Model Theory of Banach Spaces

Ivory Fronteau

sous la direction de Jorge López Abad

Rapport de stage

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Déroulé du stage

Je débute en septembre 2024 une thèse portant sur la *Théorie des modèles des représentations unitaires de groupes dénombrables, en logique continue et affine.* J'ai été introduit à la théorie des modèles métriques à l'occasion de mon mémoire de M2 [Frob], et j'ai poursuivi mon étude du domaine d'un point de vue assez général [Froa]. Il me semblait donc pertinent, avant de débuter ma thèse, d'élargir ma vision du domaine et de m'intéresser à des applications potentielles de la théorie des modèles métrique. C'est Tomás Ibarlucía, mon directeur de thèse, qui m'a parlé des travaux de Jorge López Abad sur la combinatoire des espaces de Banach et de son intérêt pour ce que peut apporter le point de vue modèle-théorique à l'étude des espaces de Banach. J'ai rapidement vu là une opportunité en cohérence avec mes envies et projets.

Je connaissais très peu la théorie des espaces de Banach avant de commencer le stage. Surtout pendant les premières semaines, Jorge et moi avons beaucoup échangé pour introduire nos domaines respectifs, et nous avons établi des pistes de recherche pour la suite. Certaines se sont avérées infructueuses, d'autres trop ambitieuses mais plusieurs ont pu être explorées et ce rapport rendra principalement compte de ce travail. En parallèle, j'ai étudié la littérature sur des sujets plus ou moins connexes comme les limites de Fraïssé d'espaces de Banach [FLAMT], la théorie des modèles des treillis L_p [BYBH] et des espaces de probabilité [BH]. En fin de stage, Jorge m'a parlé de phénomènes de concentration de la mesure en lien avec les limites de Fraïssé et les propriétés de Ramsey.

Introduction

The object of study of model theory is the relationship between syntactical objects (theories axiomatized by mathematical sentences) and semantic ones (models of a given theory i.e. the class of structures which respect the given set of axioms). Of course, one needs to give a precise sense to *structures* which are classically sets equipped with constants, functions and predicates for a given signature, and to *mathematical sentences* which are classically first-order sentences on the signature.

Back in the 60's, Chang and Keisler suggested to apply model theory to topological structures, but the framework was too general to be of interest. Later, people like Krivine and Maurey began to use ideas from model theory to study specifically Banach spaces like stable ones. A first formalism (positive bounded logic) close to classical model theory is due to Henson. During the first decade of the 21th century, a unifying framework to study metric structures in general would eventually be developed, by Ben Yaacov and Usvyatsov among others. This framework has lead to study L_p spaces viewed as lattices but not so much in the signature of Banach spaces. We will thus try to explore some aspects of this second direction.

Section 1 is a quick introduction the main notions of metric model theory that we will need. Section 2 is a presentation of how Banach spaces can be treated as metric structures and shows some interactions between Banach spaces notions and model-theoretic ones. The richer Sections 3 and 4 compile results specific to L_p spaces, some of which do not appear in the literature to our knowledge. In particular in Section 3, we give a new look to the description of the completions of the theory of L_p spaces by studying how atomicity is definable in the signature of Banach spaces. We give semantic arguments to prove that the class of L_p spaces and the theory of $L_p(0,1)$ are Π_2-axiomatizable. This allows to give a proof that the theory of $L_p(0,1)$ is always model-complete, even when $p = 4, 6, 8, \ldots$. In Section 4, we give insights on the relation between equimeasurability and equality of types in $L_p(0,1)$ spaces to enlight our view on an old result from Lusky [Lus] stating that the theory $L_p(0,1)$ is not only model-complete but admits quantifier elimination when $p \neq 2, 4, 6, \ldots$.

Acknowledgements

I would like to thank Jorge López Abad for our numerous discussions, as well as Tomás Ibarlucía who gave me the idea of working with him. I would also like to thank the UNED Departamiento de Matemáticas Fundamentales in general for their warm welcome.

Contents

1 Basics of metric model theory

Metric structures

A metric signature $\mathcal{L}$ consists of a distance symbol d, constant symbols, function symbols and predicate symbols. In addition :

- The distance symbol d has a bound D attached to it and its arity is $n_d = 2$.
- Each function symbol f comes with an arity $n_f \geq 1$ and a modulus of continuity ϖ_f.
- Each predicate symbol P comes with an arity $n_P \geq 0$, a modulus of continuity ϖ_P, and bounds $r_P \leq R_P \in \mathbb{R}$.

A (metric) $\mathcal{L}$-structure M consists of :

- A complete metric space (Ω^M, d^M)
- For each constant symbol c, an element $c^M \in \Omega^M$.
- For each function symbol f, a function $f^M : (\Omega^M)^{n_f} \to \Omega^M$ which is uniformly continuous with respect to ϖ_f.
- For each predicate symbol P (and d), a function $P^M : (\Omega^M)^{n_P} \to \mathbb{R}$ which is uniformly continuous with respect to ϖ_P and such that $r_P \leq P^M \leq R_P$.

An *isomorphism* between two structures is an isometry between the underlying spaces which preserves the interpretation of each symbol. An *automorphism* of a structure M is an isomorphism between M and itself. A *substructure* A of M is a complete subspace $A \subseteq \Omega^M$ closed under every function f^M for f function symbol of $\mathcal{L}$. We write $A \subseteq M$ if A is a substructure of M and we say that A embeds in M. If a is a set of elements of M, then we denote by $\langle a \rangle$ the smallest substructure $A \subseteq M$ containing a.

We will often identify M and its underlying set Ω^M. More generally, if x is a set of new constant symbols (i.e. a set of constant symbols c such that $c \notin \mathcal{L}$), then we denote by M^x the set of functions $a : x \to \Omega^M$.

Formulas and type spaces

Terms and their interpretations are defined exactly as for classical logic. The set of first-order formulas is defined inductively :

- If P is a predicate symbol of arity n (or the symbol d of arity 2) and $t_1, ..., t_n$ are terms, then $P(t_1, ..., t_n)$ is a first-order formula. In this case, $P(t_1, ..., t_n)$ is also called an *atomic formula.*
- If U is a continuous function $\mathbb{R}^n \to \mathbb{R}$ and $\varphi_1, ..., \varphi_n$ are first-order formulas, $U(\varphi_1, ..., \varphi_n)$ is a first-order formula. We also say that U is a connective.
- If x is a constant symbol and φ is a first-order formula, then $\inf_x \varphi$ and $\sup_x \varphi$ are first-order formulas.

We can define as for classical formulas the notion of free and bounded occurence of a variable in a (first-order) formula. We write $\varphi = \varphi(x)$ to mean that the variables with free occurences in φ are in the set x. A *(first-order) sentence* is a formula with no free variables. Quite straightforwardly and analogously to classical model theory, one can define by induction the interpretation of a formula $\varphi(x)$ in a structure M as a function $\varphi^M : M^x \to \mathbb{R}$. A quantifier-free formula is a formula where no quantifier inf or sup appears.

Let $A \subseteq M$ be a substructure of a given structure M. We say that A is an *elementary substructure* of M and write $A \preceq M$ if for every formula $\varphi(x)$ and $a \in A^x$, $\varphi^A(a) = \varphi^M(a)$.

A theory T is a set of conditions of the form $(\varphi \leq r)$ where φ is a sentence and $r \in \mathbb{R}$ which is consistent i.e. there is some M such that $\varphi^M \leq r$ for every condition $(\varphi \leq r)$ in T. Such a structure M is called a model of T and we write $M \models T$. A complete theory is a theory such that for every sentence φ and $M, N \models T$, $\varphi^M = \varphi^N$. If T, S are two theories, we write $T \models S$ if every model of T is a model of S, and $T \equiv S$ if $T \models S$ and $S \models T$. In the latter case, we can say that S is *axiomatizable* by S. If M is a structure, its *theory* is the set of conditions E such that $M \models E$ and will be denoted by $\mathrm{Th}(M)$. Finally, we write $M \equiv N$ if M, N are two structures with the same theory.

Let T be a theory, x a finite set of new constant symbols. An x-type p is a complete $\mathcal{L}(x)$-theory such that $p \models T$. Modulo $\equiv$, the set of complete x-type will be denoted by $S_x(T)$ (or $S_n(T)$ if x has n elements). Each $\mathcal{L}$-formula $\varphi(x)$ defines a function $S_x(T) \to \mathbb{R}$ which maps p to $\varphi^p := \varphi^M(a)$ for any $(M, a) \models p$. The coarsest topology on $S_x(T)$ which makes these functions continuous is called the logic topology on $S_x(T)$ and is compact Hausdorff (this fact is called the Compactness Theorem). We can also define a metric d on $S_x(T)$ which refines the logic topology which is defined by :

$$d(p, q) = \inf_{(M,a) \models p, (M,b) \models q} d^M(a, b)$$

If M is a structure and $a \in M^x$, then $\mathrm{tp}(a)$ will denote the type of a i.e. the type defined by $\mathrm{Th}(M, a)$.

