Internship Report : Multi-index regularity structures of rational regularity and SDE

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1 Context of the internship

I did my internship between February 20 and July 20 in the Max-Planck Institut for Mathematics in the Sciences in Leipzig, Germany. It is a research institute for applied (in a very broad sense) mathematics of about 200 scientists. The main research domains of the institute are geometry, non-linear algebra and analysis. I was part of a research group led by Felix Otto, consisting of around 15 people, the main interests of the group concern analysis but with a non-negligible part of probability. To give a few examples of topics actively studied in this group: homogenization, optimal transport, fluid PDEs, stochastic (partial) differential equations, and more specifically regularity structures.

The members of the institute do not have a teaching duty at the university, however there is quite an active doctoral school, offering several lectures on the research interests of the lecturer, which I had the opportunity to follow. Our group had several seminars, once or twice a week, trying to intertwine analysis and probability. The other group members are rather young (mostly post-docs and PhD students) with only one permanent position and three long-term non-permanent positions and half of the group was international.

My advisor was Felix Otto but I was simultaneously advised by Lucas Broux, a post-doc working on regularity structures. I deeply thank both of them for their very valuable time and help.

2 Motivation

Let us consider a stochastic partial differential equation (SPDE) of the form

$$(\partial_2 u - \partial_1^2)u = \lambda(\partial_1 u)^2 + \xi \tag{1}$$

for $u : x_1, x_2 \in \mathbb{R}^2 \to \mathbb{R}$, $\lambda \in \mathbb{R}$ and ξ some (random) noise. Let us consider this equation from a purely analytical point of view (forgetting the stochastic nature of ξ). We assume the noise is of Hölder regularity α.[1] For instance we can take a white noise of regularity $-\frac{3}{2}$ (we measure the regularity with a parabolic norm). Some basic regularity theory shows that with such a noise we only expect a solution of regularity $\frac{1}{2}$ and hence we can not give a sense to the (distributional) product $(\partial_1 u)^2$. Indeed, we recall that *a priori* we can multiply two distributions only if the sum of their regularity (in a Hölder, Sobolev or Besov sense) is positive. Hence, (1) does not have a well-defined meaning. One possible strategy would be to regularize the noise by convolving it with some smooth Schwartz function $\rho \in \mathcal{S}$

$$\xi_\varepsilon := \xi * \rho_\varepsilon \tag{2}$$

with $\rho_\varepsilon(x_1, x_2) := \frac{1}{\varepsilon^3}\rho(\frac{x_1}{\varepsilon^2}, \frac{x_2}{\varepsilon})$. We hope to get a meaningful solution as $\varepsilon \to 0$. Unfortunately, as the regularization ε vanishes, the solutions u_ε are not expected to stay under control but rather to diverge. We can however transform (1), as we regularize it by adding some «counter-term» C_ε

$$(\partial_2 - \partial_1^2)u_\varepsilon = \lambda(\partial_1 u_\varepsilon)^2 + \xi_\varepsilon - C_\varepsilon. \tag{3}$$

Even though the counter-term is expected to diverge as $\varepsilon \to 0$, the constructed solutions converge. This strategy has been successfully implemented by Hairer with his theory of regularity structures ([Hai14] and numerous other works). However, it involves a rather intricate hierarchy of equations indexed by trees. In a series of papers [Lin+24; LOT23; Ott+21], Otto and coauthors started to develop another version of this theory, which we call «multi-index» or «tree-free». One of the main differences is that the combinatorics is simpler and relies on some geometric intuition. A gentle introduction can be found in [BOT24; OST23], most of the content from Section 3 is adapted from the SPDE case treated in those notes. Unfortunately this approach breaks down for noises of Hölder regularity $\alpha \in \mathbb{Q}$.

This approach by regularity structure is primarily meaningful for SPDE, but we can also apply it to stochastic differential equations (SDE). In this case the regularity structure of Hairer boils down to what is called «branched rough paths» ([Bru+19] for definitions and more precise statements). In this setting counter-terms and renormalization can be understood as translation of rough paths. Infinite renormalization (with diverging constants) is more seldom but is still meaningful as shown in [Bru+19, Section 4.3]. Multi-index regularity structures applied to ODE can also be interpreted as some kind of rough paths. Hence, applying multi-index regularity structures to SDE is not only a useful way of simplifying our objects to understand them better in an easier setting but has also an intrinsic meaning. In the following, we do not formally introduce rough paths and we only use them marginally. Rough paths have been introduced by [Lyo98] and an introduction can be found in [FH20].

Hence, the goal of this internship is to study the behavior of multi-index regularity structures in the case of rational regularity $\alpha \in \mathbb{Q}$ applied to SDE in order to better understand the obstruction created by those rational regularities. In the next part, we start by briefly introducing multi-index

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[1] We do not define in which sense we exactly measure this regularity here, since we apply this only to a one-dimensional case, more details can be found in [OST23, Section 1].

regularity structures and we give some examples. In 4 we prove the existence and we discuss the uniqueness of a «model » under some suitable hypothesis, we will see that in some case we do not have uniqueness. Trying to understand this lack of uniqueness is the goal of 5 and 7, the former section focuses on an analytical point of view and the later adopts a purely algebraical perspective. In 6, we briefly describe another approach to deal with regularity structures giving us more freedom to understand ξ.

3 A brief introduction to multi-index regularity structures

3.1 General strategy

We only define all the objects in the case of SDE, since we are primarily interested in those objects. The (more interesting) case of SPDE uses the same ideas but needs more work.
For now, we consider an equation such as

$$\partial_t u = a(u)\xi, \tag{4}$$

or more generally

$$\partial_t u = \sum_{i=1}^{d} a_i(u)\xi_i. \tag{5}$$

with a_i some (non-linear) functions and ξ_i some noises. Roughly speaking, in regularity structures one considers all those equations at once, for every (reasonable) a_i. The idea is to (formally) parameterize the solution u by the a_i. For instance, if the a_i are analytic, they can be parameterized by a countable number of parameters and hence we hope that it is the case of u too. This strategy involves three steps:

1. An algebraic step which encodes the equation and the solution into some algebraic structure, corresponding to this «formal» parameterization.
2. A probabilistic step which gives some natural estimates on each component of the algebraic structure.
3. An analytic step which constructs a (real) solution from this algebraic structure using the previous estimates.

Concerning multi-index regularity structures (in the SPDE case), the first two steps are well-understood ([LOT23] for the first step and [Lin+24] for the second one for instance) and the last one is still an active ongoing work. In this internship, we also only focus on the two first steps, without trying to reconstruct a «real» solution.

3.2 Algebraizing the SDE

We start by considering (4). If we assume that a is analytic and can be written as $a(x) = \sum x^k a^{(k)}(0) = \sum x^k z_k[a]$, with $z_k[a] := a^{(k)}(0)$ then we can hope to parameterize the solutions by the family z_k, hence we (formally) hope that in some sense

$$u(t) = \sum \Pi_\beta(t) z^\beta \tag{6}$$

where $\beta : \mathbb{N} \to \mathbb{N}$ is a multi-index, i.e. $\beta(k) = 0$ except for a finite number of k, $z^\beta := \prod_k z_k^{\beta(k)}$ and Π_β is some function.
To make sense of all products let us assume for now that the noise is $\mathcal{C}^\infty$. Given some Π_β, we define the **formal** power series (hence we do not demand any convergence)

$$\Pi(t) := \sum_\beta \Pi_\beta(t) z^\beta \in \mathbb{R}[[z_k]]. \tag{7}$$

$\mathbb{R}[[z_k]]$ is endowed with its natural multiplication. Hence Π is formally playing the role of u, that's why we call it a model (see Definition 1). It plays the same role as a rough path $\mathbb{X}$.
If such a parameterization of u exists, we formally expect that it respects the SDE (4) in some sense. By using the parametrization of a, it is natural to expect that with $\Pi^- := \partial_t \Pi$,

$$\Pi^- = \sum_{k \in \mathbb{N}} z_k \Pi^k \xi. \tag{8}$$

We now demand the noise ξ to be stationary (i.e. the law is invariant by composing by a space-translation) and unless explicitly written we also assume it is centered (for every Schwartz function φ, $\mathbb{E}\langle\xi,\varphi\rangle = 0$). The goal is to add some counter-term, respecting all the symmetries of the noise, keeping as much information as possible and allowing the solution to stay under control as the regularization vanishes. First of all the counter-term (which is a functional of u and its derivatives) shouldn't depend on higher derivatives of u than those appearing in the operator, hence in this case, we demand that in only depends on u.

