|_{C^0}$, for $C' > 0$ a constant. □

It is time now to prove Lemma 4.8.

Proof. (of Lemma 4.8)
We apply first Lemma 4.9 and we now want to do slight modifications on the diffeomorphisms $\phi_p^{(-1)}$ in order to vanish the obstruction $\mathcal{A}$ on each edge.

In order to do this, we first distinguish an indice $i \in I_0$ and its vertice Δ_i^0. We will define for a vertice $j \in I_0$ the real number $C(j)$ by

$$C(j) := \sum_{p=0}^{m-1} \mathcal{A}_{i_p,i_{p+1}}((\widetilde{\phi})),$$

where $i_0 = i, i_1, \dots, i_m = j$ is a sequence of indices such that they are all linked by an edge in T. Using property (iv) of Proposition 4.7 we see that this value does not depend on the sequence $(i_p)_p$ we take. We define now $\phi_j^{(0)} = \rho_j \circ \phi_j^{(-1)}$ where ρ_j is a compactly supported diffeomorphism from the annulus $U_j^0 \backslash V_j^0$ to itself, and such that $\mathcal{O}(\rho) = C(j)$ (we identify ∂V_j^0 with $S \times \{-1\}$ to match Definition 3), note that Lemma 4.5 allows us to take ρ_j with a Lipschitz estimate. Then the diffeomophisms $\phi_j^{(0)}$ are the fragments needed in Lemma 4.8, and $\phi' = \bigcirc_j (\rho_j^{(-1)})^{-1} \widetilde{\phi}$ satisfy all the conditions needed to conclude Lemma 4.8.

□

We now work on ϕ' and fragment it too using Lemma 4.10.

Lemma 4.10 (Fragmentation on the 1-skeleton)**.** *Let ϕ' be the resulting Hamiltonian diffeomorphism after applying Lemma 4.8, then there exists a fragmentation of ϕ',*

$$\phi' = \phi_1^{(1)} \circ \phi_2^{(1)} \cdots \circ \phi_m^{(1)} \circ \phi'',$$

where $m = |I_1|$, for all $i \in I_1$, $\phi_i^{(1)} \in Ham(U_i^1)$, $\phi_q^{(1)} |_{V_i^1} = \phi'$ and the following estimate is true

$$\|\phi_p^{(1)}\|_{C^0} \le C \|\phi'\|_{C^0},$$

where $C > 0$ is a constant. The resulting ϕ'' is then supported in $\Sigma \backslash (V^0 \cup V^1)$ and satisfy a Lipschitz estimate with respect to ϕ and thus with respect to ϕ.

Proof. Let $i \in I_1$, and i_1 and i_2 are the vertices of the edge Δ_i^1. Then, since ϕ' is C^0-small ϕ' is an area-preseving embedding of V_i^1 in $V_i'^1$ being equal to the identity on $V_i^1 \cap V_{i_1}^0$ and $V_i^1 \cap V_{i_2}^0$, also the condition $\mathcal{A}_{i_1,i_2}(\phi') = 0$ is exactly the condition we need to apply Corollary 3. We have now $\phi_i^{(1)} \in \mathrm{Ham}(U_i^1)$ such that $\phi_i^{(1)} |_{V_i^1} = \phi'$, $\|\phi_i^{(1)}\|_{C^0} \le C \|\phi'\|_{C^0}$. So there exists a diffeomorphism ϕ'' such that $\phi' = \bigcirc_{i \in I_1} \phi_i^{(1)} \circ \phi''$. Then ϕ'' is a Hamiltonian diffeomorphism of Σ itself, it is compactly supported in $\Sigma \backslash (V_0 \cup V_1)$ and satisfy $\|\phi''\|_{C^0} \le \|\phi'\|_{C^0} + \sum \|\phi_i^{(1)}\|_{C^0} \le C' \|\phi\|_{C_0}$. □

Lemma 4.11 (Fragmentation on the 2-skeleton)**.** *Let ϕ'' be the resulting Hamiltonian diffeomorphism, we can fragment it in Hamiltonian diffeomorphism itself*

$$\phi'' = \phi_1^{(2)} \circ \phi_2^{(2)} \cdots \circ \phi_n^{(2)},$$

where $n = |I_2|$, for all $i \in I_2$, $\phi_i^{(2)} \in Ham(U_i^2)$ and

$$\|\phi_p^{(2)}\|_{C^0} \le C \|\phi''\|_{C^0},$$

where $C > 0$ is a constant.