Löwenheim-Skolem Theorem

The Compactness Theorem and Löwenheim-Skolem Theorem are often considered as the two fundamental tools for a model theorist. Let us state one part of the latter :

Proposition 1.1 (Downward Löwenheim-Skolem, [BBHU], Proposition 7.3)**.** *Assume that $\mathcal{L}$ is countable. Let M be a structure and suppose that A is a subspace of M which is separable. Then there exists a separable structure $N \preceq M$ containing A.*

ω-categoricity

A theory T is called *ω-categorical* if whenever we have two separable models $A, B \models T$, A and B are isomorphic. A structure M is called *ω-categorical* if its theory is ω-categorical. Ryll-Nardzewski Theorem gives characterizations of this property in terms of type spaces :

Proposition 1.2 (Ryll-Nardzewski Theorem, see e.g. [BBHU] Theorem 12.10)**.** *Let T be a complete theory in a countable signature. The following are equivalent :*

(i) T is ω-categorical.

(ii) For each $n \geq 1$, every type in $S_n(T)$ is realized in every model of T.

(iii) For each $n \geq 1$, the topology induced by d on $S_n(T)$ is the logic topology.

(iv) For each $n \geq 1$, $(S_n(T), d)$ is compact.

Separable ω-categorical structures are quite nice. In particular, we will see later that they are homogeneous.

Ultraproducts

One of the first contribution of model theory to metric structures is through the construction of ultraproducts. Let I be a set, $(M_i)_{i\in I}$ a family of $\mathcal{L}$-structures and $\mathcal{U}$ an ultrafilter on I. We define a pseudo-distance $d^{\mathcal{U}}$ on $\prod_{i\in I} \Omega^{M_i}$ by putting $d^{\mathcal{U}}(a,b) = \lim_{i\to\mathcal{U}} d^{M_i}(a_i, b_i)$. We now define $(\Omega^{M_{\mathcal{U}}}, d^{M_{\mathcal{U}}})$ as the metric space obtained from $\prod_{i\in I} \Omega^{M_i}$ by identifying elements a and b whenever $d^{\mathcal{U}}(a,b) = 0$. The equivalence class of $a \in \prod_i M_i$ in this quotient space will be denoted by $a_{\mathcal{U}}$. Then, we make $(\Omega^{M_{\mathcal{U}}}, d^{M_{\mathcal{U}}})$ an $\mathcal{L}$-structure by putting :

- For each constant symbol c, $c^{M_{\mathcal{U}}} := (c^{M_i})_{i\in I}$.
- For each function symbol f and $a_1, ..., a_{n_f} \in \prod_i M_i$,
$$f^{M_{\mathcal{U}}}(a_{1,\mathcal{U}}, ..., a_{n_f,\mathcal{U}}) := (f^{M_i}(a_{1,i}, ..., a_{n_f,i}))_{\mathcal{U}}$$
- For each predicate symbol P and $a_1, ..., a_{n_P} \in \prod_i M_i$,
$$P^{M_{\mathcal{U}}}(a_{1,\mathcal{U}}, ..., a_{n_P,\mathcal{U}}) := \lim_{i\to\mathcal{U}} P^{M_i}(a_{1,i}, ..., a_{n_P,i})$$

It is now easy to check that all these definitions make sense and that they define an $\mathcal{L}$-structure which will be denoted by $\prod_i M_i/\mathcal{U}$ or $M_{\mathcal{U}}$. If M is an $\mathcal{L}$-structure, we denote by $M^{\mathcal{U}}$ the ultrapower $\prod_{i\in I} M/\mathcal{U}$.

The importance of the ultraproduct construction lies in the two following facts stating essentially that first-order predicates are exactly the predicates which are preserved by the ultraproduct construction in the sense of Łoś Theorem (see the next section for a precise formulation of this fact).

Fact 1.3 (Łoś Theorem for continuous logic)**.** *If $\varphi(x)$ is a first-order $\mathcal{L}$-formula, $(M_i)_{i\in I}$ a family of $\mathcal{L}$-structures, $\mathcal{U}$ an ultrafilter on I and $a \in \prod_i M_i^x$, then :*

$$\varphi^{M_{\mathcal{U}}}(a_{\mathcal{U}}) = \lim_{i\to\mathcal{U}} \varphi^{M_i}(a_i)$$

Fact 1.4 (Keisler-Shelah's Theorem for continuous logic)**.** *Let M, N be two $\mathcal{L}$-structures. The following are equivalent :*

(i) $M \equiv N$

(ii) $M^{\mathcal{U}} \simeq N^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$.

Definability

Let T be a theory, x a finite set of new constant symbols, ϖ a modulus of uniform continuity and real numbers $r \leq R$. Suppose that we are given for every model M of T a function $P^M : M^x \to [r, R]$ which is uniformly continuous with respect to ϖ. We say that P is T-*definable* if for every $\varepsilon > 0$, there exists a formula $\varphi(x)$ such that for every $M \models T$ and $a \in M^x$, $|P^M(a) - \varphi^M(a)| \leq \varepsilon$.

Fact 1.5 (see e.g. [BBHU], Sections 9-10)**.** *The following are equivalent :*

(i) *P is a definable predicate.*

(ii) *There exists a function $\Phi : S_x(T) \to [r, R]$ such that for every $M \models T$ and $a \in M^x$, $P(a) = \Phi(\mathrm{tp}(a))$.*

(iii) *For every family $(M_i)_{i\in I}$ of models of T, every ultrafilter $\mathcal{U}$ on I and every $a \in \prod_i M_i^x$, we have $P^{M_\mathcal{U}}(a_\mathcal{U}) = \lim_{i\to\mathcal{U}} P^{M_i}(a_i)$.*

If M is a structure and $A \subseteq M$, we denote by $\mathrm{dcl}(A)$ the definable closure of A i.e. the set of $b \in X$ such that the function $x \to d(b, x)$ is definable in the theory $\mathrm{Th}(M, A)$.

Homogeneity, quantifier elimination

We say that a structure M is *strongly ω-homogeneous* if for every $n < \omega$, whenever $a = (a_i)_{1\leq i\leq n}, b = (b_i)_{1\leq i\leq n}$ are tuples of elements of M such that $\mathrm{tp}(a) = \mathrm{tp}(b)$, then there exists $\sigma \in \mathrm{Aut}(M)$ such that $\sigma a_i = b_i$ for every i.

Strong homogeneity is a strong assumption which will often not occur in "small" models but only in extensions. We are thus interested in weaker notions of homogeneity such as the following. We say that a structure M is *approximately ω-homogeneous* if whenever $a = (a_i)_{1\leq i\leq n}, b = (b_i)_{1\leq i\leq n}$ are tuples of elements of M which have the same type, then for every $\varepsilon > 0$ there exists $\sigma \in \mathrm{Aut}(M)$ such that $d(\sigma a_i, b_i) < \varepsilon$ for every i.

Fact 1.6 ([BBHU], Corollary 12.11)**.** *If M is the separable model of an ω-categorical theory T, then M is approximately ω-homogeneous.*

A particularly nice class of homogenous structures is the one of *approximately ultrahomogeneous* structures. A structure M is approximately ω-ultrahomogeneous if for every $n < \omega$ and $a, b \in M^n$, every isomorphism $\varphi : \langle a\rangle \simeq \langle b\rangle$ and every $\varepsilon > 0$, there exists $\sigma \in \mathrm{Aut}(M)$ such that $d(\sigma a, \varphi a) < \varepsilon$.

Ultrahomogeneity is closely related to the more syntactic notion of *quantifier elimination*. A theory T has quantifier elimination (QE) if every formula $\varphi(x_1, ..., x_n)$ can be approximated by quantifier-free formulas i.e. there exist quantifier-free formulas $(\varphi_n)_{n<\omega}$ such that for every $\varepsilon > 0$ we have for every n large enough :

$$T \models \sup_{x_1} ... \sup_{x_n} |\varphi(x) - \varphi_n(x)| < \varepsilon$$

In particular, every type is determined by the restriction to the quantifier-free type.

Fact 1.7. *If M is the separable model of an ω-categorical theory T, then M is approximately ultrahomogeneous if and only if T has (QE).*

Proof. On the one hand, Fact 1.6 together with T having (QE) clearly implies that if two tuples have the same quantifier-free type, they have the same type and thus one can be sent to the other (up to ε) by an automorphism.

On the other hand, we have the general fact that if M is approximately ultrahomogeneous and a, b have the same quantifier-free type, then there exists σ sending a to b up to ε and thus $\mathrm{tp}(a)$ and $\mathrm{tp}(b)$ are at distance less than ε. $\square$

Ultrahomogeneity is also the key to view structures as Fraïssé limit, and thus the starting point to study dynamics via combinatorial methods (see e.g. [Lup] or [FLAMT] for applications to Banach spaces).

A common task for model theorists is to know if some theory T has (QE). If not, one can add new symbols to the signature to get (QE) for an equivalent theory T' in the new signature. For instance, we can add a predicate symbol $P_\varphi(x)$ for every $\varphi(x)$ in a dense subset of all first-order formulas and consider the new theory T' adding the axioms $\sup_x |P_\varphi(x) - \varphi(x)| = 0$. Clearly, the models of T and T' are in a natural correspondence. Moreover, T' is built to have quantifier elimination. If we make the same construction for T' but we only add P_φ for every φ in a dense subset of all Π_1-formulas (i.e. of the form $\sup_x \psi$ where ψ is quantifier-free), then we say that T is *model-complete* if T' has (QE).

Fact 1.8. *A theory T is model-complete if and only if for every $M, N \models T$ such that $M \subseteq N$, we have $M \preceq N$.*

Note that obviously, every theory which has (QE) is model-complete.