The counter term $h[a]$ also depends on the non-linearity a. However, assuming we decide to shift the origin of the space $\mathbb{R}$ by v (replacing a by $a(\cdot+v)$), by stationarity of the noise, this should only shift the counter-term, hence we demand the counter-term should only depend on a functional $c[a]$ as $h[a](v) = c[a(\cdot+v)]$. This invariance propagates at the level of the parametrization. We do not justify it, but it amounts to take some $c = \sum_\beta c_\beta z^\beta$ and to define $h[a](v) := \left(\sum_l v^l D^l c\right)[a]$ with D a derivation (a linear operator on an algebra respecting the Leibniz rule) on $\mathbb{R}[[z_k]]$, given by $Dz_k = (k+1)z_{k+1}$ (and extended to $\mathbb{R}[[z_k]]$).

Finally this gives us the equation we want Π to respect

$$\Pi^-(t) = \sum_{k\in\mathbb{N}} z_k \Pi^k \xi - \sum_l \frac{1}{l!}\Pi^l D^l c. \tag{9}$$

Or at the level of a given multi-index β

$$\Pi^-_\beta(t) = \sum_k \sum_{e_k+\beta_1+\dots+\beta_k=\beta} \Pi_{\beta_1}\dots\Pi_{\beta_k}\xi - \left(\sum_l \frac{1}{l!}\Pi^l D^l c\right)_\beta . \tag{10}$$

We also make the assumption

$$\Pi_\beta(0) = 0. \tag{11}$$

We define two important quantities to study those systems of equations. Let $[\beta] := \sum_k (k-1)\beta(k)$ and $\|\beta\| := \sum_k \beta(k)$. By algebraically studying the equations (10) and (11), we can show that $\Pi_\beta = 0$ unless $[\beta] = -1$ (we call this the population condition and call such a β populated). The system (10) is then strictly triangular with respect to the norm $\|\beta\|$. Hence, given some counter-term c and a smooth noise, we can uniquely construct Π. But we want the counter-term to be uniquely defined by the equation and the noise. We impose this by adding some analytical conditions guided by the roughness of the noise.

3.3 Decentering the model

Before doing it, we need to define a larger algebraical object. Indeed, we centered the model Π at 0 (since $\Pi_\beta(0) = 0$) but we could center it at any point x. This would give us another Π_x, satisfying $\Pi_x(x) = 0$ and the equations (10) replacing Π by Π_x (for the same c). In the simpler case of SDE, this transformation can be understood algebraically.

Lemma 1. *Let* $\Gamma_{yx} = \sum_{l\in\mathbb{N}} \frac{1}{l!}\Pi^l_y(x)\mathrm{D}^l$ *an endomorphism of* $\mathbb{R}[[z_k]]$*, then*

$$\Pi^-_y = \Gamma_{yx}\Pi^-_x, \tag{12}$$

$$\Pi_y = \Gamma_{yx}\Pi_x + \Pi_y(x). \tag{13}$$

Furthermore, $\Gamma - \mathrm{Id}$ *is strictly triangular for the norm* $\|\cdot\|$ *on multi-indices.*

Sketch of proof. The strict triangularity is direct. It allows us to prove (12) by induction using (10). It is then possible to upgrade it to (13) thanks to $\Pi_{x,\beta}(x) = 0$. ☐

Contrary to the SPDE case, Γ is rather easy to understand and it actually turns out that it encodes most of the properties of Π, hence our proofs heavily rely on it. The reader familiar with rough paths, can interpret (13) as some type of Chen relationship.

3.4 Roughness and uniqueness

We do not assume anymore that the noise is smooth, but we demand it should be $(\alpha-1)$-Hölder. There are several ways of demanding this for a random object, an efficient one is to demand it respects some spectral gap inequality (see 4.1.1). We will also consider a particular noise (the derivative of the fractional Brownian motion, see 4.1.2) where we do not need in the proofs and statements to introduce any spectral gap or to formalize the regularity of this process.

With this framework, we can furthermore assume that Π reflects some rescaling properties due to the regularity of the noise and hence, it becomes formally natural (but we do not detail the heuristic justification here), to expect that for all $p < \infty$

$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x,\beta}(y)|^p \lesssim_{\beta,p} |y-x|^{|\beta|}, \tag{14}$$
$$\mathbb{E}^{\frac{1}{p}}|\Pi^-_{x,\beta,t}(y)|^p \lesssim_{\beta,p} t^{\alpha-1}(t+|y-x|)^{|\beta|-\alpha} \tag{15}$$

where $|\beta| := \alpha\|\beta\|$. Π_t is the distributional pairing $\langle \Pi, \psi_t\rangle$ for any function $\psi \in \mathcal{S}$ in the Schwartz class, where $\psi_t(x) := \frac{1}{t}\psi(\frac{x}{t})$. $\lesssim_a$ means an inequality up to some constant that depends only on a, hence we demand both estimates to hold uniformly in the semi-norms of ψ and in x, y and t. We note that (14) is reminiscent of the Hölder properties of a rough path but here we demand it should also hold for $t \gg 1$, hence contrary to the classical rough path setting, we are also interested in the large-scale behavior. This is the type of additional properties which give us uniqueness statements. Using the definition of Γ, one can check that those estimates imply (with $\Gamma^\gamma_\beta = (\Gamma z^\gamma)_\beta$)

$$\mathbb{E}^{\frac{1}{p}}|(\Gamma_{yx})^\gamma_\beta(y)|^p \lesssim_{\beta,\gamma,p} |y-x|^{|\beta|-|\gamma|}. \tag{16}$$

The purpose of the counter-term c is to balance the singularities arising out of the products of two distributions that are not regular enough. Hence if the products are regular, we do not need the c_β anymore. That's why we demand $c_\beta = 0$ for $|\beta| > 1$.
But to give a sense to the equations (10), we have to smoothen ξ by (2) and to demand (14) and (15) to hold **uniformly** in ε.

Definition 1. *Given a centered, stationary noise* ξ*, a model of regularity* α *for the equation* (4) *is a random* $\Pi_{\beta,x} : \mathbb{R} \to \mathbb{R}[[z_k]]$ *and a deterministic* $c_\beta \in \mathbb{R}[[z_k]]$ *such that* $\Pi_{\beta,x} = 0$ *if* $[\beta] \neq -1$, $c_\beta = 0$ *if* $|\beta| > 1$ *and* Π *respects (in a classical sense)* (10) *for* ξ*. Furthermore it respects the estimates* (14) *and* (15).
We call «a bounded family of models», a family $\Pi^\varepsilon, c^\varepsilon$ *of models associated with the noises* ξ_ε *such that the estimates* (14) *and* (15) *hold uniformly in* ε.

In the following, we mostly omit the ε in Π^ε. Furthermore, we notice that the interesting cases are $\alpha \leqslant \frac{1}{2}$, otherwise the SDE (5) has a well defined meaning (with the Young integral).

3.5 Several noises

Actually, the equation (4) turns out to be not very rich. Indeed, the Π_β are explicitly computable since $X^k \mathrm{d}X = \frac{1}{k+1}\mathrm{d}X^{k+1}$ and one can check that the estimates hold (see Theorem 1 for a precise statement). However it is a useful toy-model to study the influence of c_β. On the contrary the SDE (5) has a richer behavior because of products such as $X_i \mathrm{d}X_j$ for $i \neq j$. One can adapt the definition of a model to this setting.
We have to consider multi-indices $\beta = (\beta^1, \dots, \beta^d) : \mathbb{N}^d \to \mathbb{N}$, and the equation at the level of Π becomes

$$\Pi^-_\beta(t) = \sum_{l=1}^{d}\sum_{k}\sum_{e^l_k+\beta_1+\cdots+\beta_k=\beta} \Pi_{\beta_1}\dots\Pi_{\beta_k}\xi_l - \left(\sum_l \frac{1}{l!}\Pi^l D^l c\right)_\beta \tag{17}$$

with e^l_k the multi-index such that $\beta^{l'}(k') = \delta_{(l,k)=(l',k')}$. The population condition is $[\beta] := \sum_l \sum_k (k-1)\beta^l(k) = -1$ (i.e. $\Pi_\beta = 0$ if $[\beta] \neq -1$) and we define the homogeneity $|\beta| := \sum_l \alpha_l \sum_k \beta^l(k)$. Note that we can take ξ_l of different regularities α_l, however, unless specified otherwise we always assume all the α_l are the same. With those notations, the definition of a model stays the same, just replacing (10) by (17) and changing the homogeneity $|\beta|$ in the estimates.