Proof. ϕ'' has now support in $\Sigma \backslash (V^0 \cup V^1) \subset \bigcup U_i^2$. So it can be decomposed in Hamiltonian diffeomorphism with support in the U_i^2 (always disjoint from each other) and the bound is also immediate. □

Combining Lemma 4.8, 4.10 and 4.11 we obtain a fragmentation with a finitely bounded number of fragment. We describe now the procedure to give the result in the shape of Theorem 1. We want to swap the support of the diffeomorphisms, let $\psi = fg$ for two diffeomorphisms f and g such that $\mathrm{supp}(f) \subset B_1' \subset B_1$ and $\mathrm{supp}(g) \subset B_2' \subset B_2$. Then $\psi = g(g^{-1}fg)$ and we want to show now that $g^{-1}fg$ is supported inside B_1 if f and g have a small C^0-norm, this is indeed not hard to see since that if $d(x, B_1') \ge \|g\|_{C^0}$ then $g^{-1}(f(g(x))) = g^{-1}(g(x)) = x$. This procedure is the kind of commutation we needed

$$\psi = \underbrace{f}_{\text{support in } B_1} \underbrace{g}_{\text{support in } B_2} = \underbrace{g}_{\text{support in } B_2} \underbrace{g^{-1}fg}_{\text{support in } B_1}.$$

By repeatedly applying this procedure on the fragmentation obtained previously we obtain indeed a fragmentation $\phi = \phi_1 \phi_2 \cdots \phi_m$ with $\phi_i \in \mathrm{Ham}(W_i)$ and $\|\phi_i\|_{C^0} \leq C\|\phi\|_{C^0}$ for some constant $C > 0$.

□

Proof. (of Theorem 2) The proof transpose for $\mathrm{Ker}(\theta)$ by adapting directly the Corollaries 2 and 3, then the obstruction $\mathcal{O}$ and $\mathcal{A}$ and finally the Lemma 4.8, 4.9, 4.10 and 4.11. □

5 Proof of the area-preserving extension lemma

5.1 Preliminaries

We need to state three proposition about the area forms on a manifold in order to carry out the proof of Lemma 3.2. The two first proposition are already well-known. The third one is more a technical claim that will be useful.

We say that a Borel measure μ on a compact manifold X is said to be a OU (Oxtoby-Ulam) measure if it is nonatomic, of full support and is zero on the boundary. The following proposition is proven in [12].

Proposition 5.1. *Let μ and ν two OU measures on a rectangular r-cell R such that $\mu(R) = \nu(R)$, then there exists a homeomorphism h which restricts to the identity on the boundary of B such that $h^*\nu = \mu$.*

Before giving the proof of Lemma 3.2 we recall here the part of Moser's trick [11] as described in [14].

Proposition 5.2 (Moser's trick)**.** *Let M be a compact connected oriented manifold of dimension n, possibly with a non-empty boundary ∂M, and let ω_1, ω_2 be two volume forms on M. Assume that $\int_M \omega_1 = \int_M \omega_2$. If $\partial M \neq 0$, we also assume that the forms ω_1 and ω_2 coincide on ∂M.*

Then there exists a diffeomorphism $f : \Sigma \to \Sigma$, isotopic to the identity, such that $f^*\omega_2 = \omega_1$. Moreover, f can be chosen to satisfy the following properties:

(i) If $\partial M \neq 0$, then f is the identity on ∂M, and if ω_1 and ω_2 coincide near ∂M, then f is the identity near ∂M.