If M is a separable metric structure which is (strongly, approximately) ω-(ultra)homogeneous, then we simply say that M is (strongly, approximately) (ultra)homogeneous.

Arithmetical Hierarchy

We define recursively a hierarchy for first-order sentences. The set of quantifier-free formulas is denoted by Π_0 and Σ_0. Now, for every $k \geq 0$, we define by Π_{k+1} (resp. Σ_{k+1}) the set of formulas of the form $\sup_{x_1} \dots \sup_{x_n} \varphi$ (resp. $\inf_{x_1} \dots \inf_{x_n} \varphi$) where φ is a Σ_k formula (resp. Π_k formula). Let A be Σ_k or Π_k for some k. A theory is A^c-axiomatizable (resp. A^o-axiomatizable) if it is axiomatizable by conditions of the form $\varphi \leq r$ (resp. $\varphi < r$) where φ is an A-sentence and $r \in \mathbb{R}$. Note that a complete theory is A^c axiomatizable if and only if it is A^o-axiomatizable so in this case we can use the terminology A-axiomatizable with no ambiguity.

Preservation theorems allow to characterize complexity of formulas and theories in terms of their preservation properties under certain semantical construction. One instance is the following. If T is a first-order theory, let us denote by $T_\forall$ (resp. $T_\exists$) the set of Π_1^c conditions (resp. Σ_1^c conditions) which are consequences of T.

Proposition 1.9. *Let T be a first-order theory and M a metric structure. The following are equivalent :*

(i) $M \models T_\forall$

(ii) *There exists $N \models T$ such that $M \subseteq N$.*

A consequence of this preservation theorem is that Π_1^c-axiomatizable theories are exactly those for which every time $N \models T$ and $M \subseteq N$ is a substructure, $M \models T$ (this is for example the case of Banach spaces). With a simple application of Compactness Theorem (see [Froa] for examples), we can deduce that Π_1 formulas are exactly those which are preserved by substructures, a fact that will be useful later.

Let us review another example of preservation theorem. We say that a theory T is *preserved under unions of chains*[1] if whenever $(M_i)_{i<\alpha}$ is a family of models of T where α is an ordinal,

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[1] In classical model theory, we say that such a theory is *inductive*.

then $\overline{\bigcup_{i<\alpha} M_i}$ is also a model of T. It is easy to see that any Π_2^c-axiomatizable theory is preserved under unions of chains. The following gives some kind of converse (which really is a converse if T is complete).

Proposition 1.10 ([Usv] Theorem 3.6)**.** *If T is preserved under unions of chains then T is Π_2^c-axiomatizable.*

Corollarily, any model-complete theory is Π_2^c-axiomatizable by Tarski's elementary chain Theorem. Since the theory of any approximately ultrahomogeneous structure has (QE), the theory of an approximately ultrahomogeneous Banach space (for instance, the Gurarij space) is thus Π_2-axiomatizable.

2 Model theory of Banach spaces

Banach spaces as metric structures

In the following, we will view Banach spaces as metric structures. To this end, one needs to get around the unboundedness of Banach spaces. Three solutions exist :

- Ben Yaacov [BY] developed a more general framework to treat unbounded metric structures, but this would be a bit of overkill in our situation.
- One can use many-sorted structures which allows to represent balls of diameter n for each n. This is a good compromise between ease of use and straightforwardness but require to be more careful on the model-theoretic definitions.
- Finally, a Banach space is actually completely described by its unit ball, so one can consider a metric structure based on it as a metric structure. In this case, the syntax would only allow convex combinations which would technically need extra attention. We use this convention in spirit but for the sake of clarity we allow ourselves to use arbitrary linear combinations in the syntax. The reader can check that it does not affect the results.

The signature $\mathcal{L}_{BS}$ for Banach spaces will consist of a constant symbol 0, a function symbol λ_q of arity 1 for each $q \in \mathbb{Q} \cap [-1, 1]$[2], a function symbol $\dot{+}$ of arity 2 and a predicate symbol $\|.\|$ of arity 1. We do not specify the moduli of continuity and bounds but the reader can easily guess them after the next paragraph.

If X is a Banach space, we define an $\mathcal{L}_{BS}$-structure denoted by BX in the following way :

- BX is the closed ball of center 0 and diameter 1, with the usual distance $d^{BX}(x, y) = \|x - y\|$.
- 0^{BX} is 0.
- λ_q^{BX} is the scalar multiplication by q.
- $\dot{+}^X$ is defined by $x \dot{+}^X y := x/2 + y/2$.
- $\|.\|^{BX}$ is the norm.

Fact 2.1. *The class of all $\mathcal{L}_{BS}$-structures of the form BX where X is a Banach space is Π_1^c-axiomatizable. The class of all non-compact $\mathcal{L}_{BS}$-structures of the previous form is Π_2^c-axiomatizable.*

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[2] We could have also used complex scalars instead without affecting the following definitions and results.

We denote by T_{BS} the theory of unit balls of Banach spaces and $T_{BS\infty}$ the theory of non-compact unit balls. It is easy to build a Banach space SM from a model $M \models T_{BS}$ by allowing arbitrary scalar multiplications. Moreover, $B(SM)$ and M are isomorphic as metric structures, and for every Banach space X, $S(BX)$ and X are isometric. Also, $M \models T_{BS\infty}$ if and only if SM is an infinite dimensional Banach space. To keep notations simple, we will make no distinctions between these objects but bear in mind that "seen as a metric structure", a Banach space is bounded and that when interpreting a formula, the domain of sup and inf is restricted to the unit ball and not the entire vector space.

Finite representability

Here is one translation of preservation theorems in the special case of Banach spaces making a reference to the well studied notion of finite representability. Recall that a Banach space X is finitely representable in Y if for every finite dimensional subspace $E \subseteq X$ and $\varepsilon > 0$ there exists $F \subseteq Y$ and a surjective linear mapping $T : E \to F$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.

Proposition 2.2 ([Iovb], Proposition 1.16)**.** *Let X, Y be two Banach spaces. The following conditions are equivalent :*

(i) *X is finitely representable in Y.*

(ii) $\mathrm{Th}(Y) \models \mathrm{Th}(X)_\exists$

(iii) $\mathrm{Th}(X) \models \mathrm{Th}(Y)_\forall$

(iv) *There exists $Y \preceq Y'$ such that X embeds isometrically in Y'.*

Moreover, Y' in (iv) can be chosen as an ultrapower of Y.

Definable closure

Ferenczi and López Abad defined several envelopes for subspaces of Banach spaces [FLA]. We show here that one of them is very close to the model-theoretic notion of definable closure. We denote by $\mathrm{Env}(A)$ the isometric envelope of A in X i.e. the set of $x \in X$ such that whenever a net $(T_i)_{i\in I}$ in $\mathrm{Iso}(X)$ converges pointwise on A to id_A, then $(T_i(x))_{i\in I}$ converges to x.

Lemma 2.3. *Let X be a Banach space, $A \subseteq X$ a subset. Then, $\langle \mathrm{dcl}(A)\rangle \subseteq \mathrm{Env}(A)$.*

Proof. Let $d \in \mathrm{dcl}(A)$. Let $(T_i)_{i\in I}$ be a net of isometries such that for every $a \in A$, $T_i a \to a$. Let $\varepsilon > 0$. By definition, there exist a formula $\varphi(x, y)$ and a tuple $a \in A$ such that :

$$M \models \sup_x |\varphi(x, a) - \|x - d\|| \leq \varepsilon$$

We also know that for some $\delta > 0$, $M \models \sup_x |\varphi(a, x) - \varphi(b, x)| \leq \varepsilon$ as long as $\|a - b\| \leq \delta$. Now, take $i_0 \in I$ such that for every $i \geq i_0$, $\|T_i a - a\| \leq \delta$. We thus have :

$$\begin{aligned}\|T_i d - d\| &\leq \varphi(T_i d, a) + \varepsilon \\ &\leq \varphi(T_i d, T_i a) + 2\varepsilon \\ &\leq \varphi(d, a) + 2\varepsilon \\ &\leq 3\varepsilon\end{aligned}$$

Thus, $d \in \mathrm{Env}(A)$. □

Proposition 2.4. *Let X be an approximately ω-homogeneous Banach space, $A \subseteq X$ a subset. Then,* $\mathrm{Env}(A) \subseteq \langle \mathrm{dcl}(A) \rangle$.