3.6 Example

We compute the first levels of the model for 2 noises. We write $\beta = (\beta^1, \beta^2)$ and with the previous notations, we write $(2e_0, e_2)$ for $2e^1_0 + e^2_2$. To simplify the expressions we furthermore assume $c_{e_0,0} = c_{0,e_0} = 0$(this is always the case since we consider centered noises). Moreover we only write half of the multi-indices, indeed, by symmetry we can switch ξ_1 and ξ_2 to get the other ones.

$$\Pi^-_{e_0,0} = \xi_1 \tag{18}$$

$$\Pi^-_{e_0+e_1,0} = \Pi_{e_0,0}\xi_1 - c_{e_0+e_1,0} \tag{19}$$

$$\Pi^-_{e_0,e_1} = \Pi_{e_0,0}\xi_2 - c_{e_0,e_1} \tag{20}$$

$$\Pi^-_{e_0+e_1,e_1} = \Pi_{e_0,e_1}\xi_1 + \Pi_{e_0+e_1,0}\xi_2 - c_{e_0+e_1,e_1} - (c_{e_0,e_1} + c_{e_1,e_0})\Pi_{e_0,0} - c_{0,e_0+e_1}\Pi_{0,e_0} \tag{21}$$

$$\Pi^-_{2e_0,e_2} = (\Pi_{e_0,0})^2\,\xi_2 - c_{2e_0,e_2} - 2c_{e_0,e_1}\Pi_{e_0,0} \tag{22}$$

We haven't written all the Π_β for $\|\beta\| = 3$, but the other ones are of the same kind. Hence, we see that we can interpret the Π_β as linear combinations of (formal) iterated integrals (or of components of a branched rough path). Indeed, we can for instance compute

$$\Pi_{2e_0,e_2}(t) = \int_0^t \mathrm{d}u \left(\int_0^u \xi_1 \right)^2 \xi_2 - tc_{2e_0,e_2} - 2c_{e_0,e_1} \int_0^t \xi_1. \tag{23}$$

The exact relationship between those structures and (branched) rough paths has been developed in [LOT23]. We can already remark that with this interpretation we *a priori* do not have an integration rule of the type $\int X\,\mathrm{d}Y = \int Y\,\mathrm{d}X$, that's why the multi-indices are not redundant.

3.7 Role of α

In the SPDE case, with m the (initial) space dimension, if $\alpha \notin \mathbb{Q}$ for $\alpha > 1 - \frac{2+m}{4}$, under the assumptions made before there is a unique bounded family of models (for a definition of model adapted to the SPDE case). We do not state precisely this theorem ([BOT24, Theorem 1], [OST23, Theorem 1]) since we have a slightly different framework but it is the analogous of Theorem 1.
The assumption $\alpha > 1 - \frac{2+m}{4}$ is important to avoid blow-up of variances. In the SDE case it is expressed as $\alpha > \frac{1}{4}$. It is a standard condition that also arises for instance when we construct a canonical rough path over a Gaussian noise of regularity α, see section 4 for instance, we also comment on it in section 6.
The restriction $\alpha \in \mathbb{Q}$ is more surprising and is needed to make sure $|\beta| \notin \mathbb{N}$ for the multi-indices in the case of SPDE. In the SDE framework, we can see that for $x = y = 0$, (15) tells us $\lim_{t\to\infty} \mathbb{E}\Pi^-_{0,\beta,t}(0) = 0$ if $|\beta| < 1$, however for $|\beta| = 1$, we only know that it is bounded. This implies that for $|\beta| < 1$ there is at most one c_β respecting this estimate. Indeed the algebraic definition (17) of $\Pi^-_\beta - c_\beta$ only depends on γ with $|\gamma| < |\beta|$, and then we necessarily have (to respect $\lim_{t\to\infty} \mathbb{E}\Pi^-_{0,\beta,t}(0) = 0$)

$$-c_\beta \int_{\mathbb{R}} \psi := \lim_{t\to\infty} \mathbb{E}(\Pi^-_\beta - c_\beta)_t(0). \tag{24}$$

But the previous remark implies that the same argument fails for $|\beta| = 1$, since any c_β works. This is one of the reasons that make the construction and uniqueness fail in the SPDE case if $\alpha \in \mathbb{Q}$.
In the SDE setting, for some α such that $\frac{1}{\alpha} \in \mathbb{N}$, we are able to construct several models by changing the values of the c_β such that $|\beta| = 1$. This is of course an undesired property, since we would ultimately want to define a «canonical» solution theory. To better understand the meaning of the restriction $\alpha \in \mathbb{Q}$ in the SPDE case, we want to understand first what happens in the simpler SDE case. Hence in this internship we are interested in several questions: Do the different constructed (bounded families of) models have a natural interpretation? Are some models canonical? Given a (bounded family of) model, can we «translate» it to another model?

4 Existence and uniqueness of a model

In this section, we study the existence and uniqueness of a bounded family of models for the fractional Brownian motion and for a noise respecting some spectral gap inequality, which we introduce.

4.1 Some useful tools and definitions

4.1.1 Malliavin derivative

We do not define precisely the Malliavin derivative here, a brief introduction can be found in [BOT24, Section 2.4]. In the following, we just need some basic facts. Given some noise ξ and a functional $F[\xi]$ (regular enough), the Malliavin derivative is the Fréchet-derivative of F with respect to the noise in the direction $\delta\xi$ (for $\delta\xi$ nice enough[2]) $\delta F[\xi] := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F[\xi + t\delta\xi]$. Furthermore, this derivative respects the Leibniz rule.

Given some regularity α, let $s := \alpha - \frac{1}{2}$. We demand that the noise ξ respects some spectral gap in the sense that for all «reasonable» F, for all $p > 1$:

$$\mathbb{E}^{\frac{1}{p}} \left|F - \mathbb{E}F\right|^p \lesssim_p \mathbb{E}^{\frac{1}{p}} \left\| \frac{\partial F}{\partial \xi} \right\|^p_{\dot{H}^{-s}} = \sup_{\mathbb{E}^{\frac{1}{p'}} \|\delta\xi\|^{p'}_{\dot{H}^s} = 1} \mathbb{E}\delta F \tag{25}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. If ξ is Gaussian (on the space of Schwartz functions), this amounts to demand $\mathrm{Var}\langle \xi, \varphi\rangle \lesssim \|\varphi\|_{\dot{H}^{-s}}$ for $\varphi \in \mathcal{S}$.

We note that if ξ respects (25), to deduce an estimate on $F[\xi]$ we work with $\delta\xi$ which is of (Sobolev) regularity s, whereas ξ is only of regularity $\alpha - 1 = s - \frac{1}{2}$, hence we gain $\frac{1}{2}$ of regularity, which turns out to be crucial.

4.1.2 Fractional Brownian motion (fBm)

The fractional Brownian motion (fBm) is the archetypal Gaussian process of Hölder-regularity $0 < H < 1$. It is defined for $0 < H < 1$ as the centered Gaussian Process $B^H(t)$ of covariance

$$\mathbb{E}\left[B^H(s)B^H(t)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{26}$$

We briefly recall some of its properties.

- For $H = \frac{1}{2}$ it is nothing else than a Brownian motion and for $H \neq \frac{1}{2}$, B^H is neither a Markovian process nor a martingale.
- For all $\alpha < H$, almost surely a sample path is α-Hölder continuous (by Kolmogorov continuity theorem).
- It has stationary increments, $B^H(t) - B^H(s) \sim B^H(t-s)$ but they are not independent except for $H \neq \frac{1}{2}$.
- It is self similar, $B^H(at) \sim |a|^t B^H(t)$.

Hence for the interesting case $H < 1/2$, it is rougher than the Brownian motion and its increments are negatively correlated. Another important property of the fBm is that $\mathrm{d}B^H$ respects the spectral gap inequality (25) for $s = H - \frac{1}{2}$.

4.1.3 Some estimates on the fractional white noise

We define the noise $\xi := \mathrm{d}B^H$ as a distributional derivative. By the growth properties of the fBm it is well defined as a (random) tempered distribution. Some computations show that for $\varphi, \psi \in \mathcal{S}$ $\mathbb{E}\langle \xi, \psi\rangle\langle \xi, \varphi\rangle = \langle \psi, \varphi\rangle_{\dot{H}^{-s}}$ where $\dot{H}^{-s}$ is the homogenous Sobolev space. Hence, we can compute

$$K_\varepsilon(x) := \mathbb{E}\xi_\varepsilon(x)\xi_\varepsilon(0) = \mathcal{F}(|q|^{-2s}|\mathcal{F}(\rho_\varepsilon)|^2)(x). \tag{27}$$

We more generally define ${}_rK(x) := \mathcal{F}(|q|^r\varphi)(x)$ for some even $\varphi \in \mathcal{S}$. Furthermore by stationarity we also have $K_\varepsilon(x-z) = \mathbb{E}\xi_\varepsilon(x)\xi_\varepsilon(z)$. We finally notice that we have some rescaling properties $K_\varepsilon(z) = K_1(\frac{z}{\varepsilon})\varepsilon^{2s-1}$. To prove «by hand» the existence of a bounded family of models for the fBm, one of the main ingredient is to prove some estimates on K.

Lemma 2. *Let $0 < r < 1$, then for some constant c depending on φ, we have for $z \to \infty$*

$${}_rK(z) \sim cz^{-r-1}.$$

If the constant is 0, we understand that as ${}_1K(z) = o(z^{-r-1})$.

