An excursion into noncommutative Riemannian geometry and its various incarnations

M1 Internship at Universiteit Leiden
Supervised by Francesca Arici and Bram Mesland

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1 Introduction

1.1 Personal experience

I carried out a five-month internship at the Mathematisch Instituut in Universiteit Leiden, Netherlands. I was under the supervision of Francesca Arici and Bram Mesland, working in noncommutative geometry (NCG). The subject was on noncommutative Riemannian geometry and quantum Levi-Civita connections, following recent work by Bram Mesland and Adam Rennie (visiting professor at Leiden from January to May 2024). There were three main stages in my internship, with some overlap: learning the basics and getting acquainted with previous work (one month); developing the core mathematical framework for solving the problem (three months); writing, in collaboration with Bram Mesland and Adam Rennie, a paper detailing our results (two months). Our paper will soon be on arXiv.

During my internship I mostly interacted (mathematically) with my supervisors, Francesca Arici and Bram Mesland, and with Adam Rennie. I was in daily contact with the other members of the NCG team at Leiden, from whom I learned much about life as a mathematician and the challenges and rewards of academia: PhD students Yufan Ge, Yuezhao Li, Jack Thelin af Ekenstam and Torstein Ulsnaes; postdoctoral fellow Dimitris Gerontogiannis. Other mathematicians visited the team for short periods of time, often giving a talk at the team's weekly colloquium. Finally, I participated in two mathematical events: the first was the Dutch Mathematical Congress, a two-day gathering of mathematicians from across the country, and the second was a one-week conference titled "Group operator algebras and noncommutative geometry" at CIRM, Marseille. The latter especially was a fantastic experience, as I was introduced to many mathematicians in the field and had the opportunity to participate for the first time in a "professional" mathematical conference.

Acknowledgements: I want to thank heartily Francesca Arici and Bram Mesland for taking me on during an already busy semester for the Leiden team. I thank Adam Rennie, for having tutored me and offered me so much of his time, despite not being officially my supervisor. I thank Cyril Houdayer for helping me to find the internship. Finally, I thank the whole NCG team for having so fully and immediately accepted me into their midst.

1.2 Elements of noncommutative Riemannian geometry

1.2.1 General noncommutative geometry

Noncommutative geometry (NCG) emerged in the 1980s as a subbranch of the study of operator algebras, principally with the works of Connes on cyclic cohomology and the Connes-Chern character [Conn85]. It bloomed in the 1990s with the introduction of spectral triples [Conn94] and the application of Kasparov's KK-theory to the Baum-Connes conjecture [Laf02], and has been an ever-expanding area of research since. In some ways, NCG is more a mathematical philosophy than a single coherent theory or discipline. It takes its roots in the Gelfand-Naimark and Serre-Swan theorems, which we describe briefly here in order to make clearer what exactly NCG is about.

Let A be a C^*-algebra, i.e. an algebra with an involution that mimicks the adjoint of operators on a Hilbert space, and which is complete with respect to a certain norm. The Gelfand-Naimark theorem [Conw90] states that if A is commutative, then there is a locally compact Hausdorff space X, called the spectrum of A, such that A is isometrically isometric to $C_0(X)$, the C^*-algebra of complex-valued functions on X that vanish at infinity. So commutative C^*-algebras correspond exactly to locally compact Hausdorff topological spaces, and one may thus be tempted to call noncommutative C^*-algebras "noncommutative topological spaces". It turns out that this is indeed a fruitful point of view (see [GBFV00] and references therein).

A similar idea emerges when studying vector bundles over smooth manifolds. Let M be a smooth real manifold and E a smooth complex vector bundle over M. The smooth sections $\Gamma^\infty(E)$ of E form a finitely generated and projective module over the algebra $C^\infty(M)$ of smooth complex-valued functions on M. Conversely, the Serre-Swan theorem [Ser55, Swa62] states, in its differential form, that any finitely generated and projective module over $C^\infty(M)$ is of the form $\Gamma^\infty(E)$ for some vector bundle E. Hence one could view finitely generated and projective modules over a noncommutative algebra $\mathcal{A}$ as "noncommutative vector bundles".

An important question concerns the motivations and applications of noncommutative geometry. Why extend geometrical concepts in this way, despite the fact that noncommutative algebras do not represent "true" geometric spaces? An important motivation comes from theoretical physics, specifically quantum gravity. Mathematically, the problem of adapting general relativity to the framework of quantum field theory may be restated in the following way: how to incorporate a (pseudo-)Riemannian formalism into a theory based on noncommuting operators? NCG has been suggested as a possible approach, at least at the mathematical level. There are also many applications of NCG to index theory, foliation theory, condensed matter physics and number theory. We refer to [Car+14] for a more complete exposition. Note that, due to the proximity between NCG and quantum theory, the terms "quantum" and "noncommutative" are often used interchangeably in the literature (and conversely "classical" means "commutative" or "geometrical in the usual sense"). We will be no different.

1.2.2 The Riemannian case

Let us now specialise to the case of noncommutative (or quantum) Riemannian geometry, i.e. trying to extend the theory of Riemannian or pseudo-Riemannian manifolds to noncommutative algebras. Many frameworks have been proposed [AW17a, BGM20, LNW94, LRZ09, Ros13], based on noncommutative extensions of various aspects of Riemannian geometry. For example, vector fields on a smooth manifold M are, according to the Serre-Swan theorem, exactly described by finitely generated and projective modules over the algebra $C^\infty(M)$. But they are also in one-to-one correspondence with derivations on $C^\infty(M)$, i.e. maps $\Delta : C^\infty(M) \to C^\infty(M)$ that satisfy $\Delta(fg) = f\Delta(g) + \Delta(f)g$ for all $f, g \in C^\infty(M)$. Derivations on $C^\infty(M)$ form a Lie algebra, so vector fields on M inherit a Lie algebra structure. On a noncommutative algebra $\mathcal{A}$ one can still define a Lie algebra of derivations, but these no longer form an $\mathcal{A}$-module in general. Conversely, $\mathcal{A}$-modules cannot usually be given a Lie algebra structure in a canonical way. So there is an ambiguity for what the "correct" definition of noncommutative vector fields should be.

One can identify three main directions for noncommutative Riemannian geometry: heat kernel expansions, derivations, and noncommutative differential forms. The first attempts only to define a notion of scalar curvature by expanding some noncommutative version of the heat kernel $e^{-t\Delta}$, where Δ is the Laplace-Beltrami operator on a Riemannian manifold (M, g). This strategy has seen many successes [CM14, FK13, FK15], but falls short of providing a full formalism for noncommutative Riemannian geometry, in particular the Riemann and Ricci curvature tensors. The second approach uses derivations to define a notion of noncommutative vector fields. This is the route taken in [Ros13, AW17a, AW17b]. Finally, the third approach uses an abstract notion of a "differential calculus", with a bimodule Ω^1 over a noncommutative algebra mimicking the bimodule of differential 1-forms on a manifold, and a derivation $\mathrm{d} : \mathcal{A} \to \Omega^1$ playing the role of the exterior derivative on functions. This is the formalism developed in [Conn94, Lan97, BM20, BGM20] and in the latest papers by Mesland-Rennie [MR24a, MR24b, MR24c].

One would like to compare these various formalisms, to see if they are in some way equivalent or *a contrario* fundamentally different. Since the strategy using heat kernel expansions really differs in its aim and scope from the other two, we are mainly interested in comparing the formalisms of bimodule-based noncommutative differential forms and derivation-based noncommutative vector fields. During the internship we endeavoured to relate the theories developed in [AW17a] (based on derivations) and [MR24a] (based on bimodules). Our conclusion is as follows: in the case where the bimodule Ω^1_{d} of noncommutative differential forms is centred over $\mathcal{A}$, i.e. it is generated by its centre, we show that the theory developed in [AW17a] is dual to that of [MR24a].

In this exposition we will not be able to give a detailed account of both theories or of their relations here. We will focus on exposing the important ideas and results from [MR24a], which already provide a detailed description of the kind of advances made by noncommutative Riemannian geometers.

2 Basic concepts and definitions

The fundamental building blocks of our theories are $*$-algebras and inner product modules. They will respectively play the roles of smooth functions and of differential forms on a Riemannian manifold.

We start by recalling some terminology. Given a field $\mathbb{K}$, a $\mathbb{K}$-algebra $\mathcal{A}$ is a $\mathbb{K}$-vector space with a compatible ring structure. An algebra is called unital if it has a multiplicative unit $\mathbb{1}$. A right (resp. left) $\mathcal{A}$-module $\mathcal{M}$ is a $\mathbb{K}$-vector space with a $\mathbb{K}$-linear right (resp. left) action of $\mathcal{A}$. If $\mathcal{M}$ is a (right or left) $\mathcal{A}$-module with a basis $\{x_1, \dots, x_n\}$, i.e. each element $m \in \mathcal{M}$ can be written in a unique way as an $\mathcal{A}$-linear combination of the elements $x_i \in \mathcal{M}$, then we say that $\mathcal{M}$ is free with rank n (all bases then having cardinality n). If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathbb{K}$-algebras, an $\mathcal{A}$-$\mathcal{B}$-bimodule is a $\mathbb{K}$-vector space with commuting left and right actions of $\mathcal{A}$ and $\mathcal{B}$ respectively. When $\mathcal{A}$ and $\mathcal{B}$ are the same we simplify the notation, and write $\mathcal{A}$-bimodule instead of $\mathcal{A}$-$\mathcal{A}$-bimodule. We define the centre of an algebra $\mathcal{A}$, resp. of an $\mathcal{A}$-bimodule $\mathcal{M}$, to be $\mathcal{Z}(\mathcal{A}) := \{b \in \mathcal{A} \mid \forall a \in \mathcal{A}, ab = ba\}$, resp. $\mathcal{Z}(\mathcal{M}) := \{x \in \mathcal{M} \mid \forall a \in \mathcal{A}, ax = xa\}$. An algebra $\mathcal{A}$ is called commutative or abelian if $\mathcal{Z}(\mathcal{A}) = \mathcal{A}$, and an $\mathcal{A}$-bimodule $\mathcal{M}$ is called central if $\mathcal{Z}(\mathcal{M}) = \mathcal{M}$.

Morally, going from fields and vector spaces to algebras and modules is the same as going from geometry over a single point to geometry over a manifold. Indeed, the $\mathbb{C}$-algebra of functions on a point is isomorphic to $\mathbb{C}$, and a complex vector bundle over a point is just a complex vector space.