Proof. Let $d \in \mathrm{Env}(A)$ such that $\|d \leq 1\|$, $p := \mathrm{tp}(d/A)$ and $b \in X$ such that $b \models p$. By approximate homogeneity, one can find for every $n < \omega$ and tuple $a \in A$, $\sigma_{n,a} \in \mathrm{Iso}(X)$ such that $\|\sigma_{n,a}d - b\| \leq 2^{-n}$ and $\|\sigma_{n,a}a - a\| \leq 2^{-n}$. Thus, we have $\sigma_{n,a} \restriction A \to \mathrm{id}_A$ (the directed poset is defined by $(n,a) \leq (n',a')$ if $n \leq n'$ and $a \subseteq a'$). By assumption, we thus get that $\sigma_{n,a}d \to d$. But we also have $\sigma_{n,a}d \to b$ so $d = b$. Hence, d is the only realisation of p in X, d is definable over A. □

Corollary 2.5. *If X is a separable, ω-categorical Banach space and $A \subseteq X$ then*

$$\langle \mathrm{dcl}(A) \rangle = \mathrm{Env}(A)$$

ω-categoricity and reflexivity

We finish this section by presenting a small result which concerns metric structures of the form (X, X^*) showing how model-theoretic notions can interact with Banach spaces ones. To be more precise, here we are interested in structures with two sorts[3] s and s^*, and the signature consists of two copies of $\mathcal{L}_{BS}$ each assigned to a sort, and a predicate $\langle , \rangle$ of arity $s \times s^* \to \mathbb{R}$. A metric structure M on this signature is called a dual pairing if :

- The reduct of M to each sort is the unit ball of a Banach space.
- $\langle , \rangle$ is bilinear.
- For every $x \in s^M$ and $y \in s^{*M}$, $\langle x, y \rangle \leq \|x\|\|y\|$.
- $\sup_x \inf_y \min(\|x\|\|y\| - |\langle x, y \rangle|, 1 - \|x\|) \leq 0$
- $\sup_y \inf_x \min(\|x\|\|y\| - |\langle x, y \rangle|, 1 - \|y\|) \leq 0$

It is easy to check that a dual pairing is of the form $(B, C, \langle , \rangle)$ where B, C are the unit balls of two Banach spaces, respectively X and Y such that there are (not necessarily surjective) isometries $X \subseteq Y^*$ and $Y \subseteq X^*$ defined by $x \mapsto \langle x, . \rangle$ and $y \mapsto \langle ., y \rangle$. We will now write a structure of this form as (X, Y). The class of dual pairings is Σ_2^c-axiomatizable. Dual pairings might be used to study a Banach space X via the metric structure (X, X^*) (here the isometries are the canonical injection $X \subseteq X^{**}$ and the identity $X^* = X^*$).

Proposition 2.6. *If (X, Y) is a dual pairing such that X is reflexive, then $(X, Y) \simeq (X, X^*)$.*

Sketch of the proof. Suppose that (X, Y) is a dual pairing and X is reflexive. $x \mapsto \langle x, . \rangle$ is a linear isometry $X \to Y^*$. Let us show it is surjective. Take $\varphi \in Y^*$. By Hahn-Banach Theorem, φ has a continuous linear extension $\tilde{\varphi} \in X^{**} \simeq X$. We have for every $y \in Y$:

$$\varphi(y) = \tilde{\varphi}(y) = \langle \tilde{\varphi}, y \rangle$$

Hence, $X = Y^*$. Since Y^* is reflexive and Y is a Banach space, Y is also reflexive. Thus, $Y = Y^{**} = X^*$. □

Proposition 2.7. *Let X be a reflexive Banach space. The following are equivalent :*

---

[3] We do not give any details here on how to define and use many-sorted structures in metric model theory but most of what is true for one-sorted structures can be generalized, especially if there are finitely or even countably many sorts to handle.

(i) X *is* ω*-categorical.*

(ii) (X, X^*) *is* ω*-categorical.*

(iii) X^* *is* ω*-categorical.*

Proof. By symmetry, we only need to prove that (i) is equivalent to (ii). One implication is a consequence of Proposition 2.6. Now suppose (ii), then X is a reduct of the ω-categorical structure (X, X^*) and thus also ω-categorical by a standard application of Ryll-Nardzewski Theorem. ☐

3 Theory of L_p spaces

3.1 Axiomatizability of L_p spaces

Let $1 \leq p < +\infty$, $p \neq 2$[4]. An L_p *space* is a Banach space which is isometric to some $L_p(\mu)$ for some measure μ. The following interesting result is due to Henson and Krivine.

Proposition 3.1. *The class of* L_p *spaces is axiomatizable.*

We can thus talk about the theory T_p of L_p spaces. Since a Banach space is infinite dimensional if and only if its unit ball is not compact, we can also talk about the theory $T_{p,\infty}$ of infinite dimensional L_p spaces.

Proposition 3.2 ([Sip], Theorem 3.7)**.** *A Banach space* X *is an* L_p *space if and only if for all* $x_1, ..., x_n \in X$ *of norm at most 1 and for all* $N \geq 1$*, there is a subspace* $C \subseteq X$ *and* $y_1, ..., y_n \in C$ *of norm at most 1 such that* C *is isometric to* $\ell_p^{\dim C}$ *and for all* i*,* $\|x_i - y_i\| \leq 1/N$*.*

In the previous proposition, we can find a bound on the dimension of C which depends only on N and n. This allowed Sipoş to give a syntactic axiomatization of T_p in positive-bounded logic. We take another path to give precisions on the axiomatizability of L_p spaces.

Theorem 3.3. T_p *and* $T_{p,\infty}$ *are preserved under unions of chains.*

Proof. The fact of being infinite dimensional is obviously preserved under unions of chains, so we only prove the fact about T_p.

Take $(X_i)_{i<\omega}$ a chain of L_p spaces, $x_1, ..., x_n \in \overline{\bigcup_{i<\omega} X_i}$ of norm at most 1 and $N \geq 1$. We can find $y_1, ..., y_n \in \bigcup_{i<\omega} X_i$ such that $\|x_k - y_k\| \leq 1/2N$ for every k. For some $l < \omega$, we have that $y_1, ..., y_n \in X_l$. Since X_l is an L_p space, we can find $C \subseteq X$ and $z_1, ..., z_n \in C$ of norm at most 1 such that C is isometric to $\ell_p^{\dim C}$ and for all k, $\|y_k - z_k\| \leq 1/2N$. Thus for every k, $\|x_k - z_k\| \leq 1/N$ and C is a subspace of $\overline{\bigcup_{i<\omega} X_i}$ which is isometric to $\ell_p^{\dim C}$ so $\overline{\bigcup_{i<\omega} X_i}$ is an L_p space. ☐

Corollary 3.4. T_p *and* $T_{p,\infty}$ *are* Π_2^0*-axiomatizable.*

Proof. By the previous theorem and Proposition 1.10. ☐

Each $L_p(\mu)$ has a very natural structure of lattice. L_p Banach lattices have been axiomatized and studied from the beginning of metric model theory (see [BBHU] Section 17). However, even if L_p Banach spaces and L_p Banach lattices are obviously related, their model-theoretic properties are not quite the same.

---

[4] Most of what follows could be stated for $p = 2$ but would be useless. Indeed, the structure of L_2 space is much nicer than for any other p because it can define a structure of Hilbert space using the Polarization Identity. It is for instance much easier to prove categoricity and homogeneity properties for infinite dimensional Hilbert spaces than for other L_p spaces (see [BBHU] Section 15).

Proposition 3.5. *For every $1 \leq p \leq \infty$, the lattice structure of $L_p(0,1)$ is not definable in the Banach space $L_p(0,1)$.*

Proof. Take $f = 1_{(0,1/2)} - 1_{(1/2,1)}$ and $g = 1_{(1/2,1)} - 1_{(0,1/2)}$. In particular, $f \vee g = 1$ and $f \wedge g = -1$, and the two have the same type over (f,g) (use the isometry $h \mapsto -h(1-.)$). ☐

We can however use the Banach lattice structures to get some information about the underlying spaces considered as reducts. For instance, the following has been noticed by Henson but in an old formalism [Iova] . We give a proof in the formalism of metric model theory using reducts.

Theorem 3.6. $\mathrm{Th}(L_p(0,1))$ *is ω-stable.*

Proof. Consider $E := L_p(\mu)$ such that $E \equiv L_p(0,1)$ (i.e. E is atomless as we will see later) and $A \subseteq E$ of cardinality ω. The theory of the Banach lattice $E_l := (E, \vee, \wedge)$ is ω-stable (see [BBHU] Section 17) so $(S^{E_l}(A), d)$ has a countable density character. We have a natural map $S^{E_l}(A) \to S^E(A)$ which sends a type q in the language of Banach lattices to the restriction of q to the language of Banach spaces. This map is onto and 1-Lipschitz. As a consequence, the density character of $(S^E(A), d)$ is bounded by the one of $(S^{E_l}(A), d)$, which is countable. ☐

3.2 Completions of T_p and atomicity

The following is a well-known classification of separable L_p spaces, which is a consequence of Maharam's Theorem. If X, Y are two Banach space, define $X \oplus_p Y$ as the direct sum of X and Y with the norm $\|(x,y)\| := (\|x\|^p + \|y\|^p)^{1/p}$. Note that if X and Y are L_p spaces, then $X \oplus_p Y$ is also an L_p space.

Proposition 3.7 (see e.g. [Woj] III.A)**.** *Let X be a separable L_p space. Then X is isometric to one of the Banach spaces below :*

(1) The n-dimensional space $\ell_p(n)$ for some $0 \leq n < \omega$.

(2) The atomless L_p space $L_p(0,1)$.

(3) $\ell_p(n) \oplus_p L_p(0,1)$ for some $1 \leq n < \omega$.

(4) The L_p space with a countable number of atoms $\ell_p(\omega)$.

(5) $\ell_p(\omega) \oplus_p L_p(0,1)$.

From a model-theoretic point of view, we can wonder whether these different spaces can be distinguished by their first-order theory. For the finite-dimensional cases (1), the unit balls are compact so it is a general fact that they are completely determined by their first-order theory (as finite structures in classical model theory). For the others, the answer already appeared in [Hen] in the formalism of positive bounded logic. We will give a new look to this result with some precisions on the axiomatizability along the way. In particular, we will study how atomicity can be described in the language of Banach spaces.