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[2] More precisely $\delta\xi$ should be an element of the Cameron-Martin space.

4.2 Existence and uniqueness of a model

We are finally able to prove the existence of a bounded family of models for a noise respecting (25).

Theorem 1. *Let $\xi = (\xi_i)_{1 \leqslant i \leqslant d}$ be a centered stationary noise (on the Schwartz space $\mathcal{S}$) respecting the spectral gap inequality* (25) *for α.*

- *One-noise case: For $d = 1$, there exist a bounded family of models for* (4)*, for every $0 < \alpha \leqslant \frac{1}{2}$. Furthermore, it is unique if $\frac{1}{\alpha} \notin \mathbb{N}$.*
- *Several noises case: For $d > 1$, there exist a bounded family of models for* (5) *for every $\frac{1}{4} < \alpha \leqslant \frac{1}{2}$. Furthermore it is unique if $\frac{1}{\alpha} \notin \mathbb{N}$ (i.e. $\alpha \neq \frac{1}{2}, \frac{1}{3}$).*

We have already explained that the one-noise case has an explicit solution. The only non-uniqueness arises from a free choice of c_β for $|\beta| = 1$ as explained in the previous section. In the several noises case, provided $\frac{1}{\alpha} \notin \mathbb{N}$ (which implies that we never have $|\beta| = 1$) one can prove uniqueness. We focus on the existence part for the several noises case, where we treat separately the cases $|\beta| > 1$ and $|\beta| \leqslant 1$. The case $|\beta| > 1$ is easier to treat, since we have less analytical issues. Hence we start by claiming that once the cases $|\beta| \leqslant 1$ have been taken care of, the remaining ones are constructible. This result is reminiscent of Lyon extension theorem in the rough path setting.

Proposition 1. *For ξ respecting the assumptions of Theorem 1, if we already constructed $\Pi^\varepsilon_\beta, c^\varepsilon_\beta$ for $|\beta| \leqslant 1$ respecting* (17), (14) *and* (15) *then a bounded family of models exists.*

Sketch of proof. The following sketch of proof is rather intended for a reader having some basic knowledge of rough paths. In the multi-index regularity structure for SPDE case, we need two steps, a reconstruction step (giving us estimates on Π^-) and an integration step (upgrading the estimates to Π). Here, it is actually possible to give a proof using a slightly modified version of the sewing bound [CGZ, Theorem 1.9], which allows to control some $A_{st} : \mathbb{R}^2 \to \mathbb{R}$ provided we control $A_{s,u,t} := A_{st} - A_{su} - A_{ut}$ (in some Hölder norms). That is why we directly get estimates on Π. Indeed let $A_{st} := \Pi_{\beta,s}(t)$, then we can control $A_{s,u,t} = (\Gamma_{su} - 1)\Pi_u(t)$ thanks to the induction hypothesis used to control $\Gamma - 1$ (by strict triangularity). An application of (a random version of) the sewing bound (possible since $|\beta| > 1$) gives the desired estimates, which we can upgrade to Π^-. We note that we need a priori the estimate $A_{st} = o(s-t)$, but we do not need it to be uniform in ε, hence one can derive it. ☐

To conclude we only have to construct the first levels. We give two proofs, one relying on an extension of the sewing lemma proposed by Lucas Broux, the other is specific to the fractional Brownian motion.

4.2.1 Proof of existence for $|\beta| \leqslant 1$ - general version

Proposition 2 (Existence of a model, critical case $|\beta| \leqslant 1$ - general version)**.** *Given ξ respecting the assumptions of Theorem 1, Π_β, c_β respecting the definition of a bounded family of models exist for $|\beta| \leqslant 1$.*

In this report, we only give a broad outlook of the proof, a detailed version is under redaction. We crucially rely on the study of the Malliavin derivative. A similar approach also inspired by [Lin+24] has been proposed in [GK23] to construct a rough path over a Gaussian path of regularity $H > \frac{1}{4}$ satisfying a spectral gap inequality. The authors consider the traditional setting of (classical) rough paths and rely on a classical result allowing to construct a rough path over two rough paths, whose sum of regularity is strictly greater than 1. Hence an important difference is that they only need estimates over a compact time interval. Furthermore in this classical setting, there is no need to construct c but the noise does not have to be stationary. Even though the statements share some kind of similarity, the main difference seems to be the techniques used since we heavily rely on a sewing lemma instead of the strategy developed in [GK23].

Proposition 3. *Let $A : \mathbb{R}^2 \to \mathbb{R}$ smooth such that $A_{s,t} = o(t-s)$. Let $\gamma > 1/2$, $0 < \alpha < \gamma$. We assume that for $h_1 < h_2 < h_3$*

$$\left(\int_{\mathbb{R}} \mathrm{d}s |A_{s+h_1, s+h_2, s+h_3}|^2 \right)^{\frac{1}{2}} \lesssim |h_3 - h_1|^{\gamma + \frac{1}{2}}, \tag{28}$$

$$|A_{h_1,h_2,h_3}| \lesssim \min(|h_2 - h_1|, |h_3 - h_2|)^\alpha (|h_2 - h_1| + |h_3 - h_2|)^{\gamma - \alpha}. \tag{29}$$

Then

$$\left(\int_{\mathbb{R}} \mathrm{d}s |A_{s,s+h}|^2\right)^{\frac{1}{2}} \lesssim_{\gamma} |h|^{\gamma+\frac{1}{2}}, \tag{30}$$

$$|A_{s,t}| \lesssim_{\alpha,\gamma} |t-s|^{|\beta|}. \tag{31}$$

We note that [FS21] also study some Besov rough analysis, although it is written in a more abstract and general setting.

Sketch of proof. The first step is to obtain the estimate (30) by an adaptation of the proof of the sewing bound already used in the proof of Proposition 1. The main idea is to decompose $A_{s,t}$ as an infinite sum of A_{s_1,s_2,s_3}. To write such a decomposition, we need $A_{s,t} = o(t-s)$. The second step is to convert this L^2 estimate into the L^∞ estimate (31), this can be done through a «Campanato» argument. □

This lemma admits a random version if we replace $|\cdot|$ by $\mathbb{E}^{\frac{1}{q}}|\cdot|^q$ for every $q < 2$ in (28) and (30) and for every $q < \infty$ in (29) and (31). We apply it with $A_{s,t} := \delta\Pi_{s,\beta}(t)$, and $\gamma = |\beta| \geqslant 2\alpha$. Hence the hypothesis $\gamma > \frac{1}{2}$ is equivalent to $\alpha > \frac{1}{4}$.
We prove the following points in the induction loop (they all have to be uniform in ε).

$$\forall p < +\infty, \mathbb{E}^{\frac{1}{p}}|\Pi_{s,\beta}(t)|^p \lesssim_{\beta,p} |t-s|^{|\beta|} \tag{32}$$

$$\forall p < +\infty, \mathbb{E}^{\frac{1}{p}}|\delta\Pi_{s,\beta}(t)|^p \lesssim_{\beta,p} |t-s|^{|\beta|} \tag{33}$$

$$\forall q < 2, \left(\int_{\mathbb{R}} ds \mathbb{E}^{\frac{2}{q}}|\delta\Pi_{s,\beta}(s+h)|^q\right)^{\frac{1}{2}} \lesssim_{\beta,p} |h|^{|\beta|+\frac{1}{2}} \tag{34}$$

$$\Pi_{s+t}[\xi](s+u) = \Pi_t[\xi(s+\cdot)](u) \tag{35}$$

(32) also implies (15). The base case $\beta = e_0$ comes from some computations relying on the spectral gap, with an application of a Fourier isometry and Besov-Sobolev embeddings, that we do not develop here. In the induction loop the first step is to obtain an estimate on $\delta\Pi$.

Proposition 4. *If the induction hypotheses hold at level* $|\gamma| < |\beta|$*, then* (33) *and* (34) *hold.*

Sketch of proof. We recall $A_{s,t} = \delta\Pi_{s,\beta}(t)$. By the induction hypothesis, it can be checked that $A_{s,t}$ respects the assumptions of Proposition 3. Indeed we can compute

$$A_{s,u,t} = (\Gamma_{su} - \mathrm{Id})\delta\Pi_u(t) + \delta\Gamma_{su}\Pi_u(t) \tag{36}$$

with $\delta\Gamma_{su} = \sum_{l\in\mathbb{N}} \frac{1}{l!} l(\delta\Pi_s(u))\Pi_s^{l-1}(u)\mathrm{D}^l$ also strictly triangular. The estimates are obtained by applying (32) and (33) for the Hölder-like bound and (32) and (34) for the Besov-Sobolev-like bound as well as a Hölder inequality in probability (where the assumptions $q < 1$ is needed). This is inductively possible thanks to the strict triangularity of $\delta\Gamma$ and $\Gamma - \mathrm{Id}$. $A_{s,t} = o(t-s)$ can be checked almost surely for every regularization $\varepsilon > 0$ since we do not need any uniformity.
Hence, we deduce that

$$\mathbb{E}^{\frac{1}{p}}|\delta\Pi_s(t)|^p \lesssim_{\beta,p} |t-s|^{|\beta|}.$$

□

After that it is sufficient to upgrade the estimate to Π and to construct c at the same time.