(ii) If M is partitioned into polyhedron (with piecewise smooth boundaries), so that $\omega_1 - \omega_2$ is zero on the $(n-1)$-skeleton Γ of the partition and the integral of ω_1 and ω_2 on each of the polygons are equal, then f can be chosen to be the identity on Γ.

(iii) Suppose that $\omega_2 = \chi\omega_1$ for a function χ. The diffeomorphism f can be chosen to satisfy the following estimate :

$$d_{C^0}(Id, f) \leq C\|\chi - 1\|_{C^0},$$

for some $C > 0$. Here, $\|\cdot\|_{C^0}$ denotes the standard sup norm on functions.

We describe here a lemma which allows us to adjust two volume forms by a C^0-small diffeomorphism if they disagree on a small strip only. This is the biggest divergence in our proof of the extension lemma and the previous proof in [6] and [14].

Proposition 5.3. *Let $C \in \mathbb{R}$ be a constant and M^{n-1} a compact manifold equipped with a volume form ω'. Let ω and Ω two volume forms on $M^{n-1} \times [-1,1]$. Let χ be the function such that $\omega = (1+\chi)\Omega$. We assume that:*

- $\int_{M^{n-1}\times[-1,1]} \omega = \int_{M^{n_1}\times[-1,1]} \Omega.$
- $\|\chi\|_{C^0} \leq C.$

- *There exists $\delta > 0$ such that $Supp(\chi) \subset M^{n-1} \times [0, \delta]$.*

Then, there exists a constant $D \in \mathbb{R}$ independant on δ such that we can find $f \in Diff_c(M^{n-1} \times [-1, 1])$, $f \equiv Id$ on $M^{n-1} \times [-1, 0]$, $f^*\Omega = \omega$ and $\|f\|_{C^0} \leq D\delta$.

Proof. We will denote by z the last coordinate of the manifold $M^{n-1} \times [-1, 1]$ and x the coordinate on M^{n-1}. By applying Moser's trick we see that we can assume that $\omega = \omega' \wedge dz$. Let $N^n := M^{n-1} \times \mathbb{R}$ we will describe two diffeomorphisms on N^n that will combine to give what we want on $M^{n-1} \times [-1, 1]$.

Define $\rho_\delta : \mathbb{R} \to \mathbb{R}, x \mapsto 2\delta x$. We can then define two functions from N^n to N^n by:

$$\Psi_1(x, z) := \left(x, \int_0^z (1 + \chi(x, t))\, dt\right)$$

and

$$\Psi_2(x, z) := \left(x, \int_0^z (1 + \rho'_\delta(t)\chi(x, \rho_\delta(t)))\, dt\right).$$

We can then compute $d\Psi_1(x, z) := dx + (1 + \chi(x, z))dz$, and since $1 + \chi > 0$ the function Ψ_1 is a diffeomorphism, in the same manner one can show that Ψ_2 is also a diffeomorphism. A simple computation also show that $(\Psi_1)_*\omega = (1 + \chi(x, z))\omega = \Omega$ and also $(\Psi_2)_*\omega = (1 + \rho'_\delta\chi(x, \rho_\delta(z)))\omega$ wich gives

$$\left((\Psi_1 \circ (\Psi_2)^{-1})^*\Omega\right)_{(x,z)} = (1 + \rho'_\delta(z)\chi(x, \rho_\delta(z)))\omega_{(x,z)}.$$

Moreover,

$$d\left((x, z), \Psi_1(x, z)\right) = \left|\int_0^z \chi(x, \rho_\delta(t))\, dt\right| \leq \|\chi\|_{C^0}\delta \leq C\delta$$

so $\|\Psi_1\|_{C^0} \leq C\delta$ and similarly $\|\Psi_2\|_{C^0} \leq C\delta$.