Lemma 3.8. *There exist quantifier-free formulas* $\mathrm{orth}(x,y)$ *and* $\mathrm{rest}(x,y)$ *such that for every L_p space X and $f, g \in X$*

(1) $\mathrm{orth}(f,g) = 0$ *if and only if* $\mu(\mathrm{supp}(f) \cap \mathrm{supp}(g)) = 0$.

(2) $\mathrm{rest}(f,g) = 0$ *if and only if* $f = g \restriction \mathrm{supp}(f)$.

Sketch of the proof. (1) $\operatorname{orth}(x,y) := \|x+y\|^p + \|x-y\|^p - 2\|x\|^p - 2\|y\|^p$ is proved to be a candidate in [Lam] Corollary 2.1 using convexity arguments.

(2) Define $\operatorname{rest}(x,y) := \operatorname{orth}(x, y-x)$. We have $\operatorname{rest}(f,g) = 0$ if and only if $(g-f) \restriction \operatorname{supp}(f) = 0$ i.e. $g \restriction \operatorname{supp}(f) = f$. □

Let us introduce some notation. Let X be an L_p space. If $f \in X$, then its support consists of at most countably many atoms and an atomless part. Let us write in decreasing order $A_1(f) \geq A_2(f) \geq \dots$ the family of $\|f \restriction a\|$ where a are the atoms of $\operatorname{supp}(f)$. If $\operatorname{supp}(f)$ contains n atoms, then we set $0 = A_{n+1}(f) = A_{n+2}(f) = \dots$, otherwise $(A_n(f))_{n\geq 1}$ is an infinite nonincreasing sequence of positive numbers. In any case, $\sum_{n\geq 1} A_n(f)^p \leq \|f\|^p$.

Lemma 3.9. *Let $r > 0$. We can find $n < \omega$ such that if X is an L_p space and $f \in X$ has norm less than 1, then $A_1(f) \leq r$ if and only if there are $g_1, \dots, g_n \in X$ such that :*

(1) $f = \sum_{i=1}^n g_i$.

(2) $\|g_i\| \leq r$.

(3) $\mu(\operatorname{supp}(g_i) \cap \operatorname{supp}(g_j)) = 0$ whenever $i \neq j$.

Proof. Suppose that $A_1(f) \leq r$ and write $f = \sum_{i=1}^N A_i(f)a_i + b + c$ where $\|a_i\| = 1$, $\operatorname{supp}(a_i)$ is an atom, $\operatorname{supp}(b)$ has no atomless part, $\|b\| \leq r$ and $\operatorname{supp}(c)$ is atomless. Let $\sigma_{i,j} := A_i(f)^p + \dots + A_j(f)^p$. Define inductively $l_0 := 0$ and for the next k's, $l_{k+1} := \max\{i \leq N : \sigma_{l_k+1,i} \leq r^p\}$ and define K as the first natural number such that $l_{K+1} = N$. K is well defined and this sequence is increasing until K because for every i, $A_i(f) \leq r$. Moreover, for every $k < K$, $\sigma_{l_k+1,l_{k+1}} > r^p/2$ because otherwise we would have $A_{l_{k+1}+1}(f)^p \leq A_{l_{k+1}}(f)^p \leq \sigma_{l_k+1,l_{k+1}} \leq r^p/2$ and this would contradict the definition of l_{k+1}. We deduce that :

$$Kr^p/2 \leq \sum_{k=0}^{K-1} \sigma_{l_k+1,l_{k+1}} \leq \|f\|^p \leq 1$$

Thus, $K \leq 2/r^p$. Now, put $g_k := \sum_{i=l_k+1}^{l_{k+1}} A_i(f)a_i$ for $k = 0, \dots, K$, $g_{K+1} := b$. Finally, since $\operatorname{supp}(c)$ is atomless, then for $K' \geq 1/r^p$, we can find $g_{K+2}, \dots, g_{K+K'+1}$ with disjoint supports and supports contained in $\operatorname{supp}(c)$, $\|g_i\| \leq r$ such that $c = g_{K+2} + \dots + g_{K+K'+1}$. It is now easy to check that any $n \geq 3/r^p + 1$ is a good candidate thanks to the decomposition $f = g_1 + \dots + g_{K+K'+1}$.

Conversely, suppose that we can find $g_1, \dots, g_n \in X$ satisfying (1),(2) and (3) and let us show that $A_1(f) \leq r$. Let A be an atom. By (1), A must be in the support of at least one g_i. By (3), this i must be unique. Finally, $\|f \restriction A\| \leq \|f \restriction \operatorname{supp}(g_i)\| = \|g_i\| \leq r$, so necessarily $A_1(f) \leq r$. □

Proposition 3.10. *Let $(X_i)_{i\in I}$ be a familly of L_p spaces and $\mathcal{U}$ an ultrafilter. For every family $f_i \in X_i$ and $n \geq 1$,*

$$A_n(f_{\mathcal{U}}) = \lim_{i\to\mathcal{U}} A_n(f_i)$$

Consequently, A_n is T_p-definable i.e. for every $\varepsilon > 0$, there exists a formula φ such that $T_p \models \sup_f |A_n(f) - \varphi(f)| \leq \varepsilon$.

Lemma 3.11. *Let $(X_i)_{i\in I}$ be a familly of L_p spaces, $\mathcal{U}$ an ultrafilter and $a_i \in X_i$ be a family such that $\operatorname{supp}(a_i)$ are atoms. Then, the support of $a_{\mathcal{U}}$ is an atom.*

Proof. Assume that we can find $f_{\mathcal{U}} = (f_i)_{i\in I}/\mathcal{U}$ and $g_{\mathcal{U}} = (g_i)_{i\in I}/\mathcal{U}$ where $\mu(\operatorname{supp}(f_{\mathcal{U}}) \cap \operatorname{supp}(g_{\mathcal{U}})) = 0$ such that $a_{\mathcal{U}}$ can be written as $f_{\mathcal{U}} + g_{\mathcal{U}}$.

First, we can assume without loss of generality that $\mathrm{supp}(f_i) \subseteq \mathrm{supp}(a_i)$ and $\mathrm{supp}(g_i) \subseteq \mathrm{supp}(a_i)$ for every i. Indeed, let $\tilde{f}_i := f_i \restriction \mathrm{supp}(a_i)$ and $\tilde{g}_i := g_i \restriction \mathrm{supp}(a_i)$. Clearly, we have $\mu(\mathrm{supp}(\tilde{f}_\mathcal{U}) \cap \mathrm{supp}(\tilde{g}_\mathcal{U})) = 0$ using Lemma 3.8 and :

$$\begin{aligned} \|\tilde{f}_i + \tilde{g}_i - a_i\| &= \|(f_i + g_i - a_i) \restriction \mathrm{supp}(a_i)\| \\ &\leq \|f_i + g_i - a_i\| \to_{i \to \mathcal{U}} 0 \end{aligned}$$

so $a_\mathcal{U} = \tilde{f}_\mathcal{U} + \tilde{g}_\mathcal{U}$.

With this additional assumption, one of the three cases occurs :

- If $\{i : \mu(\mathrm{supp}(f_i)) = 0\} \in \mathcal{U}$, then $\|f_\mathcal{U}\| = \lim_{i \to \mathcal{U}} \|f_i\| = 0$ and $f_\mathcal{U} = 0$.
- Similarly, if $\{i : \mu(\mathrm{supp}(g_i)) = 0\} \in \mathcal{U}$, then $g_\mathcal{U} = 0$.
- Otherwise, since $\mathrm{supp}(a_i)$ is an atom we must have $J := \{i : \mathrm{supp}(g_i) = \mathrm{supp}(f_i) = \mathrm{supp}(a_i)\} \in \mathcal{U}$. But then, if we put for every $i \in J$, $f_i = \mu_i a_i$ and $g_i = \nu_i a_i$ and μ_i, ν_i arbitrary for $i \notin J$, then $f_\mathcal{U} = (\lim_{i \to \mathcal{U}} \mu_i) a_\mathcal{U}$ and $g_\mathcal{U} = (\lim_{i \to \mathcal{U}} \nu_i) a_\mathcal{U}$. ☐

Corollary 3.12. *Let $(X_i)_{i \in I}$ be a familly of L_p spaces, $\mathcal{U}$ an ultrafilter and $f \in \prod_i X_i$. For every $n \geq 1$, $A_n(f_\mathcal{U}) \geq \lim_{i \to \mathcal{U}} A_n(f_i)$.*

Proof. For every i, take $a_{1,i}, ..., a_{n,i}$ which are restrictions of f_i for which $\|a_{k,i}\| = A_k(f_i)$, and the $\mathrm{supp}(a_i)$'s are disjoint atoms. From the previous lemma, the $\mathrm{supp}(a_{k,\mathcal{U}})$'s are atoms. They are restrictions of $f_\mathcal{U}$ of disjoint supports according to Lemma 3.8. Moreover, $\|a_{k,\mathcal{U}}\| = \lim_{i \to \mathcal{U}} A_k(f_i)$. From these observations, it is clear that $A_n(f_\mathcal{U}) \geq \min(\|a_{1,\mathcal{U}}\|, ..., \|a_{n,\mathcal{U}}\|) = \lim_{i \to \mathcal{U}} A_n(f_i)$. ☐

