

# Rapport de stage de M1

Nicolas Daire, 2021

## 1 Contexte du stage

J'ai été accueilli par Fumiharu Kato à l'Université de technologie de Tokyo au Japon pour effectuer un stage de recherche en géométrie arithmétique. Le sujet du stage portait sur les dessins d'enfant et provenait d'un mémoire de master encadré par Kato l'an passé. J'ai pu être logé dans le logement international du campus à dix minutes à pieds du bâtiment principal, et ai été affecté un bureau dans une des salles de travail pour mastériens et doctorants. J'en ai aussi profité pour participer en tant qu'auditeur à un séminaire pour élève maintenu en présentiel d'introduction à la géométrie algébrique et à la géométrie rigide. Mon stage s'est conclu par un exposé de cinquante minutes en anglais sur mes travaux.

# Regular dessins whose moduli fields are nonabelian Galois extensions of $\mathbb{Q}$

Nicolas Daire, Fumiharu Kato, Yoshiaki Uchino

## Abstract

Gareth Jones asked during the 2014 SIGMAP conference for examples of regular dessins with nonabelian fields of moduli. In this paper, we first construct dessins whose moduli fields are nonabelian Galois extensions of the form  $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$ , where  $p$  is an odd prime and  $\zeta_p$  is a  $p$ th root of unity and  $q \in \mathbb{Q}$  is not a  $p$ th power, and we then show that their regular closures have the same moduli fields. Finally, in the special case  $p = q = 3$  we give another example of a regular dessin with moduli field  $\mathbb{Q}(\zeta_3, \sqrt[3]{3})$  of degree  $2^{19} \cdot 3^4$  and genus 14155777.

## 1 Introduction

Grothendieck first coined the term *Dessin d'enfant* in *Esquisse d'un Programme* [5] to denote a connected bicolored graph embedded on a compact connected oriented topological surface, such that the complement of the graph is a disjoint union of 2-cells. His motivation came from the one to one correspondance existing between this topological definition, the combinatorial data of the permutations of the edges following the cyclic ordering around vertices of both colors, and the geometric concept of coverings of  $\mathbb{P}^1$  by compact Riemann surfaces ramified at most over  $\{0, 1, \infty\}$ . Moreover, by Belyi's theorem any such covering is given the structure of an algebraic curve defined over a number field, and conversely any algebraic curve defined over a number field admits a covering of  $\mathbb{P}^1$  ramified at most over  $\{0, 1, \infty\}$ . The absolute Galois group  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  then acts naturally on the set of isomorphism classes of dessins, and a lot of the interest for dessins stems from the fact that this action is faithful. Grothendieck soon turns his attention to the family of regular dessins, characterized by the fact that their automorphism groups act transitively on their sets of edges.

To any dessin we can associate a number field we call its moduli field, which is defined as the subfield of  $\bar{\mathbb{Q}}$  fixed by all the elements of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  that send the dessin to an isomorphic dessin. Conder, Jones, Streit and Wolfart noted in [1] that all examples of regular dessins known at the time had moduli fields that were abelian Galois extensions of  $\mathbb{Q}$ . Herradón constructed in [7] an explicit equation for a regular dessin whose moduli field  $\mathbb{Q}(\sqrt[3]{2})$  is not a Galois extension of  $\mathbb{Q}$ , and Hidalgo later generalized his construction in [8] to produce regular dessins whose moduli fields are of the form  $\mathbb{Q}(\sqrt[p]{2})$  where  $p$  is an odd prime number. However there is as of yet no known example of regular dessin whose moduli field is a nonabelian Galois extension of  $\mathbb{Q}$ . This is the starting point of this paper, in which we will exhibit examples of regular dessins with moduli fields that are nonabelian Galois extensions of  $\mathbb{Q}$ .

In the present paper, we will first recall the main definitions and theorems on dessins d'enfant, then we will discuss examples of regular dessins whose moduli fields are nonabelian Galois extensions of  $\mathbb{Q}$ . We exhibit dessins whose moduli fields are of the form  $\mathbb{Q}(\zeta_3, \sqrt[3]{q})$ , where  $\zeta_3$  is a primitive third root of unity and  $q \in \mathbb{Q}$  is not a third power, and show that the regular closures of these dessins possess the same moduli fields. We then generalize this construction to show that there exist regular dessins with moduli fields  $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$ , where  $\zeta_p$  is a primitive  $p$ th root of unity and  $q \in \mathbb{Q}_{>0}$  is not a  $p$ th power. Finally, we give an example of a regular dessin with moduli field  $\mathbb{Q}(\zeta_3, \sqrt[3]{3})$  of degree 42467328 and genus 14155777.

*Notations:*

- $\mathfrak{S}_E$ : the group of self-bijections of the set  $E$ , similarly  $\mathfrak{S}_n$  is the group of permutations of a set of  $n$  elements (we favor a right action, hence we write the product  $\sigma\tau := \tau \circ \sigma$ )
- $\text{Gal}(E/F)$ : the Galois group of  $F$ -automorphisms of  $E$
- $\zeta_k$ : the  $k$ th primitive root of unity  $\exp(\frac{2i\pi}{k})$
- $F_2$ : the free group of rank 2 with generators  $(\xi, \eta)$
- Crit: the set of critical values of a function

## 2 Dessins d'enfant

Dessins d'enfant admit multiple equivalent definitions in several domains of mathematics, which makes them interesting objects to study. We give here an overview of several constructions which we will use in the following. We refer the reader to existing expositions of the theory such as [6], [11] and [9] for proofs of the presented facts and further details.

**Definition 2.1.** A *dessin* is a pair  $(X, \mathcal{D})$  where  $X$  is a compact connected oriented topological surface, and  $\mathcal{D}$  is a connected bicolored graph with black and white vertices embedded in  $X$  such that  $X \setminus \mathcal{D}$  is homeomorphic to a disjoint union of open disks (e.g. Figure 1).

Two dessins are isomorphic if there exists a homeomorphism between the underlying surfaces that induces an isomorphism on the embedded bicolored graphs.

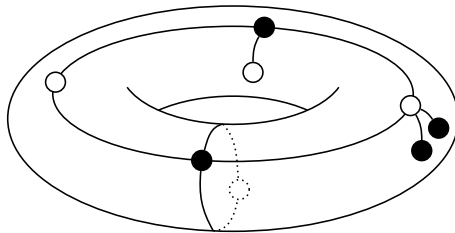


Figure 1: Example of a dessin on a surface of genus 1.

Let  $(X, \mathcal{D})$  be a dessin, and denote by  $E$  the set of edges of  $\mathcal{D}$ . Using the orientation of  $X$ , at a given vertex we can define a cyclic ordering of the incident edges by enumerating them in a counter-clockwise fashion. Let  $\sigma \in \mathfrak{S}_E$  (resp.

$\tau \in \mathfrak{S}_E$ ) be the product of those cyclic permutations around the black (resp. white) vertices of  $\mathcal{D}$ , and let  $\phi := \sigma\tau$  (e.g. Figure 2).

We call the group  $\text{Cart}(\mathcal{D}) := \langle \sigma, \tau \rangle$  the *cartographic group* of the dessin, and the pair of the distinguished elements  $(\sigma, \tau)$  its *canonical generators*. The cartographic group  $\text{Cart}(\mathcal{D})$  acts naturally on the right on  $E$ , and we denote by  $e^\omega$  the image of the edge  $e \in E$  by an element  $\omega \in \text{Cart}(\mathcal{D})$ . Conversely, the cycles of  $\sigma$  (resp.  $\tau$ , resp.  $\phi$ ) are in bijection with the black vertices (resp. white vertices, resp. faces) of  $\mathcal{D}$ . We may thus reconstruct a dessin from the action of its cartographic group on its set of edges by gluing the faces along their edges according to this data.

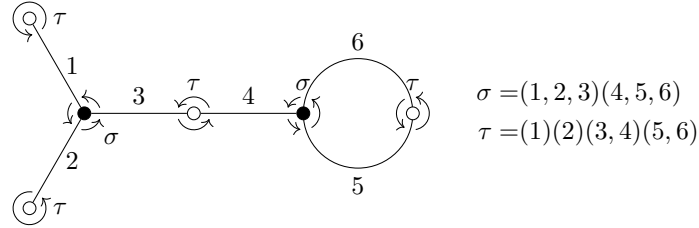


Figure 2: Example of the action of the cartographic group.

We will now describe the correspondance between dessins drawn on surfaces and coverings of  $\mathbb{P}^1$  ramified at most over  $\{0, 1, \infty\}$ .

**Definition 2.2** (Belyi pair). Let  $X$  be a compact Riemann surface, and  $\beta: X \rightarrow \mathbb{P}^1$  a holomorphic map ramified at most over  $\{0, 1, \infty\}$ . We call  $\beta$  a *Belyi function*, and  $(X, \beta)$  a *Belyi pair*.

To a Belyi pair  $(X, \beta)$  we can easily associate the dessin  $(X, \mathcal{D})$ , where the sets of black and white vertices of  $\mathcal{D}$  are defined as the preimages of 0 and 1 respectively, and the edges are defined as the connected components of the preimage of the real open interval  $]0, 1[$ .

Conversely, let  $(X, \mathcal{D})$  be a dessin. Let us add a vertex denoted by a star ( $\star$ ) in each face of  $X \setminus \mathcal{D}$ . We then triangulate  $X$  by adding edges between each star and the vertices on the corresponding face. We obtain a cover of  $X$  by lozenges made up of pairs of triangles sharing an edge with black and star vertices. By mapping those lozenges onto  $\mathbb{P}^1$  by sending the black, white and star vertices respectively to 0, 1 and  $\infty$ , and the two triangles to the two hemispheres delimited by the real line, we get a covering  $\beta: X \rightarrow \mathbb{P}^1$  ramified at most over  $\{0, 1, \infty\}$  (see Figure 3). Finally, we give  $X$  a Riemann surface structure by requiring that  $\beta$  is a holomorphic mapping.