2.1 Involutive algebras

In the abstract or noncommutative context, complex structures are better behaved both algebraically and analytically than real structures. For example, there is no analogue of the Gelfand theorem for real algebras. For this reason we use *complex* algebras as a basis for generalising *real* geometry. In order to then recover real objects (real functions, real differential forms etc.), we introduce an abstract notion of complex conjugation on our algebras.

Definition 2.1. A *$*$-algebra* $\mathcal{A}$ is an algebra over $\mathbb{C}$ with an involution $* : \mathcal{A} \to \mathcal{A}$ satisfying

$$(a + \lambda b)^* = a^* + \overline{\lambda} b^* \qquad (ab)^* = b^* a^*$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We call a^* the *adjoint* of a.

Definition 2.2. A *pre-C^*-algebra* A is a $*$-algebra with a norm $\|\cdot\|$ such that $\|a^* a\| = \|a\|^2$ for all $a \in A$. Such a norm is called a *C^*-norm*.

If A is complete with respect to $\|\cdot\|$, we say that A is a C^*-algebra.

Example 2.3. The complex field $\mathbb{C}$ is a commutative unital C^*-algebra, where adjoints are complex conjugates and the norm is given by the absolute value.

Example 2.4. Let M be a smooth compact manifold. The algebra $C^\infty(M)$ (resp. $C(M)$) of smooth (resp. continuous) complex-valued functions on M is a unital $*$-algebra (resp. a C^*-algebra), where involution is given by the pointwise complex conjugate and the norm on $C(M)$ is the supremum norm. In fact, $C(M)$ is the completion of $C^\infty(M)$ under this norm. The unit is the constant function 1. These algebras are commutative.

If M is not compact, then we can still define the algebras $C_0^\infty(M)$ and $C_0(M)$ of respectively smooth and continuous functions that vanish at infinity, i.e. all f such that for any $\varepsilon > 0$, there exists a compact subset $K \subset M$ with $|f(x)| < \varepsilon$ for all $x \notin K$. On these algebras the supremum norm is well-defined and the above relations hold. They are however non-unital, since the constant function 1 does not vanish at infinity.

The above example shows how, in noncommutative geometry, unitality plays the role of compactness. Since non-unital algebras can be significantly more difficult to handle and lead to many technicalities, for simplicity we make the following assumption.

Assumption 2.5. Throughout our exposition, all algebras are unital and all geometric spaces are compact, unless specified otherwise.

Example 2.6. The complex $N \times N$ matrices $M_N(\mathbb{C})$ form the first natural examples of noncommutative C^*-algebras. More generally, if $\mathcal{H}$ is a Hilbert space then the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ forms a C^*-algebra with the operator norm and the $*$ operation given by the adjoint T^* such that $\langle \psi \mid T\xi \rangle_{\mathcal{H}} = \langle T^* \psi \mid \xi \rangle_{\mathcal{H}}$ for all $\psi, \xi \in \mathcal{H}$. These algebras are unital with unit the identity operator.

Example 2.7. Let $\mathcal{A}$ be a unital $*$-algebra and $(\mathcal{H}, \pi)$ a $*$-representation of $\mathcal{A}$, i.e. $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an algebra map such that $\pi(a^*) = \pi(a)^*$ and $\pi(\mathbb{1}) = \mathrm{Id}$. For any $a \in \mathcal{A}$, set $\|a\| := \|\pi(a)\|_{\mathrm{op}}$. Since the operator norm is a C^*-norm and π is a $*$-algebra map, then $(\mathcal{A}, \|\cdot\|)$ is a pre-C^*-algebra. Denoting $A := \overline{\mathcal{A}}$ its completion for this norm, A is then a C^*-algebra.

Example 2.4 is a special case of this construction when M is a Riemannian manifold with metric g. Indeed, $\pi : C^\infty(M) \to \mathcal{B}(L^2(M, g))$, $(\pi(f)\xi)(x) := f(x)\xi(x)$ is a $*$-representation and $\|\pi(f)\|_{\mathrm{op}} = \sup\{|f(x)|,\ x \in M\}$. So $C(M)$ is the completion of $C^\infty(M)$ under the norm induced by π.

Remark 2.8. An important theorem, called the Gelfand-Naimark-Segal theorem or Gelfand-Naimark representation theorem, states that any C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. So, in principle at least, one could restrict oneself to study only C^*-algebras of operators. In reality, as for many other algebraic structures, it is often easier to consider C^*-algebras as abstract objects with a representation theory.

Definition 2.9. Let $\mathcal{A}$ be a $*$-algebra and $a \in \mathcal{A}$. We say that a is *hermitian* (or *real*) if $a^* = a$. We say that a is *positive* and write $a \geq 0$ if there exists $b \in \mathcal{A}$ such that $a = b^*b$. The set of positive elements in $\mathcal{A}$ forms a convex cone in $\mathcal{A}$ which we denote by $\mathcal{A}^+$.

Remark 2.10. Since for all $b \in \mathcal{A}$, $(b^*b)^* = b^*b$, then positive elements in $\mathcal{A}$ are also hermitian. In fact, as for operators on a Hilbert space one can define the spectrum of an element a in a $*$-algebra, and then a is positive if and only if it is hermitian and has nonnegative spectrum (the spectrum of a hermitian element is contained in the real line by the spectral theorem).

Remark 2.11. If $\mathcal{A}$ is a commutative $*$-algebra, then the hermitian elements of $\mathcal{A}$ form an $\mathbb{R}$-algebra since a product of two hermitian elements a and b is still hermitian: $(ab)^* = b^*a^* = ba = ab$. However, when $\mathcal{A}$ is noncommutative this is not the case. This is why we are forced to work with hermitian elements in $*$-algebras instead of just considering real algebras.

Example 2.12. The hermitian elements of $C^\infty(M)$ or $C(M)$ are the (smooth or continuous) real-valued functions on M. The positive elements are the nonnegative functions.

Example 2.13. The hermitian elements of $M_N(\mathbb{C})$ are the hermitian, i.e. conjugate symmetric, matrices. These matrices are diagonalisable by the spectral theorem, and the positive elements are those with only nonnegative eigenvalues.

2.2 Hilbert modules

Using $*$-algebras we are able to define "abstract" or "noncommutative" algebras of functions. We wish to do the same for vector bundles. Guided by the Serre-Swan theorem, it seems natural to define noncommutative vector bundles as modules over noncommutative algebras. Since we want to deal with Riemannian structures, we additionally need to assume the existence of inner products on our modules.

2.2.1 Inner product modules, completions

Definition 2.14. Let $\mathcal{A}$ be a $*$-algebra. A right *inner product module* (or *right pre-C^*-module*) $\overrightarrow{\mathcal{M}}$ over $\mathcal{A}$ is a right $\mathcal{A}$-module with a right inner product, i.e. a map $\langle\cdot|\cdot\rangle_{\mathcal{A}} : \overrightarrow{\mathcal{M}} \times \overrightarrow{\mathcal{M}} \to \mathcal{A}$ satisfying

1. $\langle x \mid y\rangle_{\mathcal{A}}^* = \langle y \mid x\rangle_{\mathcal{A}}$,
2. $\langle x \mid x\rangle_{\mathcal{A}} \geq 0$ and $\langle x \mid x\rangle_{\mathcal{A}} = 0$ iff $x = 0$,
3. $\langle xa \mid yb\rangle_{\mathcal{A}} = a^* \langle x \mid y\rangle_{\mathcal{A}}\, b$,
4. $\langle ax \mid y\rangle_{\mathcal{A}} = \langle x \mid a^*y\rangle_{\mathcal{A}}$,

for all $x, y \in \overrightarrow{\mathcal{M}}$ and $a, b \in \mathcal{A}$. A left inner product module (or left pre-C^*-module) $\overleftarrow{\mathcal{M}}$ over $\mathcal{A}$ is a left $\mathcal{A}$-module with a left inner product, i.e. a map ${}_{\mathcal{A}}\langle\cdot|\cdot\rangle : \overleftarrow{\mathcal{M}} \times \overleftarrow{\mathcal{M}} \to \mathcal{A}$ satisfying conditions 1. and 2. above, as well as ${}_{\mathcal{A}}\langle ax \mid by\rangle = a\, {}_{\mathcal{A}}\langle x \mid y\rangle b^*$ and ${}_{\mathcal{A}}\langle x \mid ya\rangle = {}_{\mathcal{A}}\langle xa^* \mid y\rangle$ instead of 3. and 4.

Remark 2.15. With our conventions, right inner products are $\mathbb{C}$-linear in the second argument and anti-linear in the first, while the opposite is true of left inner products.

Proposition 2.16. *Let A be a C^*-algebra and $\overrightarrow{\mathcal{M}}$ be a right inner product module over A. For all $x, y \in \overrightarrow{\mathcal{M}}$, an analogue to the Cauchy-Schwarz inequality holds: $\langle x \mid y\rangle_A \langle y \mid x\rangle_A \leq \| \langle y \mid y\rangle_A \| \langle x \mid x\rangle_A$. As a consequence, $\|x\| := \| \langle x \mid x\rangle_A \|^{\frac{1}{2}}$ defines a norm on $\overrightarrow{\mathcal{M}}$, called the Euclidean norm induced by $\langle\cdot|\cdot\rangle$. Analogue statements hold for left inner product modules over A.*

Definition 2.17. Let A be a C^*-algebra. A right (resp. left) *Hilbert C^*-module* over A, or *Hilbert A-module*, is a right (resp. left) inner product A-module which is complete with respect to the norm induced by the inner product.

Example 2.18. A right or left $\mathbb{C}$-module is just a complex vector space V. A Hilbert $C*$-module over $\mathbb{C}$ is a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$.

Example 2.19. Let M be a smooth compact real manifold and E a smooth complex vector bundle over M. We denote by $\Gamma^\infty(E)$ the set of smooth sections of E, i.e. smooth maps $u : M \to E$ such that $u(x) \in E_x$, $x \in M$. Then $\Gamma^\infty(E)$ can be endowed with the structure of both a right and a left module over the $*$-algebra $C^\infty(M)$. The vector space structure is clear, and right/left multiplication by functions is defined pointwise: if $f \in C^\infty(M)$ and $u \in \Gamma^\infty(E)$ then $(fu)(x) := f(x)u(x)$ and $(uf)(x) := u(x)f(x)$, $x \in M$. Suppose furthermore that E is equipped with a hermitian Riemannian metric, i.e. a smooth family of maps $g_x : E_x \times E_x \to \mathbb{C}$, $x \in M$ such that g_x is $\mathbb{C}$-linear in the second argument, hermitian and positive-definite. Then the family $(g_x)_{x\in M}$ induces a map $g : \Gamma^\infty(E) \times \Gamma^\infty(E) \to C^\infty(M)$ by $g(u, v)(x) := g_x(u(x), v(x))$. It is straightforward to check that $(\Gamma^\infty(E), g)$ is both a right and a left pre-C^*-module over $C^\infty(M)$, since all operations are performed fibrewise. The continuous sections $\Gamma(E)$ of E then form (by completion) both a right and a left Hilbert C^*-module over the C^*-algebra $C(M)$.

