

The local-global principle for torsors under a group scheme.

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Introduction

During my three months internship in Niigata University in Japan, under the supervision of Pr. Hoshi, I studied algebraic group schemes over a field and more precisely what is called the 'local-global principle' for 'G-torsors'. Let's split this definition in two and explain what is the local-global principle first.

Given a polynomial equation with coefficients in \mathbb{Q} :

$$f(x_1, \dots, x_n) = 0, \quad (*)$$

it may be difficult to decide whether or not it has a solution in \mathbb{Q}^n (for example, see Hilbert 10th problem). However, if the equation (*) has a solution in \mathbb{Q}^n then it has a solution in K^n for any field extension of \mathbb{Q} .

Here we look at extensions of \mathbb{Q} that are obtained by completion of \mathbb{Q} with respect to an absolute value :

Definition 1 – Absolute value on a field

Let k be a field. An absolute value on k is a map :

$$|\cdot| : k \rightarrow \mathbb{R}_{\geq 0}$$

Satisfying the following axioms :

1. The map $|\cdot|$ is multiplicative
2. For $x \in k$, $|x| = 0$ if and only if $x = 0$
3. There exist some constant $C > 0$ such that

$$(\forall x, y \in k) \quad |x + y| \leq C \max(|x|, |y|)$$

The absolute value is said to be non-archimedean if one can choose $C = 1$, and archimedean otherwise.

Two absolute values are said to be equivalent $|\cdot|$ and $|\cdot|'$ if there is some $\gamma > 0$ such that :

$$\forall x \in k, \quad |x|' = |x|^\gamma$$

An equivalence class of absolute values for k is called a place of k .

Ostrowski's theorem states that, up to equivalence, the only absolute values on \mathbb{Q} are the infinite absolute value and the p -adic ones, setting for p a prime and $x \in \mathbb{Q}$:

$$|x|_p = p^{-v_p(x)}$$

giving various fields extensions of \mathbb{Q} :

$$\mathbb{R} \quad \mathbb{Q}_2 \quad \mathbb{Q}_3 \quad \mathbb{Q}_5 \quad \mathbb{Q}_7 \quad \mathbb{Q}_{11} \cdots$$

called local fields¹ In the case of number field k , there is one non-archimedean valuation for each prime ideal \mathfrak{p} of the ring of integers \mathcal{O}_k of k , and the archimedean one are induced by the complex or real embeddings $k \subset \mathbb{R}$ or \mathbb{C} .

The goal of local-global principles on \mathbb{Q} for the equation (*) is to give an answer to the following problem :

Problem 1

If (*) has a solution in every completion \mathbb{R} and \mathbb{Q}_p of \mathbb{Q} for p prime, does (*) have a solution in \mathbb{Q} ?

When the answer is positive, we say that the local-global principle holds for the equation (*).

For example, we have the following classical theorem (due to Minkowski for \mathbb{Q} and Hasse for number fields) :

1. A local field is \mathbb{R}, \mathbb{C} or a finite extension of $\mathbb{Q}_p, \mathbb{F}_p((t))$.

Theorem 1 – Hasse-Minkowski theorem

The local-global principle holds for any quadratic form^a.

a. i.e when f is a sum of monomials of degree less or equal to 2.

In this intership memoir, we focus the local-global principle for a very specific class of equations, related to what is called a G-torsor.

A G-torsor is roughly speaking a ‘space-like’ object X with an action of a ‘group-like’ G such that locally, in some sense, X and G are isomorphic in a G-equivariant² way.

For instance, given an finite extension l/k of fields of degree n, we can define the multiplicative norm :

$$N_{l/k} : l^\times \rightarrow k^\times$$

thus defining a polynomial $N_{l/k}$ in n-variables by fixing a basis of l over k.

Here, the group-like object is the set G of solutions to the equation

$$N_{l/k}(x) = 1,$$

related to the so-called norm-one torus, and the space-like object is the set X of solutions of solutions to the twisted version of the equation :

$$N_{l/k}(x) = a \quad (\iota)$$

for a fixed $a \in k^\times$, related the norm hypersurfaces.

Then, there is a natural action of G on X by multiplication, because if $N_{l/k}(u) = 1$ and $N_{l/k}(x) = a$, then $N_{l/k}(xu) = a$.

Moreover, after extending the scalars (‘locally in some sense’ = making a base change by some morphism) to a sufficiently large finite extension K of k, the equation (ι) has a solution x_0 , and then the group G and X become ‘isomorphic’ because

$$x \in (l \otimes_k K)^\times \text{ satisfies } (\iota) \iff y = x_0^{-1}x \in (l \otimes_k K)^\times \text{ satisfies } N_{l/k}(y) = 1$$

This is made precise by the categorical notion of sites, Grothendieck topologies and sheaves, which are presented briefly here in the related annexe.

In our internship report, we are interested in the case where the group like object G is an affine algebraic group scheme over a global field³, the space like object X is a k-scheme, and use the étale or the fppf topology.

The notion of scheme and group scheme are briefly presented in the related annexes 5 and 6.

Given such a group scheme G over k, the obstruction to the local-global principle for the G-torsors for the étale topology is given by a set $\text{III}(G)$, the Shafarevich-Tate set of G.

When G is abelian, this set has the structure of an abelian group. The finiteness of $\text{III}(G)$ for abelian varieties is deeply linked with the more famous Swinerton-Dyer conjecture on the rank of the abelian group $A(k)$, and we have the following conjecture :

Conjecture 1 – Tate’s conjecture

If A is an abelian variety over a global field, then $\text{III}(A)$ is a finite group.

Most of the cases of this conjecture remain unsolved, and

The goal of this report is to give a proof of the finiteness of $\text{III}(G)$ when G is a linear algebraic group⁴ over a number field k.

The proof relies strongly on the reduction theory for algebraic groups, a theory developped by Armand Borel and Harish-Chandra in the real case [1], and then by Tamagawa, Godement and Weil in the adelic case [9].

The structure of this report is as follows :

The main body of the report is page is from page 9 to 29, the rest is just some annexes to complete some proofs and give an more complete introduction to the theory of sheaves, of schemes and of group schemes.

The chapter 2 is devoted to the presentation of the Shafarevish-Tate set of an arbitrary group scheme. We start by developping more intuition and examples of G-torsors that arise in nature, before giving a precise definition in terms of sheaves for some given topology, the definition of $\text{III}(G)$ and some of it’s incarnations.

2. An isomorphism compatible with the action of G on X and of G on itself
 3. number field or finite separable extension of $\mathbb{F}_p(t)$.
 4. It is the second big case besides abelian varieties, because by a theorem of Chevalley, any algebraic group over k has a maximal normal linear subgroup with quotient an abelian variety.

The study case of $G = \mathrm{PGL}_n$ follows in .We prove that in this case $\mathrm{III}(G) = 1$, a result linked with the local-global principle for a precise class of k -schemes, the Severi-Brauer varieties, that are ‘twisted’ versions of the projective space.

The goal of the part 3 of the report is to give an exposition of the proof of the finiteness of $\mathrm{III}(G)$ for G a linear algebraic group over a number field k .

We use general results on group scheme cohomology to reduce the problem to 3 base cases⁵ :

Unipotent group Finite group Reductive group

The most difficult case is the reductive one, where the strategy consists in finding a suitable geometric representation of the III developed in the last section 5.

I’d like to thank warmly my internship director Pr. Akinari Hoshi for helping me throughout my stay at Niigata university in so many ways and orienting my curiosity in the world of group schemes and cohomology, Pr. Hashizume Kenta for kindly answering my algebraic-geometry related questions, and finally thank you Iida for our shared moments! On the other side of the map, i’d like to thank sincerely my tutor Pr. Ariane Mezard for helping me finding an internship and bringing answers to my multiple doubts, and Yue Teng from the secretary of the ENS for her precious help.

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5. See the annexe on group scheme for the different definition, especially the section 5.3

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The Tate-Shafarevich set of a group scheme

1 Torsors

Motto 1

A torsor for a group-like object G is a space-like object X endowed with a G -action such that locally in some sense G and X are isomorphic.

We give two motivating examples in different contexts before giving a definition for schemes. We use the notion of sheaves developed in the annexe 4 :

Example 1 – Torsors in differential geometry and number theory

1. The first example comes from differential geometry.

Let M be a smooth manifold, and

$$p : E \rightarrow M$$

a smooth vector bundle with fibers of constant dimension d . Then, we can consider the sheaf of basis $\text{Basis}(E)$ on M :

Construction 1 – Sheaf of basis

A section of $\text{Basis}(E)$ on $U \subset M$ open is a smooth section of E^d :

$$e = (e_1, \dots, e_d) : U \rightarrow E^d$$

such that $(e_{1,x}, \dots, e_{d,x}) \in E_x^d$ is a basis of the fiber for each $x \in U$.

The sheaf $\text{Basis}(E)$ has naturally an action of the sheaf of groups $GL_{n,M} := \text{Basis}(M \times \mathbb{R}^d)$, and that a choice of trivialisation (U_i) of E give $GL_{n,M}$ -equivariants isomorphisms :

$$\text{Basis}(E)|_{U_i} \simeq GL_{n,U_i}$$

Thus, we say that $\text{Basis}(E)$ is a $GL_{n,M}$ -torsor for the open topology of M .

2. We can consider a second exemple, this time from number theory :

Let k be a field, n an integer prime to the characteristic of k and let $a \in k^\times$ any element.

For any k -algebra A , we can define :

$$\mu_n^a(A) := \{x \in A \mid x^n = a\}$$

And set $\mu_n := \mu_n^1$.

We have a natural action of the group $\mu_n(A)$ on $\mu_n^a(k)$ by multiplication. Then, μ_n^a is locally trivial in the following sense : The polynomial $f(x) = x^n - a$ is prime to it's non zero derivative nx^{n-1} , f is a separable polynomial, and thus $x^n - a$ splits in a separable extension l/k .

If we fix such solution $\alpha \in l$, we see that for any l -algebra B , $\mu_n^a(B)$ is parametrised by $\mu_n(l)$ the set on n -th roots of unity in l in a functorial way.

Thus, after 'base change' by a morphism $k \rightarrow l$, μ_n^a (instead of base change by an open immersion $U \rightarrow M$) and μ_n become isomorphic.

In this sense, μ_n^a is a μ_n -torsor for the étale topology, see ?? for more details on Grothendieck's topologies, and 4 for more details on the étale site.

Let us now focus on to the case of schemes.

We use the language of sheaves and grothendieck topologies to not be worried about representability questions and to emphasize the use of the functorial point of view, see the appendix 4 for more details.

Definition 2 – Torsor under a sheaf of groups

Let S be a scheme, and G a sheaf of groups on Sch_S for the topology $\tau \in \{\text{zar}, \text{étale}, \text{fppf}\}$. A G -torsor is a sheaf on Sch_S for the topology τ with an action of G on X such that :

1. For any S -scheme T , either $X(T) = \emptyset$, or the action of $G(T)$ on $X(T)$ is simply transitive.
2. There exist a τ -covering $\{S_i \rightarrow S\}_{i \in I}$ of X such that $X(S_i) \neq \emptyset$ for each $i \in I$.

Example 2

1. As we saw earlier, if k is a field, $a \in k$ and n is prime to the characteristic of k , the étale sheaf on k -schemes defined on k -algebras A by

$$\mu_n^a(A) = \{\text{solutions to } x^n = a\}$$

is a μ_n -torsor.

2. The sheaf of group G acts simply transitively on itself on the right by multiplication, and $G(T)$ is always non empty for T an S -scheme, as it contains the unit section. This torsor is called the trivial torsor.

The G -torsors have very rigid structure. Indeed, if X is a right- G -torsor that admits a global section $x \in X(S)$, then, defining

$$g \in G(T) \mapsto x|_T \cdot g$$

for any S -scheme T gives an isomorphism between G and X .

For more results in this sense, see 4.4.

Definition 3

The class of all torsors up to isomorphism, denoted by

$$H_\tau^1(S, G)$$

is pointed by the class of the trivial torsor.

2 Functoriality of non-abelian cohomology

The construction of $H_\tau^1(S, G)$ is functorial in S and G , using base-change and the so-called "contracted product". Let us describe both constructions.

Construction 2 – Functoriality in S

Let E a G -torsor, and $f : S' \rightarrow S$ a morphism. Then the base change $E_{S'}$ of the sheaf E is a $G_{S'}$ -torsor and the map

$$[E] \in H_\tau^1(S, G) \rightarrow [E_{S'}] \in H_\tau^1(S', G_{S'})$$

is well-defined.

Construction 3 – Functoriality in G

Let $u : G \rightarrow H$ be a morphism of τ -sheaves of groups, and E a G -torsor. Then, we define the push-forward of E by u to be the quotient :

$$u_*(E) := E \times_S H / G$$

Where G acts on the left $E \times_S H$ on T -points by :

$$g \cdot (e, h) = (e, f(g) \cdot h)$$

Then, there is a natural action of H on $u_*(E)$, induced on the quotient by the right action of H on $E \times_S H$ given by :

$$(e, h) \cdot h' = (e, hh').$$

This gives a well-defined map :

$$u_* : H^1(S, G) \rightarrow H^1(S, H).$$

Remark

1. In a more general setup, one can define the contracted product of X and Y , where X (resp. Y) is a sheaf with a right (resp. left) action on G by :

$$X \times_G Y := X \times_S Y /_G.$$

The group G acts on the fiber product by $g \cdot (x, y) := (x, g^{-1} \cdot y)$. With this notation, we see that :

$$u_*(E) = E \times_G H$$

Where G acts on the left on H by $g \cdot h := u(g) \cdot h$.

Using the functoriality in G , we can construct short exact sequences on points from any short exact sequences of sheaves¹ :

Theorem 2 – Long exact sequence in non abelian cohomology

Let S be a scheme, $\tau \in \{\text{Zar}, \text{ét}, \text{fppf}\}$ a topology on the big site Sch_S , G a sheaf of group on S for the topology τ , and H a subgroup of G defined by a monomorphism :

$$i : H \longrightarrow G$$

and $X = G /_H$ the quotient sheaf with canonical projection

$$\pi : G \longrightarrow X.$$

Then, there is a natural exact sequence of pointed sets :

$$1 \longrightarrow H(S) \longrightarrow G(S) \longrightarrow X(S) \longrightarrow H^1(S, H) \longrightarrow H^1(S, G)$$

Where $\delta(q)$ is the fiber of $q \in Q(S)$ in G defined by the following fiber product :

$$\begin{array}{ccc} \delta(q) & \longrightarrow & G \\ \downarrow & & \downarrow \pi \\ S & \xrightarrow{q} & X \end{array}$$

If additionally H is normal in G , then X has also a structure of sheaf of groups, and previous the exact sequence can be extended into :

$$1 \longrightarrow H(S) \longrightarrow G(S) \longrightarrow X(S) \longrightarrow H^1(S, H) \longrightarrow H^1(S, G) \longrightarrow H^1(S, X)$$

and if additionally H is central in G , in particular it abelian, we can define $H^2(S, N)$ ^a and the previous exact sequence can be extended once more to get :

$$1 \longrightarrow H(S) \longrightarrow G(S) \longrightarrow X(S) \longrightarrow H^1(S, H) \longrightarrow H^1(S, G) \longrightarrow H^1(S, X) \longrightarrow H^2(S, H)$$

a. using derived fonctors and homological algebra, see ?? for more details

Proof

We only prove the non-normal case in the fppf topology. For more details, see for example book by [11], proposition 11.14.

The sequence is exact at H because i is a monomorphism. The sequence is exact in G because for a S -point x of G , $\pi(x) = 1$ if and only if there exist some covering $\{S_i \rightarrow S\}_{i \in I}$ for the topology τ such that for any $i \in I$,

$$x|_{S_i} \in H(S_i).$$

But, because H is a subsheaf of G , this implies $x \in H(S)$.