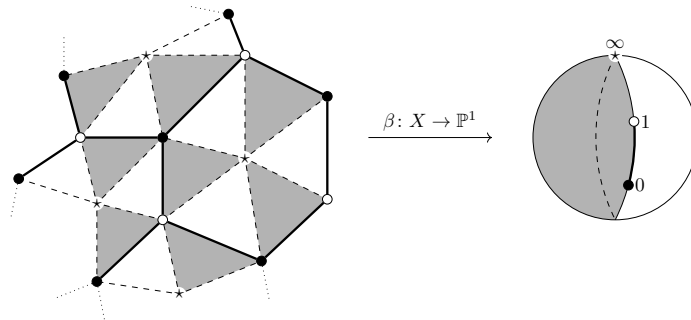


Figure 3: Construction of a ramified covering.

Let  $(X, \beta)$  be a Belyi pair. The fundamental group  $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$  is isomorphic to  $F_2$ , the free group of rank two, the covering  $\beta$  thus induces a monodromy map  $M: F_2 \rightarrow \mathfrak{S}_{\beta^{-1}(\frac{1}{2})}$ , and we call its image  $\text{Mon}(\mathcal{D}) \leq \mathfrak{S}_{\beta^{-1}(\frac{1}{2})}$  the monodromy group. This is equivalently determined by any stabilizer  $H$  of the monodromy action, which is a finite index subgroup of  $F_2$ . The fiber  $\beta^{-1}(\frac{1}{2})$  is in bijection with the set of edges  $E$  of the graph, and we choose for generators  $(\xi, \eta)$  of the fundamental group  $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$  the classes of the two loops

$$\begin{aligned} \xi: [0, 1] &\rightarrow \mathbb{P}^1 & \eta: [0, 1] &\rightarrow \mathbb{P}^1 \\ t &\mapsto \frac{1}{2}e^{2i\pi} & t &\mapsto 1 - \frac{1}{2}e^{2i\pi} \end{aligned}$$

starting at  $\frac{1}{2}$  and circling respectively around the points 0 and 1 counter-clockwise (see Figure 4). Therefore the monodromy map induces the same action on  $E$  as that of the cartographic group, and we have an isomorphism  $\text{Mon}(\mathcal{D}) \cong \text{Cart}(\mathcal{D})$  that sends the canonical generators  $(M(\xi), M(\eta))$  to the canonical generators of  $\text{Cart}(\mathcal{D})$ . Two dessins are isomorphic if and only if the corresponding monodromy maps are conjugate in  $\mathfrak{S}_E$ , or equivalently if the corresponding finite index subgroups of  $F_2$  are conjugate.

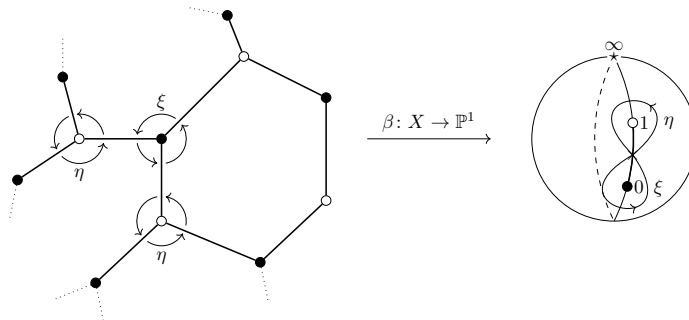


Figure 4: Example of the monodromy action on the set of edges.

Let us now introduce the central theorem of dessins d'enfants.

**Theorem 2.3** (Belyi's theorem). *Let  $C$  be a complex algebraic curve. Then  $C$  is defined over a number field if and only if there exists a rational function  $\beta: C \rightarrow \mathbb{P}^1$  ramified at most over  $\{0, 1, \infty\}$ .*

*Proof.* See [10] for a proof in the language of algebraic geometry, or [3] for a proof using a different approach.  $\square$

Thanks to this result we may now assume every Belyi pair to be a pair  $(C, \beta)$  where  $C$  is an algebraic curve and  $\beta$  is a rational function, both defined over a number field. Moreover, it implies that any algebraic curve defined over a number field is the underlying curve of a Belyi pair, revealing a profound connection between dessins and algebraic curves defined over number fields.

Instead of the topological fundamental group, we can now naturally consider the algebraic fundamental group of  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ , i.e. the profinite completion of the topological fundamental group,

$$\pi_1^{\text{alg}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \cong \hat{F}_2 = \varprojlim_{N \in \mathcal{N}} F_2/N$$

where  $\mathcal{N}$  is the set of finite index subgroups of  $F_2$ . In the same way as before, we see that a dessin is determined by an open subgroup of  $\hat{F}_2$ , and two dessins are isomorphic if and only if the corresponding open subgroups of  $\hat{F}_2$  are conjugate.

Let us summarize all the equivalent definitions of dessins d'enfant introduced so far.

**Theorem 2.4.** *A dessin is uniquely determined up to isomorphism by:*

- *an isomorphism class of Belyi pairs;*
- *an isomorphism class of transitive actions of  $F_2$  on finite sets;*
- *a conjugacy class of finite index subgroups of  $F_2$ ;*
- *a conjugacy class of open subgroups of  $\hat{F}_2$ ;*

### 3 Regular dessins

In this section we focus on a particularly interesting family of dessins, the regular dessins.

**Theorem-Definition 3.1.** *Let  $\mathcal{D}$  be a dessin. The following conditions are equivalent:*

1. *the automorphism group of  $\mathcal{D}$  acts transitively on its set of edges;*
2. *the defining pair  $(C, \beta)$  is a Galois cover, i.e.  $\beta$  is the quotient of  $C$  by a subgroup of its automorphism group;*
3. *the defining subgroup of  $F_2$  (or equivalently of  $\hat{F}_2$ ) is normal;*
4. *the order of the cartographic group of  $\mathcal{D}$  is equal to its degree;*
5. *there exists an isomorphism  $\text{Aut}(\mathcal{D}) \cong \text{Cart}(\mathcal{D})$ .*

*When those conditions hold we say that  $\mathcal{D}$  is a regular dessin.*

*Proof.* See Theorem 4.43 in [3] for the equivalence between 1, 2, 3 and 4. See section 2.9 in [3] for the equivalence with 5.  $\square$

These additional conditions lead to the following characterization of regular dessins.

**Corollary 3.2.** *Let  $\mathcal{D}$  be a regular dessin. The action of  $\text{Cart}(\mathcal{D})$  on the set of edges of  $\mathcal{D}$  is isomorphic to the natural action of  $\text{Cart}(\mathcal{D})$  on itself. The set of isomorphism classes of regular dessins is thus in bijection with the set of isomorphism classes of finite groups generated by two distinguished elements. Two regular dessins corresponding to the groups  $G = \langle \xi, \eta \rangle$  and  $G' = \langle \xi', \eta' \rangle$  are isomorphic if and only if the map  $(\xi \mapsto \xi', \eta \mapsto \eta')$  extends to an isomorphism of groups  $G \cong G'$ .*

With the following result we can canonically associate a regular dessin to any dessin.

**Theorem-Definition 3.3.** *Let  $\mathcal{D}$  be a dessin determined by the Belyi pair  $(C, \beta)$  and the subgroup  $H \leq \hat{F}_2$ . Up to isomorphism there exists a unique regular dessin  $\tilde{\mathcal{D}}$  with a morphism  $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$ , such that  $\tilde{\mathcal{D}}$  is minimal among the regular dessins with this property, i.e. any morphism  $\mathcal{D}' \rightarrow \mathcal{D}$  where  $\mathcal{D}'$  is a regular dessin factors through  $\tilde{\mathcal{D}}$ . We say that  $\tilde{\mathcal{D}}$  is the regular closure of  $\mathcal{D}$ .*

*The regular closure  $\tilde{\mathcal{D}}$  is given by the Galois closure  $(\tilde{C}, \tilde{\beta})$  and by the normal subgroup*

$$\text{Core}_{F_2}(H) := \bigcap_{g \in \hat{F}_2} gHg^{-1} \leq \hat{F}_2.$$

*Moreover, there exists an isomorphism  $\text{Cart}(\tilde{\mathcal{D}}) \cong \text{Cart}(\mathcal{D})$  that preserves the canonical generators.*

*Proof.* See section 2.9 in [3].  $\square$

## 4 Galois action and moduli field

Finally, let us describe the action of the absolute Galois group on the set of dessins, and introduce the notion of moduli field of a dessin, which plays an important role in what follows.

**Definition 4.1.** Let  $\mathcal{D}$  be a dessin determined by the Belyi pair  $(C, \beta)$ , and let  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . Recall that the Belyi pair is defined over a number field, so we can define  $\mathcal{D}^\sigma$  to be the dessin associated to the Belyi pair  $(C^\sigma, \beta^\sigma)$  obtained from  $(C, \beta)$  by applying  $\sigma$  to the coefficients of the polynomial defining  $C$  and to the coefficients of the rational function  $\beta$ . Furthermore morphisms between Belyi pairs are also rational functions, and by defining the action on morphisms in the same way we obtain an action on the category of isomorphism classes of dessins.

We say that two dessins are Galois conjugate if the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  sends one to the other.

**Lemma 4.2.** *The following are invariant under Galois action:*

1. the degree,
2. the list of ramification indices of ramified points,

3. the genus of the underlying surface,
4. the automorphism group up to isomorphism,
5. the cartographic group up to isomorphism (without necessarily identifying the canonical generators).

*Proof.* The degree and the list of ramification indices are clearly preserved by Galois action, and then so is the genus by the Riemann-Hurwitz formula.