If $z > \delta$, since χ has support in a strip we have $\Psi_2(x, z) = (x, z + c(x))$, where $c(x)$ is a function independant on z and whose value is $c(x) = \int_0^\delta \chi(x, t)\, dt$. If $z > 0.5$, we have, by the same argument, that $\Psi_2(x, z) = (x, z + d(x))$, where $d(x) = \int_0^{0.5} \rho'_\delta(t)\chi(x, \rho_\delta(t))\, dt$.

It follows that those two diffeomorphisms aren't compactly supported in $M^{n-1} \times [-1, 1]$, however since $c(x) = d(x)$ (simple change of variable in the integral) $\Psi := \Psi_1 \circ (\Psi_2)^{-1}$ can be restricted on $M^{n-1} \times [-1, 1]$ to a compactly supported diffeomorphism. Moreover,

$$(\Psi^*\Omega)_{(x,z)} = \left((\Psi_1 \circ (\Psi_2)^{-1})^*\Omega\right)_{(x,z)} = (1 + \rho'_\delta(z)\chi(x, \rho_\delta(z)))\omega_{(x,z)}$$

and

$$\|\Psi\|_{C^0} \leq \|\Psi_1\|_{C^0} + \|\Psi_2\|_{C^0} \leq 2C\delta.$$

By Proposition 5.2 one can find a function $h \in \text{Diff}_c(M^{n-1} \times [-1, 1])$ such that $h^*(\Psi^*(\Omega)) = \omega$ and there exists a constant independant on ω and Ω such that $\|h\|_{C^0} \leq D\|\rho'_\delta\chi(\cdot, \rho_\delta(\cdot))\|_{C^0} \leq 2DC\delta$. It then follows that Ψh is the diffeomorphism we needed.

□

5.2 Proof of Lemma 3.2 and Lemma 3.3

The proof of Lemma 3.2 (resp. Lemma 3.3) will go as follows, we start first to extend the area-preserving embedding ϕ by a C^0-small diffeomorphism (resp. homeomorphism) f not necessarily preserving the area form. Then working on this function f we can modify it to also have $f^*\omega$ (resp. and we have to make sure that $f^*\omega$ is an area-form) and ω being close to each other in the sense of part (iii) of Proposition 5.2. More precisely we will find the following constraints on f :

- for χ such that $f^*\omega = (1+\chi)\omega$, we want to have $\|\chi\|_{C^0}$ smaller than a constant not depending on ϕ,
- f have a compact support close to $\partial\mathbb{A}_1$.

We will then be able to apply Proposition 5.3 in order to finish the proof.

Proof. (of Lemma 3.2) We can assume without loss of generality that the area form ω is $dx \wedge dy$, where x denote the angular coordinate on S^1 and y the radius along $\mathbb{A}_3$. We start the proof as in [6] by using a diffeomorphic extension f of ϕ as stated in Lemma 5.4, we will talk about this lemma in Section 6

Lemma 5.4. *Let ϕ be a smooth embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$, isotopic to the identity, such that $d_{C^0}(Id, \phi) \leq \delta$ for some $\delta > 0$. Then there exists $f \in \mathit{Diff}_{0,c}(\mathbb{A}_2)$ such that f is supported in $\mathbb{A}_{1+C_2\delta}$, satisfy*

$$f\big|_{\mathbb{A}_{1-C_2\delta}} = \phi\big|_{\mathbb{A}_{1-C_2\delta}}$$

and

$$d_{C^0}(Id, f) \leq C_1\delta.$$

for some constant $C_1, C_2 > 0$ independant of ϕ.

Moreover, if $f = Id$ outside a quadrilateral $I \times [-1,1]$ and $f(I \times [-1,1]) \subset I \times [-2,2]$ for some arc $I \subset S^1$, then f can be chosen to be the identity outside $\mathbb{A}_2$.