Finally, the sequence is exact at X because for an element $x \in X(S)$,

$$x \in \pi_S(G(S)) \iff \pi^{-1}(x)(S) \neq \emptyset \iff [\pi^{-1}(x)] \in H^1(S, H) \text{ is trivial} \iff \delta_S(x) = 1.$$



1. [?] for the definition of exact sequences of sheaves on a site.

3 Local-global principle for G-torsors

Definition 4 – Global field

A global field is a number field or a finite separable extension of $\mathbb{F}_p(t)$ for some prime p .

Then, we may consider \mathcal{P}_k the set of places on k , and consider at each place v the completion k_v of k at the place v .

Definition 5 – Local-global principle

Let G an τ -sheaf of groups on Sch_k (e.g. a G is a k -group scheme) for the topology τ . We say that the local-global principle holds for G for the topology τ if for any étale G -torsor H ,

$$H(k) \neq \emptyset \text{ if and only if } H(k_v) \neq \emptyset \text{ for each } v \in \mathcal{P}_k$$

By the properties of torsors stated in ??, G satisfies the local-global principle for the topology τ if and only if the natural map :

$$[E] \in H_{\text{ét}}^1(k, G) \longmapsto ([E \otimes_k k_v])_{v \in \mathcal{P}_k} \prod_{v \in \mathcal{P}_k} H_{\tau}^1(k_v, G_{k_v})$$

is injective.

This suggests the following definition :

Definition 6 – Shafarevich-Tate set of a group scheme

Let G a k -sheaf of groups, and $S \subset \mathcal{P}_k$ a set of places, then, we define $\text{III}_{\tau}(G)$ to be the kernel of the natural morphism of pointed sets :

$$\text{III}_{\tau}(G) := \text{Ker} \left(H_{\tau}^1(k, G) \longrightarrow \prod_{v \in \mathcal{P}_k} H_{\tau}^1(k_v, G_{k_v}) \right)$$

This set is called the Shafarevich-Tate set of G , it is the obstruction to the local-global principle for G -torsors.

Remark

1. If G is not abelian, $\text{III}(G)$ is not a group in an obvious way. However, it is always a pointed set with marked point the trivial torsor.
2. One can also be interested in stronger versions of the local-global principle for torsors, and define for any set of places $S \subset \mathcal{P}_k$:

$$\text{III}_{S, \tau}(G) := \text{Ker} \left(H_{\tau}^1(k, G) \longrightarrow \prod_{v \notin S} H_{\tau}^1(k_v, G_v) \right)$$

3. If G is an abelian sheaf on k , then we can also consider higher order cohomology (defined as the derived functor of the global section functor) and define for $i \geq 0$ the abelian group :

$$\text{III}^i(G) := \text{Ker} \left(H^i(k, G) \longrightarrow \prod_{v \in \mathcal{P}_k} H_S^i(k_v, G_{k_v}) \right)$$

4 A first example : $\text{III}(\text{PGL}_n)$ and the local-global principle for Severi-Brauer varieties

4.1 The group scheme PGL_n

Let n be a positive integer, and k a field. There is several ways natural ways of thinking about the scheme analog of the group $\text{PGL}_n(k)$, let us describe two of them :

Construction 4 – The different definitions of PGL_n

1. Firstly, as a fppf sheaf, given by the quotient of the affine scheme (see as a fppf sheaf) $\text{GL}_{n, k}$ by the natural action of $\mathbb{G}_{m, k}$ on the left induced by the natural morphism $\text{G}_m \rightarrow \text{GL}_n$ given on T -points by :

$$a \in \mathbb{G}_m(T) \longmapsto a \cdot I_{n \times n} \in \text{GL}_n(T)$$

2. Secondly, as the fppf sheaf of automorphisms of the scheme \mathbb{P}_k^{n-1} , that is to say, for a k -scheme T , we set :

$$\mathrm{PGL}_n(T) := \underline{\mathrm{Aut}}_k(\mathbb{P}_k^{n-1})(T) = \mathrm{Aut}_{\mathrm{Sch}_T}(\mathbb{P}_T^{n-1})$$

The two definition give naturally isomorphic fppf sheaves, as we prove it in [PGL annexe].

4.2 Non abelian cohomology of PGL_n

From the exact sequence :

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow 1$$

We get, for any field extension L of k , the exact sequence of pointed sets, and because \mathbb{G}_m is central in GL_n we get :

$$1 \longrightarrow L^\times \longrightarrow \mathrm{GL}_n(L) \longrightarrow \mathrm{PGL}_n(L) \longrightarrow H^1(L, \mathbb{G}_{m,L}) \longrightarrow H^1(L, \mathrm{GL}_{n,L}) \longrightarrow H^1(L, \mathrm{PGL}_{n,L}) \longrightarrow H^2(L, \mathbb{G}_m)$$

The abelian group $H^2(L, \mathbb{G}_{m,L})$ is called the Brauer group of L , denoted by $\mathrm{Br}(L)$.

But, we know the first cohomology of \mathbb{G}_m , it is trivial :

Theorem 3 – Hilbert 90'

For any field k , then

$$H^1(k, \mathbb{G}_{m,k}) = 1$$

See for example [10, Tag 03P7] for a proof.

Thus, we have an injective morphisms of pointed sets :

$$\delta_L^2 : H^1(L, \mathrm{PGL}_{n,L}) \longrightarrow \mathrm{Br}(L)$$

that is natural in L .

4.3 $\mathrm{III}(\mathrm{PGL}_n)$

Assume k is a global field. Then, the naturality of δ_L^2 and H^1 in L yields the following commutative diagramm :

$$\begin{array}{ccccc} \mathrm{III}(\mathrm{PGL}_{n,k}) & \hookrightarrow & H^1(k, \mathrm{PGL}_{n,k}) & \longrightarrow & \prod_{v \in S} H^1(k_v, \mathrm{PGL}_{n,k_v}) \\ & & \downarrow & & \downarrow \\ ? & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \prod_{v \in S} \mathrm{Br}(k_v) \end{array}$$

from which we deduce an injective map :

$$\mathrm{III}(\mathrm{PGL}_n) \longrightarrow \mathrm{Ker} \left(\mathrm{Br}(k) \rightarrow \prod_{v \in \mathcal{P}_k} \mathrm{Br}(k_v) \right)$$

But, we know the structure of the Brauer group of a global field. More precisely we admit the following theorem, a proof of which we can find in [8] (Theorem 8.1.17) by Neukrich :

Theorem 4 – Brauer group of a global field

For any global field k , the following statements hold :

1. For each place finite place v , there is a canonical injective morphism :

$$\mathrm{res}_v : \mathrm{Br}(k_v) \rightarrow \mathbb{Q}/\mathbb{Z}$$

which is an isomorphism if v is finite, and the natural injection $0, \frac{1}{2}\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ when v is real or complex.

2. The image of $\text{Br}(k)$ by the natural map (\star) lies in the direct sum :

$$\bigoplus_{v \in \mathcal{P}_k} \text{Br}(k_v)$$

3. If we define res to be the direct sum of all the res_v , with $v \in \mathcal{P}_k$, then, we have an exact sequence of abelian groups :

$$0 \longrightarrow \text{Br}(k) \longrightarrow \bigoplus_{v \in \mathcal{P}_k} \text{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Thus, the kernel of

$$\text{Br}(k) \rightarrow \bigoplus_{v \in \mathcal{P}_k} \text{Br}(k_v)$$

is trivial, and so is $\text{III}(\text{PGL}_{n,k})$.

Corollary 1

The local-global principle holds for PGL_n .

4.4 Application : Local-global principle for Severi-Brauer varieties

In this section we establish the local-global principle for the class of so-called Severi-Brauer varieties, that are twisted form of the projective space, by using the vanishing of $\text{III}(\text{PGL}_d)$ for $d \geq 0$.

Let k be a field and fix \bar{k} an algebraic closure of k , and $k_s \subset \bar{k}$ the separable closure of k in \bar{k} .

Definition 7

A Severi-Brauer variety V is a k -scheme such that the base-change $V_{\bar{k}_s}$ is isomorphic as a k_s -scheme to a projective space $\mathbb{P}_{k_s}^n$ for some $n \geq 0$.

Remark

1. A Severi-Brauer scheme is said to be split or trivial if it is isomorphic to a projective space as a k -scheme.
2. Every-Brauer scheme admits a finite separable extension splitting field l , that is to say a field extension of k such that V_l is isomorphic to a projective space.

Then, Severi-Brauer schemes isomorphic to $\mathbb{P}_{k_s}^n$ after base change are strongly related to PGL_{n-1} -torsors via the following construction :

Construction 5 – Torsor associated to a Severi-Brauer variety

Let V a Severi-Brauer scheme such that V_{k_s} is isomorphic to \mathbb{P}^n . We can define the sheaf of isomorphisms between the projective space and V :

$$E_V := \underline{\text{Isom}}_{\text{Sch}_k}(\mathbb{P}^n, V)$$

Then PGL_{n-1} acts naturally by composition on E_V , and we see that it is a PGL_{n-1} -torsor.

Reciprocally, any PGL_{n-1} -torsor for the étale topology is representable by induced by such a Severi-Brauer variety, it is shown by V.E Voskresenskii's book [13] (even though the point of view on cohomology is with cocycles and coboundaries) in section 3.1.

Now we use the following theorem, due to Châtelet in it's classical form :

Theorem 5 – Châtelet's theorem

Let V a Severi-Brauer scheme over k . Then, the following assertions are equivalent :

1. V is split over k
2. E_V is trivial
3. V admits a k -point.

Before giving the proof, let us recall some facts about Galois actions on morphisms of schemes :

Construction 6 – Galois action on morphisms of schemes

Let l/k a Galois extension of the field k , X, Y two k -schemes. For any $\sigma \in \text{Gal}(l/k)$, we have an automorphism :

$$\text{spec}(\sigma) : \text{Spec}(l) \longrightarrow \text{Spec}(l)$$

Which induces an automorphism of k -scheme of $X_l = X \times_{\text{spec}(k)} \text{spec}(l)$ and we may define an action of $\text{Gal}(l/k)$ on l -morphisms $f : X_l \rightarrow Y_l$ by :

$$\sigma f := (1 \otimes \sigma^{-1}) \circ f \circ (1 \otimes \sigma)$$

Example 3

Let $f : \mathbb{A}_l^n \rightarrow \mathbb{A}_l^m$ defined by polynomials $(f_1, \dots, f_m) \in l[x_1, \dots, x_n]$. Then,

$$\sigma f : \mathbb{A}_l^n \rightarrow \mathbb{A}_l^m$$

is given by the polynomials $(\sigma(f_1), \dots, \sigma(f_m))$ (σ acts on the coefficients of the f_i).

Then, we have the following theorem which is a consequence of Galois descent :

Theorem 6

Let X, Y schemes over k , and l/k a finite Galois extension. Then, the l -morphisms from X_l to Y_l that are invariant under the action of $\text{Gal}(l/k)$ are precisely the base-change of k -morphisms from X to Y .

In particular,

$$X(L)^{\text{Gal}(L/k)} = X(k).$$

Let us now give a proof of Châtelet's theorem :

Proof

We have the following straightforward implications :

$$E_V \text{ is trivial} \iff \text{Isom}_{k\text{-Sch}}(\mathbb{P}^n, V) = E_V(k) \neq \emptyset \iff V \text{ splits over } k \implies V \text{ admits a } k\text{-point}$$

Thus, only 3. \implies 1. remains to be proven. Let V a Severi-Brauer variety, and assume that V admits a k -point, say $x \in V(k)$. Fix $f : V_l \rightarrow \mathbb{P}_l^n$ an isomorphism, with l/k a finite Galois extension (it is possible since V admits a separable splitting field, and it's normal closure is also a splitting field). Then, define $\Pi := \text{Gal}(l/k)$, and fix $x \in V(k)$ and let $y := f(x) \in \mathbb{P}^n(l)$, corresponds to some line $y = [v] = [v_0, \dots, v_n]$ in l^{n+1} . We may consider the isomorphism :

$$\sigma(f) : V_l \longrightarrow \mathbb{P}_l^n$$

for each $\sigma \in \text{Gal}(l/k)$. which gives an automorphism $f_\sigma := (\sigma f) \circ f^{-1}$ of the l -scheme \mathbb{P}_l^n , fitting in the following diagram :

$$\begin{array}{ccc} V_l & \xrightarrow{f} & \mathbb{P}_k^n \\ & \searrow \sigma f & \downarrow f_\sigma \\ & & \mathbb{P}_k^n \end{array}$$

And we get by direct calculation and because x is Π -invariant :

1. Firstly, for any $\sigma, \tau \in \Pi$,

$$f_{\sigma\tau} = \sigma(f_\tau) \cdot f_\sigma$$

2. Secondly, for any $\sigma \in \Pi$,

$$f_\sigma(y) = \sigma(y)$$

Thus, by 1., when choosing lifts $A_\sigma \in \text{GL}_n(l)$ of f_σ , we obtain elements $(c_{\sigma,\tau})_{\sigma,\tau \in \Pi} \in (l^\times)^{\Pi^2}$ such that

$$A_{\sigma\tau} = c_{\sigma,\tau} \sigma(A_\tau) A_\sigma$$

for all $\sigma, \tau \in \Pi$. By 2., we obtain elements $\lambda_\sigma \in l^\times$ such that :

$$\forall \sigma \in \Pi \quad A_\sigma \cdot \sigma(v) = \lambda_\sigma v$$

Combining these relations, we obtain :

$$c_{\sigma,\tau}\sigma(\lambda_\tau)\lambda_\sigma = c_{\sigma,\tau}\sigma(A_\tau)A_\sigma v = A_{\sigma\tau}\sigma(\tau(v)) = \lambda_{\sigma\tau}v$$

And thus,

$$\forall \sigma, \tau \in \Pi, \quad c_{\sigma,\tau} = \lambda_\sigma^{-1} \lambda_{\sigma\tau} \sigma(\lambda_\tau)^{-1}.$$

And, if we set $B_\sigma = A_\sigma \lambda_\sigma^{-1}$, we obtain :

$$B_{\sigma\tau} = \sigma(B_\tau) B_\sigma$$

By Hilbert 90' again 4.2 (or in Bourbaki, [3] A V.63 for example) there exist some $B \in GL_n(l)$ such that :

$$\forall \sigma \in \Pi \quad B_\sigma = \sigma(B) B^{-1}$$

and therefore, setting b the class of B in $PGL_n L$, we obtain :

$$\forall \sigma \in \Pi \quad \sigma(b^{-1} \circ f) = \sigma(b^{-1}) \circ \sigma(f) = b \circ f_\sigma^{-1} \circ \sigma(f) = b^{-1} \circ f$$

thus, there is some isomorphism $V \simeq \mathbb{P}_k^n$ inducing $b^{-1} \circ f$ by 4.4 Theorem 6 we are done. ■

Corollary 2

The class of Severi-Brauer varieties over a global field satisfy the local-global principle.