Proposition 5. *Let* $|\beta| < 1$*, we assume the induction steps for* $|\gamma| < |\beta|$ *as well as* (33)*. Then there exist a (unique)* c_β *independent of* x *such that* (32) *holds.*

Sketch of proof. To reconstruct the estimate on Π thanks to the spectral gap, we just have to control the expectation $\mathbb{E}\Pi_{x,\beta}(y)$. To do that we use (13) with $y = 0, x = t$ and we evaluate it at $2t$, we use (35) as well as the strict triangularity of $\Gamma - \mathrm{Id}$ to deduce inductively

$$|v_{2t} - v_t| \lesssim t^{|\beta|-1} \tag{37}$$

with $v_t = \frac{\mathbb{E}\Pi_{0,\beta}(t)}{t}$. Hence $v_{2^{k+1}t} - v_{2^k t}$ is summable. We deduce the existence of $-c_\beta = \lim_{k\to\infty} \frac{\mathbb{E}\Pi^-_{0,\beta}(2^k t)}{2^k t} - c_\beta$ (we recall that the right hand-size depends only on γ for $|\gamma| < |\beta|$). With some extra work one can prove that this limit is independent of t.
Furthermore, by (35), c_β does not depend on x (and hence (35) is still true at the level of β). This closes the induction loop. Let us remark that this argument fails if $|\beta| = 1$, because the series is not summable anymore. However every choice of c_β will give rise to a different family of models which respect the estimates and hence we do not have uniqueness. □

4.2.2 Proof of existence for $|\beta| \leqslant 1$ - fBm

In this part we give an alternative proof of Proposition 2 for the particular case where ξ is the derivative of a fractional Brownian motion of regularity $H > \frac{1}{4}$. We reproduce the argument here because this proof is less abstract than the general case and was done at the beginning of the internship before having the general result Proposition 2. We state it for the slightly different framework of noises of different regularity, which is useful in the following sections. To simplify the proof, we focus on $x = y = 0$. We also give a counter-example for the existence of a bounded family of models for regularities $H \leqslant \frac{1}{4}$.

Proposition 6 (Existence of a model, critical case $|\beta| \leqslant 1$ - fBm)**.** *Let* $\xi_i = \mathrm{d}B^{H_i}$ *the (distributional) derivative of independent fBm* B^{H_i} *for* $\frac{1}{4} < H_i < 1$*. Then* Π_β, c_β *respecting the definition of a bounded family of models exist for* $|\beta| \leqslant 1$*, they are unique provided* $1 \notin \{a_i H_i | a_i \in \{0,1,2,3\}, \sum a_i \leqslant 3\}$*. Furthermore, for* $|\beta| < 1$*, we have* $c_\beta = 0$*.*
For $d = 2$, $\frac{1}{2} \geq H_1 + H_2 > 0$, $\Pi^\varepsilon_{e_0,e_1}$ *does not respect the estimates* (15) *uniformly in* ε*, hence the family of models is not bounded.*

If the H_i are all the same H, the condition becomes $\frac{1}{H} \notin \mathbb{N}$.

Sketch of proof. We recall that in section 3.6 we gave the first components of the model. We furthermore remark that the components of the kind $\Pi_{(\beta,0)}$ or $\Pi_{(0,\beta)}$ boil down to the one-noise case. We should only be concerned with the cross-terms. By definition we have

$$\mathbb{E}\left(\Pi^-_{(e_0,e_1),t}\right)^2 = \\ \iint \mathrm{d}u\,\mathrm{d}u' \psi_t(u)\psi_t(u')\mathbb{E}\left[(B^{H_1}_\varepsilon(u) - B^{H_1}_\varepsilon(0))(B^{H_1}_\varepsilon(u') - B^{H_1}_\varepsilon(0))\right]\mathbb{E}\left[\xi_{2,\varepsilon}(u)\xi_{2,\varepsilon}(u')\right].$$

We can replace B_ε by B without changing convergence. Furthermore, the only critical term in $\mathbb{E}\left[B^{H_1}(u)B^{H_1}(u')\right]$ is $|u-u'|^{2H_1}$, one can check that the other terms do not influence the claimed result. Now we are left with

$$\iint \mathrm{d}u\,\mathrm{d}u' \psi_t(u)\psi_t(u') K_\varepsilon(u-u')|u-u'|^{2H_1}\,\mathrm{d}u\,\mathrm{d}u' = \int \mathrm{d}z \tilde{\psi}_t(z) K_1\left(\frac{z}{\varepsilon}\right)|z|^{2H_1}\varepsilon^{2(H_2-1)} \tag{38}$$

$$= \int \mathrm{d}z \tilde{\psi}_t(z\varepsilon) K_1(z)|z|^{2H_1}\varepsilon^{2(H_2+H_1)-1} \tag{39}$$

with $\tilde{\psi} := \psi * \psi(-\cdot)$. We can now use Lemma 2 to see that if $H_1 + H_2 < \frac{1}{2}$, $K_1(z)|z|^{2H_1}$ is integrable and by dominated convergence the integral converges but the pre-factor $\varepsilon^{2(H_2+H_1)-1}$ diverges. We furthermore claim that we can choose at least one test function ψ such that the limit is not 0 (for instance a Gaussian). Hence, the estimates (15) does not hold uniformly in ε. The case $H_2 + H_1 = \frac{1}{2}$ can be taken care of with an analysis of a second order expansion of K_1.
For the case $H_1 + H_2 > \frac{1}{2}$ and for the level 2 estimates, we apply a Fourier isometry and similar estimates allow us to conclude since we know explicitly the Fourier transform of K_1. For the third level, it is more technical to prove the estimates, however we can reduce with a pre-analysis the components to some products of the kind $\Pi_{(\beta_1,0)}\Pi_{(0,\beta_2)}$ which is easier to estimate by induction using the independence of the noises. This only gives us estimates on Π^-, however a similar argument as in the SPDE case allows us to deduce estimate on Π (we do not reproduce it here, since it is rather intricate).
Finally, the values of c_β for $|\beta| < 1$ are an immediate consequence of (24). ☐

5 Non-uniqueness of the model for fBm with $\frac{1}{H} \in \mathbb{N}$

Now that we have shown the existence of a bounded family of models, we are interested in the cases where we do not expect uniqueness. We recall that if $\frac{1}{H} \in \mathbb{N}$ one can construct several models by changing the constants c_β for $|\beta| = 1$. Hence given $\frac{1}{H} \in \mathbb{N}$ and some c_β we want to identify an integration theory for $\mathrm{d}B^H$ which coincides with the Π_β (viewed as iterated integrals as explained in 3.6).
Let us start with an example, which was the remark that motivated the subject of this internship. If we take $B^{\frac{1}{2}}$ a Brownian motion (in 1D), we can choose for instance $c_{e_0+e_1} = 0$ or $c_{e_0+e_1} = \frac{1}{2}$. In the first case we find after integration (for $\varepsilon > 0$ and then we go to the limit $\varepsilon \to 0$) $\Pi_{e_0+e_1}(t) = \frac{1}{2}\Pi^2_{e_0}(t)$ and in the second one $\Pi_{e_0+e_1}(t) = \frac{1}{2}\Pi^2_{e_0}(t) - \frac{1}{2}t$. Since $\Pi_{e_0}(t) = B^H(t)$, if we understand $\Pi_{e_0+e_1}$ as a way to give sense to $\int B\mathrm{d}B$, we find that $c_{e_0+e_1} = 0$ corresponds to the Stratonovich integral and $c_{e_0+e_1} = \frac{1}{2}$ to the Itô integral ! Hence the question we are concerned with in this part is to understand if we can generalize this statement for other H with $\frac{1}{H} \in \mathbb{N}$.

5.1 Odd and even levels

In the several noises case, we have seen that we have the constraint $H > \frac{1}{4}$ to construct a bounded family of models. Hence, the only interesting cases are $H = \frac{1}{2}, \frac{1}{3}$.
However, if we try to apply the formula (24) for $|\beta| = 1$, $H = \frac{1}{3}$, we always find $c_\beta = 0$, because the expectation cancels since the odd moments of Gaussian variables are 0. We will call «distinguished» such a choice of c_β formally given by (24) (even though $|\beta| = 1$). The same holds for Π_{e_0,e_1} and Π_{e_1,e_0} for $H = \frac{1}{2}$ a noise with independent components, where we also have a distinguished choice $c_\beta = 0$.
Of course, non-distinguished choices of c_β are still valid but the next subsection shows for instance that a similar interpretation of c_β as for even levels does not work. We try to address this issue in section 6.
If we try to apply (24) for $|\beta| = 1$, $H = \frac{1}{2}$ (or $\frac{1}{2n}$ for the one-noise case), we find

$$\lim_{t\to\infty} \mathbb{E}\left[(\Pi_\beta^{\varepsilon -} - c_\beta^\varepsilon)_t\right](0) \propto \int_{u>0} \psi(u)\mathrm{d}u - \int_{u<0} \psi(u)\mathrm{d}u. \tag{40}$$

$\propto$ means equality up to some constant depending on β. Hence (40) depends on ψ, that means, that we have *a priori* no distinguished choice for c_β^ε (except taking them equal up to some fixed proportionality constant and independent of ε). We interpret the choice of the limiting constant in the following section.
To go beyond the limit $H > \frac{1}{4}$, we focus on the one-noise case and we also look at several correlated noises.