The automorphism group is preserved up to isomorphism by functoriality of the action. To show that the cartographic group is also preserved, we use the Lemma 4.3 below. Let  $\mathcal{D}$  be a dessin. We then have the chain of isomorphisms

$$\mathrm{Cart}(\mathcal{D}^\sigma) \cong \mathrm{Cart}(\widetilde{\mathcal{D}}^\sigma) \cong \mathrm{Aut}(\widetilde{\mathcal{D}}^\sigma) \cong \mathrm{Aut}(\widetilde{\mathcal{D}}) \cong \mathrm{Cart}(\widetilde{\mathcal{D}}) \cong \mathrm{Cart}(\mathcal{D}).$$

□

**Lemma 4.3.** *The Galois action sends regular dessins to regular dessins. More precisely, if  $\mathcal{D}$  is a dessin and  $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , then  $(\widetilde{\mathcal{D}})^\sigma \cong \widetilde{(\mathcal{D}^\sigma)}$ .*

*Proof.* By Lemma 4.2 above, the Galois action preserves the degrees and the automorphism groups of dessins, so it sends regular dessins to regular dessins. The regular closures are defined by a condition of minimality with regards to morphisms, so by functoriality of the Galois action they are also preserved. □

The interest for the action of  $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on dessins arises from the following theorem.

**Theorem 4.4.** *The action of the absolute Galois group  $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on the set of isomorphism classes of dessins is faithful.*

*In fact, the action is already faithful on the set of isomorphism classes of regular dessins.*

*Proof.* For the first assertion, by Belyi's theorem it suffices to show that the action is faithful on isomorphism classes of algebraic curves defined over number fields, and we may further restrict to those of genus one. Let

$$C_\lambda: y^2 = x(x-1)(x-\lambda)$$

be the elliptic curve with parameter  $\lambda \neq 0, 1$ . Using the  $j$ -invariant we have a bijection between the set of isomorphism classes of elliptic curves over  $\mathbb{C}$  and  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  given by

$$C_\lambda \mapsto j(\lambda) := \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}.$$

Thus for  $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \setminus \{\mathrm{id}\}$  we can choose  $\lambda \in \bar{\mathbb{Q}} \setminus \{0, 1\}$  such that  $j(\sigma(\lambda)) = \sigma(j(\lambda)) \neq j(\lambda)$ , and then  $C_\lambda^\sigma \cong C_{\sigma(\lambda)} \not\cong C_\lambda$ .

See [4] for a proof of the faithfulness of the action on the set of isomorphism classes of regular dessins. □

Let us now associate number fields to dessins. Let  $\mathcal{D}$  be a dessin. We say that a number field  $k$  is a field of definition of  $\mathcal{D}$  if there exists a dessin defined over  $k$  that is isomorphic to  $\mathcal{D}$ . Any extension of a field of definition is again a field of definition, however there does not necessarily exist one smallest field of definition determined by  $\mathcal{D}$ , see for example [2]. Instead we look at the Galois action on  $\mathcal{D}$  to define the following.

**Definition 4.5.** Let  $\mathcal{D}$  be a dessin. Let  $G := \{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \mathcal{D} \cong \mathcal{D}^\sigma\}$  be the subgroup of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  consisting of those elements that fix  $\mathcal{D}$  up to isomorphism. We call *moduli field* of  $\mathcal{D}$  and note  $\mathcal{M}(\mathcal{D}) := \overline{\mathbb{Q}}^G$  the subfield of  $\overline{\mathbb{Q}}$  fixed by  $G$ .

The moduli field of a dessin is clearly contained inside any of its fields of definition, however it is not necessarily itself a field of definition as stated above. However, for regular dessins we have the following.

**Lemma 4.6.** *Regular dessins are defined over their moduli fields.*

*Proof.* See Proposition 14 in [13]. □

## 5 Constructions of regular dessins with nonabelian moduli fields

We are now ready to give examples of regular dessins whose moduli fields are non-abelian Galois extensions of  $\mathbb{Q}$ . To do so, we will first exhibit dessins with such moduli fields, and then prove that their regular closures admit the same moduli fields.

Before proceeding with the examples, let us first present a classic family of Belyi polynomials that we will use in the following constructions. For positive integers  $m, n \in \mathbb{N}$  we define the polynomial

$$B_{m,n} := \frac{(m+n)^{m+n}}{m^m n^n} X^m (1-X)^n \in \mathbb{Q}[X].$$

By computing the derivative  $B'_{m,n} = \frac{(m+n)^{m+n}}{m^m n^n} X^{m-1} (1-X)^{n-1} (m - (m+n)X)$  we verify that  $B_{m,n} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a Belyi function that ramifies only at  $0, 1, \infty$  and  $\frac{m}{m+n}$  with ramification indices  $m, n, m+n$  and  $2$  respectively, and  $B_{m,n}(0) = 0$ ,  $B_{m,n}(1) = 0$ ,  $B_{m,n}(\infty) = \infty$  and  $B_{m,n}(\frac{m}{m+n}) = 1$  (see Figure 5).

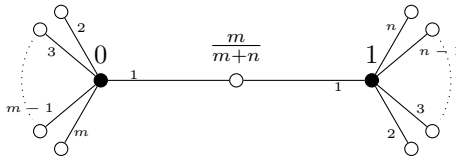


Figure 5: Dessin corresponding to the Belyi pair  $(\mathbb{P}^1, B_{m,n})$ .

## 5.1 Regular dessins with moduli fields of the form $\mathbb{Q}(\zeta_3, \sqrt[3]{q})$

Let  $q \in \mathbb{Q}_{>0}$  be a positive rational number that is not a third power.

Let  $m, n \in \mathbb{N}$  be coprime positive integers such that  $\frac{27}{27+q^2} = \frac{m}{m+n}$ , and let

$$C: y^2 = x(x - (1 - \zeta_3))(x - \sqrt[3]{q}),$$

$$\beta: C \rightarrow \mathbb{P}^1, (x, y) \mapsto \frac{1}{27^m q^{2n}} (x^6 + 27)^m (q^2 - x^6)^n.$$

The function  $\beta$  is given by the composition  $\beta = \beta_1 \circ \beta_0 \circ \pi$  of the following maps.

1.  $\pi: C \rightarrow \mathbb{P}^1$  is the projection on the coordinate  $x$ , which is ramified over  $\{0, 1 - \zeta_3, \sqrt[3]{q}, \infty\}$ .
2.  $\beta_0 := X^6 \in \mathbb{Q}[X]$ ,  $\text{Crit}(\beta_0) = \{0\}$  so  $\beta_1 \circ \pi$  ramifies over  $\{0, (1 - \zeta_3)^6 = -27, q^2, \infty\}$ .
3.  $\beta_1 := B_{m,n}(\frac{X+27}{q^2+27})$ , so  $\beta = \beta_1 \circ \beta_0 \circ \pi$  ramifies over  $\{0, 1, \infty\}$ .

The pair  $(C, \beta)$  is thus a Belyi pair, and we call  $\mathcal{D}$  the corresponding dessin. The dessin  $\mathcal{D}$  is defined over  $\mathbb{Q}(\zeta_3, \sqrt[3]{q})$ , so its moduli field is one of its subfields, and the same holds for  $\tilde{\mathcal{D}}$ . We will prove that  $\mathcal{M}(\tilde{\mathcal{D}})$  is exactly  $\mathbb{Q}(\zeta_3, \sqrt[3]{q})$ , which is a nonabelian Galois extension of  $\mathbb{Q}$  with Galois group

$$\text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{q})/\mathbb{Q}) \cong \mathfrak{S}_3.$$

To that end we must show that an automorphism  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  fixes  $\tilde{\mathcal{D}}$  if and only if it fixes  $\zeta_3$  and  $\sqrt[3]{q}$ , or equivalently that  $\text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{q})/\mathbb{Q})$  acts freely on the orbit of  $\tilde{\mathcal{D}}$ .

Let  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , the Galois conjugate  $\mathcal{D}^\sigma$  is given by the Belyi pair  $(C^\sigma, \beta^\sigma)$ , where

$$C^\sigma: y^2 = x(x - (1 - \sigma(\zeta_3)))(x - \sigma(\sqrt[3]{q})),$$

and  $\beta^\sigma$  has the same expression as  $\beta$  because all of its coefficients are rational. The orbit of the pair  $(\zeta_3, \sqrt[3]{q})$  by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  is  $\{\zeta_3^i, \zeta_3^j \sqrt[3]{q}\}_{1 \leq i \leq 2, 0 \leq j \leq 2}$ . Elliptic curves given by equations of the form  $y^2 = (x - a)(x - b)(x - c)$  are isomorphic if and only if the cross-ratios of the tuples  $(a, b, c, \infty)$  coincide. We verify that the cross-ratios are all distinct, so the orbit of  $\mathcal{D}$  is given by the six dessins  $\mathcal{D}^\sigma$  for  $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{q})/\mathbb{Q})$ . As a consequence  $\mathcal{M}(\mathcal{D}) = \mathbb{Q}(\zeta_3, \sqrt[3]{q})$ . To prove that the regular closures  $\tilde{\mathcal{D}}^\sigma$  constituting the orbit of  $\tilde{\mathcal{D}}$  are also non isomorphic, we must first draw the dessins  $\mathcal{D}^\sigma$  to compute their cartographic groups.

Let us first draw the dessin  $\mathcal{D}_0$  corresponding to the Belyi pair  $(\mathbb{P}^1, \beta_1 \circ \beta_0)$  (see Figure 7). The dessin  $\mathcal{D}_0$  is defined over  $\mathbb{Q}$ , so the dessins  $\mathcal{D}^\sigma$  in the orbit are then obtained by lifting  $\mathcal{D}_0$  to the curves  $C^\sigma$ . To simplify the graphical representations of the dessins, we will use the notation in Figure 6 for consecutive edges incident to a vertex.

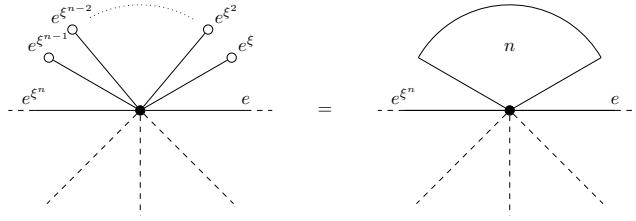


Figure 6: Notation for consecutive edges.