Proof

Indeed, let V a Severi-Brauer variety. Assume that V has a k_v -point for each $v \in \mathcal{P}_k$. Then, $E_V \in \text{III}(PGL_n) = 1$, that is to say, E_V admits a k -point, which is equivalent to $V(k) \neq \emptyset$. ■

Finiteness results for the III of a group scheme

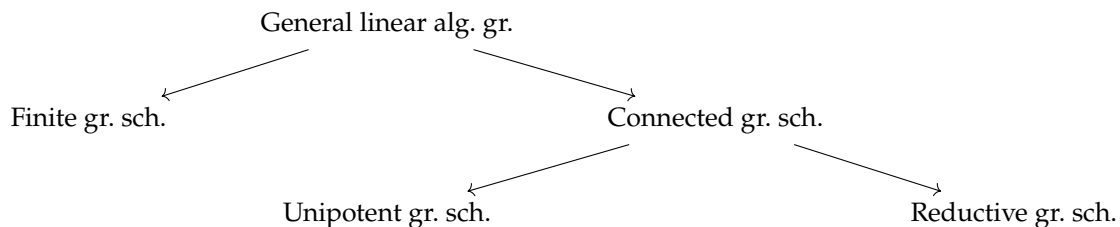
1 Introduction

In this part, we give a proof of the following theorem :

Theorem 7

If G a linear algebraic group over a number field, $\text{III}(G)$ is finite.

Let us explain the strategy to prove the theorem. In the first section, we reduce to several bases cases, more precisely, we proceed to the following reductions :



Then, in the following sections, we proceed to treat the basic cases, that is to say G is finite, unipotent or reductive. The most advanced case is the reductive case, where we use several results on actions of reductive groups on arithmetic groups.

2 Reduction to the basic cases

2.1 The exact sequences

To reduce to the basic cases, we admit the following structure theorems for linear algebraic group schemes over a field.

Theorem 8 – Quotient by the identity component

Let G a linear algebraic group over a field k . Then there is an exact sequence of group schemes over k :

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$

Where $\pi_0(G)$ is a finite etale algebraic group and G^0 is the connected component of the identity.

For a proof of this theorem, see [7], proposition 1.31.

The second exact sequence used to reduce this type to the reductive and unipotent case from the connected case :

Theorem 9 – Levi's decomposition

Let G be a connected linear algebraic group scheme over a perfect field k . Then, there is a split-exact sequence of group schemes over k :

$$1 \longrightarrow U \longrightarrow G \longrightarrow R \longrightarrow 1$$

Where U is unipotent and R is reductive.

For a proof of this theorem, see the chapter 8, theorem 4.3 of [6] by Gerhard P. Hochschild.

Thus, before treating these case, we must place ourselves in the following general setting : G is a linear algebraic group, N a normal closed subgroup of G , and Q the quotient of G by N .

That is to say, we have an exact sequence of affine algebraic groups :

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$$

In this setting, we want to find conditions on N and Q to assure that $\text{III}(G)$ is finite. By functoriality of III , we have a map of sets :

$$\text{III}(p) : \text{III}(G) \rightarrow \text{III}(Q)$$

Thus we have :

$$\text{III}(G) \text{ is finite} \iff \text{III}(Q) \text{ is finite and } \text{III}(p) \text{ has finite fiber.}$$

Let's try to understand the fiber of $\text{III}(p)$. To do this, we will use a technic called 'twisting'.

2.2 Fibers of the III of a quotient map

We present the technique of twisting to identify the fiber of $\text{III}(p)$ at the point $p_*([\mu])$ for a G -torsor μ . For more details and proofs see the section 4.6.

From the action of G on N by conjugation, we get a morphism of fppf-sheaves :

$$G \rightarrow \underline{\text{Aut}}_{k\text{-gr}}(N)$$

and thus a map :

$$H^1(k, G) \rightarrow H^1(k, \underline{\text{Aut}}_{k\text{-gr}}(N))$$

of pointed sets.

The right-hand side classifies k -forms of the group scheme N .

Thus, we get for each G -torsor $[\mu] \in H^1(k, G)$ a twisted form ${}_{\mu}N$ of N (defined up to isomorphism). Similarly the action of G on itself by conjugation gives a twisted form ${}_{\mu}G$ of G .

Then, there is a closed immersion ${}_{\mu}i$ that makes ${}_{\mu}N$ a closed normal subgroup of ${}_{\mu}G$ and a bijection :

$$\sigma_{\mu} : H^1(k, {}_{\mu}G) \rightarrow H^1(k, G)$$

sending ${}_{\mu}G$ to $[\mu]$ such that the following proposition holds :

Proposition 1 – Fibers of p_* are H^1

The fiber of the map

$$p_* : H^1(k, G) \rightarrow H^1(k, Q)$$

at the point $p_*([\mu])$ is precisely the image of the composition :

$$H^1(k, {}_{\mu}N) \xrightarrow{{}_{\mu}i_*} H^1(k, {}_{\mu}G) \xrightarrow{{}_{\mu}\sigma} H^1(k, G)$$

for a proof of theorem, see the section 4.6 on twisting.

Then, the fiber of $\text{III}(p)$ at $p_*(\mu)$ is controlled by the so-called Selmer set of the twisted groups ${}_{\mu}N$ and ${}_{\mu}G$:

Definition 8 – Selmer set

Let N a closed subgroup of a k -group scheme G defined by the closed immersion $i : N \rightarrow G$. We define :

$$\text{Sel}(N, G) := \text{Ker}(H^1(k, N) \rightarrow \prod_{v \in S} H^1(k_v, G_v)).$$

Corollary 3 – Fibers of $\text{III}(p)$

There is a bijection between the fiber of $\text{III}(p)$ at the point $\text{III}(p)([\mu])$ and

$${}_{\mu}i_* (H^1(k, {}_{\mu}N)) \cap \text{III}({}_{\mu}G) = {}_{\mu}i_* (\text{Sel}({}_{\mu}N, {}_{\mu}G))$$

Proof

Indeed, it is a consequence of the previous proposition, the fact that σ_μ is a bijection and the commutative diagram :

$$\begin{array}{ccccccc} H^1(k, {}_\mu N) & \xrightarrow{\mu i_*} & H^1(k, {}_\mu G) & \xrightarrow{\sigma_\mu} & H^1(k, G) & \xrightarrow{P_*} & H^1(k, Q) \\ & & & & \uparrow & & \uparrow \\ & & & & \text{III}(G) & \xrightarrow{\text{III}(p)} & \text{III}(Q) \end{array}$$

■

From these remarks, we obtain the following criteria of finiteness of III :

Corollary 4 – Criteria for finiteness of III(G)

Let G a group scheme over k, N a normal closed subgroup of G, and Q a scheme representing the quotient of G by N. Assume :

1. III(Q) is finite
2. $\text{Sel}({}_\mu N, {}_\mu G)$ is finite for all G-torsor $\mu \in \text{III}(G)$.

Then III(G) is finite.

Proof

Indeed, because III(Q) is finite, it is sufficient to show that III(p) has finite fibers. But, if $[\mu] \in \text{III}(G)$ is a cohomology class of a G-torsor, then, according to the corollary 4, μi_* induces a surjection of $\text{Sel}({}_\mu N, {}_\mu G)$ on the fiber of III(p) at the point $\text{III}(p)([\mu])$, which is therefore finite, because $\text{Sel}({}_\mu N, {}_\mu G)$ is by hypothesis. Thus, III(p) has finite fibers and the proof is complete. ■

2.3 The reduction theorem

We are ready to state and prove the theorem that permits us to reduce the ‘finiteness of III problem’ to base cases from exact sequence.

It is based on the following criteria for the finiteness of the selmer set of a pair (N, G). During the proof, we admit the following non-trivial theorem :

Theorem 10 – Finiteness of local cohomology

A linear algebraic group G on a local field has finite H^1 .

The criteria is as follows :

Proposition 2 – Criteria for the finiteness of the Selmer set

Let N be a closed subgroup of an linear algebraic group G. Then, if all k-forms of N have finite III, and if for all but finitely many places $v \in S$, the natural map :

$$H^1(k_v, N_v) \longrightarrow H^1(k_v, G_v)$$

has trivial kernel, then $\text{Sel}(N, G)$ is finite.

Proof

Using again the twisting technique, the fact that every k-form of N has finite III implies that the canonical map :

$$s : H^1(k, N) \longrightarrow \prod_{v \in S} H^1(k_v, N_v)$$

has finite fibers.

Recall that $\text{Sel}(N, G)$ is the kernel of the composition :

$$H^1(k, N) \xrightarrow{s} \prod_{v \in S} H^1(k_v, N_v) \xrightarrow{I} \prod_{v \in S} H^1(k_v, G_v)$$

But, almost all the $i_{v,*}$ have trivial kernel, and the rest of them have finite kernel by theorem 10. Thus, the map I have finite kernel, and

$$\text{Sel}(N, G) = s^{-1}(\text{Ker}(I))$$

is finite because s has finite fibers. ■

Remark

According to the long exact sequence in non-abelian cohomology, the condition on the H^1 at the place v is equivalent to the surjectivity of the map :

$$p(k_v) : G(k_v) \longrightarrow Q(k_v)$$

Let us deduce the general theorem from the previous criteria :

Theorem 11 – Reduction theorem of the III-problem for exact sequences

Let N, G, Q linear algebraic groups fitting in an exact sequence of linear algebraic groups :

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$$

Then, under the following conditions :

1. For each G -torsor μ , the natural map of pointed sets

$${}_{\mu}G(k_v) \longrightarrow {}_{\mu}Q(k_v)$$

is surjective for all but finitely many $v \in S$.

2. All k -forms of N have a finite III.

3. III(Q) is finite.

the set III(G) is also finite.

Proof

Because of the corollary 4 (2.2) and the finiteness of III(Q) it is sufficient to prove that for any G -torsor μ , the selmer set of the pair $({}_{\mu}N, {}_{\mu}G)$ is finite.

This is true because the assumptions of the criteria for the finiteness of the selmer group (2.3) are satisfied for $({}_{\mu}N, {}_{\mu}G)$ by the hypothesis 1. and 2. combined with the previous remark. ■

We apply the reduction theorem to the exact sequences presented in 2.1. This is done using the following lemma :

Lemma 1

Let N a closed normal subgroup of G with quotient Q and projection π .

Then, the map

$$\pi_{k_v} : G(k_v) \longrightarrow Q(k_v)$$

is surjective for all but finitely many places v in the following cases :

1. Q is finite and N is connected.
2. The exact sequence is split.

The split case is easy, because in this case π admits a section.

We admit the proof of the finite case of this lemma by lack of space. One can find a proof in [12] (Lemma 6.12).

From all this we reduce to the base cases :

Proposition 3

Assume that III(G) is finite for all finite, unipotent or reductive groups, then III(G) is finite for all linear algebraic groups.

Proof

First assume that G is connected. In this case, because of theorem 9(2.1), G fits in an exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$$

with N unipotent and Q reductive.

Then, $\text{III}(Q)$ is finite by hypothesis, the assumption 1. of the reduction theorem holds by the previous lemma (the exact sequence is split), and all the k -forms of N are also unipotent and thus have finite III. Therefore $\text{III}(G)$ is finite by theorem 11. Now let G be a general linear algebraic group. By theorem 8 (2.1), G fits in another exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1$$

this time with N connected and Q finite. Assumption 1. of the criteria for finiteness of III (2.2) is satisfied by the previous lemma, and assumption 3. holds because Q is finite. Thus, it is sufficient to show that the III of all k -form of N is finite. But k -forms of a connected scheme is connected. Thus any group scheme that is a k -form of N has finite III.

Therefore by the reduction theorem (2.2), $\text{III}(G)$ is finite.

■ Now we proceed to treat the three base cases, from the easiest (unipotent groups) to the most technical (reductive groups).

3 The unipotent case

In this section, we give a proof that $\text{III}(G)$ is finite when G is a unipotent group. In fact, this has nothing to do with a local-global principle nor the set of places k , and it is a consequence of the stronger statement :

Proposition 4

Let G be a unipotent group over a field k of characteristic 0. Then,

$$H^1(k, G) = 1$$

In order to prove this theorem, we use the following structure theorem of unipotent group in characteristic 0. One can find a proof of the statements of the theorem in [7], proposition 15.31 and proposition 15.23]

Proposition 5

Let G be a unipotent group scheme over a field k of characteristic 0. Then,

1. There is an isomorphism of schemes :

$$\exp : \text{Lie}(G) \longrightarrow G$$

which is an isomorphism of group schemes when G is commutative.

2. The group G admits a central normal serie with successive quotient \mathbb{G}_a .

Now let us give a proof of the proposition 4 (3) :

Proof

According to the 3, we have an exact sequence of group schemes :

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbb{G}_a \longrightarrow 1$$

With N central in G . In particular, N is commutative and by 3 again, N is isomorphic to $\text{Lie}(N) \simeq \mathbb{A}_k^d$ as a group scheme, where $d = \dim(N)$.

Thus, G fits in an exact sequence :

$$1 \longrightarrow \mathbb{A}_k^d \longrightarrow G \longrightarrow \mathbb{G}_a \longrightarrow 1$$

and the proposition follows from the fact that

$$H^1(k, \mathbb{A}_k^d) = 1$$

for any field k and any integer $d \geq 0$ (it's the additive version of Hilbert 90', see [10, Tag 03QV] for a proof). combined with the long exact sequence for H^1 .

■

4 The finite case

In this section, we want to prove the following theorem :

Theorem 12

The set $\text{III}(G)$ is finite when G is a finite group^a over a number field k .

a. G is affine and $\Gamma(G, \mathcal{O}_G)$ is a finite k -algebra

We fix such a group G for the rest of the section, \bar{k} an algebraic closure of k (it is also a separable closure because k is of characteristic 0). The absolute galois group of k is denoted by \mathcal{G}_k and M is the set $G(\bar{k})$. It is naturally a \mathcal{G}_k -set, and recall that there is a natural bijection :

$$H^1(k, G) = H^1(\mathcal{G}_k, M)$$

Where the left side is étale cohomology and the right side is Galois cohomology¹.

The first step is to reduce to the case where all \bar{k} -points of G are defined over k , that is to say when \mathcal{G}_k acts trivially on $G(\bar{k})$.

Indeed, in this case we can use the following proposition :

Proposition 6

Let G a finite algebraic group such that all point of \mathcal{G}_k acts trivially on $G(\bar{k})$. Then,

$$\text{III}(G) = 1.$$

Proof

Because \mathcal{G}_k acts trivially on $G(\bar{k})$, we have

$$H^1(k, G) = H^1(\mathcal{G}_k, M) = \text{Hom}_{\text{grp}}(\mathcal{G}_k, M)$$

Moreover, we have the following commutative diagram :

$$\begin{array}{ccc} H^1(k, G) & \longrightarrow & H^1(\mathcal{G}_k, M) \\ \downarrow & & \downarrow \\ \prod_{v \in S} H^1(k_v, G_v) & \longrightarrow & \prod_{v \in S} H^1(\mathcal{G}_{k_v}, M) \end{array}$$

Therefore we obtain the following description of $\text{III}(G)$ in term of morphisms :

$$\text{III}(G) \simeq \{ \varphi : \mathcal{G}_k \rightarrow M \text{ continuous morphism} \mid \varphi_v = 1 \text{ for each place } v \in S \}.$$

Where φ_v denotes the composition $\mathcal{G}_{k_v} \rightarrow \mathcal{G}_k \rightarrow M$.²

Let φ such a morphism. The kernel of φ corresponds to a finite Galois extension l'/k because M is finite and discrete.

We use the following result, which is a consequence of Chebotarev's density theorem :

Theorem 13

Let l/k be a galois extension of global fields. If

$$l_w \simeq k_v$$

for any place w lying above v , then $l = k$.

Let's finish the proof.

1. See [8] for more detail on the Galois cohomology point of view in terms of cocycles and coboundaries

2. The morphism $\mathcal{G}_{k_v} \rightarrow \mathcal{G}_k$ is induced by the commutative square :

$$\begin{array}{ccc} \bar{k} & \longrightarrow & \bar{k}_v \\ \uparrow & & \uparrow \\ k & \longrightarrow & k_v \end{array}$$

Fix $v \in S$ and $\sigma \in \mathcal{G}_{k_v}$. Then, by hypothesis,

$$\varphi(\sigma|_{\bar{k}}) = \varphi_v(\sigma) = 1$$

and we obtain that σ fixes $l \subset \bar{k}_v$ and k_v . As a consequence, it also fixes the compositum $k_v \cdot l$ of k_v and l in \bar{k}_v . Because $\sigma \in \mathcal{G}_{k_v}$ is arbitrary, we conclude that $k_v \cdot l = k_v$ for any place $v \in S$.