5.2 Integration scheme for $H = \frac{1}{2n}$

To be able to break the $H > \frac{1}{4}$ barrier we only work in the one-noise case in this part. We recall that the Itô integral $\int_0^t F(B_s)\mathrm{d}B_s$ can be defined as the limit in probability (for suitable F) of

$$\sum_{[s,t]\in\pi_n} F(B_s)(B_t - B_s)$$

for π_n a partition with mesh going to 0 as $n \to \infty$. In the same fashion, $\sum_{[s,t]\in\pi_n} F\left(B_{\frac{s+t}{2}}\right)(B_t - B_s)$ gives rise to the Stratonovich integral. Hence we can hope that for smaller H, more precise integration schemes still converge and give us an interpretation for the different families of models. The most natural guess is to try to generalize the Newton-Cotes scheme, which is exact on every polynomial of small degree.
We recall that for $H = \frac{1}{n}$, $|\beta| \leqslant 1$ in the one-noise case, we have (since $c_\beta = 0$ for $|\beta| < 1$ by Proposition 6)

$$\Pi_\beta^-(x) \propto (B^H(x))^{\|\beta\|-1}\xi - c_\beta\delta_{|\beta|=1}. \tag{41}$$

We also recall the interpretation of Π_β as iterated integral. Thus we aim to define an integration theory having the same behavior as the previous equations.
Let us take ν a probability measure on $[0,1]$ that coincides with the Lebesgue measure on every polynomial of degree $\leqslant n$, i.e. $\int_0^1 x^k \nu(\mathrm{d}x) = \frac{1}{k+1}$ for $k \leqslant n$. For simplicity we consider here ν a discrete measure (a sum of Diracs). Then, we can define the integration of $f(X)$ against X with the scheme ν [See Gra+05, definition 3.1 with $m = 1$] as the limit in probability (provided it exists) for $\varepsilon \to 0$ of

$$\int_0^t (X_{u+\varepsilon} - X_u) \int_0^1 f(X_u + \alpha(X_{u+\varepsilon} - X_u))\nu(\mathrm{d}\alpha). \tag{42}$$

If it exists we denote it by $\int_0^t f(X_u)\mathrm{d}^\nu X_u$. Furthermore, if ν is symmetric (i.e. ν is left invariant by the transformation $t \mapsto 1 - t$) then:

Theorem 2 ([Gra+05], Theorem 4.4). *If $H > \frac{1}{2(2r+1)}$ for $r \in \mathbb{N}$, $r \geqslant 2$, $f \in \mathbb{C}^{4r+2}$, ν symmetric respecting the hypothesis above for $n = 2(r-1)$, then $\int_0^t f'(B_u^H)\mathrm{d}^\nu B_u^H$ exists and we have*

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)\mathrm{d}^\nu B_u^H.$$

If we drop the symmetry hypothesis, we can prove that this result only holds (with the same hypothesis on the moment of ν) for $H > \frac{1}{2r}$. However, combining a modified version of [Gra+05, Proposition 3.6] to allow for a non-symmetric measure as well as a direct application of the Theorem 4.1. allows to write:

Theorem 3. *If $H = \frac{1}{2r}$ for $r \in \mathbb{N}$, $r \geqslant 2$, $f \in \mathbb{C}^{2r+1}$, ν respecting the hypothesis above for $n = 2(r-1)$, then $\int_0^t f'(B_u^H)\mathrm{d}^\nu B_u^H$ exists and we have*

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)\mathrm{d}^\nu B_u^H + c\int_0^t f^{(2r)}(B_u^H)\mathrm{d}u$$

for c some explicit constant.

By applying this to $f(x) = x^n$, for $H = \frac{1}{2r}$ in the 1D case we have hence defined some integration theory reflecting the property (41). Hence, we can interpret the various families of models as a choice in some integration scheme, generalizing the Itô-Stratonovich difference.
Furthermore, a minor modification of [Gra+05, Proposition 3.6] allows to replace $\int_0^t f(X_u)\mathrm{d}^\nu X_u$ with a similar discrete natural integration scheme (which would give exactly Itô formula for $\nu = \delta_0$ and Stratonovich for $\nu = \delta_{\frac{1}{2}}$, instead of this slightly different «continuous» version). To make a similar proof for such an integration scheme, we have to check that an equivalent version of Theorem 4.1 remains true, which we have not done. However, we believe it is possible.

5.3 Two correlated noises

We recall that in 3.5 we also defined a model for noises of different regularity. In such a framework, the results of Theorem 1 remain true if $1 \in \bigoplus H_i\mathbb{N}$ and $H_i > \frac{1}{4}$.
Here, we consider the following setup, $\xi_1 := \mathrm{d}B^{\frac{1}{2}+\eta}$ for $0 < \eta < \frac{1}{6}$, $\xi_2 := \partial_{|x|}^{2\eta}\xi_1$ with $\partial_{|x|}^{2\eta}$ the fractional partial derivative of order 2η (i.e. the Fourier multiplier $|q|^{2\eta}$). We denote by B_i their respective integral. We have $\frac{1}{2} + \eta + \frac{1}{2} - \eta = 1$, that means that the only potential difficulties to construct a bounded family of models are $\beta_1 := (e_0, e_1)$ and $\beta_2 := (e_1, e_0)$, both of homogeneity $|\beta| = 1$. We can note that $\mathbb{E}\langle\xi_1, \varphi\rangle, \langle\xi_2, \psi\rangle = \langle\varphi, \psi\rangle_{L^2}$, hence an immediate adaptation of the previous results shows:

Lemma 3. *If we define ξ_1, ξ_2 as above, we have a bounded family of models of regularity $(\frac{1}{2} + \eta, \frac{1}{2} - \eta)$ but it is not unique. The non-uniqueness is parameterized by a choice of c_{β_1} and c_{β_2}.*

Furthermore, (24) does not give us a distinguished choice, except taking $c_{\beta_1} = c_{\beta_2}$, which we assume throughout the following.
We just briefly state the results of this part without trying to formally define all the objects. This subsection is rather intended for a reader having some broad familiarity with rough paths. It is well-known that Gaussian processes with independent components X^i of regularity $> \frac{1}{4}$ have a canonical rough path lift, see for instance [FH20, Theorem 10.4]. Hence, we can define an integration against this rough path. However, we can wonder what happens when we drop the assumption that the components should be independent. We recall that this rough path lift can be defined as a limit of discrete approximations $\sum_{(u,s)\in\pi} X_{0,r}^1 X_{u,s}^2$. We can take inspiration from the techniques used in exercise 10.18 of the same book, to check that if we do not assume anymore independence, the value of $r \in [u, s]$ starts to matter in the limiting result. In our framework, we can check that the convergence gives us some Itô formula (in the spirit of chapter 5.3 and the so-called «Föllmer's calculus» of [FH20]) $(B_1B_2)(t) = \int_0^t B_1\mathrm{d}B_2 + \int_0^t B_2\mathrm{d}B_1 + ct$ with c some constant associated with the choice of r. Hence with $c = c_{\beta_1} + c_{\beta_2}$, we can interpret $\Pi_{\beta_1}(t)$ as $\int_0^t B_2\mathrm{d}B_1$ given by this integration theory (and similarly for β_2), the choice of the counter-term corresponding to the choice of r which is also some type of Itô-Stratonovich choice. Indeed, a brief computation (at the level of regularization $\varepsilon > 0$ before going to the limit), gives us the formula: $B_1(t)B_2(t) = \Pi_{e_0,0}(t)\Pi_{0,e_0}(t) = \Pi_{e_0,e_1}(t) + \Pi_{e_1,e_0}(t) + (c_{\beta_1} + c_{\beta_2})t$

5.4 Summary

Finally, we can summarize the results of this section in the following table.