Lemma 3.13. *Let $(X_i)_{i \in I}$ be a familly of L_p spaces, $\mathcal{U}$ an ultrafilter and $f \in \prod_i X_i$. Then, $A_1(f_\mathcal{U}) \leq \lim_{i \to \mathcal{U}} A_1(f_i)$.*

Proof. Let $\varepsilon > 0$ and $L := \lim_{i \to \mathcal{U}} A_1(f_i)$, let us prove that $A_1(f_\mathcal{U})^p \leq L^p + \varepsilon$. Define $J := \{i : A_1(f_i) \leq L + \varepsilon\} \in \mathcal{U}$. From Lemma 3.9 we can write for every $i \in J$ f_i as $f_i = \sum_{k=1}^N g_{k,i}$ such that $\|g_{k,i}\| \leq L + \varepsilon$, $b_{k,i}$ and $b_{k',i}$ have disjoint supports whenever $k \neq k'$ and N does not depend on i. Put $b_{k,i} = 0$ for $i \notin J$, in order to get :

$$f_\mathcal{U} = \sum_{k=1}^{N} b_{k,\mathcal{U}}$$

By Lemma 3.8, the $b_{k,\mathcal{U}}$'s have disjoint supports. Moreover, $\|b_{k,\mathcal{U}}\| \leq L + \varepsilon$. Using Lemma 3.9 once again, we have $A_1(f_\mathcal{U}) \leq L + \varepsilon$, and we can conclude that $A_1(f_\mathcal{U}) \leq \lim_{i \to \mathcal{U}} A_1(f_i)$. ☐

Proof of Proposition 3.10. Let us prove by induction that for every $n < \omega$ and $f \in \prod_i X_i$, $A_n(f_\mathcal{U}) = \lim_{i \to \mathcal{U}} A_n(f_i)$. The case $n = 1$ is already covered by the previous lemmas. Now, if $f \in \prod X_i$, take disjoint atoms $A_{1,i}, ..., a_{n,i}$ such that $\|f_i \restriction A_{k,i}\| = A_k(f_i)$ for every $k \leq n$ and put a_i the restriction of f_i to the complement of $A_1 \cup ... \cup A_n$. We have $\|a_{k,\mathcal{U}}\| = A_k(f_\mathcal{U})$ by assumption, and from Lemma 3.8 the $a_{k,\mathcal{U}}$'s are restrictions of $f_\mathcal{U}$ thus we can write :

$$A_{n+1}(f_\mathcal{U}) = A_1(f_\mathcal{U} - \sum_{k=1}^{n} a_{k,\mathcal{U}}) \leq \lim_{i \to \mathcal{U}} A_1(f_i - \sum_{k=1}^{n} a_{k,i}) = \lim_{i \to \mathcal{U}} A_{n+1}(f_i)$$

The other inequality comes from Corollary 3.12.

The fact that A_n is T_p-definable directly comes from its preservation under ultraproducts and ultrafactors (see e.g. [Froa]). ☐

Lemma 3.14. *There exists $\ell_p(\omega) \equiv X$ such that X is separable and contains $L_p(0,1)$.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on ω, let us show that for some $f_{\mathcal{U}} \in \ell_p(\omega)^{\mathcal{U}}$, $\operatorname{supp}(f_{\mathcal{U}})$ is atomless. For $i < \omega$, let $f_i := \sum_{k=1}^{i} i^{-1/p} A_{\{k\}}$. We have $A_1(f_i) = i^{-1/p} \to_{i \to \mathcal{U}} 0$. Thus, $A_1(f_{\mathcal{U}}) = 0$ but $f_{\mathcal{U}} \neq 0$ so $f_{\mathcal{U}}$ has atomless support. Now, using Löwenheim-Skolem's Theorem, we build a separable space $X \preceq \ell_p(\omega)^{\mathcal{U}}$ containing $L_p(0,1)$. ☐

The A_n's are not only definable predicates, but also of low complexity. This can be proved by semantic arguments.

Proposition 3.15. *A_n is approximable by Σ_1 formulas.*

Sketch of a proof. It is easy to see that for every $M \models T_p$ and $N \models T_p$ such that $M \subseteq N$ and every $f \in M$, we have $A_n^N(f) \leq A_n^M(f)$. It is now a standard fact that this characterizes Σ_1 formulas. ☐

Proposition 3.16. *Let X be an L_p space and $n < \omega$. We have the following :*

(1) X has at least n atoms if and only if $X \models \sup_x A_n(x) \geq n^{-1/p}$.

(2) X has strictly less than n atoms if and only if $X \models \sup_x A_n(x) \leq 0$.

Proof. (1) Suppose that the underlying measure space of M has at least n atoms $A_1, ..., A_n$. Then, $f := \sum_{k=1}^{n} n^{-1/p} 1_{A_k}$ satisfies $A_n(f) \geq n^{-1/p}$. Conversely, if $M \models \sup_x A_n(x) \geq n^{-1/p}$, then in particular there is $f \in M$ such that $A_n(f) > 0$. Thus, there are at least n atoms contained in $\operatorname{supp}(f)$ and in M.

(2) Obvious. ☐

Theorem 3.17. *The completions of $T_{p,\infty}$ are the following :*

1. *$T_p^* = \operatorname{Th}(L_p(0,1))$ which is ω-categorical and Π_2^c-axiomatizable.*
2. *For every $1 \leq n < \omega$, $T_p^n = \operatorname{Th}(L_p(0,1) \oplus_p \ell_p(n))$ which is ω-categorical.*
3. *$T_p^\infty := \operatorname{Th}(\ell_p(\omega))$ which has exactly two separable models, $\ell_p(\omega)$ and $\ell_p(\omega) \oplus_p L_p(0,1)$.*

Proof. (1) and (2) are consequences of Proposition 3.7 and Proposition 3.16. The fact that T_p^* is Π_2-axiomatizable directly comes from Proposition 3.16 together with the observation of Proposition 3.15 and because T_p is itself Π_2^c-axiomatizable.

Let us prove (3). Because of Proposition 3.7, (1) and (2), it suffices to prove that the theory of $\ell_p(\omega)$ is not ω-categorical, which is a direct consequence of Lemma 3.14. ☐

Notice that (3) gives the example of a theory with exactly two separable models which would not be possible with discrete structures. Also, due to ω-categoricity we have the following :

Corollary 3.18. *$L_p(0,1)$ is approximately homogeneous. $L_p(0,1) \oplus_p \ell_p(n)$ is approximately homogeneous for every $n \geq 1$.*

Finally, because $\ell_p(\omega)$ is elementarily embeddable in L_p spaces having an atomless part, we can use Proposition 2.2 to deduce this known fact about $\ell_p(\omega)$:

Corollary 3.19. *$L_p(0,1)$ is finitely representable in $\ell_p(\omega)$.*

3.3 Model-completeness and model-companionship

Let $1 \leq p < \infty$, $p \neq 2$. The aim of this section is to prove the following :

Theorem 3.20. *T_p^* is model-complete.*

To this end, we notice that it is a direct consequence of the following Theorem, whose classical analogue is due to Lindström. We fix here $\mathcal{L}$ a countable signature and T an $\mathcal{L}$-theory.

Theorem 3.21. *Suppose that T is ω-categorical, preserved under unions of chains and with no compact models. Then T is model-complete.*

To prove this theorem in the context of metric model theory, we use existentially closed models as in classical model theory. A model $M \models T$ is *existentially closed* if for every Σ_1 formula $\varphi(x)$ (also called existential formula, justifying the terminology), $a \in M$, and every embedding $M \subseteq N \models T$, we have $\varphi^M(a) = \varphi^N(a)$. Note that if T is model-complete, then every model is existentially closed. As a matter of fact, the converse is also true. The idea to prove this theorem will be to show that for an ω-categorical theory, it suffices to know that the separable model is existentially closed.

Lemma 3.22. *Suppose that T is preserved under unions of chains and M is a non-compact model of T. M can be extended to an existentially closed model of T with the same density character.*

Proof. The proof is written for countable models but nothing changes if we replace ω by any infinite cardinal. Let $A \models T$ be a separable model of T, $\tilde{A}$ a countable dense subset of A and $\{\varphi_i\}_{i<\omega}$ be a dense subset of the existential $L(\tilde{A})$-sentences (each appearing infinitely many times in the enumeration). We can build a sequence $A = A_0 \subseteq A_1 \subseteq \ldots$ of separable models of T such that $\varphi_i^{A_{i+1}} \leq \inf_{A_i \subseteq B \models T} \varphi_i^B + 2^{-i}$ for every i. Define then $U(A) := \overline{\bigcup_{i<\omega} A_i}$. $U(A)$ is a model of T. We claim that for every existential $L(\tilde{A})$-sentence φ :

$$\varphi^{U(A)} \leq \inf_{U(A) \subseteq B \models T} \varphi^B \tag{1}$$

By assumption on the enumeration, we can find $i < \omega$ such that $2^{-i} < \varepsilon$ and $\models |\varphi - \varphi_i| < \varepsilon$. Now, for $U(A) \subseteq B \models T$, we have :

$$\varphi^{U(A)} \leq \varphi_i^{U(A)} + \varepsilon \leq \varphi_i^{A_i} + \varepsilon \leq \varphi_i^B + 2\varepsilon$$

It is now easy to deduce Eq. (1).