But, we know that the different embedding of l in the completions l_w (w extending v) induce an isomorphism :

$$l \otimes_k k_v \simeq \prod_{w|v} l_w$$

Because $k_v \cdot l$ is of the form :

$$k_v \otimes_k l_{\mathfrak{p}}$$

for some prime ideal $\mathfrak{p} \in \text{Spec}(k_v \otimes_k l)$, this implies that the composition $k_v \cdot l = k_v$ is one of the l_w . But, because l/k is Galois, all of these completions are isomorphic, and in the end we obtain :

$$\forall w | v \quad k_v = l_w.$$

We can now apply the theorem 13 to get $l = k$ and thus $\text{Ker}(\varphi) = \mathcal{G}_k$. ■

To reduce to this case we do as follows :

Lemma 2 – Cutting the Galois group

There is a smallest finite galois extension l/k with $l \subset \bar{k}$ such that \mathcal{G}_k acts trivially on the $G(\bar{k})$.

Proof

When M is given the discrete topology, the natural action of the absolute galois group \mathcal{G}_k of k on M is continuous, and induces a continuous map :

$$\mathcal{G}_k \longrightarrow \text{Aut}_{\text{grp}}(M) \quad (\star)$$

when the righthand-side is given the topology induced by the product topology.

But as M is finite and discrete (because G is a finite group scheme), the topology on $\text{Aut}(M)$ is also the discrete topology! Thus, the map (\star) has a closed-open kernel, corresponding via Galois fundamental theorem to a suitable finite galois extension l/k that satisfies the problem. ■

Thus, we found an galois extension l/k such that G_l satisfies the conditions of the proposition 6.

Now we need to go from III(G_l) to III(G).

Construction 7 – Link between III(G) and III(G_l)

Let T denote the set of places of l . Then, for any place $v \in S$ of k , there is a natural morphism of pointed spaces :

$$H^1(k_v, G_v) \rightarrow \prod_{w|v} H^1(l_w, G_w)(\star)$$

induced by the commutative diagram, for any place w of l lying above v :

$$\begin{array}{ccc} G_w & \longrightarrow & G_v \\ \downarrow & & \downarrow \\ \text{Spec}(l_w) & \longrightarrow & \text{Spec}(k_v) \end{array}$$

From the maps (\star) we get a commutative diagram of pointed spaces :

$$\begin{array}{ccc}
 \text{III}(G) & \overset{(*)}{\dashrightarrow} & \text{III}(G_l) \\
 \downarrow & & \downarrow \\
 H^1(k, G) & \longrightarrow & H^1(l, G_l) \\
 \downarrow & & \downarrow \\
 \prod_{v \in S} H^1(k_v, G_v) & \longrightarrow & \prod_{w \in T} H^1(l_w, G_w)
 \end{array}$$

and a unique map $(*)$ from $\text{III}(G)$ to $\text{III}(G_l)$ that completes the diagram.

Using twisting, we prove the following lemma :

Lemma 3 – Fibers and base change

Let G a finite algebraic k -group. Then, the natural map :

$$[\mu] \in H^1(k, G) \rightarrow [\mu_l] \in H^1(l, G_l)$$

has finite fibers.

Proof

Let $[\mu]$ be the cohomology class of G -torsor. We use μ to twist G and μ_l to twist G_l . As twisting commutes with base change, we obtain a commutative diagram of pointed sets :

$$\begin{array}{ccc}
 H^1(k, G) & \longrightarrow & H^1(l, G_l) \\
 \downarrow \wr & & \downarrow \wr \\
 H^1(k, {}_\mu G) & \longrightarrow & H^1(l, ({}_\mu G)_l)
 \end{array}$$

Because ${}_\mu G$ is also a finite group over k by descent, we are reduced to the case where $\mu = 1$ is the trivial G -torsor. We need to show that :

$$\text{Ker} (H^1(k, G) \longrightarrow H^1(l, G_l)) \quad (*)$$

is finite. Recall that $H^1(k, G)$ can be naturally identified with the galois cohomology set $H^1(\mathcal{G}_k, M)$. In this context, we have the following exact sequence of pointed sets :

$$1 \longrightarrow H^1(\text{Gal}(l/k), M^{\mathcal{G}_l}) \longrightarrow H^1(\mathcal{G}_k, M) \longrightarrow H^1(\mathcal{G}_l, M).$$

But, as \mathcal{G}_l acts trivially on M , the kernel $(*)$ can be identified with the set

$$H^1(\text{Gal}(l/k), M)$$

which is finite because M and $\text{Gal}(l/k)$ are. ■

Now let us finish the proof of theorem 4.

To do so, we observe the following commutative diagram in the eyes :

$$\begin{array}{ccc}
 \text{III}(G) & \longrightarrow & \text{III}(G_l) \\
 \downarrow & & \downarrow \\
 H^1(k, G) & \longrightarrow & H^1(l, G_l)
 \end{array}$$

The bottom arrow has finite fibers, and $\text{III}(G_l)$ is finite, thus $\text{III}(G)$ is also finite, which concludes the proof.

5 The reductive case

In this section, we want to treat the final case of the proof of the finiteness of III(G) for G a reductive linear algebraic group.

In the first subsection, we explain a geometric interpretation of III(G) when given a realisation

$$\rho : G \hookrightarrow GL_{n,k}$$

as a linear group.

5.1 Geometric interpretation of III(G)

Fix G a linear algebraic group and :

$$\rho : G \hookrightarrow GL_{n,k}$$

a morphism of group schemes over k. Let X be the quotient space of $GL_{n,k}$ by G. Recall Hilbert' 90 :

Theorem 14 – Hilbert 90'

Let L be a field, then, for any $n \geq 0$,

$$H^1(L, GL_{n,L}) = 1$$

Remark

In this chapter, we drop the subscript 'ét' or 'fppf' because our groups are affine and smooths, the two H^1 coincides (because in this case G-torsors for the fppf topology are also smooth affine schemes and thus admits local sections for the étale topology).

Therefore, for any field L, from the exact sequence of fppf sheaves

$$1 \longrightarrow G \xrightarrow{\rho} GL_{n,k} \xrightarrow{\pi} X \longrightarrow 1$$

we obtain the following exact sequence of pointed sets :

$$1 \longrightarrow G(L) \longrightarrow GL_n(L) \longrightarrow X(L) \xrightarrow{\delta_L} H^1(L, G_L) \longrightarrow 1$$

Using the surjective map δ_L , we can obtain a geometric interpretation of $H^1(L, G_L)$ as a quotient of $X(L)$. More precisely, as a fppf sheaf, X is the sheafification of the naive quotient presheaf :

$$X^{\text{pre}} : T \in \text{Sch}_{\text{fppf},k} \longmapsto GL_{n,k}(T) /_G(T) \in \text{Sets}$$

of the right action of G on $GL_{n,k}$ (using ρ). On this presheaf, there is a natural action on the left of sheaf of groups G, given by :

$$(a, bG(T)) \in GL_{n,k}(T) \times X^{\text{pre}}(T) \longmapsto (a \cdot b)G(T) \in X^{\text{pre}}$$

Which induces a $GL_{n,k}$ -equivariant left-action of GL_n on X after sheafification.

Now we can prove (following the same proof of exactness of the long exact sequence in non-abelian cohomology) the following fact :

Proposition 7 – Fibers of δ_L are orbits

The fiber of δ_L at the class of the G-torsor $\delta_L(x) = [\pi^{-1}(x)]$ is precisely the orbit of x under the left action of $GL_n(L)$.

We use this proposition to give a description of III(G). Indeed, using the different functorialities we have the following commutative diagram :

$$\begin{array}{ccccc} & & & & \text{III}(G) \\ & & & & \downarrow \\ & & & & X(k) \xrightarrow{\delta_k} H^1(k, G) \\ & & & & \downarrow \\ \prod_{v \in S} GL_n(k_v) & \xrightarrow{\pi} & \prod_{v \in S} X(k_v) & \twoheadrightarrow & \prod_{v \in S} H^1(k_v, G_v) \end{array}$$

Which gives a bijection :

$$\text{III}(G) \simeq \text{GL}(k) \backslash \left(\prod_{v \in S} \text{GL}_n(k_v) \right)$$

We can give a more elegant description using the ring of adeles and the adelic points of a scheme.

5.2 Adelic description of the Shafarevich-Tate set

We discuss here the notion of adelic point of affine schemes over a global field k and how to topologise this set. We use the article by Brian Conrad [4] as a reference.

Firstly, we use the following proposition to give a topology to $X(R)$ for X an affine R -scheme of finite type, and R a topological ring :

Proposition 8

Let R an Hausdorff topological ring. Then, there is a unique way to topologize $X(R)$ for any affine R -scheme of finite type in order to satisfy the following conditions :

1. For any R -scheme morphism $f : X \rightarrow Y$, the map induced on the R -points is continuous.
2. The affine line is sent to the topological ring R .
3. The construction commutes with fiber products.
4. Closed immersions of R -schemes are sent to closed immersions of topological spaces.

Moreover, we have the following properties :

- ▮ If R is locally compact, then so is $X(R)$.
- ▮ If R^\times is open in R and the inverse is continuous (for example when R is a local field) then, open immersions are also carried to open immersions.

Thus, for any place v of k , and any affine k -scheme X of finite type, there is a natural topology on :

$$X(k_v) = (X \otimes_k k_v)(k_v).$$

Now let's turn to the adèle points. The adèle ring glues many informations given at the different places of k into one object :

Definition 9 – Adèle ring

The ring of adèle of a global field k is the subring of the product of all k_v , $v \in \mathcal{P}_k$ defined by :

$$\mathbb{A}_k := \{x = (x_v) \mid x_v \in \mathcal{O}_{k_v} \text{ for almost all places } v.\}.$$

It is a locally compact topological ring for the product topology.

Looking at the definition, we have a natural embedding :

$$x \in k \longmapsto (x_v)_{v \in S} \in \mathbb{A}_k$$

It is well defined because $v(x) = 0$ for almost all places v of k .

Therefore, for any affine k -scheme X of finite type, there is a natural topology on $X(\mathbb{A}_k)$, the set of adelic points of X .

We can give a more explicit description of $X(\mathbb{A}_k)$ as a product, using the notion of integer form :

Definition 10 – Integer form of a scheme

Let X be a scheme over a global field k , and S be a set of places of k . An S -integer form of an finite type affine X is a $\mathcal{O}_{k,S}$ -scheme affine scheme \tilde{X} of finite type together with an isomorphism :

$$\varphi : \tilde{X} \otimes_{\mathcal{O}_{k,S}} k \longrightarrow X$$

In other words, \tilde{X} is a $\mathcal{O}_{k,S}$ -scheme with generic fiber X .

Any finite type affine scheme X over k admits a S -integer form with S finite. Indeed :

Construction 8 – Integer form for free

Choose a presentation :

$$X = V(I) \hookrightarrow \mathbb{A}_k^n$$

into an affine space, with $I = (f_1, \dots, f_n)$ an ideal of $k[x_1, \dots, x_n]$. Then, all the f_i lies in

$$O_{k,S}[x_1, \dots, x_n]$$

for some large but finite set $S \subset \mathcal{P}_k$. Then, we see that the closed subscheme of $\mathbb{A}_{O_{k,S}}^n$ defined by the equations f_1, \dots, f_n is a S -form of X .

Then, we have the following proposition :

Proposition 9

We have homeomorphism :

$$X(\mathbb{A}_k) = \tilde{X}(\mathbb{A}_k) = \{x = (x_v)_{v \in S} \mid x_v \in \tilde{X}(O_{k,v}) \text{ for almost all } v \notin S\}$$

where the last set is induced by the product topology.

For a proof, see the theorem 3.6 in [4] in Brian Conrad's article.

Let us use this description to give a new description of $\text{III}(G)$. This part uses the technique called spreading out, which is presented in detail in the article [4], see theorem 3.4. First, from the proposition [], we can find S sufficiently large and S -forms \tilde{G}, \tilde{X} of G, X fitting in an exact sequence

$$1 \xrightarrow{\tilde{i}} \tilde{G} \longrightarrow \text{GL}_{n, O_{k,S}} \xrightarrow{\tilde{\pi}} \tilde{X} \longrightarrow 1$$

and inducing the exact sequence

$$1 \xrightarrow{i} G \longrightarrow \text{GL}_{n,k} \longrightarrow X \xrightarrow{\pi} 1$$

when we look at the generic fiber.

Then, we apply [] and extend S to have, for all $v \notin S$:

$$\tilde{\pi}(\text{GL}_n(O_{k,S})) = \tilde{X}(O_{k,S})$$

Therefore, using the description 5.2 of $X(\mathbb{A}_k)$ and $\text{GL}_n(\mathbb{A}_k)$, we obtain :

$$X(k) \cap \prod_{v \in \mathcal{P}_k} \pi(\text{GL}_n(k_v)) = X(k) \cap \pi(\text{GL}_n(\mathbb{A}_k)) \quad (\star)$$

From this fact and this identity (\star) following description of $\text{III}(G)$:

Theorem 15 – III as a set of orbits

Let $G \leq \text{GL}_{n,k}$ a connected linear algebraic group, and X the quotient space $\text{GL}_{n,k}/G$. Then,

$$\text{III}(G) \simeq \text{GL}_n(k) \backslash X(k) \cap \pi(\text{GL}_n(\mathbb{A}_k))$$

Now the goal is to find a nice representation of G and X , more precisely :

1. A nice choice of faithful representation $i : G \rightarrow \text{GL}_{n,k}$ to make G well-situated in $\text{GL}_{n,k}$.
2. A suitable representation of the scheme X as a quasi-projective algebraic scheme (in fact affine).

This will be done using a variation of Chevalley's theorem, that states that any subgroup H of an affine algebraic group G arises as the stabiliser of a line in some finite dimensional representation of G .

5.3 Representing the quotient X

To simplify the exposition of the proofs, we assume from now on that $k = \mathbb{Q}$, to reduce to this case, we need to use the so-called Weil-restriction of scalars for schemes, and it replaces $GL_{n,\mathbb{Q}}$ in the proof by a \mathbb{Q} -group $R_{k/\mathbb{Q}}(GL_n)$ which is also reductive, and thus for the general case, the full power of reduction theory for all reductive groups, not only for $GL_{n,\mathbb{Q}}$. This is done in the book [12] in the section 5.2.

By nice choice of faithful representation of G , we mean that we can choose an arbitrary faithful \mathbb{Q} -representation of G , and then use $a \in GL_n(\mathbb{R})$ to conjugate $G_{\mathbb{R}}$ and get a 'well-placed' subgroup $a^{-1}G(\mathbb{C})a$ of $GL_n(\mathbb{C})$, more precisely, one such that $G(\mathbb{C})$ is auto-adjoint, in the following sense :

Definition 11 – Auto-adjoint group

An abstract subgroup $H \leq GL_n(\mathbb{C})$ is said to be auto-adjoint if and only if

$$\forall A \in H \quad {}^t A \in H$$

We will use several results from Platonov and Rapinchuk book [12] without diving into the technical proofs :

Theorem 16 – Mostow

Given a real reductive algebraic group $G \leq GL_{n,\mathbb{R}}$, then, there is an element $a \in GL_n(\mathbb{R})$ such that :

$$a^{-1}G(\mathbb{C})a \leq GL_n(\mathbb{C})$$

is autoadjoint.