Number of noises	H	Free constants	Distinguished choice	Interpretation
1	$\frac{1}{2}$	$c_{e_0+e_1}$	No	Itô-Stratonovich
1	$\frac{1}{2n}$	c_β for $\lvert\beta\rvert = 1$	No, but c_β proportional and converge	Integration scheme
1	$\frac{1}{2n+1}$	c_β for $\lvert\beta\rvert = 1$	Yes, $c_\beta = 0$	
2	$H_i = \frac{1}{2} \pm \eta$, $0 < \eta < \frac{1}{6}$ correlated	c_{e_1,e_0}, c_{e_0,e_1}	No, but $c_{e_1,e_0} = c_{e_0,e_1}$ and converge	Rough integration (Itô-Stratonovich)

6 Malliavin derivative and non-smooth model

In this part, we derive a stable formula for $\delta\Pi^-$ without singular products. This allows us to give sense to (17) without needing a regularization procedure and hence we can study non-smooth approximations of a stochastic process. More precisely, we are able to study non-Gaussian (and non-symmetric) approximations of a fBm, hoping to create an interesting behavior at the third level.

6.1 Non-smooth models

In [Tem24] a general definition of models (as well as uniqueness, existence and convergence results) is proposed in a broader framework, where we do not necessary approximate the noises by convolution. In the SDE case, the procedure to construct such a model turns out to be simpler since the regularity is high enough to use a more direct approach. We only sketchs the idea behind it.

The goal is to find an expression of the kind (17) for $\delta\Pi^-$, without singular products. As previously we only focus on the several noises case, but since we have the limit $\alpha > \frac{1}{4}$, we only have to construct the first three levels. For simplicity we assume $c_{e_0^l} = 0$ (it is always the case, since we consider centered noises).

We start by (formally) applying the Malliavin derivative to (17). For instance for Π_{e_0,e_1} computed in (20) we get

$$\delta\Pi^-_{e_0,e_1} = \delta\Pi_{e_0,0}\xi_2 + \Pi_{e_0,0}\delta\xi_2. \tag{43}$$

We first note that the constant c_{e_0,e_1} disappears, furthermore if we consider for instance $\delta\Pi_{e_0,0}\xi_2$, we have a product of two distributions, one of (Sobolev)-regularity $H + \frac{1}{2}$ and one of (Hölder)-regularity $H - 1$, hence the distributional product is well-defined since $H + \frac{1}{2} + H - 1 > 0$ by the hypothesis $H > \frac{1}{4}$ and is of regularity $H - 1$. We can apply δ to all the second-level equations which turn out to be well-defined. Moreover, this gives us a nice heuristic argument of why the limit $\frac{1}{4}$ arises.

We can follow the same strategy for the third level equations such as $\Pi_{e_0+e_1,e_1}$ computed in (21). This gives us

$$\begin{aligned}\delta\Pi^-_{e_0+e_1,e_1} =& \delta\Pi_{e_0,e_1}\xi_1 + \Pi_{e_0,e_1}\delta\xi_1 + \delta\Pi_{e_0+e_1,0}\xi_2 + \Pi_{e_0+e_1,0}\delta\xi_2 \\ &- (c_{e_0,e_1} + c_{e_1,e_0})\delta\Pi_{e_0,0} - c_{0,e_0+e_1}\delta\Pi_{e_0,0}.\end{aligned} \tag{44}$$

Unfortunately we still have some singular products, such as $\delta\Pi_{e_0,e_1}\xi_1$ which does not make sense because ξ_1 is of regularity $H - 1$ and $\delta\Pi_{e_0,e_1}$ of regularity H (by the analysis of the second-level case) and $H + H - 1 < 0$ for $H < \frac{1}{2}$.

We make the (natural by looking at (20)) Ansatz $\delta\Pi_{e_0,e_1} \sim \delta\Pi_{e_0,0}\Pi_{0,e_0}$, let us call Δ_1 the difference then $\mathrm{d}\Delta_1 = \Pi_{e_0,0}\delta\xi_2 - \delta_1\Pi^-_{e_0,0}\Pi_{0,e_0}$ and all the products appearing in $\mathrm{d}\Delta_1$ are regular. Hence $\delta\Pi_{e_0,e_1}\xi_1 = \Delta_1\xi_1 + \delta\Pi_{e_0,0}\Pi_{0,e_0}\xi_1 = \Delta_1\xi_1 + \delta\Pi_{e_0,0}(\Pi^-_{e_1,e_0} + c_{e_1,e_0})$. Since $\delta\Pi_{e_0+e_1,0} = \delta\Pi_{e_0,0}\Pi_{e_0,0}$ the analysis is even simpler for this case and we have $\delta\Pi_{e_0+e_1,0}\xi_2 = \delta\Pi_{e_0,0}\Pi_{e_0,0}\xi_2 = \delta\Pi_{e_0,0}(\Pi^-_{e_0,e_1} + c_{e_0,e_1})$. Finally, we obtain

$$\delta\Pi^-_{e_0+e_1,e_1} = \Delta_1\xi_1 + \delta\Pi_{e_0,0}\Pi^-_{e_1,e_0} + \Pi_{e_0,e_1}\delta\xi_1 + \delta\Pi_{e_0,0}\Pi^-_{e_0,e_1} + \Pi_{e_0+e_1,0}\delta\xi_2 - c_{0,e_0+e_1}\delta\Pi_{e_0,0}.$$

In the previous step we already gave sense to $\Pi^-_{e_1,e_0}$ of regularity $H - 1$. We get rid of c_{0,e_0+e_1} by plugging in the previous equation. Hence, we see that we have an expression of $\delta\Pi^-_{e_0+e_1,e_1}$ as a sum of regular products. We can apply this procedure to all the equations and kill off all the singular products and the constants (which arise again in the reconstruction $\delta\Pi \to \Pi$). This is the starting point to write a definition of a model where only regular products arise. This could also have been the starting point of a proof of Proposition 2, however we bypassed it by directly working at the level of Π thanks to the sewing lemma.

6.2 An attempt to construct non-Gaussian approximations

Now that we can directly define models without needing to smoothen them, we want to define a family of (non-Gaussian) models that converges to the fBm $B^{\frac{1}{3}}$ but such that some $c_\beta \neq 0$ for $|\beta| = 1$ in the limit. Furthermore, we would like to construct a family such that a natural integration theory is associated with it, allowing to interpret c_β. Actually we only consider it in the one-noise case, where the previous work is not really needed.

Unfortunately we did not manage to construct such a family, but we still present some family of interest we were interested in, however we do not give any proofs or even formal statements.

We consider a «shot noise». Given a Poisson point process η on $\mathbb{R}$ of homogenous intensity $\lambda \in \mathbb{R}_+^*$ (see [Str10]), we consider the process:

$$X_T := \frac{\eta([0,T]) - \lambda T}{\sqrt{\lambda}} \tag{45}$$

which converges weakly to a Brownian motion as $\lambda \to \infty$. To reach the desired regularity, we rather consider its fractional derivative (or the Fourier multiplier $|\xi|^{\frac{1}{2}-H}$) Y_T of order $\frac{1}{2} - H$. We define a model at the level of λ and go to the limit $\lambda \to \infty$ in the same fashion as for the regularization parameter ε previously. We can show that for $|\beta| = 1$, $c_\beta \neq 0$ for $\lambda > 0$. Unfortunately $c_\beta^\lambda \to 0$ as $\lambda \to \infty$.

Another process, which is known to converge (weakly) to the fBm is constructed in [DJ00], we do not define it precisely here, however the idea is to use the so-called Voltera representation of the fBm as a Wiener integral $B^H(t) = \int_0^1 K(t,r)\,\mathrm{d}W_r$ for some K. We can define $B_\lambda := \int_0^1 K(t,r)(-1)^{\eta([0,r])}\,\mathrm{d}r$, it can be shown that under a suitable rescaling, B_λ converges weakly to B^H as $\lambda \to \infty$. However, the same behavior arises as for the shot noise, namely a trivial limit for c_β^λ at the third level.

Hence, we are still willing to know if we can construct a meaningful approximation of $B^{\frac{1}{3}}$ leading at the limit to a non-trivial third level.

7 Algebraic change of model

In this section, we adopt another point of view and rather try to describe on a purely algebraical level how c affects the hierarchy of equations. We recall that we have three types of renormalization constants c_β for $\frac{1}{\alpha} \in \mathbb{N}$.

1. $c_\beta = 0$ for $|\beta| > 1$ by an algebraic condition.
2. c_β for $|\beta| < 1$ are uniquely fixed by an analytic condition.
3. c_β for $|\beta| = 1$ are «free» constants.

We have already justified that changing the last c_β gives rise to different bounded families of models that we studied analytically. The goal now is to understand how we can pass from one model to another by changing c.