If the vector bundle E is trivial with dimension n, i.e. it is isomorphic to the product $M \times \mathbb{C}^n$, then the $C^\infty(M)$-module $\Gamma^\infty(E)$ is free with rank n. In the general case, however, the module is not free and it is not so straightforward to relate it to the dimension of M.

Example 2.20. The space of N-dimensional column (resp. row) vectors is a left (resp. right) $M_N(\mathbb{C})$-module by standard multiplication of matrices. It is generated by any non-zero vector X since span$\{MX,\ M \in M_N(\mathbb{C})\} = \mathbb{C}^N$, but is not free because we can have $MX = 0$ with M non-zero. Define ${}_{M_N(\mathbb{C})}\langle X \mid Y\rangle := XY^* \in M_N(\mathbb{C})$ for column vectors X, Y, where as usual $*$ means conjugate transpose. For all X, Y and $M, N \in M_N(\mathbb{C})$,

1. ${}_{M_N(\mathbb{C})}\langle X \mid Y\rangle^* = (XY^*)^* = YX^* = {}_{M_N(\mathbb{C})}\langle Y \mid X\rangle$;
2. ${}_{M_N(\mathbb{C})}\langle X \mid X\rangle = XX^*$ is a hermitian matrix with nonnegative spectrum, and which is non-zero whenever $X \neq 0$;
3. ${}_{M_N(\mathbb{C})}\langle MX \mid NY\rangle = MX(NY)^* = MXY^*N^* = M\, {}_{M_N(\mathbb{C})}\langle X \mid Y\rangle N^*$.

C^N is clearly complete with respect to the associated Euclidean norm, so the pair $(C^N, {}_{M_N(\mathbb{C})}\langle\cdot|\cdot\rangle)$ is a left Hilbert C^*-module over the (noncommutative) C^*-algebra $M_N(\mathbb{C})$.

2.2.2 Frames

It is not in general possible to find a basis for a module. However, on inner product modules there is a looser notion of a *frame* which is useful for computational purposes.

Definition 2.21. Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ a (right) inner product $\mathcal{A}$-module. A (right) *frame* on $\mathcal{M}$ is a countable collection of elements $v := (m_i) \subset \mathcal{M}$ such that for all $m \in \mathcal{M}$ we have

$$m = \sum_j m_j \left\langle m_j \mid m \right\rangle_{\mathcal{A}}.$$

An analogous definition exists for left inner products. It is an important fact that frames always exist on countably generated Hilbert C^*-modules and that a Hilbert C^*-module is finitely generated if and only if it contains finite frames [RW98].

2.2.3 †-bimodules

Example 2.19 shows how one might extend classical geometry in several inequivalent ways to the noncommutative world. The sections of a vector bundle can be viewed as a right $C^\infty(M)$-module, a left $C^\infty(M)$-module or a $C^\infty(M)$-bimodule. Classically, considering only the right module structure is enough because the right and left actions of functions on sections are identical: $fu = uf$. As a bimodule, this means that $\Gamma^\infty(E)$ is central. However, central bimodules over noncommutative algebras behave very badly. Indeed, if $\mathcal{M}$ is a central $\mathcal{A}$-bimodule, then for all $a, b \in \mathcal{A}$ and $x \in \mathcal{M}$, $ax \in \mathcal{M}$ hence $abx = (ax)b = b(ax) = bax$. If for instance $\mathcal{A}$ is generated by commutators $[a, b] := ab - ba$, then the action of $\mathcal{A}$ on $\mathcal{M}$ is identically zero. It is therefore not reasonable to consider identical left and right module actions of noncommutative algebras, motivating us to work explicitly with general bimodules which have more structure than simply right or left modules.

In the bimodule case, it is natural to require more compatibility between the $*$ operation and the bimodule structure, and with inner products when they arise. We warn that definitions here vary in names and exact assumptions between authors.

Definition 2.22. Let $\mathcal{A}$ be a $*$-algebra. A *†-bimodule* over $\mathcal{A}$ is a triple $\left(\mathcal{M}, \dagger, \langle\cdot|\cdot\rangle_{\mathcal{A}}\right)$ where

1. $\mathcal{M}$ is an $\mathcal{A}$-bimodule,
2. $\dagger : \mathcal{M} \to \mathcal{M}$ is an $\mathbb{R}$-linear involution such that $(axb)^\dagger = b^* x^\dagger a^*$,
3. $\langle\cdot|\cdot\rangle_{\mathcal{A}} : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ is a right inner product such that $\left(\mathcal{M}, \langle\cdot|\cdot\rangle_{\mathcal{A}}\right)$ is a right inner product $\mathcal{A}$-module and for all $x, y \in \mathcal{M}$, $a \in \mathcal{A}$, we have $\langle ax \mid y\rangle_{\mathcal{A}} = \langle x \mid a^* y\rangle$.

If $\left(\mathcal{M}, \dagger, \langle\cdot|\cdot\rangle_{\mathcal{A}}\right)$ is a †-bimodule over $\mathcal{A}$, then for all $a, b \in \mathcal{A}$ and $x, y \in \mathcal{M}$, $\left\langle (ax)^\dagger \mid (by)^\dagger \right\rangle_{\mathcal{A}} = \left\langle x^\dagger a^* \mid y^\dagger b^* \right\rangle_{\mathcal{A}} = a \left\langle x^\dagger \mid y^\dagger \right\rangle_{\mathcal{A}} b^*$. So setting ${}_{\mathcal{A}}\langle x \mid y\rangle := \left\langle x^\dagger \mid y^\dagger \right\rangle_{\mathcal{A}}$ defines a left inner product on $\mathcal{M}$. This is why in the definition we only assume existence of a right inner product.

Definition 2.23. Let A be a C^*-algebra. A *Hilbert †-bimodule* over A is a †-A-bimodule which is complete with respect to the Euclidean norm induced by the right inner product.

Example 2.24. A separable Hilbert space $\mathcal{H}$ is naturally a central Hilbert †-bimodule over $\mathbb{C}$, where the dagger is coordinate-wise complex conjugation (identifying $\mathcal{H}$ with $l^2(\mathbb{N})$).

Example 2.25. With the notations of Example 2.19, $\Gamma^\infty(E)$ is a central †-bimodule over $C^\infty(M)$ where $u^\dagger(x) := \overline{u(x)}$, $u \in \Gamma^\infty(E)$. In local coordinates, one can write the right inner product as $g(u, v) = g^{\mu\nu} \overline{u}_\mu v_\nu$ using Einstein notation (summing over repeated indices), where $g^{\mu\nu}$ are locally defined real-valued functions on M such that $g^{\nu\mu} = g^{\mu\nu}$. Then the left inner product is $g(u^\dagger, v^\dagger) = g^{\mu\nu} u_\mu \overline{v}_\nu$. By completion $\Gamma(E)$ is a central Hilbert †-bimodule over $C(M)$.

2.2.4 Tensor products

If inner product modules constitute a good generalisation of classical vector fields or differential forms, it should be possible to construct generalised higher-order covariant or contravariant tensors, i.e. to take tensor powers of inner product modules. This is easily done using the $(\dagger)$-bimodule formalism.

Definition 2.26. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be $\mathbb{K}$-algebras. Let $\mathcal{M}$ and $\mathcal{N}$ be respectively an $\mathcal{A}$-$\mathcal{B}$-bimodule and a $\mathcal{B}$-$\mathcal{C}$-bimodule. Then the (algebraic) *tensor product* of $\mathcal{M}$ and $\mathcal{N}$ over the algebra $\mathcal{B}$ is defined as the quotient of the vector space tensor product $\mathcal{M} \otimes \mathcal{N}$ by the subspace $\{mb \otimes n - m \otimes bn,\ m, n \in \mathcal{M}, b \in \mathcal{B}\}$. We denote this quotient by $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$ and see immediately that it is spanned by elements of the form $m \otimes_{\mathcal{B}} n$, where by definition $mb \otimes_{\mathcal{B}} n = m \otimes_{\mathcal{B}} bn$. The tensor product is naturally equipped with the structure of an $\mathcal{A}$-$\mathcal{C}$-bimodule given by $a(m \otimes_{\mathcal{B}} n)c = am \otimes_{\mathcal{B}} nc$.

This definition is quite general, but we will immediately restrict ourselves to the case of $\dagger$-bimodules since those will be the structures of interest in the rest of our exposition. Recall that by $\mathcal{A}$-bimodule we mean an $\mathcal{A}$-$\mathcal{A}$-bimodule.