Then, because $GL_{n,\mathbb{Q}}$ and G are both reductive, the quotient X can be realised as an affine group scheme, more precisely, we have the following theorem (see theorem 2.15 in [12] for the proof of Platonov and Rapinchuk) :

Theorem 17 – Chevalley’s theorem, reductive version

Let G be a reductive subgroup of a linear algebraic group H over k , a field of characteristic 0. Then, there is a representation :

$$\rho : H \longrightarrow GL_{m,k}$$

such that for the induced left action of H on \mathbb{A}_k^m by matrix multiplication, there is a point $v \in k^m$ satisfying :

1. The stabiliser of v for the action ρ is G
2. The orbit of v for the action ρ is closed in \mathbb{A}_k^m .

Thus, from now on, using the previous two theorems, we fix :

1. A faithful representation of the group G :

$$i : G \longrightarrow GL_{n,\mathbb{Q}}$$

2. A choice of $a \in GL_n(\mathbb{R})$ such that $a^{-1}G(\mathbb{C})a$ is auto-adjoint.

3. A representation

$$\rho : GL_{n,\mathbb{Q}} \longrightarrow GL_{m,\mathbb{Q}}$$

of $GL_{n,\mathbb{Q}}$ as in 5.3 and a suitable point $v \in \mathbb{Q}^m$ with stabiliser G and a closed orbit \mathcal{O}_v under the right action³ of $GL_{n,\mathbb{Q}}$ on $\mathbb{A}_{\mathbb{Q}}^m$

Thus, $X := \mathcal{O}_v \hookrightarrow \mathbb{A}_{\mathbb{Q}}^m$ is an (affine) quotient of $GL_{n,\mathbb{Q}}$ by G , we have a $GL_{n,\mathbb{Q}}$ -equivariant⁴ closed immersion :

$$X \hookrightarrow \mathbb{A}_{\mathbb{Q}}^m$$

and a commutative diagram :

$$\begin{array}{ccc} GL_{n,\mathbb{Q}} & \xrightarrow{\pi} & X \\ & \searrow \lambda \mapsto \rho(A) \cdot v & \downarrow \\ & & \mathbb{A}_{\mathbb{Q}}^m \end{array}$$

3. By multiplication, seeing point of $\mathbb{A}_{\mathbb{Q}}^m$ as line vectors.

4. $GL_{n,\mathbb{Q}}$ acts on the left on X the usual way, and acts on the right on $\mathbb{A}_{\mathbb{Q}}^m$ via ρ

from which we obtain :

$$\begin{array}{ccc} X(\mathbb{Q}) & \longleftrightarrow & \mathbb{Q}^m \\ \uparrow & & \uparrow \\ X(\mathbb{Q}) \cap \pi(\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^n)) & \longleftrightarrow & \mathbb{Q}^m \cap \nu \cdot \rho(\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^n)) \end{array}$$

. Thus, readily from the diagram, we have :

Corollary 5

To show that III(G) is finite, it is sufficient to show that

$$\mathbb{Q}^m \cap \nu \cdot \rho(\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^n))$$

has only a finite number of orbits under the action of $\mathrm{GL}_n(\mathbb{Q})$.

This suggests a more thorough study of the embedding :

$$\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}).$$

This is presented in the next section, using ‘reduction theory’.

5.4 Reduction theory for GL_n

5.4.a Real reduction theory

The goal of ‘real’-reduction theory for $\mathrm{GL}_n(\mathbb{R})$ is to construct so-called fundamental sets for arithmetic groups.

Definition 12 – Arithmetic subgroup

A subgroup Γ of $\mathrm{GL}_n(\mathbb{R})$ is called an arithmetic subgroup if both :

$$[\Gamma \cap \mathrm{GL}_n(\mathbb{Z}) : \Gamma] \text{ and } [\Gamma \cap \mathrm{GL}_n(\mathbb{Z}) : \mathrm{GL}_n(\mathbb{Z})]$$

are finite.

Remark

Two such groups are called commensurable. A fundamental set for an arithmetic group is a subset $\Sigma \subset \mathrm{GL}_n(\mathbb{R})$ sufficiently large to meet every orbit of Γ on $\mathrm{GL}_n(\mathbb{R})$, but not too large in the following sense :

Definition 13 – Fundamental set

A subset $\Sigma \subset \mathrm{GL}_n(\mathbb{R})$ is called a fundamental set for an arithmetic subgroup $\Gamma \leq \mathrm{GL}_n(\mathbb{R})$ if and only if :

1. There exists some maximal compact subgroup K of $\mathrm{GL}_n(\mathbb{R})$ such that $K\Sigma = \Sigma$
2. $\Sigma \cdot \Gamma = \mathrm{GL}_n(\mathbb{R})$
3. For all $x, y \in \mathrm{GL}_n(\mathbb{Q})$, the intersection

$$\Sigma \cdot \Sigma^{-1} \cap x \cdot \Gamma \cdot y$$

is finite.

Remark

The notion of fundamental set is weaker than the notion of fundamental domain, which usually requires Σ to be open, $\bar{\Sigma}$ to meet every orbit, and the stronger condition :

$$\Sigma \cap \gamma \Sigma = \emptyset$$

when $\gamma \neq 1$.

Historically, fundamental sets for $\mathrm{GL}_n(\mathbb{Z})$ for $\mathrm{GL}_n(\mathbb{R})$ have been constructed using what is now called Siegel sets :

Definition 14 – Siegel sets

Let $u, t > 0$. The Siegel set $\Sigma_{t,u}$ is defined as :

$$\Sigma_{t,u} := O_n(\mathbb{R}) \cdot D_t \cdot U_u \leq GL_n(\mathbb{R})$$

Where :

$$D_t := \{a \in D_n^a \mid a_{i,i} \leq ta_{i+1,i+1} \text{ for all } i \in \{1, \dots, n-1\}\}$$

and

$$U_u := \{m \in U_n^b \mid |m_{i,j}| \leq u \text{ for all } i < j\}$$

- a. the group of diagonal matrices.
b. The group of unipotent matrices

Indeed, we have the following theorem, referred as the theorem 4.7 (by Borel, Harish-Chandra) in the reference [12] :

Theorem 18

The Siegel set $\Sigma_{t,u}$ is fundamental domain for $GL_n(\mathbb{Z})$ for all $t \geq \frac{2}{\sqrt{3}}$ and $u \geq \frac{1}{2}$.

5.4.b Review of adelic reduction theory

We now explain the adelic counterpart of the construction of the previous section.

The analog of the previous pair $GL_n(\mathbb{Z}) \leq GL_n(\mathbb{R})$ is the natural diagonal emmbeding :

$$GL_n(\mathbb{Q}) \rightarrow GL_n(\mathbb{A}_{\mathbb{Q}})$$

We can define the analog of a fundamental set in this setting :

Definition 15

Let G an algebraic group over a global field k . A fundamental set for $G(\mathbb{A}_k)$ with respect to $G(k)$ is a set $\Sigma \subset G(\mathbb{A}_k)$ such that :

1. $\Sigma G(k) = G(\mathbb{A}_k)$
2. $\Sigma^{-1} \cdot \Sigma \cap G(k)$ is finite.

Then, we can also use the Siegel sets to construct a fondamental set for $GL_n(\mathbb{A}_{\mathbb{Q}})$ with respect to $GL_n(\mathbb{Q})$:

Theorem 19

Let Ω a fundamental set for $GL_n(\mathbb{R})$ with respect to $GL_n(\mathbb{Z})$. Then,

$$\Sigma := \Omega \times \prod_{v \in \mathcal{P}_{\mathbb{Q}} \setminus \{\infty\}} GL_n(\mathbb{Z}_p)$$

is a fundamental set for $GL_n(\mathbb{A}_{\mathbb{Q}})$ with respect to $GL_n(\mathbb{Q})$.

In particular, for $t \geq \frac{2}{\sqrt{3}}$ and $u \geq \frac{1}{2}$,

$$\Sigma := \Sigma_{t,u} \times \prod_{v \in \mathcal{P}_{\mathbb{Q}} \setminus \{\infty\}} GL_n(\mathbb{Z}_p)$$

is a fundamental domain for $GL_n(\mathbb{A}_{\mathbb{Q}})$ with respect to $GL_n(\mathbb{Q})$.

5.4.c Application of reduction theory to the finiteness of III(G)

We want to prove that there is only a finite number of orbits for the right action of $GL_n(\mathbb{Q})$ on the set :

$$\mathbb{Q}^m \cap v \cdot \rho(GL_n(\mathbb{A}_{\mathbb{Q}}) \subset \mathbb{Q}^m(\star)$$

Choose a fundamental set for $GL_n(\mathbb{A}_{\mathbb{Q}})$ with respect to $GL_n(\mathbb{Q})$ of the form :

$$\Sigma := \Sigma_{t,u} \times \prod_{p \in \mathcal{P}_{\mathbb{Q}} \setminus \{\infty\}} GL_n(\mathbb{Z}_p)$$

and see a as an element of $GL_n(\mathbb{A}_Q)$ via the injection :

$$A \in GL_n(\mathbb{R}) \mapsto A_\infty := (A, 1, 1, \dots) \in GL_n(\mathbb{A}_Q)$$

Then, from the identity :

$$a_\infty \cdot \Sigma \cdot GL_n(\mathbb{Q}) = GL_n(\mathbb{A}_Q)$$

we obtain :

$$v \cdot \rho(GL_n(\mathbb{A}_Q) \cap \mathbb{Q}^m = v \cdot \rho(a_\infty \Sigma) \cdot \rho(GL_m(\mathbb{Q})).$$

As a consequence, any orbit of (\star) for the right action of $GL_n(\mathbb{Q})$ has a representant in the set

$$\mathbb{Q}^m \cap v \cdot \rho(a_\infty \cdot \Sigma) \subset \mathbb{Q}^m(\star)$$

and we want to show that this is a finite set. Let us explain how to replace \mathbb{Q}^m by a discrete version $l^{-1}\mathbb{Z}$ for some suitable $l \in \mathbb{Z} \setminus \{0\}$:

Definition 16 – p-adic norm

For $x = (x_1, \dots, x_m) \in \mathbb{Q}^m$, we set :

$$|x|_p = \max_{1 \leq i \leq m} |x_i|_p$$

the p-adic valuation of x .

Then, because

$$\rho : A \in GL_{n,\mathbb{Q}} \mapsto \rho(A) \in GL_{m,\mathbb{Q}}$$

is a morphism defined by polynomials in the components of A and $\det(A)^{-1}$, and with rational coefficients, we see that we can make a choice of constants $\alpha_p > 0$ such that the following holds :

1. For all prime p and matrix $A \in GL_n(\mathbb{Z}_p)$:

$$|\rho(A)|_p \leq \alpha_p$$

2. Almost all α_p equals 1.

Then, for a prime p , and $A \in GL_n(\mathbb{Z}_p)$, we get :

$$|v \cdot \rho(A)|_p \leq |v| \alpha_p$$

Then, it suffices to fix $l \in \mathbb{Z}$ such that for all prime p ,

$$|l|_p < |v|_p^{-1} \alpha_p^{-1}$$

to obtain, for any $x = (x_1, \dots, x_m) \in \mathbb{Q}^m$ in the intersection (\star) and prime p :

$$x \in v \cdot \rho(GL_n(\mathbb{Z}_p))$$

which leads to

$$|x|_p \leq |l|_p \alpha_p |v|_p \leq 1.$$

Because p is arbitrary, we conclude that $lx \in \mathbb{Z}^m$.

Therefore, to show that (\star) is finite, it is sufficient to show that

$$\frac{1}{l} \mathbb{Z}^m \cap v \cdot \rho(a) \rho(\Sigma_{t,u})$$

is finite!

Set $v^a = v \cdot \rho(a)$. Then, we know from abstract representation theory that :

1. The abstract orbit of v^a is just :

$$a^{-1}v \cdot GL_n(\mathbb{C})a$$

2. The stabiliser of v in $GL_n(\mathbb{C})$ is the self-adjoint group :

$$a^{-1}G(\mathbb{C})a$$

To conclude the proof we apply the following technical lemma. We admit it here, but to have a proof of this lemma, see [2] (*Un lemme de finitude*, or Lemma 6.2) by Armand Borel, or [12] (Proposition 4.5 of chapter 4) :

Lemma 4

Let $\rho : GL_n(\mathbb{C}) \rightarrow GL_m(\mathbb{Q}^m)$ a representation and $v \in \mathbb{R}^m$ satisfying :

1. The stabiliser of v is self-adjoint.
2. The abstract orbit $v \cdot \rho(GL_n(\mathbb{C})) \subset \mathbb{C}^m$ is Zariski closed.

Then, for any lattice $L \subset \mathbb{Q}^m$, and $t \geq \frac{2}{\sqrt{3}}$, $u \geq \frac{1}{2}$ the intersection :

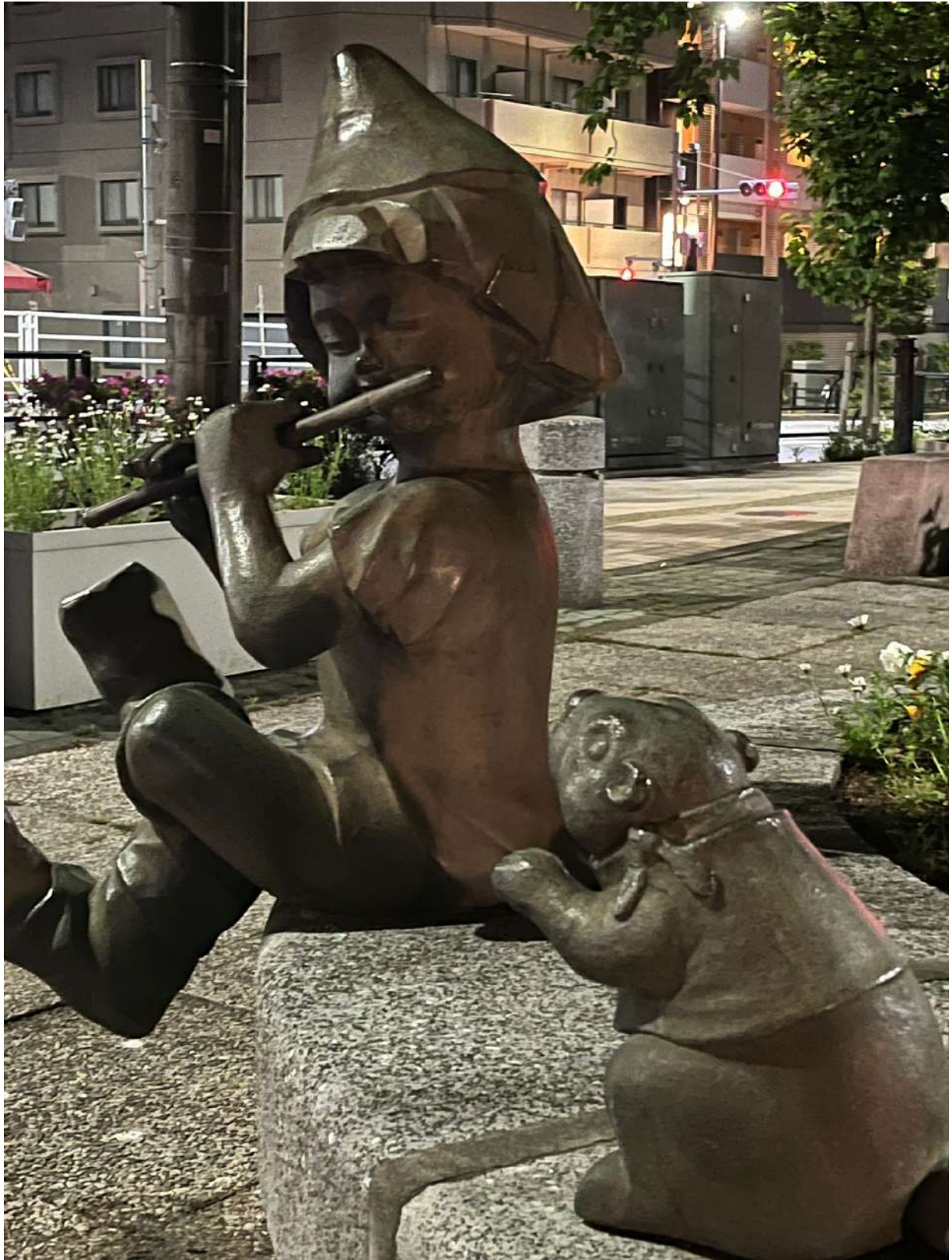
$$v\rho(\Sigma_{t,u}) \cap L$$

is finite.