Denote $\Omega := f^*\omega$ and define $\mathbb{A}_- := S^1 \times [-2,0]$ and $\mathbb{A}_+ := S^1 \times [0,2]$. By the condition (1) the following equalities hold :

$$\int_{\mathbb{A}_-} \omega = \int_{\mathbb{A}_-} \Omega, \quad \int_{\mathbb{A}_+} \omega = \int_{\mathbb{A}_+} \Omega.$$

We are going to adjust f by constructing $h \in \mathrm{Diff}_{0,c}(\mathbb{A}_2)$ such that $h\big|_{\mathbb{A}_{1-D\delta}} = Id$, $h^*\Omega = \omega$ and $d_{C^0}(Id, h) \leq C_3 d_{C^0}(Id, \phi)$ for some $C_3 > 0$. To do so we just need to solve the case of $\mathbb{A}_+$ the other will then follow by symmetry. The diffeomorphism will be constructed bit by bit as said previously. Once we have such a diffeomorphism fh, it can be extended by the identity on $\mathbb{A}_3$ to obtain $\Psi \in \mathrm{Symp}_c(\mathbb{A}_3)$. Ψ may not be a Hamiltonian diffeomorphism but after a Lipschitz C^0-adjustment (see Lemma 4.5) on $\mathbb{A}_3 \backslash \mathbb{A}_2$ the resulting diffeomorphism will be Hamiltonian.

In the next paragraph, we find a constant bound on Ω.

Modifying the area form Ω to find a constant estimate

Divide the annulus $S^1 \times [1-(1+C_1+C_2)\delta, 1+(1+C_1+C_2)\delta]$ in N squares $R_1, \ldots, R_N$ of side length $2(1+C_1+C_2)\delta$ and denote by Γ the 1-skeleton of the partition. Then we can find $h_1 \in \mathrm{Diff}_c(\mathbb{A}_2)$ as C^0-small as we want such that $h_1^*\Omega$ is equal to ω on Γ (Such a construction is made in the paragraph **Adjusting Ω on Γ** of [6]). Denote

$$\Omega' := h_1^*\Omega,$$

and assume $d(fh_1, Id) \leq (1+C_1)\delta$ by asking $d(Id, h_1) \leq \delta$. Note that here we have

$$\int_{\mathbb{A}_+} \Omega' = \int_{\mathbb{A}_+} \omega.$$

We will use the same method as in the paragraph **Adjusting the areas of the squares** of [6] the only twist is that we obtain a constant bound and not a bound in $\delta^{3/4}$. Indeed, on a square R_i we obtain the following inequality by considering the fact that $\|fh_1\|_{C^0} \leq (1+C_1)\delta$

so the image of R_i is inside a square of side length $(4+4C_1+2C_2)\delta$ and outside a square of side length $2C_2\delta$

$$\frac{4(C_2)^2\delta^2}{4(1+C_1+C_2)^2\delta^2} \leq \frac{\int_{R_i}\Omega'}{\int_{R_i}\omega} \leq \frac{4(2+2C_1+C_2)^2\delta^2}{4(1+C_1+C_2)^2\delta^2}$$

and after simplification,

$$\frac{(C_2)^2}{(1+C_1+C_2)^2} \leq \frac{\int_{R_i}\Omega'}{\int_{R_i}\omega} \leq \frac{(2+2C_1+C_2)^2}{(1+C_1+C_2)^2}.$$

By renaming the constant on the left-hand side and right-hand side of the inequation, and setting $s_i := \int_{R_i}\Omega'$, $r_i := \int_{R_i}\omega$ we can eventually rewrite this as :

$$0 < 1 - A \leq \frac{s_i}{r_i} \leq 1 + A, \tag{5}$$

for some constant $1 > A > 0$. Set $t_i := \frac{s_i}{r_i} - 1$. By (5),

$$|t_i| \leq A. \tag{6}$$

For each i choose a nonnegative fonction $\overline{\rho}_i$ supported in the interior of R_i such that $\int_{R_i}\overline{\rho}_i\omega = r_i$ and