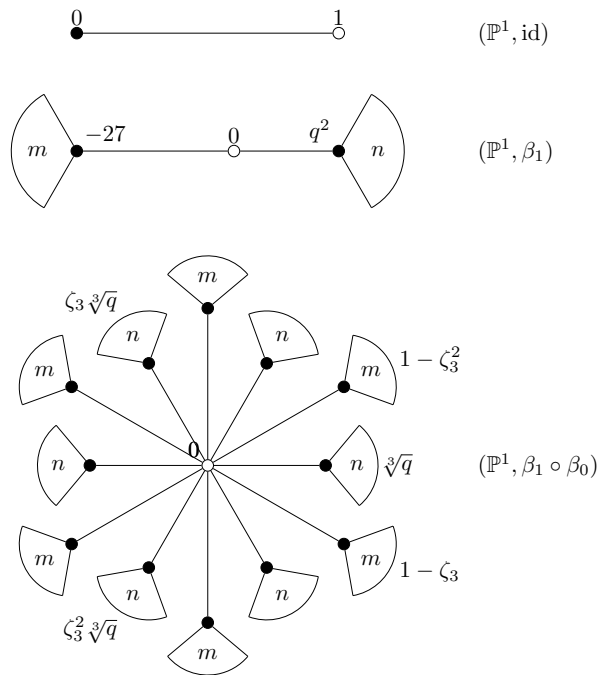
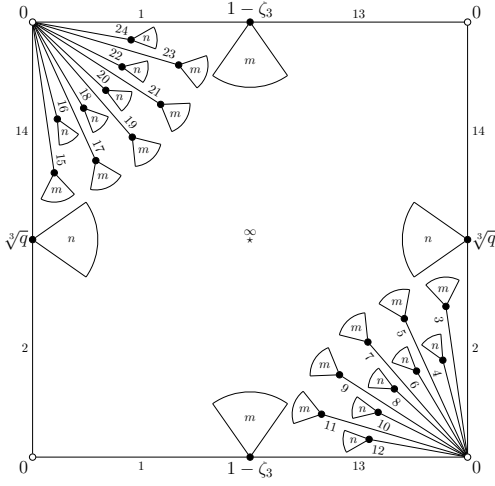
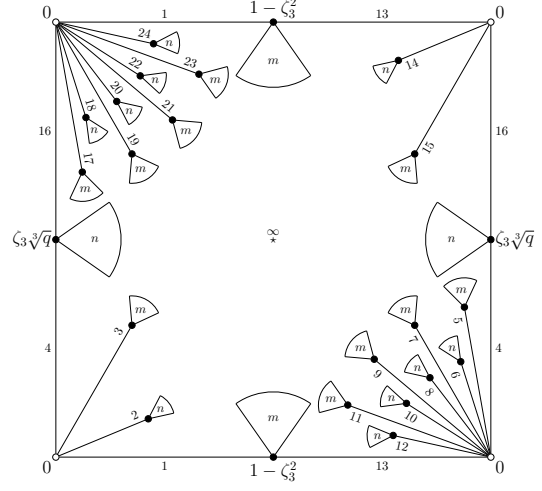


Figure 7: Construction of  $\mathcal{D}_0$ .

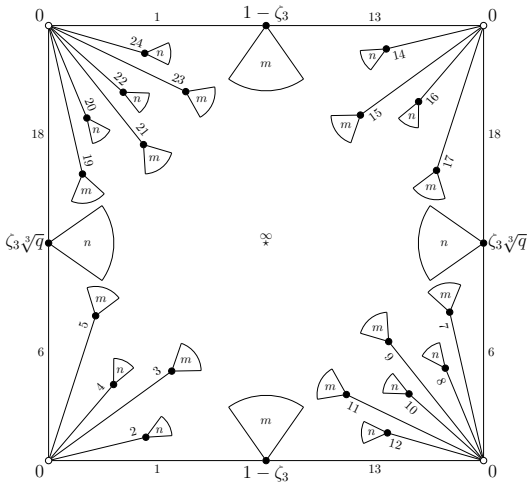
The dessins  $\mathcal{D}_1, \dots, \mathcal{D}_6$  conjugate to  $\mathcal{D}$  are embedded on a torus, so in the representations in Figure 8 we will identify the outermost edges on opposite sides.



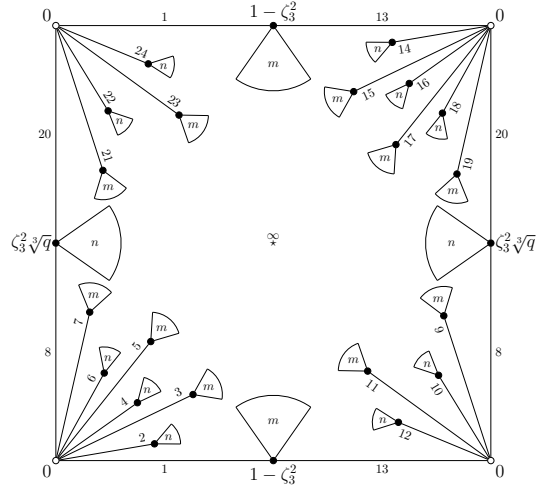
(a)  $\mathcal{D}_1 := \mathcal{D}$



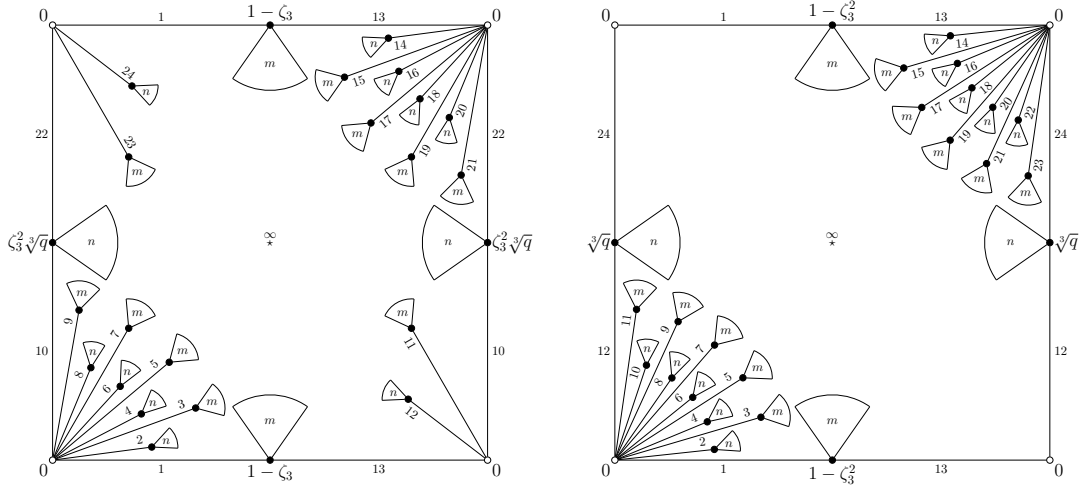
(b)  $\mathcal{D}_2 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3^2, \zeta_3 \sqrt[3]{q})$



(c)  $\mathcal{D}_3 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3, \zeta_3 \sqrt[3]{q})$



(d)  $\mathcal{D}_4 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3^2, \zeta_3^2 \sqrt[3]{q})$



(e)  $\mathcal{D}_5 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3, \zeta_3^2 \sqrt[3]{q})$

(f)  $\mathcal{D}_6 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3^2, \sqrt[3]{q})$

Figure 8: Dessins  $\mathcal{D}_1, \dots, \mathcal{D}_6$  in the Galois orbit of  $\mathcal{D}$ .

We will now establish that  $\widetilde{\mathcal{D}}_1$  is not isomorphic to  $\widetilde{\mathcal{D}}_2, \dots, \widetilde{\mathcal{D}}_6$ . To that end it suffices to show that there is no isomorphism between the cartographic groups fixing the canonical generators. We shall therefore exhibit an element  $\omega \in F_2 = \langle \xi, \eta \rangle$  such that  $M_k(\omega)$  commutes with  $M_k(\eta^2)$  only when  $k = 1$ , where  $M_k$  is the monodromy map of  $\mathcal{D}_k$ .

We have defined  $m$  and  $n$  to be positive coprime integers such that  $\frac{27}{27+q^2} = \frac{m}{m+n}$ , so we cannot have  $m = n = 1$ . We will treat the case where  $m \neq 1$  does not divide  $n$ , the other case being treated similarly. Let

$$\omega := \xi^n \eta^{-1} \xi^{m-n} \eta \xi^n.$$

We shall show that  $M_k(\omega)$  commutes with  $M_k(\eta^2)$  only when  $k = 1$ .

Let  $E_k := \{1, 2, \dots, 24\}$  be the set of edges of  $\mathcal{D}_k$  incident to 0. The action of  $\eta$  fixes the set  $E_k$  on which it induces the cyclic permutation  $(1, 2, \dots, 24)$ , and every white vertex except 0 has degree one so the action of  $\eta$  is trivial on the complement of  $E_k$ .

We can write  $E_k = E_k^{\text{odd}} \sqcup E_k^{\text{even}}$  as the disjoint union of the sets of respectively odd and even numbered edges incident to 0, such that  $\eta$  sends one to the other. The black vertices of  $E_k^{\text{odd}}$  are of degree  $m$  except for the two black vertices of the edges 1 and 13 that are of degree  $2m$ . Therefore if  $m$  does not divide some integer  $l$  then  $\xi^l$  sends every edge of  $E_k^{\text{odd}}$  to the complement of  $E_k$ , and otherwise the action of  $\xi^m$  on  $E_k^{\text{odd}}$  corresponds to the sole transposition  $(1, 13)$ . Similarly if  $n$  does not divide  $l$  then  $\xi^l$  sends every edge of  $E_k^{\text{even}}$  to the complement of  $E_k$ , and the action of  $\xi^n$  on  $E_k^{\text{even}}$  is the transposition  $(2k, 2k + 12)$ .