To apply the lemma we just need to prove that $v \cdot GL_n(\mathbb{C})$ is Zariski closed in \mathbb{C}^m . But, we have a factorisation :

$$GL_n(\mathbb{C}) \xrightarrow{\pi_{\mathbb{C}}} X(\mathbb{C}) \xrightarrow[\begin{smallmatrix} \text{Zariski closed} \\ g \mapsto v \cdot \rho(g) \end{smallmatrix}]{\quad} \mathbb{C}^m$$

with $\pi_{\mathbb{C}}$ surjective as \mathbb{C} is algebraically closed so $H^1(\mathbb{C}, G) = 1$, and $X(\mathbb{C}) \subset \mathbb{C}^m$ is Zariski closed because $X \hookrightarrow \mathbb{A}_{\mathbb{Q}}^m$ is a closed immersion, which concludes the proof.



Annex of sheaf theory on sites

1 Sites

1.1 Grothendieck topologies and morphisms of topologies

In this section, we define sites and Grothendieck's topologies, a generalization of usual topologies, for locally small categories having some fiber products.

Let \mathcal{C} a category. We want to see objects of \mathcal{C} as open subsets for some topology, and maps as inclusions between these subsets. According to this point of view, if $U, V \in \mathcal{C}$ and if there is some morphisms $U, V \rightarrow X$, the fiber product $U \times_X V \in \mathcal{C}$ (if it exists) is interpreted as the intersection of U and V in X .

Let $f : U \rightarrow X$ be a morphism in \mathcal{C} , and \mathcal{G} be a set of morphisms with target fixed target X . Let $g \in \mathcal{G}$. If the base change of g by f exist, we denote it by $g|_{U,f}$ (or $g|_U$ if f is implicit). In the same way, if all $g|_U$ exist, we denote by $\mathcal{G}|_{U,f}$ (or $\mathcal{G}|_U$) the set of morphism with source D obtained by base change with f :

$$\mathcal{G}|_U = \{g|_U \mid g \in \mathcal{G}\}.$$

It is a family of morphisms with fixed target U .

Definition 17 – Grothendieck topology

A Grothendieck topology τ on \mathcal{C} is the data of, for each $C \in \mathcal{C}$, of a class $\text{cov}(C)$, whose elements are sets of morphisms

$$\{C_i \rightarrow C\}_{i \in I}$$

called coverings of C for the topology τ , and satisfying the following 3 additional axioms :

1. (TGI) If $f : C \rightarrow D$ is an isomorphism, then $\{f\}$ is a covering of D .
2. (TGII) If Λ is a covering of C , and $f : D \rightarrow C$ is a morphism, then all the base changes of elements of \mathcal{F} by f are well-defined, and $\mathcal{F}|_D$ is a covering of D
3. (TGIII) Let Λ be a covering of C , and for each morphism $f : D \rightarrow C$ in Λ , let Λ^f be a covering of D . Then the set of morphisms with target C of the form

$$E \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{f} \end{array} D \xrightarrow{f} C$$

with $g \in \Lambda^f$ form a covering of C .

A category endowed with a Grothendieck topology is called a site.

Remark

1. The existence of bases changes for coverings in (TGII) is automatically satisfied for categories admitting fiber products.
2. The three axioms are the categorical counterparts of the following facts for a topological space X , with the usual notion of covering :
 - a. (TGI) An open subset covers itself.
 - b. (TGII) If $\{U_i\}_{i \in I}$ covers U , and $V \subset U$, then $\{U_i \cap V\}$ covers V .
 - c. (TGIII) If $\{U_i\}_{i \in I}$ is a covering, and for each $i \in I$, we have a covering $\{U_{i,j}\}_{j \in J_i}$, then the set

$$\{U_{i,j} \mid i \in I, j \in J_i\}$$

form a covering of U .

Example 4 – The site of open subsets of a topological space

The previous remark gives us the first example of a site, the category $\mathcal{O}(X)$ of open subsets of a topological space,

with inclusions of open subsets as morphisms, and with

$$\text{cov}(\mathbf{U}) = \{ \{ \mathbf{U}_i \hookrightarrow \mathbf{U} \}_{i \in I} \mid \bigcup_{i \in I} \mathbf{U}_i = \mathbf{U} \}$$

2 Sheaves of sets on a site

In this section (\mathcal{C}, τ) is a fixed site. A presheaf of sets on \mathcal{C} is a functor

$$\mathcal{F} : \mathcal{C}^{\text{op}} \longrightarrow \text{Sets}.$$

Elements of $\mathcal{F}(C)$ are called sections of $\mathcal{F}(C)$ defined over C , or local sections if C is not mentioned. And if $f : V \rightarrow U$ is a morphism by functoriality we have a restriction map :

$$\mathcal{F}(\mathbf{U}) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(\mathbf{V}).$$

To emphasise the topological analogy, if $s \in \mathcal{F}(\mathbf{U})$ we often denote $s|_V$ instead of $\mathcal{F}(f)(s)$ when f is implicit.

The terminology 'section' used is justified by the following example :

Example 5

In this paragraph \mathcal{C} denotes the site $\mathcal{O}(X)$ associated to a topological space X . Let $Y \xrightarrow{p} X$ be a space over X . We define the presheaf of sections of p on X :

$$\begin{aligned} \mathcal{S}_p : \quad \mathcal{O}(X)^{\text{op}} &\longrightarrow \text{Set} \\ \mathbf{U} &\longmapsto \mathcal{S}_p(\mathbf{U}) = \{ s : \mathbf{U} \rightarrow p^{-1}(\mathbf{U}) \mid p \circ s = \text{id}_{\mathbf{U}} \} \end{aligned}$$

that is to say, elements of $\mathcal{S}_p(\mathbf{U})$ are precisely sections of p defined over the open subset \mathbf{U} of X .

Definition 18

A morphism f between presheaves \mathcal{F} and \mathcal{G} is a natural transformation between the two functors.

These morphisms define a category $\text{Psh}(\mathcal{C}, \text{Sets})$ of presheaves on \mathcal{C} , that does not depend on the topology τ .

Let \mathcal{F} be a presheaf of sets on \mathcal{C} and \mathbf{U} be an object of \mathcal{C} . As elements $\mathcal{F}(\mathbf{U})$ represent local objects defined over \mathbf{U} , one can ask whether these objects can be glued together, because we have a topology τ on \mathcal{C} .

More precisely, fix a covering $\{ \mathbf{U}_i \rightarrow \mathbf{U} \}$ in the object $\mathbf{U} \in \mathcal{C}$ for the topology τ . We say that a family of sections $s_i \in \mathcal{F}(\mathbf{U}_i)$ is matching if for all $i, j \in I$, we have

$$s_i|_{\mathbf{U}_i \times_{\mathbf{U}} \mathbf{U}_j} = s_j|_{\mathbf{U}_i \times_{\mathbf{U}} \mathbf{U}_j}$$

Given such a matching set of sections one can ask two things :

1. Does the s_i can be glued? That is to say, is there a section s defined over \mathbf{U} such that for each $i \in I$, we have

$$s|_{\mathbf{U}_i} = s_i$$

2. If the s_i can be glued, is the gluing s unique?

For example, both questions have positive answer for the presheaf \mathcal{S}_p .

Because every section s defined \mathbf{U} is the gluing of the sections $s_i = s|_{\mathbf{U}_i}$, the second condition is equivalent to the following condition :

1. If s, s' are two sections defined on \mathbf{U} and $s|_{\mathbf{U}_i} = s'|_{\mathbf{U}_i}$ for each $i \in I$, then $s = s'$.

A presheaf satisfying this condition is called a separated presheaf.

Definition 19 – Sheaves on a site

A sheaf of sets on \mathcal{C} is a presheaf \mathcal{F} such that for every covering, every set of compatible sections can be glued in a unique way. A morphism of sheaves is a morphism between the underlying presheaves.

Example 6

1. If F is a sheaf, and $X \in \mathcal{C}$, then the base change F_X to \mathcal{C}/X is also a sheaf.
2. For any sheaves F, G , we can define the internal Hom sheaf $\underline{\text{Hom}}(F, G)$ defined by :

$$\underline{\text{Hom}}(F, G)(X) = \text{Hom}(F_X, G_X)$$

We denote by $\text{Sh}(\mathcal{C}, \text{Sets})$ the category of sheaves of sets on \mathcal{C} .

In the same way, we can define the sheaf of isomorphism $F \rightarrow G$ and the sheaf of automorphisms of F , denoted respectively by $\underline{\text{Isom}}(F, G)$ and $\underline{\text{Aut}}(F)$.

3 Associated sheaf functor

We admit the following theorem, a proof of which we can find in [10, Tag 00W1]

Theorem 20 – Sheafification

If \mathcal{C} is small, then the inclusion functor :

$$\text{Sh}(\mathcal{C}) \xrightarrow{i} \text{Psh}(\mathcal{C})$$

has a left-adjoint

$$\text{Psh}(\mathcal{C}) \xrightarrow{\#} \text{Sh}(\mathcal{C})$$

called the sheafification functor, and $\#$ is a left inverse of i .

For a presheaf \mathcal{F} , the sheaf $\#\mathcal{F}$ is called the sheaf associated to \mathcal{F} , or the sheafification of \mathcal{F} and we know from the construction of $\#\mathcal{F}$ that for $s_0, s_1 \in F(X)$, then,

$$\#_{\mathcal{F}}(s_0) = \#_{\mathcal{F}}(s_1)$$

if and only if there exist some covering $\{f_i : U_i \rightarrow U\}$ of U such that

$$\forall i \in I \quad s_0|_{U_i} = s_1|_{U_i}.$$

It permits to construct quotients and images of morphisms of sheaves :

Definition 20 – Image of a morphism

Let $f : X \rightarrow Y$ be a morphism of sheaves. Then, the image of f is the sheafification of the naive presheaf :

$$T \in \mathcal{C} \longmapsto f_T(X(T)) \subset Y(T).$$

It is naturally a subsheaf of Y .

4 Non-abelian cohomology of group sheaf

4.1 Group sheaves and G-torsors

Definition 21 – Group sheaf

A sheaf of groups on a site (\mathcal{C}, τ) is a sheaf F on \mathcal{C} for the topology τ together with a group structure on $F(X)$ for any $X \in \mathcal{C}$ that is natural in X .

Then we can define actions of a group sheaf on another sheaf :

Definition 22 – Action of a group sheaf

An action of a group sheaf G on a sheaf F is a morphism :

$$G \times F \rightarrow F$$

such the induced map :

$$G(X) \times F(X) \rightarrow F(X)$$

is an action of $G(X)$ on $F(X)$.

From now on, we assume that \mathcal{C} admits a final element S .

We define in all generality the notion of G -torsor for a group sheaf G on \mathcal{C} :

Definition 23 – G -torsor

A G -torsor for a topology τ on \mathcal{C} is a sheaf F on \mathcal{C} together with a right G -action such that :

1. For any $X \in \mathcal{C}$, either $F(X) = \emptyset$, or $G(X)$ acts simply transitively on $F(X)$
2. There exist a τ -covering $\{S_i \rightarrow S\}$ of S such that

$$\forall i \in I \quad F(S_i) \neq \emptyset.$$

The second condition asserts that F admits local sections for the topology τ .

Definition 24 – H^1 of a group sheaf

Let G a group sheaf on \mathcal{C} for the topology τ . The class of all G -torsors up to isomorphism is denoted by :

$$H^1_\tau(S, G).$$

4.2 Quotients of group sheaves and exact sequences of group sheaves

Let H, G be two group sheaves. The quotient G/H of G by H is the sheafification of the naive presheaf :

$$T \in \mathcal{C} \longmapsto G(T)/_{H(T)}$$

If H is normal in G , then G/H is naturally a group sheaf again.

Definition 25 – Exact sequence of group sheaves

A sequence of morphisms of group sheaves :

$$1 \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \rightarrow 1$$

is said to be exact if and only if i is a monomorphism, the image of π is Q itself, and $\text{Im}(i) = \text{Ker}(\pi)$.

Remark

In this case, π induces an isomorphism :

$$\bar{\pi} : G/H \rightarrow Q$$

4.3 The contracted product

Let us recall the definition of the so-called contracted product, that can be use to construct new-torsors and objects :

Construction 9 – Contracted product of two sheaves

Let X and Y be two sheaves on \mathcal{C} for the topology τ , and G, H two sheaves of groups. Assume that G acts on the right on X , on the left on Y , that H acts on the right on Y , and the two actions on H commutes.

Then, the right action of H on Y induces a right action of H on the quotient :

$$X \times_G Y := X \times Y /_G$$

Where G acts on the right on the product (on local sections) :

$$(x, y) \cdot g = (x \cdot g, g^{-1} \cdot y).$$

The H -sheaf obtained $X \times_G Y$ is called the contracted product of X and Y .

4.4 Rigidity of G -torsors

The G -torsors have a rigid structure :

Proposition 10

Let E, E' be two G -torsors. Then,

1. The natural map :

$$\varphi \in \text{Hom}_{G\text{-Sh}}(G, E) \rightarrow \varphi(1_S) \in E(S)$$

is a bijection.

2. Any G -equivariant morphism from E to E' is an isomorphism.
3. The torsor E is trivial if and only if $E(S) \neq \emptyset$.

Proof

1. Any $x \in E(S)$ define a unique morphism :

$$\varphi_x : G \longrightarrow X$$

defined on local section by :

$$\varphi_x(s) = x \cdot g$$

which gives the inverse of the map $\varphi \mapsto \varphi(1_E)$.

2. Because any equivariant morphism between two sets with a simply transitive action is a bijection.
3. Is a consequence of 1. and 2. as :

$$\text{Isom}_{G\text{-Sh}}(G, E) = \text{Hom}_{G\text{-Sh}}(G, E) = E(S)$$

■

4.5 Cohomology of the automorphism sheaf

To understand the H^1 of the automorphism sheaf of some sheaf F , we need the notion of form of a sheaf :

Definition 26 – Form of a sheaf

Let F a sheaf for the topology τ on \mathcal{C} .

A form of F for the topology τ is a sheaf F^{tw} such that there exist a τ -covering $\{S_i \rightarrow S\}_{i \in I}$ of S such that $F|_{S_i}$ and $F^{\text{tw}}|_{S_i}$ are isomorphic for each $i \in I$.

In short, a form of F is a sheaf locally isomorphic to F for the topology τ .

Definition 27

Denote by $\mathcal{F}(S, F)$ the class of all forms of F up to isomorphisms.

Construction 10 – Link between forms of F and Aut-torsors :

1. For any k -form F^{tw} of F , the sheaf :

$$\underline{\text{Isom}}(F, F^{\text{tw}})$$

has a right action of $\underline{\text{Aut}}(F)$ which makes it a $\underline{\text{Aut}}(F)$ -torsor.

2. Reciprocally, given an $\underline{\text{Aut}}(F)$ -torsor, say E , we can construct the form of F using the contracted product of E and F ($\underline{\text{Aut}}(F)$ acts naturally on the left on F).