$$\|\overline{\rho}_i\| \leq C_4 < \frac{1}{A}, \tag{7}$$

for some constant C_4 independent of δ. Define a function ϱ on $\mathbb{A}_{1+C_1+C_2}$ by

$$\varrho = 1 + \sum_i t_i\overline{\rho}_i.$$

By (6) and (7), we see that ϱ is positive. Moreover, ϱ is equal to 1 over Γ and the two area forms $\varrho\omega$ and Ω' have the same integral on each R_i. Let us apply part (iii) of Proposition 5.2 to the forms Ω' and $\varrho\omega$ on $S^1\times[1-(1+C_1+C_2)\delta, 1+(1+C_1+C_2)\delta]$: these forms coincides near the boundary of $S^1\times[1-(1+C_1+C_2)\delta, 1+(1+C_1+C_2)\delta]$, therefore there exists h_2 a diffeomorphism with compact support in $S^1\times[1-(1+C_1+C_2)\delta, 1+(1+C_1+C_2)\delta]$ such that $h_2^*\Omega = \varrho\omega$ and $d(Id, h_2) \leq C_5\delta$ for some constant $C_5 > 0$.

Note that

$$\int_{\mathbb{A}_+}\varrho\omega = \int_{\mathbb{A}_+}\Omega' = \int_{\mathbb{A}_+}\omega.$$

What we have done yet is finding a function $fh_1h_2 \in \mathrm{Diff}_c(\mathbb{A}_{1+(1+C_1+C_2)\delta})$ such that :

- $d(fh_1h_2, Id) \leq (1+C_1+C_5)\delta$,
- $fh_1h_2|_{\mathbb{A}_{1-(1+C_1+C_2)\delta}} = \phi$,
- $\varrho\omega = (fh_1h_2)^*\omega$ and ω are not too far.

Let extend fh_1h_2 by the identity to obtain g, a compactly supported diffeomorphism on $\mathbb{A}_2$. And define

$$\Omega'' := g^*\omega.$$

We can now apply Proposition 5.3 with $M = S^1$ and the two area forms ω and Ω'' in order to finish the proof.

□

We now prove the area-preserving extension lemma for the homeomorphisms. We will only describe the noticeable changes in the previous proof. The overall idea stays the same in both proofs.

Proof. (of Lemma 3.3) We first extend the continuous area-preserving embedding ϕ to a global homeomorphism of $\mathbb{A}_2$ with the help of Lemma 5.5

Lemma 5.5. *Let ϕ be a continuous embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$, isotopic to the identity, such that $d_{C^0}(Id, \phi) \leq \delta$ for some $\delta > 0$. Then there exists $f \in Homeo_{0,c}(\mathbb{A}_2)$ such that f is supported in $\mathbb{A}_{1+C_2\delta}$, satisfy*

$$f\big|_{\mathbb{A}_{1-C_2\delta}} = \phi\big|_{\mathbb{A}_{1-C_2\delta}}$$

and

$$d_{C^0}(Id, f) \leq C_1\delta.$$

for some constant $C_1, C_2 > 0$ independant of ϕ.

Moreover, if $f = Id$ outside a quadrilateral $I \times [-1, 1]$ and $f(I \times [-1, 1]) \subset I \times [-2, 2]$ for some arc $I \subset S^1$, then f can be chosen to be the identity outside $\mathbb{A}_2$.

Denote $\nu := f^*\omega$ (ν is then an OU measure) then, by the condition 2 the following equalities hold :

$$\omega(\mathbb{A}_-) = \nu(\mathbb{A}_-), \quad \omega(\mathbb{A}_+) = \nu(\mathbb{A}_+).$$

We are going to adjust f by constructing $h \in \text{Homeo}_{0,c}(\mathbb{A}_2)$ such that $h\big|_{\mathbb{A}_{1-D\delta}} = Id$, $h^*\nu = \omega$ and $d_{C^0}(Id, h) \leq C_3 d_{C^0}(Id, \phi)$ for some $C_3 > 0$.