In particular, by hypothesis  $n$  is not a multiple of  $m$ , so  $m - n$  is not a multiple of  $m$  either, hence both  $\xi^n$  and  $\xi^{m-n}$  send the edges of  $E_k^{\text{odd}}$  to the complement of  $E_k$ . However  $\eta$  acts trivially on the latter, so  $\xi^n \eta^{-1} \xi^{m-n}$  and  $\xi^{m-n} \eta \xi^n$  both fix the set  $E_k^{\text{odd}}$  on which they induce the same action as  $\xi^m$ , i.e. the transposition  $(1, 13)$ .

Therefore the action of  $\omega = \xi^n \eta^{-1} \xi^{m-n} \eta \xi^n$  is the same as that of  $\xi^m \eta \xi^n$  on  $E_k^{\text{odd}}$  and the same as that of  $\xi^n \eta^{-1} \xi^m$  on  $E_k^{\text{even}}$ . See Figure 9.

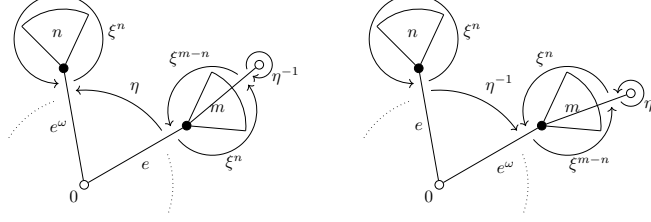


Figure 9: Action of  $\omega$  on  $E_k^{\text{odd}} \setminus \{1, 2k-1\}$  and on  $E_k^{\text{even}} \setminus \{2, 2k\}$ .

The action of  $\omega$  fixes the set  $E_k$  on which it induces the permutation

$$M_k(\omega)|_{E_k} = (1, 13)(2k, 2k+12) \cdot (1, 2)(3, 4) \cdots (23, 24) \cdot (1, 13)(2k, 2k+12)$$

Therefore for  $k = 1$ ,

$$\begin{aligned} M_1(\omega)|_{E_1} &= (1, 13)(2, 14) \cdot (1, 2)(3, 4) \cdots (23, 24) \cdot (1, 13)(2, 14) \\ &= (1, 2)(3, 4) \cdots (23, 24) \end{aligned}$$

so  $\omega$  and  $\eta^2$  commute on  $E_1$ . Moreover  $\eta$  acts trivially on the complement of  $E_1$  so  $M_1(\omega)|_{\mathcal{D}_1 \setminus E_1}$  and  $M_1(\eta^2)|_{\mathcal{D}_1 \setminus E_1}$  automatically commute. Finally, we obtain that  $M_1(\omega)$  and  $M_1(\eta^2)$  commute.

For  $k = 2$ , we observe that  $4^{\omega \eta^2} = 15^{\eta^2} = 17$  but  $4^{\eta^2 \omega} = 6^{\omega} = 5$ . Similarly, for  $3 \leq k \leq 6$ , we observe that  $1^{\omega \eta^2} = 14^{\eta^2} = 16$  but  $1^{\eta^2 \omega} = 3^{\omega} = 4$ . We have thus shown that  $M_k(\omega)$  and  $M_k(\eta^2)$  commute only for  $k = 1$ .

This concludes the proof that  $\tilde{\mathcal{D}}$  is a regular dessin with moduli field  $\mathbb{Q}(\zeta_3, \sqrt[3]{q})$ .

## 5.2 Regular dessins with moduli fields of the form $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$

Let  $p$  be an odd prime, and  $q \in \mathbb{Q}_{>0}$  a positive rational number that is not a  $p$ th power. In this example we will need an additional parameter  $\gamma \in \mathbb{Q} \setminus \{0\}$ . Let

$$C: y^2 = x(x - (1 - \zeta_p))(x - \gamma \sqrt[p]{q}).$$

We construct the Belyi function  $\beta: C \rightarrow \mathbb{P}^1$  as the composition  $\beta = \beta_2 \circ \beta_1 \circ \beta_0 \circ \pi$  of the following maps.

1.  $\pi: C \rightarrow \mathbb{P}^1$  is the projection on the coordinate  $x$ , which ramifies over  $\{0, 1 - \zeta_p, \gamma \sqrt[p]{q}, \infty\}$ .
2.  $\beta_0 := X^{2p} \in \mathbb{Q}[X]$ , and  $\text{Crit}(\beta_0) = \{0, \infty\}$  so  $\beta_0 \circ \pi$  ramifies over  $\{0, (1 - \zeta_p)^{2p}, \gamma^{2p} q^2, \infty\}$ .
3.  $\beta_1 \in \mathbb{Q}[X]$  is chosen independently of  $\gamma$  such that  $\text{Crit}(\beta_1) \cup \{\beta_1((1 - \zeta_p)^{2p})\} = \{0, 1, \infty\}$ ,  $\beta_1((1 - \zeta_p)^{2p}) = 0 < \beta_1(0) < 1$  and  $\beta_1'(0) > 0$ . The existence of  $\beta_1$  verifying those conditions is assured by Proposition 5.1 below. Under those assumptions  $\beta_1 \circ \beta_0 \circ \pi$  ramifies over  $\{0, 1, \beta_1(0), \beta_1(\gamma^{2p} q^2), \infty\}$ .

4.  $\gamma \in \mathbb{Q}_{>0}$  is then chosen small enough so that  $\beta'_1 > 0$  on  $[0, \gamma^{2p}q^2]$ . This guarantees us that we have  $0 < \beta_1(0) < \beta_1(\gamma^{2p}q^2) < 1$ .
5.  $\beta_2 := B_{r,s} \circ B_{m,n}$ , where  $(m, n)$  and  $(r, s)$  are pairs of coprime positive integers such that  $\beta_1(\gamma^{2p}q^2) = \frac{m}{m+n}$  and  $B_{m,n}(\beta_1(0)) = \frac{r}{r+s}$ . Finally,  $\beta = \beta_2 \circ \beta_1 \circ \beta_0 \circ \pi$  ramifies over  $\{0, 1, \infty\}$ .

The pair  $(C, \beta)$  is thus a Belyi pair, and we call  $\mathcal{D}$  the corresponding dessin. With the same arguments as before, the moduli field of  $\mathcal{D}$  is  $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$ , which is a nonabelian Galois extension of  $\mathbb{Q}$  with Galois group

$$\text{Gal}(\mathbb{Q}(\zeta_p, \sqrt[p]{q})/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^\times$$

generated by  $\sigma: \zeta_p^i \sqrt[p]{q} \mapsto \zeta_p^{i+1} \sqrt[p]{q}$  and  $\tau: \zeta_p^i \sqrt[p]{q} \mapsto \zeta_p^{gi} \sqrt[p]{q}$  where  $g$  generates  $(\mathbb{Z}/p\mathbb{Z})^\times$ . We shall show that there exists  $\gamma \in \mathbb{Q} \setminus \{0\}$  such that the regular closure of the dessin  $\mathcal{D}$  thus obtained also has moduli field  $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$ .

*Remark.* In the previous subsection we treated the case  $p = 3$ . In that specific case we gave a simpler expression for  $\beta$ , mainly due to the fact that  $\beta_0 \circ \pi$  already had all of its critical values in  $\mathbb{Q} \cup \{\infty\}$ . However in the general case we must use the intermediate map  $\beta_1$  as well as the parameter  $\gamma$  to conclude the proof.

Let us first prove the existence of  $\beta_1$ .

**Proposition 5.1.** *Let  $E \subset \bar{\mathbb{Q}} \cap \mathbb{R} \setminus \{0\}$  be a finite set. Then there exists  $P \in \mathbb{Q}[X]$  such that  $P(E) \subseteq \{0\}$ ,  $\text{Crit}(P) \subseteq \{0, 1\}$ ,  $0 < P(0) < 1$  and  $P'(0) > 0$ .*

*Remark.* In the context of this proposition we only deal with polynomials so for  $P \in \mathbb{Q}[X]$  we define  $\text{Crit}(P) := \{P(z) \mid z \in \mathbb{C}, P'(z) = 0\}$ , which does not include the point at infinity to simplify notations.

*Proof.* To show this we will proceed similarly as in the proof of the *only if* part of Belyi's theorem, by applying additional transformations to ensure that  $0 < P(0) < 1$ . Let us first prove that we can reduce to the case where  $E$  is a subset of rational numbers.

**Lemma 5.2.** *Let  $E \subset \bar{\mathbb{Q}} \cap \mathbb{R} \setminus \{0\}$  be a finite set fixed by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . Then there exists  $P \in \mathbb{Q}[X]$  such that  $P(0) = 0$  and  $\text{Crit}(P) \cup P(E) \subset \mathbb{Q} \setminus \{0\}$ .*

*Proof.* Let  $\{a_1, \dots, a_m\} = E \cap \mathbb{Q}$  and  $\{b_1, \dots, b_n\} = E \setminus \mathbb{Q}$ . We construct  $P$  by induction on the number  $n$  of non rational elements of  $E$ .

For  $\alpha \in \mathbb{Q}$ , define  $F_\alpha, G_\alpha \in \mathbb{Q}[X]$  by

$$F_\alpha := \prod_{j=1}^n (X - (b_j - \alpha)^2) \quad \text{and} \quad G_\alpha := F_\alpha((X - \alpha)^2) = \prod_{j=1}^n (X - b_j)(X + b_j - 2\alpha).$$

Let us first assume that there exists  $\alpha \in \mathbb{Q}$  such that  $G_\alpha(0) \notin \text{Crit}(G_\alpha) \cup G_\alpha(E)$ . Define  $P_1(X) := G_\alpha(X) - G_\alpha(0) \in \mathbb{Q}[X]$ , then  $P_1(0) = 0 \notin E'$ ,  $E' := \text{Crit}(P_1) \cup P_1(E) \subset \bar{\mathbb{Q}} \cap \mathbb{R} \setminus \{0\}$ . Note that  $E'$  is stable under the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , and  $|E' \setminus \mathbb{Q}| = |\text{Crit}(F_\alpha) \cup F_\alpha(0) \setminus \mathbb{Q}| = |\text{Crit}(F_\alpha) \setminus \mathbb{Q}| < \deg F_\alpha = n$ . By induction, there exists  $P_2 \in \mathbb{Q}[X]$  such that  $P_2(0) = 0$  and  $\text{Crit}(P_2) \cup P_2(E') \subset \mathbb{Q} \setminus \{0\}$ . Now  $P := P_2 \circ P_1$  has the desired properties, since  $P(0) = 0$  and  $\text{Crit}(P) \cup P(E) = \text{Crit}(P_2) \cup P_2(\text{Crit}(P_1)) \cup P_2(P_1(E)) = \text{Crit}(P_2) \cup P_2(E') \subset \mathbb{Q} \setminus \{0\}$ .