By functoriality of the formation of contracted product and isomorphism sheaf, the two constructions depend on F^{tw} and E only up to isomorphism.

These construction are even inverse to each other :

Theorem 21

The forms of a sheaf F are classified by $H^1_\tau(S, \underline{\text{Aut}}(F))$. More precisely, the map :

$$[F^{\text{tw}}] \in \mathcal{F}(S, F) \longmapsto [\underline{\text{Isom}}(F, F^{\text{tw}})] \in H^1(S, \underline{\text{Aut}}(F))$$

and

$$[E] \in H^1(S, \underline{\text{Aut}}(F)) \longmapsto [E \times_{\underline{\text{Aut}}(F)} F] \in \mathcal{F}(S, F)$$

are inverse to each other.

Proof

Let E be an $\underline{\text{Aut}}(F)$ -torsor and F^{tw} a form of F . Denote $\underline{\text{Aut}}(F)$ by G . By lemma 4.4, it is sufficient to find :

1. An $\underline{\text{Aut}}(F)$ -equivariant morphism :

$$E \rightarrow \underline{\text{Isom}}(F, F \times_G E)$$

2. An isomorphism of sheaves :

$$\underline{\text{Isom}}(F, F^{\text{tw}}) \times_G F \longrightarrow F^{\text{tw}}$$

Let's do this :

1. For 1., we can define, for each $X \in \mathcal{C}$ and $x \in E(X)$, an isomorphism :

$$F \longrightarrow F \times_G E$$

defined by the composition :

$$\varphi_x : F \xrightarrow{f \mapsto (f, e)} F \times E \xrightarrow{\text{can}} F \times_G E$$

Then, we obtain a G -equivariant morphism as we wanted to.

2. For 2., we have a natural (in F^{tw}) morphism :

$$\underline{\text{Isom}}(F, F^{\text{tw}}) \times_G F \longrightarrow F^{\text{tw}}(\star)$$

induced by the pairing :

$$\underline{\text{Isom}}(F, F^{\text{tw}}) \times F \longrightarrow F^{\text{tw}}$$

defined on local sections by :

$$(\varphi, f) \mapsto \varphi(f)$$

because locally F and F^{tw} are isomorphic, and because the map (\star) is natural in F^{tw} , we obtain that (\star) is an isomorphism. ■

4.6 Twisting

Let G a group sheaf on a site \mathcal{C} for the topology τ , and μ a G -torsor. We may define $G_\mu := \underline{\text{Aut}}_{G\text{-tors}}(E)$. It is a sheaf of groups on \mathcal{C} for the topology τ , and any G -torsor gives a G_μ -torsor using the following construction :

Construction 11 – Change of base point

Let E a G -torsor. Then,

$$E_\mu := \underline{\text{Isom}}_{G\text{-torsor}}(\mu, E)$$

with the natural $G_\mu = \underline{\text{Aut}}(\mu)$ right-action is a G_μ -torsor.

Moreover, we have the following theorem, we skip the proof that is very similar to the one of the theorem 20 :

Theorem 22 – Change of base point

The map :

$$\sigma_\mu : [E] \in H^1(S, G) \longrightarrow [E_\mu] \in H^1(S, G)$$

is a well-defined bijection.

If N is a normal subgroup of G , then, G acts by conjugation on N , and thus we have an induced morphism :

$$G \longrightarrow \underline{\text{Aut}}(N)$$

And, by functoriality of H^1 , we get for each G -torsor μ a τ -form of N associated to the $\underline{\text{Aut}}(N)$ -torsor $\mu \times_G \underline{\text{Aut}}(N)$.

Let i be the inclusion $N \hookrightarrow G$, and $\pi : G \rightarrow Q$ the projection on the quotient sheaf $Q := G/Q$. Then, we have a natural composition :

$$i_* : H^1(S, {}_\mu N) \rightarrow H^1(S, {}_\mu G) \simeq H^1(S, G).$$

And the claim is the following :

Proposition 11

The image of i_* is precisely the fiber of π_* at $\pi_*([\mu])$.

Proof

We can argue as follows : define $[\mu'] = \pi_*([\mu])$. Then, as explained, we can use μ to twist N and G , and μ' to twist Q . This construction is compatible with the maps i and π , so we obtain a sequence :

$$1 \longrightarrow {}_\mu N \longrightarrow {}_\mu G \longrightarrow {}_{\mu'} Q \longrightarrow 1$$

In fact, it is an exact sequence, because locally it is isomorphic to the exact sequence :

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

thus, the fact follows from the long exact sequence in non-abelian cohomology applied to the twisted exact sequence, combined with the 'change of base-point' theorem. ■

5 Abelian sheaves on a site

5.1 The category of abelian sheaves on a site

We introduce the category of abelian sheaves and presheaves on a site, and show that this category is abelian when \mathcal{C} is small.

Definition 28 – Abelian (pre)-sheaf

An abelian presheaf on \mathcal{C} is a functor

$$F : \mathcal{C} \longrightarrow \text{Ab}$$

An abelian sheaf on \mathcal{C} is an abelian presheaf whose underlying presheaf of sets is a sheaf.

We denote the associated categories by $\text{Psh}(\mathcal{C})$ and $\text{Sh}(\mathcal{C})$. Because $\text{Psh}(\mathcal{C})$ has limits and colimits in $\text{Psh}(\mathcal{C})$ and those are calculated pointwise, and because Ab is abelian, the category $\underline{\text{Psh}}(\mathcal{C})$ is abelian.

Using sheafification, we see that :

Proposition 12 – (Co)-Limits in $\text{Sh}(\mathcal{C})$

Let I be a small category and $F : I \rightarrow \text{Sh}(\mathcal{C})$ be a functor and $F^{\text{pre}} = i \circ F$.
Then the limit of F^{pre} is a sheaf which is the limit of F , and the sheaf associated to the colimit of F^{pre} is the colimit of F .

From which we deduce the following fact :

Fact 1

If the site \mathcal{C} is small, then the category $\text{Sh}_{\text{Ab}}(\mathcal{C})$ is abelian.

Moreover, we have :

Theorem 23

If the site \mathcal{C} is small, then the category $\text{Sh}_{\text{Ab}}(\mathcal{C})$ has enough injectives.

Thus, we may use the 'machinerie' from [5] to define the n -right-derived functor of the left-exact functor :

$$\Gamma(\mathcal{U}, \cdot) : \text{Sh}_{\text{Ab}}(\mathcal{C}) \rightarrow \text{Ab}$$

for any $\mathcal{U} \in \mathcal{C}$, and we denote it by $H_t^n(\mathcal{U}, F)$ for a sheaf F on \mathcal{C} .

Annex of scheme theory

In this part, we just recall some definition and results from scheme theory without giving proofs.

1 Affine schemes

In this section, we show how to associate to each ring A (commutative with unit) a topological space Y whose points are prime ideals of A , and a sheaf of rings on Y , denoted by \mathcal{O}_Y , that permits to see A as the regular functions on a space Y . The couple (Y, \mathcal{O}_Y) is called an affine scheme.

1.1 The spectrum of a ring

Fix A a ring.

Definition 29

The spectrum of A is the set of prime ideals of A , denoted by $Y = \text{Spec}(A)$.

For each prime ideal $\mathfrak{y} \in \text{Spec}(A)$, we denote by $\mathfrak{p}_{\mathfrak{y}}$ the associated prime. For each $f \in A$, we denote by $f(\mathfrak{y})$ the class of f in the quotient ring $A_{\mathfrak{p}_{\mathfrak{y}}}$, it is the value of f at \mathfrak{y} .

Thus, for each set F of elements of A , we may define the set $V(F)$ where the elements of F vanishes :

$$V(F) := \{\mathfrak{y} \in Y \mid \forall f \in F \quad f(\mathfrak{y}) = 0\}$$

Thus, we have

$$V(F) = V(\mathfrak{a})$$

Where \mathfrak{a} is the ideal of A generated by F .

Definition 30 – Zariski topology

The set of $V(\mathfrak{a})$, with $\mathfrak{a} \subset A$ an ideal of A , form the closed subsets for a topology on Y , called the Zariski topology.

The fact that it is a topology is a consequence of the following properties :

Proposition 13 – Properties of V

Let $\{\mathfrak{a}_i\}_{i \in I}$ a set of ideals of A . Then,

1.

$$V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$$

2. And if I is finite,

$$V\left(\bigcap_{i \in I} \mathfrak{a}_i\right) = \bigcup_{i \in I} V(\mathfrak{a}_i).$$

3. If $\mathfrak{a} \subset \mathfrak{b}$, then,

$$V(\mathfrak{b}) \subset V(\mathfrak{a})$$

For each ideal subset F of A , we set $D(F) = V(F)^c$ the open subset of Y of points where elements of F do not vanish. Then, we see that :

$$\bigcup_{f \in F} D(f) = D(F)$$

And, thus, the open subsets of the form $D(f)$, $f \in A$, form a basis of the Zariski topology on Y , these open subsets are called principal open subsets.

Remark

1. Principal open subsets are also spectrum of a ring, indeed, we have a bijection :

$$y \in D(f) \rightarrow (\mathfrak{p}_y A_f) \in \text{Spec}(A_f)$$

induced by the canonical map $A \rightarrow A_f$.

2. Let \mathfrak{a} an ideal. Then,

$$D(\mathfrak{a}) = Y$$

If and only if no prime ideal of A contains \mathfrak{a} , that is to say if and only if $\mathfrak{a} = A$.

3. From the previous remark, X is quasi-compact. Indeed, if

$$\bigcup_{i \in I} D(\alpha_i) = Y$$

Then,

$$D\left(\sum_{i \in I} \alpha_i\right) = Y$$

And thus there exist $I_0 \subset I$ finite such that $\sum_{i \in I_0} \alpha_i = A$, and therefore we get

$$\bigcup_{i \in I_0} D(\alpha_i) = D(A) = Y.$$

4. We have $D(f) \subset D(g)$ if and only if there is no prime ideal \mathfrak{p} of A that contains g but not f . That is the case if and only if $fA_g = A$, that is to say if and only if g is invertible in A_f .

The construction of the spectrum is natural in A , in the following sense :

Definition 31

Let A, B be two rings, with respective spectrum Y, X . Then every ring morphism :

$$\varphi : A \rightarrow B$$

induces a continuous map between the spectrum :

$$f : X \rightarrow Y$$

denoted by $\text{Spec}(\varphi)$, and defined by :

$$f(x) = \varphi^{-1}(\mathfrak{p}_x)$$

Moreover this construction is compatible with composition.

Therefore, we have a well-defined contravariant functor :

$$\text{Spec} : \text{Ring}^{\text{op}} \rightarrow \text{Top}$$

Many properties of the ring A reflects in his spectrum, but this functor is not faithful or full. Indeed, for example, the spectrum of every field is a just a point for it's unique prime ideal 0 , and there is no morphisms form \mathbb{F}_p to \mathbb{Q} for a prime p .

Thus, we may add a structure of ringed space on Y , to recover the A from the spectrum.

1.2 The structure sheaf

The construction of the structure sheaf of Y is given by :

Construction 12 – The structure sheaf

Let A be a ring, and $Y = \text{Spec}(A)$. There exist a unique sheaf of rings \mathcal{O}_Y on the spectrum Y of A , such that for every principal open subset $D(f) \subset Y$,

$$\Gamma(D(f), \mathcal{O}_Y) = A_f$$

And, if $D(f) \subset D(g)$ for $f, g \in A$, then, g is invertible in A_f and the restriction maps

$$\Gamma(D(g), \mathcal{O}_Y) \rightarrow \Gamma(D(f), \mathcal{O}_Y)$$

is given by the canonical map :

$$A_g \rightarrow A_f.$$

Let us calculate the stalks of O_Y at some point $y \in Y$. By the remark 1.1, a basis of open neighbourhood of y is given by the

$$D(f), \quad f \notin \mathfrak{p}_y$$

And, thus, we have :

$$O_{Y,y} = \lim_{f \notin \mathfrak{p}_y} O_X(D(f)) = \lim_{f \notin \mathfrak{p}_y} A_f = A_{\mathfrak{p}_y}$$

Thus, we proved the following :

Proposition 14

The ringed space (Y, O_Y) associated to a ring A is a locally ringed space, and the stalk of O_X at x is given by the localisation of A at the prime ideal \mathfrak{p}_x .

The construction of (Y, O_Y) is functorial in the following sense :

Definition 32

Let A, B two rings, $(Y, O_Y), (X, O_X)$ their respective associated locally ringed spaces, and $\varphi : A \rightarrow B$ a morphism of rings and $f = \text{Spec}(\varphi) : Y \rightarrow X$ is the continuous map deduced by φ between the spectrum of A and B . Then, there exist a unique morphism of locally ringed spaces :

$$\text{Spec}(\varphi) = (f, f^\#) : (Y, O_Y) \rightarrow (X, O_X)$$

such that for any point $y \in Y$, the induced map

$$f_y^\# : A_y = O_{X, f(y)} \rightarrow O_{Y,y} = B_y$$

is the canonical map induced by $\varphi : A \rightarrow B$.

Because of the uniqueness property, this construction is functorial in A , and we get a functor :

$$\text{Spec} : A \in \text{Ring}^{\text{op}} \mapsto (Y, O_Y) \in \text{LocRingSp}$$

Theorem 24

The functor Spec is right-adjoint to the global section functor :

$$\Gamma : (X, O_X) \in \text{LocRingSp} \mapsto \Gamma(X, O_X) \in \text{Ring}^{\text{op}}$$

Moreover, the unit of the adjunction is the isomorphism :

$$\text{id}_A : A \rightarrow \Gamma(\text{Spec}(A), O_{\text{Spec}(A)}) = A$$

Corollary 6

The functor Spec is fully faithful.

Proof

Indeed, we have for A, B rings, with respective spectrum Y, X

$$\text{Hom}_{\text{LocRingSp}}(X, Y) = \text{Hom}_{\text{Ring}}(A, \Gamma(X, O_X)) = \text{Hom}_{\text{Ring}}(A, B)$$

Denote by AffSch the category of locally ringed spaces isomorphic to $(\text{Spec}(A), O_{\text{Spec}(A)})$ for some ring A . Such spaces are called affine schemes, and by the previous result we have :

Theorem 25 – Equivalence rings and affine schemes

The category of affine schemes is anti-equivalent to the category of rings via the functor Spec.

2 Schemes as locally ringed spaces

Definition 33 – Scheme

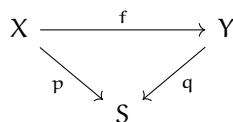
A scheme is a locally ringed space^a (X, \mathcal{O}_X) such that there exist an open covering $(U_i)_{i \in I}$ of X such that

$$(U_i, \mathcal{O}_X|_{U_i})$$

is an affine scheme for every $i \in I$.

a. Topological space with a sheaf of ring and with local stalks.

We denote by Sch the category of schemes. A morphism between schemes is a morphism of locally ringed spaces. If S is a scheme, we denote by Sch_S the category whose object are couples (X, p) where $p : X \rightarrow S$ is a scheme morphism. A morphism in this category between (X, p) and (Y, q) is a scheme morphism $f : X \rightarrow Y$ such that the following diagram commute :



3 The functorial point of view

Let S be a scheme. For any S -schemes X, T , we define :

$$X(T) := \text{Hom}_{Sch_S}(T, X)$$

the set of T -points of X .