Now, put $M_0 := M$ and $M_{i+1} := U(M_i)$ for every $i < \omega$ and then $N := \overline{\bigcup_{i<\omega} M_i}$. Clearly, $N \models T$. Let us now prove that N is existentially closed. Let $N \subseteq B \models T$, $\varphi(x)$ an existential L-formula, $n \in N^x$ and $\varepsilon > 0$. Take $\delta > 0$ such that $d(x,y) < \delta \models |\varphi(x) - \varphi(y)| < \varepsilon$. There exist $i < \omega$ and $m \in M_i$ such that $d^N(n,m) < \delta$. We thus get :

$$\varphi^N(n) \leq \varphi^N(m) + \varepsilon \leq \varphi^{M_{i+1}}(m) + \varepsilon \leq \varphi^B(m) + \varepsilon \leq \varphi^B(n) + 2\varepsilon \qquad \square$$

Proof of Theorem 3.21. Let M be the separable model of T. By assumption, M is not compact. T is preserved under unions of chains. By Lemma 3.22 and ω-categoricity, M is thus existentially closed.

Now, let $A \subseteq B$ be two models of T, $\varphi(x)$ an existential formula and $a \in A^x$. By Löwenheim-Skolem, we get some separable model $A' \preceq A$ containing a. By our previous remark, A' is existentially closed so we have :

$$\varphi^A(a) = \varphi^{A'}(a) = \varphi^B(a)$$

By a standard preservation theorem, this means that every Σ_1 formula can be approximated by Π_1 formulas modulo T. By a simple induction, every formula can be approximated by Π_1 formulas. We can conclude that $A \preceq B$. $\square$

The fact that T_p^* is model-complete allows for a nice model-theoretic characterization of $L_p(0,1)$. Let T, T^* be two theories. We say that T^* is the *model-companion* of T if :

- Every model of T can be embedded in a model of T^*.
- Every model of T^* can be embedded in a model of T.
- T^* is model-complete.

If moreover T^* has quantifier elimination, then it is called the *model-completion* of T. Model-companions can be interpreted as generic or random models in a precise sense [Usv]. One example of model-completion in the context of Banach spaces is the theory of the Gurarij space :

Theorem 3.23 ([BYH], Theorem 2.3)**.** *The theory of the Gurarij space is the model-completion of the theory of Banach spaces.*

We get a similar result for L_p spaces.

Theorem 3.24. *T_p^* is the model-companion of T_p and the model-companion of* $\mathrm{Th}(\ell_p)$.

Proof. For the first statement, we already know that T_p^* is model-complete and that every model of T_p^* can be embedded in a model of T_p (itself). It is also easy to embed any model of T_p in a model of T_p^* by replacing atoms with copies of $[0,1]$. For the second statement, one needs to check that $\mathrm{Th}(\ell_p)_\forall = (T_p^*)_\forall$, in other words that spaces which are finitely representable in $L_p(0,1)$ and ℓ_p are the same (by Proposition 2.2). First, ℓ_p embeds isometrically in $L_p(0,1)$ so clearly, every space wich is finitely representable in ℓ_p is finitely representable in $L_p(0,1)$. The converse is consequential of Corollary 3.19 i.e. that $L_p(0,1)$ is finitely representable in ℓ_p. $\square$

Corollary 3.25. *$(T_p)_\forall$, $(T_p^*)_\forall$ and $\mathrm{Th}(\ell_p)_\forall$ are equivalent. This is the common theory of the subspaces of $L_p(0,1)$.*

Proof. The first part is just a part of Theorem 3.24. For the second part, let T be the common theory of the subspaces of $L_p(0,1)$. It is clear that $T \models (T_p)_\forall$ since every subspace of $L_p(0,1)$ is finitely representable in $L_p(0,1)$. Now, let $X \models (T_p^*)_\forall$ and let us show that $X \models T$. By Downward Löwenheim-Skolem, find X' countable such that $X' \equiv X$. Since $X' \models (T_p^*)_\forall$, one can find $L_p(0,1) \preceq Y$ such that X' embeds isometrically in Y. Now, we use Downward Löwenheim-Skolem arguments and ω-categoricity of $L_p(0,1)$ to find an embedding $X' \subseteq L_p(0,1)$. Thus, $X' \models T$ and $X \models T$ too. $\square$

4 Types in atomless L_p spaces

4.1 Equimeasurability

Let $1 \leq p < \infty$, $p \neq 2$.

Lemma 4.1. *Let λ be a probability measure, $d < \omega$, $\mathfrak{P}(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$ with the Lévy-Prokhorov distance π defined by :*

$$\pi(\nu,\nu') := \inf\left\{\varepsilon > 0 : \nu(B) < \nu'(B_\varepsilon) + \varepsilon \textit{ for every Borel set } B \subseteq \mathbb{R}^d\right\}$$

where $B_\varepsilon := \{x \in \mathbb{R}^d : d(x,B) < \varepsilon\}$. The map $f = (f_1, ..., f_d) \in L_p(\lambda, \mathbb{R}^d) \mapsto f_*\lambda \in \mathfrak{P}(\mathbb{R}^d)$ is Hölder.

Proof. Let $f, g \in L_p(\lambda, \mathbb{R}^d)$. Let $B \subseteq \mathbb{R}^d$ be a Borel set, $\varepsilon > 0$. First, notice that :

$$\|f - g\|^p \geq \|f_i - g_i\|_p^p \geq \int_{|f_i - g_i| \geq \varepsilon} |f_i(x) - g_i(x)|^p d\lambda(x) \geq \varepsilon^p \lambda(|f_i - g_i| \geq \varepsilon)$$

We deduce the following :

$$\begin{aligned} f_*\lambda(B) &= \lambda(f \in B) \\ &= \lambda(f \in B \cap g \in B_\varepsilon) + \lambda(f \in B \cap g \notin B_\varepsilon) \\ &\leq \lambda(g \in B_\varepsilon) + \lambda(\|f - g\| \geq \varepsilon) \\ &\leq g_*\lambda(B_\varepsilon) + \sum_{i=1}^{d} \lambda(|f_i - g_i| \geq \varepsilon) \\ &\leq g_*\lambda(B_\varepsilon) + d\|f - g\|^p / \varepsilon^p \end{aligned}$$

We can now deduce that $\pi(f_*\lambda, g_*\lambda) \leq d^{1/(p+1)} \|f - g\|^{\frac{p}{p+1}}$. ☐

Let $(\Omega, \Sigma, \lambda)$ be a measured space. A *regular set isomorphism* [Lam] is a map $T : \Sigma \to \Sigma$ defined modulo sets of measure zero satisfying for every $A, A_n \in \Sigma$:

- $T(\Omega \setminus A) = T\Omega \setminus TA$
- $T(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} TA_n$ for disjoint A_n
- $\mu(TA) = 0$ if and only if $\mu(A) = 0$

Let us recall the Banach-Lamperti's Theorem characterizing isometries of $L_p(\lambda)$. A regular set isomorphism induces a linear transformation T on the set of measurable functions, which is defined by $T1_A := 1_{TA}$ for every $A \in \Sigma$.

Theorem 4.2 (Banach-Lamperti, [Lam] Theorem 3.1)**.** *If $\sigma \in \mathrm{Iso}(L_p(\lambda))$ then there are a regular set isomorphism T and a function $h : \Omega \to \Omega$ such that $\sigma f = h.(Tf)$ λ-almost everywhere. Moreover, if $\lambda'(A) := \lambda(T^{-1}A)$ then $|h|^p = \frac{d\lambda'}{d\lambda}$ λ-almost everywhere.*

Lemma 4.3. *Let λ be a probability measure on Ω and $\varepsilon > 0$. There exists $\delta > 0$ such that whenever $\sigma \in \mathrm{Iso}(L_p(\lambda))$ satisfies $\|\sigma(1) - 1\|_p \leq \delta$, then for every measurable set $A \subseteq \Omega$, we have :*

$$|\lambda(TA) - \lambda(A)| \leq \varepsilon$$

where T is a regular set isomorphism defining σ.