Let us now prove that there exists  $\alpha \in \mathbb{Q}$  such that  $G_\alpha(0) \notin \text{Crit}(G_\alpha) \cup G_\alpha(E)$ . Let us first treat the case where  $0 < b_1 < b_2, \dots, b_n$ . When  $\alpha$  approaches  $\frac{1}{2}$ ,  $G_\alpha(0) = \prod_{j=1}^n -b_j(b_j - 2\alpha)$  approaches 0 but the critical values of  $G_\alpha$  do not. Indeed,  $\text{Crit}(G_\alpha) = \text{Crit} F_\alpha \cup F_\alpha(\text{Crit}((X - \alpha)^2)) = \text{Crit}(F_\alpha) \cup \{F_\alpha(0)\}$ ;  $F_\alpha(0)$  approaches  $F_{\frac{1}{2}}(0) \neq 0$ , and since  $F_{\frac{1}{2}}$  does not have multiple roots, the critical values of  $F_\alpha$  approach the critical values of  $F_{\frac{1}{2}}$  which are all non zero. Therefore for  $\alpha \neq \frac{b_1}{2}$  in the neighborhood of  $\frac{b_1}{2}$  we have  $G_\alpha(0) \notin \text{Crit}(G_\alpha)$ . Moreover  $G_\alpha(0), G_\alpha(a_1), \dots, G_\alpha(a_m)$  are all distinct polynomials in the indeterminate  $\alpha$ , so they coincide at only finitely many points. In particular for  $\alpha \neq \frac{b_1}{2}$  in the neighborhood of  $\frac{b_1}{2}$  we have  $G_\alpha(0) \notin \{G_\alpha(a_1), \dots, G_\alpha(a_m)\}$ . Since  $\alpha \in \mathbb{Q}$  we also have  $G_\alpha(0) \neq 0 = G_\alpha(b_1) = \dots = G_\alpha(b_n)$  hence  $G_\alpha(0) \notin G_\alpha(E)$ , proving the existence of  $\alpha$  as desired.

Let us now treat the general case where  $b_1, \dots, b_n$  are not assumed to be positive by reducing it to the previous case. For  $\alpha' \in \mathbb{Q}$ , define  $H_{\alpha'} \in \mathbb{Q}[X]$  by

$$H_{\alpha'} := (X - \alpha')^2 - \alpha'^2 \in \mathbb{Q}[X].$$

Note that  $\text{Crit}(H_{\alpha'}) = \{-\alpha'\}$ . For  $\alpha' > 0$  sufficiently small we have  $-\alpha'^2 < H_{\alpha'}(0) = 0 < H_{\alpha'}(a_1), \dots, H_{\alpha'}(a_m), H_{\alpha'}(b_1), \dots, H_{\alpha'}(b_n)$ . Let  $E'' := \text{Crit}(H_{\alpha'}) \cup H_{\alpha'}(E)$ . The set  $E''$  is a finite subset of  $\mathbb{Q} \cap \mathbb{R} \setminus \{0\}$  fixed by  $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ , and  $E''$  has at most  $n$  non rational elements, which are all positive. By the above, there exists  $P_3 \in \mathbb{Q}[X]$  such that  $\text{Crit}(P_3) \cup P_3(E'') \subset \mathbb{Q} \setminus \{0\}$  and  $P_3(0) = 0$ . Then  $P := P_3 \circ H_{\alpha'}$  has the desired properties, since  $P(0) = 0$  and  $\text{Crit}(P) \cup P(E) = \text{Crit}(P_3) \cup P_3(\text{Crit}(H_{\alpha'})) \cup P_3(H_{\alpha'}(E)) = \text{Crit}(P_3) \cup P_3(E'') \subset \mathbb{Q} \setminus \{0\}$ .  $\square$

Let us denote by  $P_1$  the polynomial obtained using this lemma, which verifies  $P_1(0) = 0$  and  $E' := \text{Crit}(P_1) \cup P_1(E) \subset \mathbb{Q} \setminus \{0\}$ . We can further assume that  $P_1'(0) > 0$  by taking  $(-P_1)$  if necessary. We now send the points  $E'$  to  $\{0, 1\}$ .

**Lemma 5.3.** *Let  $E \subset \mathbb{Q} \setminus \{0\}$  a finite set. Then there exists  $P \in \mathbb{Q}[X]$  such that  $P(E) \subseteq \{0\}$ ,  $\text{Crit}(P) \subseteq \{0, 1\}$ ,  $0 < P(0) < 1$  and  $P'(0) > 0$ .*

*Proof.* For  $\alpha \in \mathbb{Q}$ , let  $F_\alpha := (X - \alpha)^2 \in \mathbb{Q}[X]$ , and note that  $\text{Crit}(F_\alpha) = \{0\}$ . There exists  $\alpha < 0$  sufficiently small such that  $0 < F_\alpha(0) < F_\alpha(a)$  for all  $a \in E$ . We take

$$F := \frac{F_\alpha}{\max_{a \in E} F_\alpha(a)}.$$

Let  $\{a_1, \dots, a_l\} = F(E)$  such that  $0 < F(0) < a_1 < \dots < a_l = 1$ . We also add a rational point  $a_0 \in \mathbb{Q}$  such that  $F(0) < a_0 < a_1$ .

Let  $m$  and  $n$  be the coprime positive integers such that  $a_{l-1} = \frac{m}{m+n}$ . We recall that  $B_{m,n}$  verifies  $\text{Crit}(B_{m,n}) = \{0, 1\}$ ,  $B_{m,n}(0) = B_{m,n}(1) = 0$ ,  $B_{m,n}(\frac{m}{m+n}) = 1$ , and  $B_{m,n}$  is strictly increasing between 0 and  $\frac{m}{m+n}$ . Let  $P_1 := B_{m,n}$ , then  $\text{Crit}(P_1) = \{0, 1\}$  and  $0 < P_1 \circ F(0) < P_1(a_0) < \dots < P_1(a_{l-1}) = 1$ . There is one point fewer than before, so we can iteratively construct  $P_2, \dots, P_l$  in the same way, so that  $P := P_l \circ \dots \circ P_1$  verifies  $\text{Crit}(P) \subseteq \{0, 1\}$ ,  $P(a_1) = \dots = P(a_l) = 0 < P(F(0)) < 1 = P(a_0)$  and  $P'(F(0)) > 0$ . Therefore  $P \circ F$  has the desired properties.  $\square$

Let us denote by  $P_2$  the polynomial obtained using this lemma with the finite set  $E'$  obtained previously. Then the polynomial  $P := P_2 \circ P_1$  verifies  $P(E) \subseteq \{0\}$ ,  $\text{Crit}(P) \subseteq \{0, 1\}$ ,  $0 < P(0) < 1$  and  $P'(0) > 0$ , thus concluding the proof of Proposition 5.1.  $\square$

We can now use Proposition 5.1 with the finite set

$$E := \{(1 - \zeta_p^k)^{2p}\}_{1 \leq k \leq \frac{p-1}{2}}$$

to obtain the map  $\beta_1$  as desired. For  $1 \leq k \leq \frac{p-1}{2}$  we have  $(1 - \zeta_p^k)^{2p} = (|1 - \zeta_p^k| \zeta_p^{2k-1})^{2p} = |1 - \zeta_p^k|^{2p} \in \mathbb{R}$  so  $E \subset \bar{\mathbb{Q}} \cap \mathbb{R}$ , and the set  $E$  is the Galois orbit of  $(1 - \zeta_p)^{2p}$  so it is fixed by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , hence  $E$  verifies the conditions of Proposition 5.1.

Let us denote by  $\mathcal{D}(\beta_1)$  the dessin corresponding to the Belyi pair  $(\mathbb{P}^1, \beta_1)$ . The Belyi pair  $(\mathbb{P}^1, \beta_1)$  is fixed by the action of the complex conjugation, so the embedding of  $\mathcal{D}(\beta_1)$  on  $\mathbb{P}^1$  admits a symmetry along the real line. Moreover the Belyi function  $\beta_1$  is a polynomial, so  $\mathcal{D}(\beta_1) \cap \mathbb{R}$  is a (graph theoretic) path. Let  $v_l < \dots < v_1$  be the negative vertices on the path, and let  $e_k$  denote the edge  $(v_{k-1}, v_k)$ . By hypothesis  $\beta_1'(0) > 0$  so  $v_1$  is a black vertex, and for  $k < l$ , the vertex  $v_k$  is of even degree  $2d_k$ . We then have  $e_k^{\xi^{d_k}} = e_{k+1}$  and  $e_{k+1}^{\xi^{d_k}} = e_k$  if  $k$  is odd, or  $e_k^{\eta^{d_k}} = e_{k+1}$  and  $e_{k+1}^{\eta^{d_k}} = e_k$  if  $k$  is even. See Figure 10.