Definition 2.27. Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}, \mathcal{N}$ two $\dagger$-$\mathcal{A}$-bimodules. We define a $\dagger$-$\mathcal{A}$-bimodule structure on the algebraic tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ by the following:

$$(m \otimes_{\mathcal{A}} n)^\dagger := n^\dagger \otimes_{\mathcal{A}} m^\dagger \qquad \langle m_1 \otimes_{\mathcal{A}} n_1 \mid m_2 \otimes_{\mathcal{A}} n_2 \rangle_{\mathcal{A}} := \langle n_1 \mid \langle m_1 \mid m_2 \rangle_{\mathcal{A}} n_2 \rangle_{\mathcal{A}}.$$

Proof. We check that these are well-defined operations on the quotient $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ and that the conditions of Definition 2.22 hold. First, for all $a, b \in \mathcal{A}$ we have

$$(ma \otimes_{\mathcal{A}} n)^\dagger = n^\dagger \otimes_{\mathcal{A}} a^* m^\dagger = n^\dagger a^* \otimes_{\mathcal{A}} m^\dagger = (m \otimes_{\mathcal{A}} an)^\dagger$$

so the dagger is well-defined, and

$$(am \otimes_{\mathcal{A}} nb)^\dagger = b^* n^\dagger \otimes_{\mathcal{A}} m^\dagger a^* = b^* (m \otimes_{\mathcal{A}} n)^\dagger a^*.$$

Second,

$$\begin{aligned}\langle m_1 a \otimes_{\mathcal{A}} n_1 \mid m_2 b \otimes_{\mathcal{A}} n_2 \rangle_{\mathcal{A}} &= \langle n_1 \mid \langle m_1 a \mid m_2 b \rangle_{\mathcal{A}} n_2 \rangle_{\mathcal{A}} = \langle n_1 \mid a^* \langle m_1 \mid m_2 \rangle_{\mathcal{A}} b n_2 \rangle_{\mathcal{A}} \\ &= \langle a n_1 \mid \langle m_1 \mid m_2 \rangle_{\mathcal{A}} b n_2 \rangle_{\mathcal{A}}\end{aligned}$$

so the inner product is well-defined, and

$$\langle m_1 \otimes_{\mathcal{A}} n_1 a \mid m_2 \otimes_{\mathcal{A}} n_2 b \rangle_{\mathcal{A}} = \langle n_1 a \mid \langle m_1 \mid m_2 \rangle_{\mathcal{A}} n_2 b \rangle_{\mathcal{A}} = a^* \langle m_1 \otimes_{\mathcal{A}} n_1 \mid m_2 \otimes_{\mathcal{A}} n_2 \rangle_{\mathcal{A}} b.$$

□

Remark 2.28. If A is a C^*-algebra and M, N are Hilbert $\dagger$-bimodules, then there is a way to complete the algebraic tensor product $M \otimes_A N$ into a Hilbert $\dagger$-bimodule. We do not give the precise construction here.

3 Differential calculi and spectral triples

Involutive algebras and Hilbert modules are a good starting point for noncommutative Riemannian geometry. There is, however, an important element that is yet missing: a notion of derivation. Indeed, it is a crucial fact that tangent vector fields on manifolds act as derivations on smooth scalar-valued functions on the manifold, while differential forms are given their full structure once one takes into account the differential $\mathrm{d} : C^\infty(M) \to \Gamma^\infty(T^*M)$ and its higher-order extensions, which are (graded) derivations. In this section, we explain how one may define an abstract notion of a differential map d and thus try to extend differential forms to noncommutative algebras.

Classical references for this section are [Conn94], [Lan97], [GBFV00]. We mostly follow the modern exposition found in [BM20] and introduce the concepts laid out in [MR24a].

3.1 First-order differential calculi

Definition 3.1. [BM20, Definition 1.4.] Let $\mathcal{A}$ be a complex algebra. A *first-order differential calculus* over $\mathcal{A}$ is a pair $(\Omega^1_{\mathrm{d}}, \mathrm{d})$ with

1. Ω^1_{d} an $\mathcal{A}$-bimodule;
2. $\mathrm{d} : \mathcal{A} \to \Omega^1_{\mathrm{d}}$ a derivation, i.e. a $\mathbb{C}$-linear map satisfying $\mathrm{d}(ab) = \mathrm{d}(a)\, b + a\, \mathrm{d}(b)$, $a, b \in \mathcal{A}$;
3. $\Omega^1_{\mathrm{d}} = \mathrm{span}_{\mathbb{C}}\{a\mathrm{d}b,\ a, b \in \mathcal{A}\}$.

If the calculus also satisfies $\operatorname{Ker} \mathrm{d} = \mathbb{C} \cdot \mathbb{1}$, we say it is *connected.* We call d the *exterior derivative* or *first-order differential*, and an element of Ω^1_{d} is a *differential 1-form.*

Definition 3.2. Let $\mathcal{A}$ be a $*$-algebra. A *first order differential* $\dagger$*-calculus* over $\mathcal{A}$ is a first order differential calculus $(\Omega^1_{\mathrm{d}}, \mathrm{d})$ with an involution $\dagger : \Omega^1_{\mathrm{d}} \to \Omega^1_{\mathrm{d}}$ such that $(\Omega^1_{\mathrm{d}}, \dagger)$ is a $\dagger$-$\mathcal{A}$-bimodule and $d(a^*) = -d(a)^\dagger$, $a \in \mathcal{A}$.

The latter concept is called a $*$-calculus in [BM20], but we call it a $\dagger$-calculus to be more in line with the terminology of $\dagger$-bimodules in [MR24a]. The choice $\mathrm{d}(a^*) = -\mathrm{d}(a)^\dagger$ is conventional and many authors require $\mathrm{d}(a^*) = \mathrm{d}(a)^\dagger$ instead. We adopt the convention that fits most naturally with the spectral triple approach (see Section 3.2.), but of course setting $\mathrm{d}'(\cdot) := i\mathrm{d}(\cdot)$ yields $\mathrm{d}'(a^*) = -i\mathrm{d}(a)^\dagger = \mathrm{d}'(a)^\dagger$.

Remark 3.3. The above definitions extend classical differential geometry, even in the case where $\mathcal{A}$ is commutative. Indeed, from the point of view developed here, classical geometry is the study of central calculi over commutative algebras. But even over commutative algebras we may have calculi with non-central bimodules, meaning there is a distinction between right and left multiplication by functions. Indeed, in Example 3.5 below, we see that interesting non-central calculi exist over (commutative) geometric spaces.

Example 3.4. Let M be a smooth compact manifold and $\Gamma^\infty(T^*M)$ the space of smooth *complexified* differential 1-forms over M. Formally $\Gamma^\infty(T^*M) := \Gamma^\infty_{\mathbb{R}}(T^*M) \otimes_{\mathbb{R}} \mathbb{C}$. According to Example 2.25, letting $\dagger$ be pointwise complex conjugation on $\Gamma^\infty(T^*M)$, we have that $\big(\Gamma^\infty(T^*M), \dagger\big)$ is a $\dagger$-$C^\infty(M)$-bimodule. The classical differential is a derivation $\mathrm{d} : C^\infty_{\mathbb{R}}(M) \to \Gamma^\infty_{\mathbb{R}}(T^*M)$, which is extended to $\mathrm{d} : C^\infty(M) \to \Gamma^\infty(T^*M)$ by setting $\mathrm{d}\big(\overline{f}\big) := -(\mathrm{d}f)^\dagger$. We thus obtain a first-order differential calculus over $C^\infty(M)$, since clearly $\Gamma^\infty(T^*M)$ is spanned by expressions of the form $f\mathrm{d}h$ with $f, h \in C^\infty(M)$.

Example 3.5. Consider $A := \mathbb{C} \oplus \mathbb{C}$ the C^*-algebra of functions on two points. Let Ω^1_{d} be the space of anti-diagonal 2×2 matrices

$$\Omega^1_{\mathrm{d}} := \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix},\ x, y \in \mathbb{C} \right\} \subset M_2(\mathbb{C})$$

and let A act from the right and the left on Ω^1_{d} by

$$(a_1, a_2) \cdot \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \cdot (b_1, b_2) := \begin{pmatrix} 0 & a_1 x b_2 \\ a_2 y b_1 & 0 \end{pmatrix}, \quad a_1, a_2, b_1, b_2, x, y \in \mathbb{C}.$$

Define a differential $\mathrm{d} : A \to \Omega^1_{\mathrm{d}}$ by

$$\mathrm{d}(a_1, a_2) := \begin{pmatrix} 0 & a_2 - a_1 \\ a_1 - a_2 & 0 \end{pmatrix}.$$

It is a derivation, since

$$(a_1, a_2)\mathrm{d}(b_1, b_2) = \begin{pmatrix} 0 & a_1(b_2 - b_1) \\ a_2(b_1 - b_2) & 0 \end{pmatrix}, \qquad \big[\mathrm{d}(b_1, b_2)\big](a_1, a_2) = \begin{pmatrix} 0 & a_2(b_2 - b_1) \\ a_1(b_1 - b_2) & 0 \end{pmatrix},$$

hence

$$
\begin{aligned}
(a_1,a_2)\mathrm{d}(b_1,b_2)+\big[\mathrm{d}(a_1,a_2)\big](b_1,b_2) &= \begin{pmatrix} 0 & a_1(b_2-b_1)+b_2(a_2-a_1) \\ a_2(b_1-b_2)+b_1(a_1-a_2) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_2b_2-a_1b_1 \\ a_1b_1-a_2b_2 & 0 \end{pmatrix} \\
&= \mathrm{d}\big[(a_1,a_2)\cdot(b_1,b_2)\big].
\end{aligned}
$$

The space spanned by 1-forms $a\mathrm{d}b$ is

$$
\operatorname{span}_{\mathbb{C}}\left\{\begin{pmatrix} 0 & a_1(b_2-b_1) \\ a_2(b_1-b_2) & 0 \end{pmatrix},\ a_1,a_2,b_1,b_2\in\mathbb{C}\right\} = \left\{\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix},\ x,y\in\mathbb{C}\right\} = \Omega^1_{\mathrm{d}}.
$$

Therefore $(\Omega^1_{\mathrm{d}}, \mathrm{d})$ is a first order differential calculus over A. It is connected, since $d(a_1,a_2)=0$ if and only if $a_1=a_2$, i.e. $(a_1,a_2)\in\mathbb{C}\cdot\mathbb{1}$. Letting $M^\dagger$ represent the conjugate transpose of a matrix M, we see that

$$
\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & \overline{y} \\ \overline{x} & 0 \end{pmatrix} \in \Omega^1_{\mathrm{d}}
$$

so $\dagger$ is an $\mathbb{R}$-linear involution on Ω^1_{d}. It satisfies

$$
\begin{aligned}
\left[(a_1,a_2)\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}(b_1,b_2)\right]^\dagger &= \begin{pmatrix} 0 & a_1xb_2 \\ a_2yb_1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & \overline{a_2yb_1} \\ \overline{a_1xb_2} & 0 \end{pmatrix} \\
&= (\overline{b}_1,\overline{b}_1)\begin{pmatrix} 0 & \overline{y} \\ \overline{x} & 0 \end{pmatrix}(\overline{a}_1,\overline{a}_2) = (b_1,b_2)^*\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}^\dagger (a_1,a_2)^*
\end{aligned}
$$

and

$$
\mathrm{d}\big[(a_1,a_2)^*\big] = \begin{pmatrix} 0 & \overline{a}_2-\overline{a}_1 \\ \overline{a}_1-\overline{a}_2 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \overline{a_1-a_2} \\ \overline{a_2-a_1} & 0 \end{pmatrix} = -\big[\mathrm{d}(a_1,a_2)\big]^\dagger.
$$

Hence $(\Omega^1_{\mathrm{d}}, \dagger)$ (the differential d implied) is a $\dagger$-calculus over A. We note that any 1-form can be written

$$
\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = (x,-y)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (x,-y)\mathrm{d}(1,0).
$$

So $\mathrm{d}(1,0)$ is a left basis for Ω^1_{d} (the above decomposition being clearly unique).