We state that the paragraph **Modifying the area form Ω to find a constant estimate** is actually now easier. Indeed everything transpose at one exception, since it is not mandatory to obtain a diffeomorphism we don't have to adjust the Ω on the skeleton Γ. We just have to apply Proposition 5.1 directly on squares alongside the neighborhood of $\mathbb{A}_1$. The resulting C^0-small homeomorphism h_1 is such that $h_1^*\nu = \varrho\omega$ is an area-form, we can then apply the Proposition 5.3 and finish the proof of Lemma 3.3. □

5.3 A generalization of the area-preserving extension lemma

By a modification of the function χ_δ described in the proof of Lemma 5.3, we can easily make a more general statement this is the point of the Lemma 5.6 which describes more precisely how much place is needed for a C^0-small area-preserving extension.

Lemma 5.6 (An other area-preserving extension lemma for annuli). *Let equipped $\mathbb{A}_3$ with an area form ω. Let ϕ be an area-preserving embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$.*

We assume furthermore that for some $y \in [-1, 1]$ (and hence for all) $S^1 \times y$ and $\phi(S^1 \times y)$ are homotopic in $\mathbb{A}_2$ and

the area in $\mathbb{A}_2$ bounded by $S^1 \times y$ and $\phi(S^1 \times y)$ is zero.

Let $\delta := d_{C^0}(Id, \phi)$. Then if δ is sufficiently small, for all $\alpha \in [0, 1]$, there exists D (independent of δ and α) and $\psi \in Ham_c(\mathbb{A}_{1+D\delta^\alpha})$ such that $\psi\big|_{\mathbb{A}_{1-D\delta}} = \phi\big|_{\mathbb{A}_{1-D\delta}}$ and

$$d_{C^0}(Id, \psi) \leq C\delta^{1-\alpha},$$

for some constant $C > 0$.

Moreover, if for some arc $I \subset S^1$ we have that $\phi = Id$ outside a quadrilateral $I \times [-1, 1]$ and $\phi(I \times [-1, 1]) \subset I \times [-2, 2]$, then ψ can be chosen to be the identity outside $I \times [-2, 2]$.

Remark 5.7. *It can be shown that the exponent* $1-\alpha$ *in the last inequality of Lemma 5.6 can't be made sharper. Indeed, one can show that the translation by a vector of norm* δ *in any direction can't be extended by a symplectomorphism of* C^0*-norm smaller than* $C'\delta^{1-\alpha}$ *for some constant* $C' > 0$*.*

Remark 5.8. *The proof in [6] of this lemma extension actually work (after some modification) for* $\alpha \in [1/2, 1]$ *but fail if* α *is smaller than 1/2.*

The case $\alpha = 0$ *in the Lemma 5.6 is Lemma 3.2 and* $\alpha = 1$ *is our first step in the proof of the area-preserving extension lemma (**Modifying the area form*** Ω ***to find a constant estimate**).*

6 Proof of the extension lemmas

The proof of Lemma 5.4 can be found, after some small adjustement, in [6]. We present instead a proof of Lemma 5.5, the proof is an adaptation to the continuous case of Lemma 6.6 in [6]. For this we will need an adaptation of the Appendix of Michael Khanevsky from the same paper.

Lemma 6.1. *Set* $L = S \times 0$ *in* $\mathbb{A}$*. Assume that* ϕ *is a continuous embedding of an open neighborhood of* L *in* $\mathbb{A}$*, so that* L *is homotopic to* $\phi(L)$ *and* $\|\phi\| \leq \varepsilon$ *for some* $\epsilon \ll 1$*.*

Then there exists a diffeomorphism $\psi \in Diff_{0,c}(\mathbb{A}_{D\varepsilon})$ *such that* $\psi = \phi$ *on* L *and* $\|\psi\| \leq C\varepsilon$ *for some* $D, C > 0$ *indenpendant on* ϕ*.*

Moreover, if $\phi = Id$ *outside some arc* $I \subset L$ *and* $\phi(I) \subset I \times [-1, 1]$*, then* ψ *can be made the identity outside* $I \times [-1, 1]$*.*

Remark 6.2. *We added an extra result on the support of* ψ *that wasn't made by Michael Khanevsky but doesn't require more effort and is needed in the proof of the sharp area-preserving extension lemma.*

We describe the change we need to make in order to prove this version of the lemma.