Proof. Write $\sigma f = h.(Tf)$ as in Banach-Lamperti's Theorem. Take $0 < \eta < 1$. First, notice that as in the proof of Lemma 4.1:

$$\eta^p \lambda(|h - 1| > \eta) \leq \|\sigma 1 - 1\|^p$$

We deduce that :

$$\begin{aligned}
1 = \|\sigma 1\| &= \int_{\Omega} |h|^p d\lambda \\
&= \int_{|h-1|\leq \eta} |h|^p d\lambda + \int_{|h-1|>\eta} |h|^p d\lambda \\
&\geq \lambda(|h-1| \leq \eta)(1-\eta)^p + \int_{|h-1|>\eta} |h|^p d\lambda \\
&\geq (1-\eta)^p \left(1 - \|\sigma 1 - 1\|^p/\eta^p\right) + \int_{|h-1|>\eta} |h|^p d\lambda \\
&\geq (1-\eta)^p - \|\sigma 1 - 1\|^p/\eta^p + \int_{|h-1|>\eta} |h|^p d\lambda
\end{aligned}$$

Using the two previous equations we get :

$$\begin{aligned}
|\lambda(A) - \lambda(TA)| &= \left| \int_{TA} |h|^p - 1 d\lambda \right| \\
&\leq \int_{(|h-1|>\eta)} |h|^p + 1 d\lambda + \int_{TA\cap(1\leq h\leq 1+\eta)} |h|^p - 1 d\lambda + \int_{TA\cap(1-\eta\leq h\leq 1)} 1 - |h|^p d\lambda \\
&\leq 2\|\sigma 1 - 1\|^p/\eta^p + 1 + (1+\eta)^p - 2(1-\eta)^p
\end{aligned}$$

For η small enough, we have $|\lambda(A) - \lambda(TA)| \leq 2\|\sigma 1 - 1\|^p/\eta^p + \varepsilon/2$, thus $\delta := (\varepsilon/4)^{1/p}/\eta$ is as required. □

Fact 4.4. *Let* (Ω, Σ) *be a measurable space,* $f = (f_1, ..., f_d) : \Omega \to \mathbb{R}^d$ *be a measurable function and* T *a regular set isomorphism. Define* $Tf = (Tf_1, ..., Tf_d)$*. Then for every Borel* $B \in \mathbb{R}^d$*,* $(Tf)^{-1}(B) = T(f^{-1}(B))$*.*

Lemma 4.5. *Let* λ *be a probability measure on* Ω*,* $f_1, ...f_d, g_1, ...g_d \in L_p(\lambda)$*. Assume that for every* $\varepsilon > 0$*, we can find* $\sigma \in \mathrm{Iso}(L_p(\lambda))$ *such that* $\|\sigma 1 - 1\| < \varepsilon$ *and* $\|\sigma f_i - g_i\| < \varepsilon$ *for every* $1 \leq i \leq d$*. Then* $f = (f_1, ..., f_d)$ *and* $g = (g_1, ..., g_d)$ *are equimeasurable.*

Proof. Let $\varepsilon > 0$, $B \subseteq \mathbb{R}^d$ Borel, $K > 0$ and $0 < \eta < 1/2$. Take $\sigma \in \mathrm{Iso}(L_p(\lambda))$ and T a regular set isomorphism defining it.

$$\lambda(f \in B) \leq |\lambda(f \in B) - \lambda(Tf \in B)| + \lambda(Tf \in B)$$

From Fact 4.4 and Lemma 4.3, we get $\lambda(f \in B) \leq \varepsilon/4 + \lambda(Tf \in B)$ if $\|\sigma 1 - 1\|$ is small enough. Now, on the other hand :

$$\begin{aligned}
\lambda(Tf \in B) &\leq \lambda(Tf \in B \cap (|\sigma f_1|, ..., |\sigma f_d| \leq K \cap |h-1| \leq \eta) \\
&\quad + \lambda(|\sigma f_1|, ..., |\sigma f_d| > K) + \lambda(|h-1| > \eta) \\
&\leq \lambda(\sigma f \in B_{2\eta K}) + d\frac{\|\sigma f\|^p}{K^p} + \frac{\|\sigma 1 - 1\|^p}{\eta^p}
\end{aligned}$$

Now if one fixes a large enough K, we get :

$$\lambda(f \in B) \leq \lambda(\sigma f \in B_{2\eta K}) + \varepsilon/2 + \frac{\|\sigma 1 - 1\|^p}{\eta^p}$$

Consequently, if one fixes a small enough η we get as long as $\|\sigma 1 - 1\|^p \le \varepsilon\eta^p/2$:

$$\lambda(f \in B) \le \lambda(\sigma f \in B_\varepsilon) + \varepsilon$$

Finally, we can deduce using Lemma 4.1 that for some $\delta > 0$, as long as $\|\sigma f - g\| \le \delta$:

$$\lambda(f \in B) < \lambda(g \in B_{2\varepsilon}) + 2\varepsilon$$

Hence, $\pi(f_*\lambda, g_*\lambda) \le 2\varepsilon$, f and g are equimeasurable. □

Proposition 4.6. *Let $f = (f_1, ..., f_d) \in L_p(0,1)^d$ and $g = (g_1, ..., g_d) \in L_p(0,1)^d$. The following are equivalent :*

- *(i) f and g are equimeasurable.*
- *(ii) For every $\varepsilon > 0$ there is $\sigma \in \mathrm{Iso}(L_p(0,1))$ such that $\sigma 1 = 1$ and $\|\sigma f - g\| < \varepsilon$.*
- *(iii)* $\mathrm{tp}(f/1) = \mathrm{tp}(g/1)$

Proof. (ii) $\Rightarrow$ (iii) is obvious, and (iii) $\Rightarrow$ (i) is Lemma 4.5 combined with the fact that $L_p(0,1)$ is approximately homogeneous.

Suppose (i) and without loss of generality, we might as well assume $\|f\| = \|g\| = 1$. Let $\varepsilon > 0$. First, take a compact space $K \subseteq \mathbb{R}^d$ such that $\|f \restriction K\| > 1 - \varepsilon$. Split this compact space into k many Borel pieces $K_1, ...K_k$ of diameters less than $\varepsilon^{1/p}$ and pick $z_i \in K_i$ for each i. Now define $\tilde{f} \in L_p(0,1)^d$ by the following :

$$\tilde{f}(t) = \begin{cases} z_i & \text{if } t \in f^{-1}(K_i) \\ 0 & \text{if } t \notin f^{-1}(K) \end{cases}$$

Notice that for every $1 \le j \le d$:

$$\begin{aligned} \|f_j - \tilde{f}_j\| &\le \|f \restriction (\mathbb{R}^d \setminus K)\| + \sum_{1 \le i \le k} \int_{f^{-1}(K_i)} |f_j - z_{ij}|^p d\lambda \\ &< \varepsilon + \sum_{1 \le i \le k} \lambda(f \in K_i)\varepsilon \\ &< 2\varepsilon \end{aligned}$$

from which we deduce that $\|f - \tilde{f}\| < 2\varepsilon$. The same is done with g instead of f to get $\tilde{g} \in L_p(0,1)^d$ and similarly, $\|g - \tilde{g}\| < 2\varepsilon$. Now, take T an automorphism of the probability algebra of $[0,1]$ sending $f^{-1}(K_i)$ to $g^{-1}(K_i)$ for each i (T exists by equimeasurability of f and g and because the probability algebra considered is strongly homogeneous, see [BH]). T induces a linear isometry $\sigma : h \mapsto Th$ between $L_p(0,1)$ and itself. This isometry fixes 1 and sends $\tilde{f}$ to $\tilde{g}$. We thus get :

$$\|\sigma f - g\| \le \|f - \tilde{f}\| + \|g - \tilde{g}\| \le 4\varepsilon$$

□

4.2 Case where p is not even

In the case where p is not even, one can say much more. Let $1 \le p < +\infty$ not even.

Proposition 4.7 (Rudin, [Rud] Theorem I). *Let $f = (f_1, ..., f_d) \in L_p(0,1)^d$ and $g = (g_1, ..., g_d) \in L_p(0,1)^d$ and $T : \langle 1, f\rangle \to \langle 1, g\rangle$ be an isometry such that $T1 = 1$. Then f and g are equimeasurable.*

With the results of the previous section, this theorem tells us that if $(1, f_1, ..., f_d)$ and $(1, g_1, ..., g_d)$ have the same quantifier-free type, then they have the same type. In other words, the expansion $(L_p(0,1), 1)$ of $L_p(0,1)$ has quantifier elimination. With a bit more work, we can actually prove the following :

Theorem 4.8 (Lusky, [Lus] Proof of Theorem 3)**.** *$L_p(0,1)$ is approximately ultrahomogeneous and thus T_p^* has quantifier elimination. Hence, T_p^* is model complete and is the model completion of its universal part.*

Corollary 4.9. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a continuous map, $A_1, ..., A_d \in \mathbb{R}^k$ and $b_1, ..., b_d \in \mathbb{R}$. For every $\varepsilon > 0$, there are $n \geq 0$, $G : \mathbb{R}^n \to \mathbb{R}$ continuous and $C_1, ..., C_n \in \mathbb{R}^k$ such that for all $f = (f_1, ..., f_k)$ where $f_i \in L_p(0,1)$, $\|f_1\|, ..., \|f_k\| \leq 1$ we have :*

$$\left| \inf_{\|g\| \leq 1} F(\|A_1 f + b_1 g\|, ..., \|A_d f + b_d g\|) - G(\|C_1 f\|, ..., \|C_n f\|) \right| < \varepsilon$$

Approximate ultrahomogeneity of $L_p(0,1)$ for $p \neq 4, 6, 8, ...$ is actually the best we can get as already noticed by Rudin [Rud]. Indeed, assume that f and g have the same quantifier-free type in $L_p(0,1)$ where $p = 2m$, $m = 1, 2, 3, ...$ This exactly means that for every $\lambda \in \mathbb{R}$, $\|\lambda + f\|^{2m} = \|\lambda + g\|^{2m}$. These are two polynomials in λ, so this is also equivalent to :

$$\int_0^1 f(t)^k dt = \int_0^1 g(t)^k dt$$

for every $k = 1, 2, ..., p$. But in general, this does not imply that f and g are equimeasurable. The failure of approximate ultrahomogeneity seems to be related with how the subspaces of $L_p(0,1)$ are complemented (see the proof of Proposition 2.1 in [FLAMT] and [Ran]). There is still much to learn about this surprising dichotomy.

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