In the example above, the bimodule of 1-forms is free with rank 1 as a left module over the algebra. In the classical case, a manifold is called parallelisable whenever its cotangent module $\Gamma^\infty(T^*M)$ is free, in which case the rank of $\Gamma^\infty(T^*M)$ is the dimension of M. Therefore it makes sense to call the calculus above *left parallelisable with cotangent dimension 1*. It is in fact also right parallelisable with the same dimension. In general, however, a noncommutative calculus may be left parallelisable but not right parallelisable, or may have two distinct cotangent dimensions.

3.2 Spectral triples

The concept of a spectral triple is one of the best-known inventions of noncommutative geometry. It has been extensively studied in the last decades and is useful for many areas of NCG, including but not limited to K-homology, index theory and abstract pseudo-differential calculus [Car11]. A theorem of Connes [Conn94], extended later by Rennie and Varilly, states that spectral triples over commutative algebras satisfying some extra assumptions are exactly those associated to spin manifolds, and that all of the geometric structure of the manifold can be recovered from the spectral triple. These theorems justify the assertion that spectral triples constitute a form of "noncommutative spin geometry".

Spectral triples form in fact such a wide area of research, that we are forced in our presentation to make some strong choices. They are the following. We do not follow expositions given in most classical texts on the subject. We do not introduce spin geometry. We focus on algebraic rather than analytic properties. We explore neither homological nor index theoretic considerations. From our point of view, spectral triples will be nothing more than a useful tool for constructing noncommutative differential calculi.

Definition 3.6. A spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is a $*$-algebra with a $*$-representation of $\mathcal{A}$ by bounded operators on $\mathcal{H}$, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and D is a densely-defined self-adjoint operator on $\operatorname{dom} D \subset \mathcal{H}$ such that

1. $\pi(\mathcal{A}) \operatorname{dom} D \subset \operatorname{dom} D$;
2. the commutator $[D, \pi(a)]$ extends for all $a \in \mathcal{A}$ to a bounded operator on $\mathcal{H}$.

By abuse of notation we will usually omit the representation π and view $\mathcal{A}$ as a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, identifying it with its image $\pi(\mathcal{A})$.

Remark 3.7. The first condition $\mathcal{A} \operatorname{dom} D \subset \operatorname{dom} D$ (identifying $\mathcal{A}$ with $\pi(\mathcal{A})$) is technical and exists in order for the second to make sense. The core assumption is that $\|[D, a]\| < +\infty$, which as we will see below corresponds classically to the fact that D induces differentiation on functions.

Most authors also require that $(D + i)^{-1}$ is a compact operator, i.e. D has compact resolvent, and that $a(1 + D^2)^{1/2}$ is compact for all $a \in \mathcal{A}$. These assumptions ensure that D behaves "geometrically", as a noncommutative Dirac operator. Since they play no role in defining noncommutative differential calculi over a spectral triple, we omit them from our definition.

In all infinite-dimensional examples, D will be an unbounded operator (this is in fact necessary if one assumes D to have compact resolvent). Technical difficulties arise when dealing with unbounded operators; even defining self-adjointness requires some functional analysis. For this reason we do not give precise definitions in the unbounded case. Suffice it to say that in the finite-dimensional case D is a hermitian matrix, and in the infinite-dimensional case it is some analogue thereof. Also, in finite-dimensional examples conditions 1. and 2. above are always trivially satisfied.

Example 3.8. Let $A := \mathbb{C} \oplus \mathbb{C}$. We represent A on $\mathbb{C}^2$ by

$$(a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Let $D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $(A, \mathbb{C}^2, D)$ is a spectral triple.

Example 3.9. Let $\mathcal{A} := C^\infty(\mathbb{S}^1)$ be represented on $L^2(\mathbb{S}^1)$ by left multiplication: $(f \cdot \psi)(\theta) := f(\theta)\psi(\theta)$, $f \in C^\infty(\mathbb{S}^1)$, $\psi \in L^2(\mathbb{S}^1)$. Consider the (unbounded) differential operator $D := -i\frac{\mathrm{d}}{\mathrm{d}\theta}$ defined on the Sobolev space

$$H^1(\mathbb{S}^1) = \left\{ \psi \in L^2(\mathbb{S}^1) : \frac{\mathrm{d}}{\mathrm{d}\theta}\psi \in L^2(\mathbb{S}^1) \right\},$$

where the derivative is taken in the sense of distributions. Choosing a right-linear convention for the inner product $\langle \cdot | \cdot \rangle$ on $L^2(\mathbb{S}^1)$, we use integration by parts to show that

$$\langle \varphi \mid D\psi \rangle = \int_{\mathbb{S}^1} \overline{\varphi(\theta)} \left(-i \frac{\mathrm{d}}{\mathrm{d}\theta} \psi(\theta) \right) \mathrm{d}\theta = \int_{\mathbb{S}^1} \left(i \frac{\mathrm{d}}{\mathrm{d}\theta} \overline{\varphi(\theta)} \right) \psi(\theta) \mathrm{d}\theta = \langle D\varphi \mid \psi \rangle, \quad \varphi, \psi \in H^1(\mathbb{S}^1).$$

Thus D is symmetric. We will accept without proof that $H^1(\mathbb{S}^1)$ is dense in $L^2(\mathbb{S}^1)$ and that D is self-adjoint over $H^1(\mathbb{S}^1)$ [Lew22]. It is clear that $H^1(\mathbb{S}^1)$ is preserved by the action of $\mathcal{A}$; remains to show the second condition of Definition 3.6. For all $f \in C^\infty(\mathbb{S}^1)$ and $\psi \in H^1(\mathbb{S}^1)$,

$$[D, f]\psi = -i \frac{\mathrm{d}}{\mathrm{d}\theta}(f\psi) + i f \frac{\mathrm{d}}{\mathrm{d}\theta}\psi = \left(-i \frac{\mathrm{d}}{\mathrm{d}\theta} f \right) \psi.$$

So $[D, f]$ acts on $H^1(\mathbb{S}^1)$ as a multiplication operator and is in particular bounded by the supremum norm of $-i\frac{\mathrm{d}}{\mathrm{d}\theta} f$ over $\mathbb{S}^1$. Hence $[D, f]$ extends to a bounded operator on $L^2(\mathbb{S}^1)$. We conclude that $(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), D)$ is a spectral triple.

As mentioned above, spectral triples are useful to us because they provide easily many examples of interesting differential calculi. In fact, although it is possible to construct differential calculi without spectral triples, all examples we have thus far used are of this form.

Proposition 3.10. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Consider the subspace*

$$\Omega^1_D(\mathcal{A}) := \mathrm{span}_{\mathbb{C}}\{a[D, b],\ a, b \in \mathcal{A}\} \subset \mathcal{B}(\mathcal{H})$$

and the map $\mathrm{d} : a \in \mathcal{A} \mapsto [D, a] \in \Omega^1_D(\mathcal{A})$. Then $(\Omega^1_D(\mathcal{A}), \mathrm{d})$ is a first-order differential calculus over $\mathcal{A}$.

Proof. The bounded operators on $\mathcal{H}$ are equipped with a natural $\mathcal{A}$-bimodule structure (by left and right composition of operators). The module $\Omega^1_D(\mathcal{A})$ is then a sub-$\mathcal{A}$-bimodule of $\mathcal{B}(\mathcal{H})$ since, for all $a, b, c \in \mathcal{A}$,

$$\begin{aligned}
&c\big(a[D, b]\big) = ca[D, b] \in \Omega^1_D(\mathcal{A})\\
&\big(a[D, b]\big)c = a(Dbc - bDc) = a(Dbc - bcD) - a(bDc - bcD) = a[D, bc] - ab[D, c] \in \Omega^1_D(\mathcal{A}).
\end{aligned}$$

The above calculation also shows that d is a derivation. Finally, $\Omega^1_D(\mathcal{A})$ is by definition spanned by elements of the form $a\mathrm{d}b$, $a, b \in \mathcal{A}$. □

The motivation behind this construction comes from Example 3.9 and generalisations to compact spin manifolds, where D is the associated Dirac operator on L^2-spinors and one can show that $[D, f]$ for $f \in C^\infty(M)$ is exactly the classical differential $\mathrm{d}f$ acting on spinors by Clifford multiplication. Explaining any of these facts and concepts in detail would take us far too long, so we merely mention them and move on.

Example 3.11. The first-order differential calculus described in Example 3.5 is the one associated to the spectral triple $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, D)$ from Example 3.8, where $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Indeed,

$$\mathrm{d}(a_1, a_2) = \left[D, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_2 - a_1 \\ a_1 - a_2 & 0 \end{pmatrix}.$$

3.3 Second-order differential calculi

In the classical setting, the next step after constructing differential 1-forms on a manifold M of dimension n is to extend them to a full exterior algebra $\Lambda^*(M)$. There are several ways to do this, not all equally suited to be adapted to noncommutative algebras. We continue to follow the exposition in [MR24a].

Begin by constructing the (full) tensor algebra

$$T^*(M) := \bigoplus_{k \geq 1} T^k(M), \quad T^k(M) := \underbrace{\Omega^1(M) \otimes_{C^\infty(M)} \ldots \otimes_{C^\infty(M)} \Omega^1(M)}_{\text{k times}},$$

where $\Omega^1(M) := \Gamma^\infty(T^*M)$ is the $C^\infty(M)$-bimodule of complexified differential forms over M.

Notation 3.12. To avoid clutter we will omit the subscript on most tensor products, but they should all be understood to be over the relevant algebras, e.g. $C^\infty(M)$ for geometric situations. Likewise we will write "bimodule" instead of "$\mathcal{A}$-bimodule" when there is no ambiguity. Also note that we are using similar but different notations for the cotangent bundle T^*M and the full tensor algebra $T^*(M)$.