Proof. (of Lemma 6.1)
The proof is divided in 4 steps, we only modify the **Step 3**. The construction of the diffeomorphism $\psi_{3,i} \in \mathrm{Diff}_{c,0}(B_i')$ is now obtained by the Jordan-Schoenflies Theorem [4]. The resulting homeomorphism is then still C^0-small. ☐

This is the tool we need to prove Lemma 5.5. Notice here that in contrary of the proof of Lemma 6.6 in [6] the proof is very short, indeed we don't need to care more about the extension since we care about obtaining a diffeomorphism and not only a homeomorphism.

Proof. (of Lemma 5.5) We apply Lemma 6.1 to the curve $S^1 \times \{\pm 1\}$ in $\mathbb{A}_2$ and there image under ϕ. We can find $\psi \in \mathrm{Diff}_{0,c}(\mathbb{A}_2)$ supported in $\mathbb{A}_{1+C_2\delta} \backslash \mathbb{A}_{1-C_2\delta}$ and agreeing with ϕ^{-1} on $\phi(S \times \{\pm 1\})$. Now when we extend $\psi' = \psi\phi$ by the identity outside of $\mathbb{A}$ we get the required result. ☐

References

[1] A. Banyaga, *Sur la structure de certains groupes de difféomorphismes*, Publications du Département de Mathématiques de Lyon, **13** (1976), 57–61.

[2] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comm. Math. Helv. **53** (1978), 174–227.

[3] A. Banyaga, *The structure of classical diffeomorphism groups*, Mathematics and its applications **400**, Kluwer Academic Publishers Group, Dordrecht, 1997.

[4] S. Cairns, *An Elementary Proof of the Jordan-Schoenflies Theorem*, Proceedings of the American Mathematical Society , **2**, No. 6, (1951), 860–867.

[5] D. Cristofaro-Gardiner, V. Humilière, S. Seyfaddini, *Prouf of the simplicity conjecture*, 2020.

[6] M. Entov, L. Polterovich, P. Py, *On continuity of quasimorphisms for symplectic maps*, Progress in Math. **296** (2012), 169–197.

[7] A. Fathi, *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Annales scientifiques de l'É.N.S. (4), **13**, No. 1 (1980), 45–93.

[8] F. Le Roux, *On closed subgroups of the group of homeomorphism of a manifold*, 2018.

[9] D. McDuff, D. Salamon, *Introduction to symplectic topology*, 2nd edition, Oxford University Press, Oxford, 1998.

[10] J.M. Mather, *Commutators of diffeomorphisms* Comment. Math. Helv. **49** (1974) 512–528, et **50** (1975) 33–40.

[11] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 288–294.

[12] J. C. Oxtoby, S. M. Ulam, *Measure-Preserving Homeomorphisms and Metrical Transitivity*, Annals of Mathematics, Second Series, **42**, (4) (1941), 874–920.

[13] S, Schwartzman, *Asymptotic Cycles*, Annals of Mathematics , Second Series, **66**, No. 2 (1957), 270–284.

[14] S. Seyfaddini, *C^0-limits of hamiltonian paths and the Oh-Schwarz spectral invariants*, nt. Math. Res. Not. **21** (2013), 4920–4960.

[15] W. Thurston *Foliations and group of diffeomorphisms*, Bull. American Math. Soc. **80** (1974) 304–307.

[16] W. Thurston, *On the structure of the volume preserving diffeomorphisms*, unpublished.