Then, by Yoneda’s lemma, the bi-functorial pairing :

$$Sch_S \times_{(T, X)} Sch_S \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \begin{array}{c} \text{Set} \\ X(T) \end{array}$$

gives rise to a fully faithful functor :

$$X \in Sch_S \longmapsto X(\cdot) \in \text{Set}^{Sch_S^{op}}.$$

Let us specify the functoriality in X and T . Let Y, U be two S -schemes, and $f : X \rightarrow Y, g : U \rightarrow T$. Then, f gives rise to a natural transformation in T :

$$f_T(\cdot) : x \in X(T) \rightarrow f \circ x =: f(x) \in Y(T)$$

And, g gives rise to a natural map of restriction of T -points :

$$x \in X(T) \rightarrow x \circ g =: x|_g \in X(U)$$

and these two operations are compatible in the sense that :

$$f_U(x|_g) = f_T(x)|_g$$

And thus, we may see any S -scheme as a functor

$$F : Sch_S \longrightarrow \text{Sets}$$

Any functor isomorphic to $X(\cdot)$ for some S -scheme is said to be representable.

4 Etale and smooth morphisms

Definition 34 – Etale morphism

Let $f : X \rightarrow S$ be a morphism of schemes. We say that X is étale over S (or that f is étale) if f is flat, locally of finite presentation and if for each point $x \in S$, the fiber $f^{-1}(x)$

Definition 35 – Smooth morphisms of relative dimension d

Let $f : X \rightarrow S$ be a morphism of schemes. It is said to be smooth of relative dimension

$$d : X \rightarrow \mathbb{N} \quad (\text{locally constant map})$$

if 'étale locally' it is of the form of a canonical projection :

$$\mathbb{A}_V^m \rightarrow Y$$

More precisely, if for each $x \in X$, there is open neighborhoods $x \in U \subset X$ and $f(x) \in V \subset S$ and a commutative diagram :

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \uparrow & & \uparrow \\ U & \xrightarrow{\text{étale}} \mathbb{A}_V^d \xrightarrow{\text{can}} & V \end{array}$$

5 Sites associated to a scheme

In this section, we fix S a scheme, we define sites associated to X , such as the small and big Zariski, étale, fppf sites. We first define such coverings, and then we will describe all the sites associated with these coverings :

Definition 36 – Different types of coverings for scheme

Let $\{f_i : X_i \rightarrow X\}_{i \in I}$ be a family of S -morphisms. We say that it is a :

1. Zariski or open covering (resp. étale covering, fppf covering) if each f_i is an open immersion (resp. étale, flat of finite presentation), and the open subsets $f_i(X_i)$ covers X .

Remark

The name fppf stands for 'fidèlement plat de présentation finie'.
For each type of covering, we get a different site on the category of S -schemes.

Definition 37 – Big sites associated to a scheme

The different kinds of coverings defined earlier form a topology on the category Sch_S of schemes over S . The resulting site is denoted by $\text{Sch}_{S,\tau}$, if $\tau \in \{\text{Zar}, \text{ét}, \text{fppf}\}$.

We can restrict objects and morphisms of Sch_τ to get a "small" site :

Definition 38 – Small sites associated to a scheme

Then, for $\tau = \text{Zar}$ (resp. ét , fppf) denote by S_τ the subcategory of $\text{Sch}_{S,\tau}$ with objects S -schemes such that the structural morphisms and the morphisms between objects are open immersions (resp. étale, flat of finite presentation). Then, the coverings given by τ form a topology on S_τ . The resulting site is called the small Zariski (resp. étale, fppf) site on S .

Remark

This site is said to be small because, given a scheme S , there exist a set \mathcal{U} of fppf-schemes over S such that any fppf scheme over S is isomorphic to an element of \mathcal{U} . Therefore, the 'small' site S_{fppf} is equivalent to the site whose objects are elements of \mathcal{U} and morphisms flat morphisms of finite presentation, which is a small category. Thus, we can apply sheafification construction and results from the annexe on sheaf theory to the fppf,étale and Zariski site.

Using the functorial point of view, any S -scheme X define a presheaf for the Sch_S . In fact, we have the following theorem, a reference for a proof is proposition 14.76 in [11] by Görtz and Wedhorn :

Theorem 26

For any S -scheme X , the presheaf defined by X on Sch_S is a sheaf for the fppf, étale and Zariski topology.

6 Representability of torsors under a group scheme

As a consequence of descent theory for quasi-coherent sheaves and fppf-coverings, see proposition 14.82 of [11] for the proofs, we have the following result :

Theorem 27

Let G a group scheme affine over a scheme S . Then, any G -torsor X for the fppf topology is representable by an affine S -scheme.

Annex of group scheme theory

1 The category of group schemes

In this section S denote an arbitrary scheme.

Definition 39 – Group schemes

A S presheaf of groups is a functor :

$$G : \text{Sch}_S^{\text{op}} \longrightarrow \text{Grp}.$$

A group scheme is a group functor such that the induced functor :

$$G : \text{Sch}_S^{\text{op}} \longrightarrow \text{Set}$$

is representable by a scheme.

Remarks

☞ The data of a group scheme over S is equivalent the data of a representable functor :

$$G : \text{Sch}_S^{\text{op}} \longrightarrow \text{Set}$$

together with group structure on $G(T)$ for each S -scheme, that is natural in T .

☞ This is turn is equivalent to giving natural transformations :

1. Unit : $e : S \rightarrow G$

2. Multiplication : $m : G \times_S G \rightarrow G$

3. Inverse : $i : G \rightarrow G$

Such that for any T -points X, Y, Z ,

$$m(m(X, Y), Z) = m(X, (Y, Z)) \quad m(X, i(X)) = m(i(X), X) = e(\text{id}_S) \quad m(e(\text{id}_S), X) = m(X, e(\text{id}_S)) = X$$

This in turn is equivalent to have the following commutative diagrams :

$$\begin{array}{ccc}
 G \times_S G \times_S G & \xrightarrow{\text{id}_G \times_S m} & G \times_S G \\
 m \times_S \text{id}_G \downarrow & & \downarrow m \\
 G \times_S G & \xrightarrow{m} & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{(\text{id}_G, i)} & \\
 G & \xrightarrow{(i, \text{id}_G)} & G \times_S G \\
 & \searrow & \downarrow m \\
 & & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & G \\
 & \swarrow (e, \text{id}_G) & \parallel \\
 G \times_S G & \xrightarrow{m} & G
 \end{array}$$

☞ A group scheme is just a group scheme over $\text{Spec}(\mathbb{Z})$.

Definition 40 – Morphism of group schemes

A morphism of between two- S -group schemes H, G is a scheme-morphism $f : H \rightarrow G$ such that for any S -scheme T , the induced map :

$$f_T : H(T) \rightarrow G(T)$$

is a group morphism.

Remark

It is equivalent to give a natural transformation between G, H seen as set valued functors that respects the group structure, that is to say :

$$f_T(m_H(X, Y)) = m_G(f_T(X), f_T(Y))$$

for any T -points X and Y .

Definition 41 – Subgroup

A subgroup of G is a group scheme H with a morphism of group schemes

$$i : H \hookrightarrow G$$

such that i is injective on T -points for all S -scheme T . It is said to be normal in G if moreover $H(T)$ is normal in $G(T)$ of all S -schemes T .

Definition 42 – Kernel of group scheme

Let $f : G \rightarrow Q$ a morphism of group schemes. We can define the kernel $\text{Ker}(f)$ of f by the functor :

$$\text{Ker}(f)(T) := \text{Ker}(f_T) \leq G(T).$$

with the natural group structure induced by $G(T)$ it is representable by a subgroup of G .

Proof

Indeed, the kernel $\text{Ker}(f)$ of f fits in the cartesian square :

$$\begin{array}{ccc} \text{Ker}(f) & \longrightarrow & H \\ \downarrow & & \downarrow f \\ S & \xrightarrow{e} & G \end{array}$$

Thus, the presheaf $\text{Ker}(f)$ is representable. ■

2 Affine group schemes and Hopf algebras

Because the category of affines schemes over a ring k is equivalent to $k - \text{Alg}^{\text{op}}$, the notion of k -group scheme shall give an equivalent (but contravariant notion) of k -algebra with additional structure, called Hopf algebras.

Let $S = \text{Spec}(k)$ an affine scheme, where k is a ring, and G an S -group scheme (or \bar{k} -group scheme) that is representable by an affine scheme, say

$$G = \text{Spec}(A)$$

Then, the operations on G given by (m, e, i) give three k -algebras morphisms by the Yoneda lemma :

1. counit : $\varepsilon : A \rightarrow k$
2. comultiplication : $\Delta : A \rightarrow A \otimes_k A$
3. inverse : $\iota : A \rightarrow A$

And according to the commutative diagrams 1, these k -algebra morphisms must satisfy the following commutative diagrams :

$$\begin{array}{ccc} A \otimes_k A \otimes_k A & \longleftarrow & A \otimes_k A \\ \uparrow & & \uparrow \\ A \otimes_k A & \longleftarrow & A \end{array} \qquad \begin{array}{ccc} & \longleftarrow 1_A \otimes_k \iota & \\ A & \longleftarrow \iota \otimes_k 1_A & A \otimes_k A \\ & \searrow & \uparrow \Delta \\ & & A \end{array} \qquad \begin{array}{ccc} & & A \\ & \nearrow a \otimes b \mapsto e(a)b & \parallel \\ A \otimes_k A & \longleftarrow & A \\ & \Delta & \end{array}$$

Such a data $(A, \varepsilon, \Delta, \iota)$ is called an Hopf- k -algebra. Reciprocally, any Hopf- k -algebra define an affine k -group scheme G^A , with underlying scheme $\text{Spec}(A)$, unit $\text{Spec}(\varepsilon)$, multiplication $\text{Spec}(m)$ and inverse map $\text{Spec}(\iota)$.

Definition 43 – Morphism of Hopf algebras

A morphism between two Hopf- k -algebras $(A, \varepsilon_A, \Delta_A, \iota_A)$ and $(B, \varepsilon_B, \Delta_B, \iota_B)$ is a k -algebras morphism $\varphi : A \rightarrow B$

such that the following diagram commutes :

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \Delta_A \downarrow & & \downarrow \\ A \otimes_k A & \xrightarrow{\varphi \otimes_k \varphi} & B \otimes_k B \end{array}$$

Of course, a k -algebra morphism $\varphi : A \rightarrow B$ is an Hopf- k -algebras morphisms if and only if the induced morphism of k -schemes :

$$\text{Spec}(\varphi) : G^B \rightarrow G^A$$

is a morphism of k -group schemes.

We may consider the category of Hopf- k -algebras, denoted by $k\text{-Hopf}$.

In the end, from the equivalence $k\text{-Alg}^{\text{op}}$ and AffSch_k we get an equivalence of categories :

$$(A, \varepsilon, \Delta, \iota) \in k\text{-Hopf}^{\text{op}} \mapsto G^A \in \text{AffGrp}$$

Remark

If we replace the base $\text{Spec}(k)$ by an arbitrary scheme S , then the construction made translates verbatim if we replace k -algebras with \mathcal{O}_S -quasi-coherent algebras. Thus, we get a notion of quasi-coherent \mathcal{O}_S -Hopf-algebra and an equivalence of categories between the category of \mathcal{O}_S -Hopf-algebras and S -affine group schemes.

3 Group action

In this section, we present some results used on actions of groups on schemes.

Definition 44 – Action of a group scheme

Let S be a scheme, G a S -group scheme and X be a S -scheme. An action of G on X is a morphism

$$G \times_S X \rightarrow X$$

that induces a group action of $G(T)$ on $X(T)$ for any S -scheme T .

4 Quotient of algebraic groups

Let H, G be two group scheme over S such that H is a subgroup of G . Then, H acts naturally on the right on G by multiplication, and we have a naïve quotient presheaf :

$$\left(G/H \right)^{\text{naive}} : T \in \text{Sch}_S \mapsto G(T)/_{H(T)}.$$

In general, this is NOT a representable functor, and not even a sheaf for the fpqc-topology.

But, we can consider the fppf-sheafification of the naïve presheaf quotient, which we denote by G/H . Then, quotients of G by H are defined as follows :

Definition 45

A quotient of G by H is a scheme X with a G -action on the left, a point $o \in X(k)$, together with a G -equivariant morphism :

$$\pi : G \rightarrow X$$

that $\pi(H) = o$ and that the induced morphism of fppf-sheaves :

$$\bar{\pi} : G/H \rightarrow X$$

is an isomorphism.

From this definition, the triple (X, π, o) is unique up to unique isomorphism.

Let us list some of the results we use implicitly in some parts of the report :

Proposition 15

Let $H \leq G$ be a smooth linear algebraic groups. Then, the quotient G/H exists and is a quasi-projective variety. If moreover H is normal in G , then the quotient is representable by affine algebraic group scheme.

For a proof, see [7], theorem 5.28.

5 Linear algebraic groups over a field

We present in this section the principal object of our study, the affine algebraic (= of finite type) group over a field.

5.1 The linear group $GL_{n,k}$

Definition 46 – Linear group

Let $n \geq 1$. We define the group scheme $mmGL_n$ by :

$$GL_{n,S}(T) = \{\text{Matrix with coefficient in } \Gamma(T, \mathcal{O}_T) \text{ with invertible determinant.}\}$$

for any S -scheme with the group structure induced by the matrix multiplication.

For $n = 1$ we have :

$$GL_{1,S}(T) = \Gamma(T, \mathcal{O}_T)^\times$$

for any S -scheme. The group scheme obtained is also called the multiplicative group, and denoted by G_m .

The group scheme GL_n is representable by the affine S -scheme given by the quasi-coherent \mathcal{O}_S -algebra : Where M is the matrix $(m_{i,j})_{1 \leq i,j \leq n}$. Indeed, we have natural bijections :

$$\begin{aligned} \text{Hom}_S(T, \text{Spec}_S(\mathcal{O}_S[m_{i,j}, \det(m_{i,j})^{-1}])) &= \text{Hom}_{\mathcal{O}_S\text{-Alg}}(\mathcal{O}_S[m_{i,j}, \det(m_{i,j})^{-1}], \mathcal{O}_T) \\ &= \{(m_{i,j}) \mid M = (m_{i,j})_{1 \leq i,j \leq n} \text{ with } m_{i,j} \in \Gamma(T, \mathcal{O}_T) \text{ and } \det(M) \in \Gamma(T, \mathcal{O}_T)^\times\} \\ &= GL_n(T) \end{aligned}$$

Now let $S = \text{Spec}(k)$ be the spectrum of a field.

Definition 47 – Algebraic group

An algebraic group scheme G over k is a group scheme of finite type over k .

We often call affine algebraic groups linear algebraic group, because of the following theorem, a proof of which we can find in [7] (corollary 4.10 :

Theorem 28

Any affine algebraic group G can be realised as a closed subgroup of a linear group $GL_{n,k}$ for some $n \geq 0$.

5.2 Smooth group schemes

We quote these unusual two theorems, due to the special properties of group schemes (they acts on themselves) whose proofs can be find in [Milne] :

Proposition 16 – Smoothness of affine group schemes

Let G be an affine algebraic group over a field k . Then, the following assertion are equivalent :

1. G is smooth
2. G is geometrically reduced

Proposition 17 – Automatic smoothness

Any algebraic group scheme G over a field of characteristic 0 is smooth.

5.3 Unipotent and reductive linear algebraic groups

Definition 48

An affine algebraic group G is said to be unipotent if there is some faithful representation :

$$\rho : G \longrightarrow \mathrm{GL}_{n,k}$$

such that for any k -scheme X and X -point $x \in G(X)$, $\rho(g)$ is a unipotent matrix^a.

^a. Upper triangular with 1 on the diagonal.

Now let's define reductive groups. These are the linear algebraic group the most far from the unipotent ones :

Definition 49

A linear algebraic group scheme G is said to be pseudo-reductive if it connected and has no non-trivial normal unipotent subgroup. It is said to be reductive if $G_{\bar{k}}$ is reductive.

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