Naively, we could try to extend the differential $\mathrm{d} : C^\infty(M) \to \Omega^1(M)$ to a map $\mathrm{d}_2 : \Omega^1(M) \to T^2(M)$ by setting

$$\mathrm{d}_2(f\mathrm{d}h) := \mathrm{d}f \otimes \mathrm{d}h, \quad f, h \in C^\infty(M)$$

and extending to $\Omega^1(M)$ by linearity. Unfortunately, such a map is ill-defined: it is clear by definition that $\mathrm{d}_2(\mathrm{d}(fh)) = 0$, but $\mathrm{d}_2(f\mathrm{d}h + h\mathrm{d}f) = \mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f$ does not vanish in general. The problem seems to come from the symmetric 2-tensors: $\mathrm{d}f \otimes \mathrm{d}h + \mathrm{d}h \otimes \mathrm{d}f$ is symmetric, and indeed to build differential 2-forms we want symmetric tensors to be sent to zero. To do this we introduce the projection $\Psi : T^2(M) \to T^2(M)$, $\Psi^2 = \Psi$ given by symmetrising 2-tensors:

$$\Psi(\omega \otimes \eta) := \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega)$$

We then let differential 2-forms be the antisymmetric 2-tensors, i.e. $\Lambda^2(M) := \operatorname{Ker}\Psi \subset T^2(M)$. Setting $\mathrm{d}_\Psi := (1-\Psi)\circ\mathrm{d}_2$ yields a map from $\Omega^1(M)$ to $\Lambda^2(M)$ which vanishes on expressions of the form $f\mathrm{d}h+h\mathrm{d}f$. We will accept without proof that d_Ψ is in fact well-defined on all of $\Omega^1(M)$ and satisfies $\mathrm{d}_\Psi(\omega^\dagger) = (\mathrm{d}_\Psi\omega)^\dagger$ and $\mathrm{d}_\Psi \circ \mathrm{d} = 0$ (the latter is immediate from the definition of d_2).

For higher orders we proceed in a similar way, by differentiating naively then projecting onto totally antisymmetric tensors. We obtain a chain

$$C^\infty(M) =: \Lambda^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) =: \Lambda^1(M) \xrightarrow{\mathrm{d}} \Lambda^2(M) \xrightarrow{\mathrm{d}} \Lambda^3(M) \xrightarrow{\mathrm{d}} \dots$$

which one can then show terminates at order n.

Having constructed differential forms of all orders on a manifold M, we can try to copy the procedure for a noncommutative first-order differential calculus $(\mathcal{A}, \Omega^1_\mathrm{d}(\mathcal{A}))$. It is, however, not so straightforward. The tensor algebra is easily defined:

$$T^*_\mathrm{d}(\mathcal{A}) := \bigoplus_{k\geq 1} T^k_\mathrm{d}(\mathcal{A}), \quad T^k_\mathrm{d}(\mathcal{A}) := \underbrace{\Omega^1_\mathrm{d}(\mathcal{A}) \otimes_\mathcal{A} \dots \otimes_\mathcal{A} \Omega^1_\mathrm{d}(\mathcal{A})}_{\text{k times}}.$$

If we try to define a second order differential in the naive way by setting $\mathrm{d}(a\mathrm{d}b) = \mathrm{d}a \otimes \mathrm{d}b$, $a, b \in \mathcal{A}$, we see that we have the same problem as in the commutative case. Indeed, there are algebra elements a_i, b_i such that $\sum_i a_i\mathrm{d}b_i = 0$ but $\sum_i \mathrm{d}a_i \otimes \mathrm{d}b_i \neq 0$. The 2-tensors $\sum_i \mathrm{d}a_i \otimes \mathrm{d}b_i$ thus obtained are called *junk forms.*

Definition 3.13. Let $(\mathcal{A}, \Omega^1_\mathrm{d}(\mathcal{A}))$ be a first-order differential calculus. We define the subbimodule of *junk forms* $JT^2_\mathrm{d}(\mathcal{A}) \subset T^2_\mathrm{d}(\mathcal{A})$ by

$$JT^2_\mathrm{d}(\mathcal{A}) := \left\{ \sum_i \mathrm{d}a_i \otimes \mathrm{d}b_i \;\middle|\; \forall a_i, b_i \in \mathcal{A} \text{ such that } \sum_i a_i \mathrm{d}b_i = 0 \right\}.$$

Example 3.14. The bimodule of junk forms on a manifold M is the $C^\infty(M)$-bimodule of symmetric 2-tensors on M. The symmetrisation map Ψ is the projection onto JT^2M.

In order to define a second order differential we wish to introduce a projection Ψ onto $JT^2_\mathrm{d}(\mathcal{A})$. In the classical case this was done by symmetrising, i.e. by setting $\Psi := \frac{1}{2}(1+\sigma)$ where σ is the flip map $\sigma(\omega \otimes \eta) := \eta \otimes \omega$. However, in a noncommutative context σ does not necessarily exist. In order to have a well-defined flip on $T^2_\mathrm{d}(\mathcal{A})$, we must ensure that $\sigma(\omega a \otimes \eta) = \sigma(\omega \otimes a\eta)$ for all $\omega, \eta \in \Omega^1_\mathrm{d}(\mathcal{A})$ and $a \in \mathcal{A}$. But this implies that $\eta \otimes \omega a = a\eta \otimes \omega$, meaning the bimodule of 2-tensors has to be central ($T^2_\mathrm{d} = \mathcal{Z}(T^2_\mathrm{d})$). This is of course not the general case, hence in general there is no well-defined flip. For this reason we have no systematic way to define a projection Ψ for noncommutative calculi, and must assume its existence as an axiom.

Finally, for technical reasons that we will not explore, it is necessary to allow the image of Ψ to *contain* the junk forms, i.e. requiring $JT^2_\mathrm{d}(\mathcal{A}) \subset \operatorname{Im}\Psi$ instead of $JT^2_\mathrm{d}(\mathcal{A}) = \operatorname{Im}\Psi$. We are thus led to the following definition.

Definition 3.15. [MR24a, Definition 3.4.] Let $\mathcal{A}$ be a complex algebra. A *second-order differential calculus* over $\mathcal{A}$ is $\mathrm{d} : \mathcal{A} \to \Omega^1_{\mathrm{d}}(\mathcal{A})$ a first-order differential calculus such that there exists an $\mathcal{A}$-bilinear idempotent $\Psi : T^2_{\mathrm{d}}(\mathcal{A}) \to T^2_{\mathrm{d}}(\mathcal{A})$ with $JT^2_{\mathrm{d}}(\mathcal{A}) \subset \operatorname{Im}\Psi$.

If $\mathcal{A}$ is a $*$-algebra and $(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger)$ is a first order $\dagger$-calculus, then a *second-order differential* $\dagger$*-calculus* over $\mathcal{A}$ is a second-order calculus with $\Psi \circ \dagger = \dagger \circ \Psi$.

Definition 3.16. [MR24a, Definition 4.1.] Let A be a C^*-algebra and $\mathcal{A}$ a dense $*$-subalgebra. Suppose $\mathcal{A}$ is local in A (see below). A *hermitian second-order differential calculus* is a quadruple $(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}})$ with

1. $(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \langle\cdot|\cdot\rangle_{\mathcal{A}})$ a $\dagger$-bimodule over $\mathcal{A}$ such that its C^* completion $(\Omega^1_{\mathrm{d}}(A), \dagger, \langle\cdot|\cdot\rangle_{A})$, with $\Omega^1_{\mathrm{d}}(A) := \overline{\Omega^1_{\mathrm{d}}(\mathcal{A})}$ and $\dagger$, $\langle\cdot|\cdot\rangle$ extended by continuity, is a Hilbert $\dagger$-bimodule over A.
2. $\Omega^1_{\mathrm{d}}(\mathcal{A})$ finitely generated and projective as a right $\mathcal{A}$-module;
3. $(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi)$ a second-order differential $\dagger$-calculus;
4. Ψ right adjointable as an operator on $T^2_{\mathrm{d}}(\mathcal{A})$ with its induced right $\mathcal{A}$-valued inner product, and $\Psi = \Psi^*$.

Remark 3.17. Some details missing from the above definition are:

- $\mathcal{A}$ local in A means that for all integers n, the $*$-subalgebra $M_n(\mathcal{A}) \subset M_n(A)$ is spectral invariant, i.e. the invertible elements of $M_n(\mathcal{A})$ are precisely those elements of $M_n(\mathcal{A})$ which are invertible in $M_n(A)$. This condition ensures that the spectrum of a matrix $M \in M_n(\mathcal{A})$ is exactly its spectrum as an element of the C^*-algebra $M_n(A)$.
- Ψ right adjointable means that there exists an adjoint Ψ^* for the right inner product (this is not automatic on Hilbert modules).
- It is proven in [MR24a] that if $(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}})$ is a hermitian second-order calculus, then $\Omega^1_{\mathrm{d}}(\mathcal{A})$ is finitely generated and projective as a left $\mathcal{A}$-module and Ψ is left adjointable.

4 Quantum Levi-Civita connections

4.1 Affine connections

A fundamental property of the metric g on a Riemannian manifold (M, g) is that it provides a natural notion of the length of a curve, and by extension of curves of minimal length between two given points. These are called geodesics. Intuitively, a geodesic is obtained by starting from a point and always walking forward in a straight line, never varying one's velocity. The velocity of a particle moving along a path is a vector field along that path, and never varying the velocity means that we require the vector field to be constant along that path. However, such a condition involves being able to compare vectors living in neighbouring tangent spaces. This is not possible *a priori* from the definition of tangent spaces on a smooth manifold. We need an additional piece of data, a *connection*.

Analytically, a connection on a manifold can be seen as a way to define a derivative for vector fields and differential forms which is *covariant*, i.e. invariant under a change of coordinates. One way to do this is with the notion of an *affine connection*: a $\mathbb{C}$-linear map $\nabla : \Gamma^\infty(TM) \otimes \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ satisfying a similar condition to the Leibniz rule, which ensure that it behaves as the directional derivative of the first vector field in the direction of the second, the result being a new vector field.

The notion of an affine connection is purely an algebraic one, meaning it can readily be extended to noncommutative calculi. Since we are working with differential forms rather than vector fields, we let the reader convince themselves that the map ∇ above induces a map $\widetilde{\nabla} : \Gamma^\infty(T^*M) \to \Gamma^\infty(T^*M) \otimes \Gamma^\infty(T^*M)$. We are thus led to the following definition.