As remarked earlier, the Galois orbit of  $(1 - \zeta_p)^{2p}$  is  $\{(1 - \zeta_p^k)^{2p}\}_{1 \leq k \leq \frac{p-1}{2}} \subset \mathbb{R}_-$ , and  $(1 - \zeta_p^{\frac{p-1}{2}})^{2p} < \dots < (1 - \zeta_p)^{2p} < 0$ . By construction  $\beta_1((1 - \zeta_p)^{2p}) = 0$ , so  $(1 - \zeta_p)^{2p}$  and all its Galois conjugates are black vertices of  $\mathcal{D}(\beta_1)$  lying on the path  $(v_1, \dots, v_l)$ . Let  $t > 0$  be the index such that  $v_t = (1 - \zeta_p)^{2p}$ , and  $v_t$  is a black vertex so  $t$  is odd. Then

$$\mu_0 := \xi^{d_1} \eta^{d_2} \dots \eta^{d_{t-1}} \xi^{2d_t} \eta^{d_{t-1}} \dots \eta^{d_2} \xi^{d_1}$$

fixes the edge  $e_1$  (Figure 10).

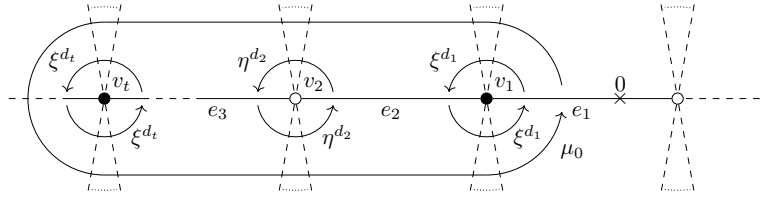


Figure 10: Dessin  $\mathcal{D}(\beta_1)$  corresponding to  $(\mathbb{P}^1, \beta_1)$ .

Let  $\gamma > 0$  small enough so that  $\beta_1' > 0$  on  $[0, \gamma^{2p} q^2]$ . Let us next draw the dessin  $\mathcal{D}(\beta_2)$  corresponding to the Belyi pair  $(\mathbb{P}^1, \beta_2 = B_{r,s} \circ B_{m,n})$ . See Figure 11.

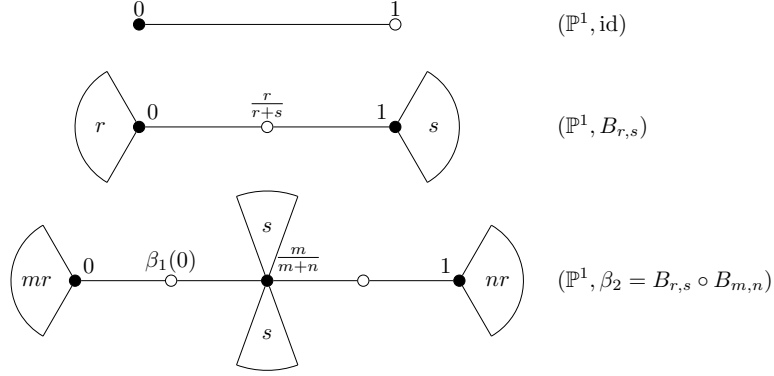


Figure 11: Dessin  $\mathcal{D}(\beta_2)$  corresponding to  $(\mathbb{P}^1, \beta_2)$ .

By lifting the dessin  $\mathcal{D}(\beta_2)$  along  $\beta_1$  we obtain the dessin  $\mathcal{D}(\beta_2 \circ \beta_1)$  corresponding to the Belyi pair  $(\mathbb{P}^1, \beta_2 \circ \beta_1)$ . This amounts to replacing each edge of  $\mathcal{D}(\beta_1)$  by a copy of  $\mathcal{D}(\beta_2)$ . Note that the degrees of the black and white vertices are thus multiplied by  $mr$  and  $nr$ , respectively. Analogously to  $\mu_0$  we define

$$\mu := (\xi^{mrd_1} \eta \xi^s \eta) (\xi^{nrd_2} \eta \xi^s \eta) \cdots (\xi^{mrd_{t-2}} \eta \xi^s \eta) (\xi^{nrd_{t-1}} \eta \xi^s \eta) \cdot (\xi^{2mrd_t} \eta \xi^s \eta) (\xi^{nrd_{t-1}} \eta \xi^s \eta) (\xi^{mrd_{t-2}} \eta \xi^s \eta) \cdots (\xi^{nrd_2} \eta \xi^s \eta) \xi^{mrd_1}$$

and we verify again that  $\mu$  fixes the edge  $(0, v_1)$ . Note also that  $\xi^{2s}$  fixes the edge  $(0, \gamma \sqrt[2p]{q})$ . See Figure 12.

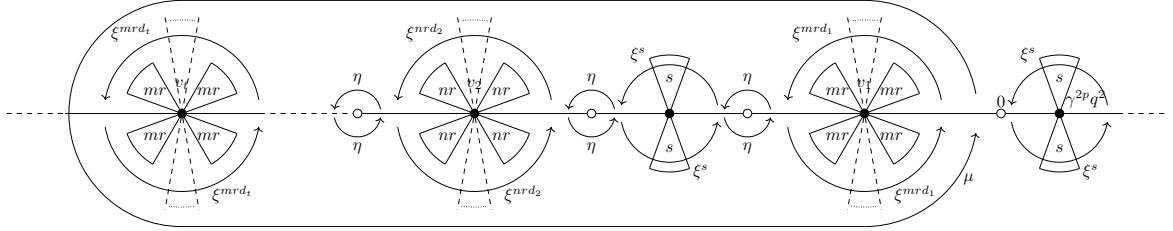


Figure 12: Dessin  $\mathcal{D}(\beta_2 \circ \beta_1)$  corresponding to  $(\mathbb{P}^1, \beta_2 \circ \beta_1)$ .

Let  $\mathcal{D}_0$  be the dessin corresponding to the Belyi pair  $(\mathbb{P}^1, \beta_2 \circ \beta_1 \circ \beta_0)$ . To simplify the representations of the dessins we only show the vertices  $0$ ,  $\zeta_{2p}^k(1 - \zeta_p)$ ,  $1 - \zeta_p^k$ , and  $\zeta_{2p}^k \gamma \sqrt[2p]{q}$ . We decorate the vertices  $\zeta_{2p}^k(1 - \zeta_p)$  (which map to  $(1 - \zeta_p)^{2p} \in \mathbb{R}_-$  by  $\beta_0$ ) and  $\zeta_{2p}^k \gamma \sqrt[2p]{q}$  (which map to  $\gamma^{2p} q^2 \in \mathbb{R}_+$  by  $\beta_0$ ) respectively with the symbols  $\ominus$  and  $\oplus$  to distinguish them. See Figure 13.

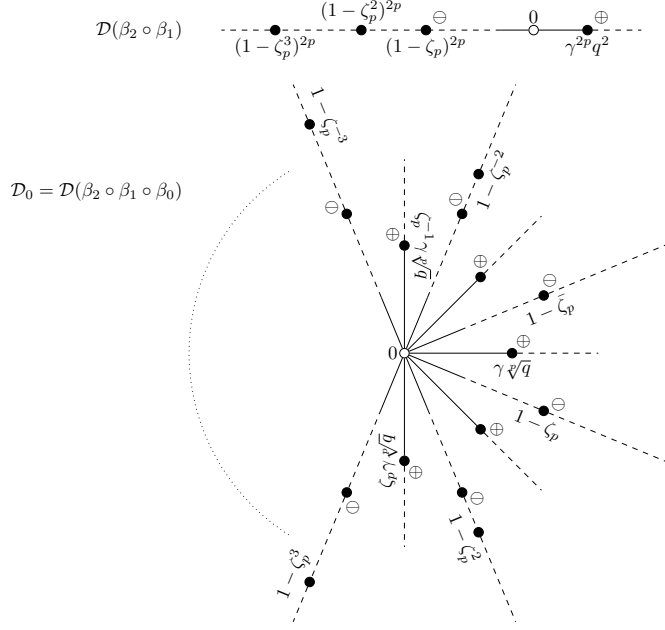


Figure 13: Dessin  $\mathcal{D}_0$  corresponding to  $(\mathbb{P}^1, \beta_2 \circ \beta_1 \circ \beta_0)$ .

We may now draw the Galois conjugates  $\mathcal{D}^\sigma$  for  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  by lifting the dessin  $\mathcal{D}_0$  along the projection  $\pi$ , by treating separately the cases  $\sigma(\zeta_p) \in \{\zeta_p, \bar{\zeta}_p\}$  and  $\sigma(\zeta_p) \in \{\zeta_p^2, \dots, \zeta_p^{p-2}\}$ . We call the dessins respectively  $\mathcal{D}_k$  and  $\mathcal{D}_k^j$ , see Figure 14. We identify the outermost edges on opposite sides in the representations.

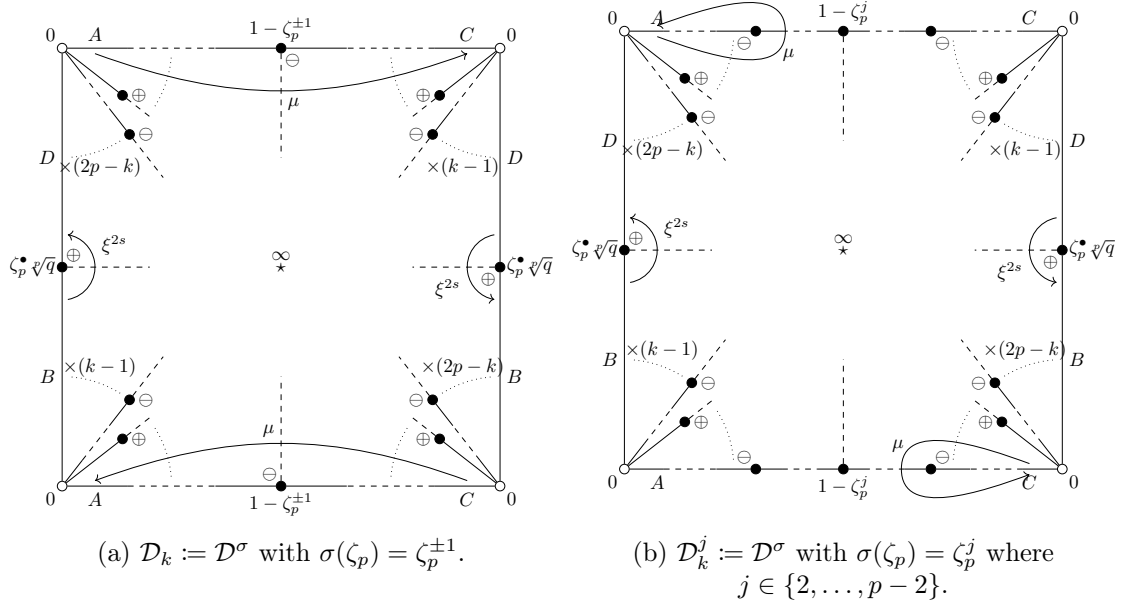


Figure 14: Dessins (a)  $\mathcal{D}_k$  and (b)  $\mathcal{D}_k^j$  in the Galois orbit of  $\mathcal{D}$ .