7.1 The algebraic structures behind Π

In this section, we assume some basic knowledge on rough paths from the reader. However, it is only needed for some justifications which remain vague and not the for the statements of the few results. For simplicity we state our results for one noise, even though one can generalize them. We are somewhat vague in the formulation of some intermediate statements, since this avoids us to define heavy notations, that we do not need for final statements. We start by describing briefly some of the underlying algebraic structures behind the multi-index setting. A detailed account can be found in [Lin23]. If we forget for now the long-range estimates (14) and (15) for $t \gg 1$, the familiar reader can notice that we boil down to rough paths (over some Hopf Algebra introduced later). We remark that the derivation D endows the family z^β for β populated (i.e. $[\beta] = -1$) with a pre-Lie structure, that is if $T_1 = \mathrm{Span}(z^\beta | [\beta] = -1)$, T_1 is an algebra for D and we have for all $a, b, c \in T_1$

$$aD(bDc) - (aDb)Dc = bD(aDc) - (bDa)Dc. \tag{46}$$

This terminology of pre-Lie algebra comes from the fact that $[a, b] := aDb - bDa$ defines a Lie bracket. The structure (T_1, D) has been identified in [Lin23, Lemma 3.4]. The grading $|\beta|$ defines a natural grading on T_1 with the additive property that $|\beta| \geqslant \alpha$ for all β, hence general results allow us to define a Hopf algebra of multi-indices, which we do not describe here, even though it turns out to be a fundamental object.

7.2 Change of model

The renormalization procedure consisting in adding a given $\mathbf{c}$ can be interpreted as a «translation» of rough paths (a phenomenon already noticed in the setting of regularity structures due to Hairer, with its one-dimensional counter-part called branched rough paths, see [Bru+19]). In the SDE case, adding some $\mathbf{c}$ is the same thing as acting «externally» on the noise:

$$\partial_t u = a_1(u)\xi - \mathbf{c}[a_1](u). \tag{47}$$

But equivalently we can write this down as (with $\cdot$ the standard scalar product)

$$\begin{aligned} \partial_t u =&(a_0(u), a_1(u)) \cdot (1, \xi) - (\mathbf{c}[a_1](u), 0) \qquad (48)\\ =&(a_0(u) - \mathbf{c}[a_1](u), a_1(u)) \cdot (1, \xi). \qquad (49)\end{aligned}$$

By taking $a_0 = 0$, we find the same equation. We recall that we defined a model for non-homogenous noises (with $d > 1$). Here we consider models of regularity of $(1, \alpha)$. Let us call $T_2 = \mathrm{Span}(z^{\beta_0,\beta_1} | [(\beta_0, \beta_1)] = -1)$. We also use the notation (e_0, e_1) already introduced. Let $z^0_k[a_0] = a_0^{(k)}(0)$ the parameters associated with a_0 and $z^1_k[a_1]$ the parameters associated with a_1.
Hence increasing the dimension allows us to write this action of $\mathbf{c}$ as a translation of the non-linearity a_0, which allows us to use the framework of [Lin23]. In comparison with a general translation of rough paths, we notice that we act only on the coordinate 0 by a $\mathbf{c}$ depending only on the coordinate 1 (we recall that $\mathbf{c}$ is parametrized by z^1_k). We are also interested in the special case where $\mathbf{c}$ depends only on β for $|\beta| = 1$ i.e. $\mathbf{c} = \sum_{\|\beta\|=n,[\beta]=-1} c_\beta z^\beta$, since we want to understand the effect of the «free» constants. We call such a $\mathbf{c}$ homogenous of degree n. Furthermore, we consider only $a_0 \equiv 0$.
We want to understand if the general results of [Lin23] can be simplified in the SDE setting, with the several assumptions made above. In the following, when writing Π_β we always refer to the model defined with (47) and Π_{β_0,β_1} to the «enhanced» model (48). The first step is to increase the dimension of the model by adding this smooth noise 1. Since the new noise is smooth, we can produce a new model Π_{β_0,β_1} respecting the equations (17) and hence being an extension of Π_β (in the sense that $\Pi_{0,\beta_1} = \Pi_{\beta_1}$, $\Pi_{\beta_0,0} \propto t^{\|\beta_0\|}$). However the extension map is in some sense very analytic, and in the setting we used, we do not have a nice algebraic definition. Hence, we do not write it as map $T_1 \to T_2$ which would automatically upgrade to the model like $T\Pi = \sum_\beta \Pi_\beta(t) T z^\beta$ but we should rather write it as a map $i : \tilde{T_1} \to \tilde{T_2}$ between the spaces of models of dimension 1 and 2. For this reason, we consider Π_{β_0,β_1} as algebraic black-boxes, even though one can understand them analytically.
On the contrary, we have a nice description of the algebraic effect of T_c on the model from [Lin23]. In vague words, translating the above equation by c amounts to apply a certain pre-Lie morphism $T_c : T_2 \to T_2$, which is explicitly described. And T_c extends instantly to $\tilde{T_2} \to \tilde{T_2}$.
Finally, since we want to take $a_0 = 0$ we consider the projection $p : T_2 \to T_1$ which conserves only the elements of the form z^{0,β_1}. Hence, in the SDE setting we are considering the following map

$$\tilde{T_1} \xrightarrow{i} \tilde{T_2} \xrightarrow{T_c} \tilde{T_2} \xrightarrow{p} \tilde{T_1}. \tag{50}$$

The last two maps are induced by

$$T_2 \xrightarrow{T_c} T_2 \xrightarrow{p} T_1. \tag{51}$$

Now, we can specialize the general results and some formulas of [Lin23] to the specific case we are considering. For simplicity, we only state it for $\mathbf{c} = cz^1_0 z^1_1$ (we recall that this $\mathbf{c}$ encodes the Itô-Stratonovich transformation $\partial_t u = a(u)\xi - ca(u)a'(u)$).

Proposition 7 (Itô-Stratonovich formula)**.** *Let* Π *be a model of* (47) *of regularity* $\frac{1}{2}$ *with* $\mathbf{c_0} = 0$, *let* $\mathbf{c_1} = cz^1_0 z^1_1$. *We have the relationship*

$$[p \circ T_{\mathbf{c_1}} \circ i(\Pi)]_\beta = \Pi_\beta + \sum_{l \in \mathbb{N}^*} \frac{c^l}{l!} \sum_{\substack{a_1,\dots,a_l \in \mathbb{N} \\ b_1,\dots,b_l \in \mathbb{N}}} \prod_{j=1}^{l} \frac{a_j!(b_j+1)!}{(a_j+b_j)!} C(\gamma_1, (a_j+b_j)_j) \Pi_{\gamma_0,\gamma_1}. \tag{52}$$

With $\gamma_0 = \sum_{j=1}^{l} e^0_{a_j+b_j}$ $\gamma_1 = \beta - \sum_{j=1}^{l} e^1_{a_j} + e^1_{b_j+1}$ *and* $C(\gamma, (a_j+b_j)_j) = \prod_k \frac{\gamma(k)!}{(\gamma(k)-\#\{j|a_j+b_j=k\})!}$ *being some combinatorial constant.*

Even though this expression is not very enlightening, it has an heuristic explanation, since the operation $(\beta) \to (e_{a+b}, \beta - (e_a + e_{b+1}))$ is the multi-index counter-part of the similar operation consisting in collapsing an edge to a single point in the tree setting, where a similar formula exists [Bru+19, Proposition 22].

One can write similar formulas for $\mathbf{c}$ homogenous of higher degree and more noise (we recall that for $n \geqslant 4$, it does not make sense with $d \geqslant 2$ noises, but the behavior in the one-noise case is still interesting). However, they do not really help to understand $p \circ T_c$. Hence, we did not manage to fully understand $p \circ T_c$, we can however make the following direct remark (the first part being true without restrictions on $\mathbf{c}$).

Lemma 4. *For* $\mathbf{c}$ *homogenous of degree* n*,* $p \circ T_c : T_2 \to T_1$ *is a pre-Lie morphism, it is furthermore graded for the grading* $n|\beta_0| + |\beta_1|$ *on* T_2 *and* $|\beta|$ *on* T_1*.*

Of course, this does not fully describe $p \circ T_c$ in a satisfactory way. Hence, adding those new z_k^0 parameters to understand the algebraic behavior of renormalization is maybe not the best strategy to provide an algebraic description of renormalization, since it greatly increases the complexity of the model. It would probably have been more efficient to try to give such a description inside the algebra $\mathbb{R}[[z_k^1, c_\beta]]$ where we added the desired c_β to the parameters, however this is still work to do.

8 Conclusion

To conclude, this internship was an opportunity to discover stochastic and rough analysis through the example of multi-index regularity structures. We first proved that we can also apply those methods to SDE with a lot of simplifications thanks to some adaptations of a sewing lemma. Once we have those tools at our disposal, we can try to understand the specific behavior of noises of regularity α with $\frac{1}{\alpha} \in \mathbb{N}$. For some specific noise of regularity $\frac{1}{2n}$, we were able to interpret those behaviors, however several questions remain open. For instance, we lack an efficient algebraic description of this behavior. Even though it remains outside of the scope of this internship, we can also wonder if those results can help us to better understand our initial motivation, namely the SPDE case.

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