Definition 4.1. Let $\mathrm{d} : \mathcal{A} \to \Omega^1_{\mathrm{d}}(\mathcal{A})$ be a first-order differential calculus over a complex algebra $\mathcal{A}$. A right (resp. left) *affine connection* on $\Omega^1_{\mathrm{d}}(\mathcal{A})$ is a $\mathbb{C}$-linear map $\nabla : \Omega^1_{\mathrm{d}}(\mathcal{A}) \to \Omega^1_{\mathrm{d}}(\mathcal{A}) \otimes \Omega^1_{\mathrm{d}}(\mathcal{A})$ such that $\nabla(\omega a) = \nabla(\omega)a + \omega \otimes \mathrm{d}a$ (resp. $\nabla(a\omega) = a\nabla(\omega) + \mathrm{d}a \otimes \omega$) for all $\omega \in \Omega^1_{\mathrm{d}}(\mathcal{A})$ and $a \in \mathcal{A}$.

Remark 4.2. The above definition makes no use of any Riemannian structure, and in particular is valid on smooth manifolds with no extra data. Since we are particularly interested in the Riemannian case, we will not seek to investigate this notion in its full generality.

On a smooth manifold there are many affine connections and no canonical choice *a priori*. However, the power of Riemannian geometry lies in that on a Riemannian manifold (M, g), there is a unique connection that satisfies certain compatibility conditions with the Riemannian structure. It is called the Levi-Civita connection. It has many nice properties, in particular that the paths whose velocity vector has everywhere vanishing covariant derivative (as measured by the Levi-Civita connection) are local distance-minimisers, i.e. are exactly the geodesics. This justifies the intuition that walking forward in a straight line minimises the distance covered.

A fundamental question in noncommutative Riemannian geometry is: does there exist similarly a unique Levi-Civita connection on noncommutative differential calculi? Before attempting to answer this question, we must first state more precisely what compatibility conditions the Levi-Civita connection satisfies.

4.2 Metric compatibility and torsion

There are two properties that characterise the Levi-Civita connection on a Riemannian manifold. The first is explicitly a compatibility condition between the connection and the metric or inner product, while the second requires a certain tensor, the torsion tensor, to vanish. We provide definitions for these concepts directly in the context of (possibly noncommutative) second-order differential structures.

In all this section we fix A a C^*-algebra, $\mathcal{A} \subset A$ a local $*$-subalgebra and $\left(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}}\right)$ a hermitian second-order differential calculus over $\mathcal{A}$.

Definition 4.3. Let $\overrightarrow{\nabla}$ be a right affine connection on $\Omega^1_{\mathrm{d}}(\mathcal{A})$. Then $\overleftarrow{\nabla} := -\dagger \circ \overrightarrow{\nabla} \circ \dagger$ is a left affine connection called the *conjugate connection*.

Proof. This is immediate from the definitions, since $\mathrm{d}(a^*) = -(\mathrm{d}a)^\dagger$ and the dagger inverts the order in tensor products: $(\omega \otimes \eta)^\dagger = \eta^\dagger \otimes \omega^\dagger$. ☐

Notation 4.4. For $\omega, \eta, \rho \in \Omega^1_{\mathrm{d}}(\mathcal{A})$, set

$$\langle \omega \mid \eta \otimes \rho\rangle_{\Omega^1_{\mathrm{d}}(\mathcal{A})} := \langle \omega \mid \eta\rangle_{\mathcal{A}}\, \rho \qquad \langle \eta \otimes \rho \mid \omega\rangle_{\Omega^1_{\mathrm{d}}(\mathcal{A})} := \rho^\dagger \langle \eta \mid \omega\rangle_{\mathcal{A}}\,,$$

then extend by linearity to all of $\Omega^1_{\mathrm{d}}(\mathcal{A}) \times (\Omega^1_{\mathrm{d}}(\mathcal{A}) \otimes \Omega^1_{\mathrm{d}}(\mathcal{A}))$ and $(\Omega^1_{\mathrm{d}}(\mathcal{A}) \otimes \Omega^1_{\mathrm{d}}(\mathcal{A})) \times \Omega^1_{\mathrm{d}}(\mathcal{A})$ respectively. Likewise for the left inner product.

Definition 4.5. [MR24a, Definition 2.23.] Let $\overrightarrow{\nabla}$, resp. $\overleftarrow{\nabla}$, be a right, resp. left, affine connection on $\Omega^1_{\mathrm{d}}(\mathcal{A})$. We say that they are *hermitian* or *metric compatible* if for all $\omega, \eta \in \Omega^1_{\mathrm{d}}(\mathcal{A})$ we have

$$\mathrm{d}\, \langle \omega \mid \eta\rangle_{\mathcal{A}} = \left\langle \omega \mid \overrightarrow{\nabla}\eta \right\rangle_{\Omega^1_{\mathrm{d}}(\mathcal{A})} - \left\langle \overrightarrow{\nabla}\omega \mid \eta \right\rangle_{\Omega^1_{\mathrm{d}}(\mathcal{A})},$$

resp.

$$\mathrm{d}\, {}_{\mathcal{A}}\langle \omega \mid \eta\rangle = {}_{\Omega^1_{\mathrm{d}}(\mathcal{A})}\left\langle \overleftarrow{\nabla}\omega \mid \eta \right\rangle - {}_{\Omega^1_{\mathrm{d}}(\mathcal{A})}\left\langle \omega \mid \overleftarrow{\nabla}\eta \right\rangle.$$

The sign difference with the classical definition comes from the fact that we are using complex differential forms with $\mathrm{d}(a)^\dagger = -\,\mathrm{d}(a^*)$.

Remark 4.6. A right affine connection is hermitian if and only if its conjugate left connection is hermitian ([MR24a, Corollary 2.25.]).

Recall from Section 2.2. that a (right) frame is a collection $v := (\omega_j)$ such that any 1-form ω can be decomposed as

$$\omega = \sum_j \omega_j \left\langle \omega_j \mid \omega \right\rangle_{\mathcal{A}}$$

and that finite frames always exist on finitely generated Hilbert C^*-modules. Corollary 2.11. in [MR24a] moreover states that this is still true for $\Omega^1_{\mathrm{d}}(\mathcal{A})$ finitely generated over $\mathcal{A} \subset A$ a local $*$-subalgebra. Therefore finite frames exist for hermitian second-order calculi.

Lemma 4.7. *Let (ω_j) be a finite frame for $\Omega^1_{\mathrm{d}}(\mathcal{A})$. Then the 2-tensor*

$$G := \sum_j \omega_j \otimes \omega_j^\dagger \in T^2_{\mathrm{d}}(\mathcal{A})$$

is independent of the choice of frame. We call G the quantum metric *associated to the second-order calculus.*

Proof. We only give an outline of the proof. The inner product on $\Omega^1_{\mathrm{d}}(\mathcal{A})$ induces an identification between 1-forms and linear $\mathcal{A}$-valued functionals on $\Omega^1_{\mathrm{d}}(\mathcal{A})$; classically this is refered to as raising and lowering indices. These isomorphisms can then be extended to higher-order tensors. The tensor G is thus associated to the identity map Id on $\Omega^1_{\mathrm{d}}(\mathcal{A})$. Indeed, in ket-bra notation we can write $\mathrm{Id} = \sum_j |\omega_j\rangle \langle\omega_j|$ and each ket-bra $|\omega_j\rangle \langle\omega_j|$ is associated with the tensor $\omega_j \otimes \omega_j^\dagger$ (recall that the inner product is left anti-linear). Since the identity map is independent of the choice of frame, then so is G. ☐

Example 4.8. Let (M, g) be a compact oriented Riemannian manifold. We accept without proof that all the conditions of Definition 3.16 are met, so that $\big(\Gamma^\infty(T^*M), \dagger, \Psi, g\big)$, with Ψ the symmetrisation map and g extended to a hermitian metric on complexified differential forms, is a hermitian second-order differential calculus. Then a finite frame for $\Gamma^\infty(T^*M)$ is given by $(\omega_{j\alpha})$ with $\omega_{j\alpha} := \sqrt{\varphi_\alpha} e_j$, where we have chosen a chart (U_α), a partition of unity (φ_α) and a local orthonormal frame (e_j) on each open set. In local coordinates (x^μ) the quantum metric thus becomes

$$G = \sum_{\alpha\mu\nu} \varphi_\alpha g_{\mu\nu} \mathrm{d}x^\mu \otimes (\mathrm{d}x^\nu)^\dagger.$$

Notation 4.9. We denote by $P, Q : T^3_{\mathrm{d}}(\mathcal{A}) \to T^3_{\mathrm{d}}(\mathcal{A})$ the $\mathcal{A}$-bimodule projections $P := \Psi \otimes 1$ and $Q := 1 \otimes \Psi$.

Classically the maps P and Q correspond to symmetrising in respectively the first two and last two components of a pure tensor $\omega \otimes \eta \otimes \rho$. They are useful for defining the torsion and curvature maps of an affine connection, and for stating the existence condition in Theorem 4.21.

Definition 4.10. [MR24a, Definition 4.2.] Let $\overrightarrow{\nabla}$, resp. $\overleftarrow{\nabla}$, be a right, resp. left, affine connection on $\Omega^1_{\mathrm{d}}(\mathcal{A})$. The following expression yields a well defined 3-tensor called the right, resp. left, *torsion tensor*:

$$\overrightarrow{T}(\overrightarrow{\nabla}) := (1 - Q)(\overrightarrow{\nabla} \otimes 1 + 1 \otimes \mathrm{d}_\Psi)(G) \in T^3_{\mathrm{d}}(\mathcal{A}),$$

resp.

$$\overleftarrow{T}(\overleftarrow{\nabla}) := (1 - P)(-1 \otimes \overleftarrow{\nabla} + \mathrm{d}_\Psi \otimes 1)(G) \in T^3_{\mathrm{d}}(\mathcal{A}).$$

The torsion tensor can be interpreted classically as the covariant derivative of the identity on $\Omega^1_{\mathrm{d}}(\mathcal{A})$, hence the role played by the quantum metric G. This is equivalent to other, more standard definitions of torsion on a manifold, which do not generalise as easily to noncommutative calculi. Note that while the hermitian property of connections only requires a first-order calculus, in the definition of torsion we use the full strength of the hermitian second-order formalism.

Definition 4.11. A right, resp. left, affine connection $\overrightarrow{\nabla}$, resp. $\overleftarrow{\nabla}$, is called *torsion-free* if its torsion tensor vanishes: $\overrightarrow{T}(\overrightarrow{\nabla}) = 0$, resp. $\overleftarrow{T}(\overleftarrow{\nabla}) = 0$.

Remark 4.12. If $\overrightarrow{\nabla}$ and $\overleftarrow{\nabla}$ are conjugate connections, then $\overrightarrow{T}(\overrightarrow{\nabla}) = 0$ if and only if $\overleftarrow{T}(\overleftarrow{\nabla}) = 0$.

Definition 4.13. Let $\overrightarrow{\nabla}$, resp. $\overleftarrow{\nabla}$, be a right, resp. left, affine connection on $\Omega^1_{\mathrm{d}}(\mathcal{A})$. We say that it is a *quantum Levi-Civita connection* over $\Omega^1_{\mathrm{d}}(\mathcal{A})$ if it is both hermitian and torsion-free.

Example 4.14. A generic right affine connection on a classical Riemannian calculus $\big(\Gamma^\infty(T^*M), \dagger, \Psi, g\big)$ is given in local coordinates by

$$(\overrightarrow{\nabla}\omega)_{\mu\nu} = \partial_\nu \omega_\mu - \omega_\rho \Gamma^\rho_{\mu\nu},$$

while its conjugate left connection is

$$(\overleftarrow{\nabla}\omega)_{\mu\nu} = \partial_\mu \omega_\nu - \omega_\rho \Gamma^\rho_{\nu\mu}.$$

Here we are using Einstein convention for implicit summation. The components $\Gamma^\rho_{\mu\nu}$ of the 2-1-tensor Γ are called the *Christoffel symbols*. One can check using the definitions above that the connection $\overrightarrow{\nabla}$ is hermitian if and only if the symbols satisfy $\partial_k g^{ij} + g^{il}\Gamma^j_{lk} + g^{lj}\overline{\Gamma}^i_{lk} = 0$, and is torsion-free if and only if $\Gamma^i_{jk} = \Gamma^i_{kj}$. These are the classical equations for hermitianness and torsion-freeness, with a sign difference for the former due to the convention $(\mathrm{d}f)^\dagger = -\,\mathrm{d}\big(\overline{f}\big)$. The Levi-Civita theorem [LC17, Wey18] states that there is a unique hermitian and torsion-free connection on every Riemannian manifold.

4.3 Existence and uniqueness of quantum Levi-Civita connections

Using the formalism of hermitian second-order differential calculi we can state a necessary and sufficient condition for the existence of a quantum Levi-Civita connection, and a sufficient condition for its uniqueness. There is, however, to this day no single existence and uniqueness theorem of similar generality as the Levi-Civita theorem for manifolds (say, one applicable to all "reasonable" calculi derived from spectral triples).

We still consider $\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}}\big)$ a hermitian second-order differential calculus over $\mathcal{A} \subset A$ a local $*$-algebra.

Notation 4.15. For a right frame (ω_j) on $\Omega^1_{\mathrm{d}}(\mathcal{A})$ we set

$$W := \sum_j \mathrm{d}_\Psi(\omega_j) \otimes \omega_j^\dagger, \qquad W^\dagger = \sum_j \omega_j \otimes \mathrm{d}_\Psi(\omega_j^\dagger).$$

Recall that $P := \Psi \otimes 1$ and $Q := 1 \otimes \Psi$ are projections on $T^3_{\mathrm{d}}(\mathcal{A})$. It is a known fact that two connections always differ by a 3-tensor. Hence if one fixes a reference connection $\overrightarrow{\nabla}^v$ (which depends on a finite frame v), any connection can be written $\overrightarrow{\nabla} = \overrightarrow{\nabla}^v + A_v$ with $A_v \in T^3_{\mathrm{d}}(\mathcal{A})$ the *connection form* (identifying 3-tensors with $\mathcal{A}$-bilinear maps $\Omega^1_{\mathrm{d}}(\mathcal{A}) \to T^3_{\mathrm{d}}(\mathcal{A})$ using the right inner product). One can then show that a hermitian connection $\overrightarrow{\nabla}$ is torsion-free if and only if $(1-P)A_v = -W$, if and only if $(1-Q)A_v = -W^\dagger$. Combining these equations and iterating yields for every $n \geq 1$

$$(1-(PQ)^n)A_v = -\sum_{k=0}^{n-1}(PQ)^k(W + PW^\dagger), \quad (1-(QP)^n)A_v = -\sum_{k=0}^{n-1}(QP)^k(W^\dagger + QW). \tag{4.1}$$

We are thus led to examine the convergence properties of the sequence $\big((PQ)^n\big)_n$.

Definition 4.16. A pair $(\mathcal{M}, \mathcal{N})$ of $\mathcal{A}$-subbimodules of $\Omega^1_{\mathrm{d}}(\mathcal{A})$ is called *concordant* if there is a decomposition $\Omega^1_{\mathrm{d}}(\mathcal{A}) = (\mathcal{M} \cap \mathcal{N}) \oplus (\mathcal{M}^\perp + \mathcal{N}^\perp)$, where the orthogonal complement is defined by

$$\mathcal{M}^\perp := \{\omega \in \Omega^1_{\mathrm{d}}(\mathcal{A}) : \langle m \mid \omega\rangle_{\mathcal{A}} = 0 \ \forall m \in \mathcal{M}\}.$$

Proposition 4.17. *(cf. [MR22, Proposition 3.12.]) The subbimodules* $\operatorname{Im} P$ *and* $\operatorname{Im} Q$ *are concordant if and only if the sequence* $\big((PQ)^n\big)_n$ *converges in operator norm. When this is the case we set* $\Pi := \lim_{n\to\infty} (PQ)^n$ *the limit projection.*

Example 4.18. On a smooth manifold M, since P and Q act on pure 3-tensors as symmetrisation respectively in the first two and last two arguments, then the limit projection Π acts as total symmetrisation, i.e. it is the projection onto symmetric 3-tensors.

Definition 4.19. [MR24a, Definition 4.30.] The calculus $\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}}\big)$ is called $\dagger$-*concordant* if $\operatorname{Im} P$ and $\operatorname{Im} Q$ are concordant and if there exists a frame (ω_j) such that the following equation holds:

$$(1 + \Pi - PQ)^{-1}(W + PW^{\dagger}) = (1 + \Pi - QP)^{-1}(W^{\dagger} + QW). \tag{4.2}$$

Remark 4.20. We accept without proof that Equation 4.2 is frame-independent. Therefore if the calculus is $\dagger$-concordant then the equation holds for any frame.

Theorem 4.21. *[MR24a, Theorem 4.34.] There exists a (right) quantum Levi-Civita connection* $\overrightarrow{\nabla} : \Omega^1_{\mathrm{d}}(\mathcal{A}) \to T^2_{\mathrm{d}}(\mathcal{A})$ *on* $\Omega^1_{\mathrm{d}}(\mathcal{A})$ *if and only if the calculus* $\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}}\big)$ *is* $\dagger$*-concordant.*

In order to state a uniqueness theorem in our context, we need an additional piece of data which plays the role of the flip map $\omega \otimes \eta \mapsto \eta \otimes \omega$ on classical differential forms. The following definitions can be found in [MR24a, BM20, BGM20].

Definition 4.22. A *braiding* on the calculus $\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}}\big)$ is an invertible $\mathcal{A}$-bilinear map $\sigma : T^2_{\mathrm{d}}(\mathcal{A}) \otimes T^2_{\mathrm{d}}(\mathcal{A})$ satisfying $\sigma^{-1} \circ \dagger = \dagger \circ \sigma$.

Definition 4.23. A *bimodule connection* over $\Omega^1_{\mathrm{d}}(\mathcal{A})$ is a pair $(\overrightarrow{\nabla}, \sigma)$ where $\overrightarrow{\nabla}$ is a right connection and σ is a braiding such that $\sigma \circ \overrightarrow{\nabla} = \overleftarrow{\nabla}$, with $\overleftarrow{\nabla} = -\dagger \circ \overrightarrow{\nabla} \circ \dagger$ the conjugate connection.

We now formulate a sufficient condition for uniqueness of the quantum Levi-Civita connection on a hermitian calculus.

Notation 4.24. The definition below uses the objects defined as follows:

- The inner products induce two separate isomorphisms $\overrightarrow{\alpha}$, resp. $\overleftarrow{\alpha}$, between $T^3_{\mathrm{d}}(\mathcal{A})$ and the right, resp. left, adjointable bimodule homomorphisms $\overrightarrow{\mathrm{Hom}}^*_{\mathcal{A}}\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), T^2_{\mathrm{d}}(\mathcal{A})\big)$, resp. $\overleftarrow{\mathrm{Hom}}^*_{\mathcal{A}}\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), T^2_{\mathrm{d}}(\mathcal{A})\big)$.
- For a $\dagger$-concordant calculus we define $\mathrm{Sym}^3_{\mathrm{d}}(\mathcal{A}) := \operatorname{Im} \Pi$ the symmetric 3-tensors (cf. Example 4.18).
- We let $\overleftrightarrow{\mathrm{Hom}}^*_{\mathcal{A}}\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), T^2_{\mathrm{d}}(\mathcal{A})\big)$ be the space of both right and left adjointable bimodule homomorphisms.

Theorem 4.25. *[MR24a, Theorem 5.14.] Suppose* $\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), \dagger, \Psi, \langle\cdot|\cdot\rangle_{\mathcal{A}}\big)$ *is* $\dagger$*-concordant with a braiding* σ *such that*

$$\overrightarrow{\alpha} + \sigma^{-1} \circ \overleftarrow{\alpha} : \mathcal{Z}(\mathrm{Sym}^3_{\mathrm{d}}(\mathcal{A})) \longrightarrow \overleftrightarrow{\mathrm{Hom}}^*_{\mathcal{A}}\big(\Omega^1_{\mathrm{d}}(\mathcal{A}), T^2_{\mathrm{d}}(\mathcal{A})\big) \tag{4.3}$$

is injective. Suppose that $(\overrightarrow{\nabla}, \sigma)$ *and* $(\overleftarrow{\nabla}, \sigma)$ *are two bimodule connections on* $\Omega^1_{\mathrm{d}}(\mathcal{A})$*. Then* $\overrightarrow{\nabla} = \overleftarrow{\nabla}$*.*

Once one has existence and uniqueness of a quantum Levi-Civita connection, one can define the associated Riemannian curvature tensor and its contractions, the Ricci and scalar curvature. Thus one may speak of the curvature of a spectral triple or any other suitable noncommutative differential calculus. Computations of curvature tensors on explicit examples can be found in [MR24b, MR24c], including the computation of scalar curvature on the Podles sphere, or noncommutative 2-sphere, which depends in a fully explicit way on the quantum deformation parameter.

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