For all  $k$ , we have in fact  $\mathcal{D}_{2k-1} = \mathcal{D}^\sigma$  where  $\sigma: (\zeta_p, \sqrt[p]{q}) \mapsto (\zeta_p, \zeta_p^{k-1} \sqrt[p]{q})$ , and  $\mathcal{D}_{2k} = \mathcal{D}^\sigma$  where  $\sigma: (\zeta_p, \sqrt[p]{q}) \mapsto (\bar{\zeta}_p, \zeta_p^k \sqrt[p]{q})$ . We have similar expressions for the dessins  $\mathcal{D}_k^j$ .

Let  $k$  be fixed, and let us consider the dessin  $\mathcal{D}_k$ . Let  $A$  denote one of the two edges incident to 0 and on the path to the ramification point  $1 - \zeta_p^{\pm 1}$ . We also call  $B := A\eta^{2k-1}$ ,  $C := A^{4p}$ ,  $D := C\eta^{2k-1}$  (See Figure 14a). Let  $E$  denote the set of edges incident to 0. The action of  $\eta$  induces the cyclic permutation of the edges of  $E = \{A\eta^i\}_{0 \leq i < 8p}$ . Furthermore by construction every white vertex aside from 0 has degree 1 or 2, so  $\eta^2$  fixes every edge in the complement of  $E$ . We can write  $E = E^\ominus \sqcup E^\oplus$  as the disjoint union of  $E^\ominus := \{A^{2i}\}_{0 \leq i < 4p}$  and  $E^\oplus := \{B^{2i}\}_{0 \leq i < 4p}$ , such that  $\eta$  sends one to the other. The action of  $\mu$  on  $E^\ominus$  is the transposition  $(A, C)$ , and similarly the action of  $\xi^{2s}$  on  $E^\oplus$  is the transposition  $(B, D)$ .

We do the same for the dessins of the form  $\mathcal{D}_k^j$ , with the only difference that this time the action of  $\mu$  on  $E^\ominus$  is trivial, including on the edges  $A$  and  $C$  (see Figure 14b).

We are almost in the same configuration as in the first example. We define analogously

$$\omega := \mu\eta\mu^{-1}\xi^{2s}\eta^{-1}\mu,$$

and we shall prove that for some choices of  $\gamma$ , the actions of  $\omega$  and of  $\eta^2$  commute only for  $\mathcal{D}_1$ . To reproduce the proof in the first example we need only show that for some choice of  $\gamma$  the actions of  $\mu\eta\mu^{-1}\xi^{2s}$  and  $\mu^{-1}\xi^{2s}\eta^{-1}\mu$  on the set  $E^\oplus$  is the same as that of  $\xi^{2s}$ .

Note that for any edge  $e \in E^\oplus$ , the edge  $e^{\xi^i}$  is fixed by  $\eta$  if  $i$  is not a multiple of  $s$ . To that end we shall show that for some choice of  $\gamma$  the action of  $\mu$  on  $E^\oplus$  is the same as that of  $\xi^\delta$ , where  $\delta$  is the number of occurrences of  $\xi$  in the word  $\mu$ , and then that  $\delta$  is not a multiple of  $s$ .

We define the words  $\rho_1, \rho'_1, \rho_2, \rho'_2, \dots, \rho_{2t-2}, \rho'_{2t-2} \in F_2$  to be the increasing subsequence of the prefixes ending in  $\eta$  of the word  $\mu$  defined above, such that  $\rho_1 := \xi^{mrd_1}\eta$ ,  $\rho'_1 := \rho_1\xi^s\eta$ ,  $\rho_2 := \rho'_1\xi^{nrd_2}\eta$ ,  $\rho'_2 := \rho_2\xi^s\eta$ , etc., and  $\mu = \rho'_{2t-2}\xi^{mrd_1}$ . We shall show by induction that for some choice of  $\gamma$  the action of  $\rho_i$  (resp.  $\rho'_i$ ) is the same as the action of  $\xi^{\delta_i}$  (resp.  $\xi^{\delta'_i}$ ), where  $\delta_i$  (resp.  $\delta'_i$ ) is the number of occurrences of  $\xi$  in the word  $\rho_i$  (resp.  $\rho'_i$ ). By induction it suffices to show that  $\delta_i, \delta'_i$  are not multiples of  $s$ . Modulo  $s$  we have  $\delta_i \equiv \delta'_i$  equal to the non empty partial sum of

$$mrd_1 + nrd_2 + \dots + mrd_{t-2} + nrd_{t-1} + 2mrd_t + nrd_{t-1} + mrd_{t-2} + \dots + nrd_2 + mrd_1$$

consisting of the first  $i$  terms.

To proceed we shall use the following result, but let us first introduce some notations. Let  $P = \sum_{i=0}^d c_i X^i \in \mathbb{Z}[X]$  and  $c \in \mathbb{Z}_{>0}$  such that  $\beta_1 = \frac{P}{c}$ . Note that  $P$  and  $c$  do not depend on the choice of  $\gamma$ , and  $0 < \beta_1(0) = \frac{P(0)}{c} < 1$  so  $0 < c_0, c - c_0$ . We define

$$\alpha := v_2(\gamma^{2p}q^2), \quad \nu := v_2(c_0) + v_2(c - c_0),$$

where  $v_2$  denotes the 2-valuation.

**Lemma 5.4.** *If  $\alpha > \nu$ , then there exists  $e \in \mathbb{Z}$  such that  $em \equiv c_0 \pmod{2^\alpha}$  and  $en \equiv c - c_0 \pmod{2^\alpha}$ , and  $v_2(s) \geq \alpha - \nu$ .*

*Proof.* Let  $a, b \in \mathbb{Z}$  coprime such that  $\gamma^{2p}q^2 = \frac{a}{b}2^\alpha$ .

Firstly,

$$\frac{m}{m+n} = \beta_1\left(\frac{a}{b}2^\alpha\right) = \frac{P\left(\frac{a}{b}2^\alpha\right)}{c} = \frac{\sum_{i=0}^d c_i a^i 2^{\alpha i} b^{d-i}}{b^d c},$$

so there exists  $f \in \mathbb{Z}$  such that  $fm = \sum_{i=0}^d c_i a^i 2^{\alpha i} b^{d-i}$  and  $f(m+n) = b^d c$ , so

$$em \equiv c_0 \pmod{2^\alpha}, \quad en \equiv c - c_0 \pmod{2^\alpha}$$

for  $e \in \mathbb{Z}$  such that  $eb^d \equiv f \pmod{2^\alpha}$ .

Secondly,

$$\frac{r}{r+s} = B_{m,n}(\beta_1(0)) = \frac{\beta_1(0)^m (1 - \beta_1(0))^n}{\beta_1\left(\frac{a}{b}2^\alpha\right)^m (1 - \beta_1\left(\frac{a}{b}2^\alpha\right))^n} = \frac{b^{d(m+n)} c_0^m (c - c_0)^n}{(b^d P\left(\frac{a}{b}2^\alpha\right))^m (b^d c - b^d P\left(\frac{a}{b}2^\alpha\right))^n},$$

so there exists  $g \in \mathbb{Z}$  such that  $gr = b^{d(m+n)} c_0^m (c - c_0)^n$  and  $g(r+s) = (b^d P\left(\frac{a}{b}2^\alpha\right))^m (b^d c - b^d P\left(\frac{a}{b}2^\alpha\right))^n$ . In the expansion of  $(b^d P\left(\frac{a}{b}2^\alpha\right))^m$ , aside from the constant term  $b^{dm} c_0^m$ , every other term is a multiple of an integer of the form  $c_0^i 2^{\alpha j}$  with  $i \leq m-1$  and  $j \geq m-i$ . By hypothesis  $\alpha > \nu \geq v_2(c_0)$ , so those other terms are all multiples of  $2^{\alpha+(m-1)v_2(c_0)}$ , hence there exists  $A \in \mathbb{Z}$  such that  $(b^d P\left(\frac{a}{b}2^\alpha\right))^m = b^{dm} c_0^m + A2^{\alpha+(m-1)v_2(c_0)}$ . Similarly there exists  $B \in \mathbb{Z}$  such that  $(b^d c - b^d P\left(\frac{a}{b}2^\alpha\right))^n = b^{dn} (c - c_0)^n + B2^{\alpha+(n-1)v_2(c-c_0)}$ . Then  $g(r+s) = b^{d(m+n)} c_0^m (c - c_0)^n + C2^{\alpha+(m-1)v_2(c_0)+(n-1)v_2(c-c_0)}$  for some  $C \in \mathbb{Z}$ , so  $gr = b^{d(m+n)} c_0^m (c - c_0)^n$  and  $gs = C2^{\alpha+(m-1)v_2(c_0)+(n-1)v_2(c-c_0)}$ . The integers  $r$  and  $s$  are coprime, so after dividing  $gr$  and  $gs$  by their greatest common divisor we obtain that

$$v_2(s) \geq \alpha - \nu > 0.$$

□

Using this lemma, we know that if  $\alpha > \nu$ , then there exists  $e \in \mathbb{Z}$  such that  $em \equiv c_0 \pmod{2^\alpha}$  and  $en \equiv c - c_0 \pmod{2^\alpha}$ ,  $v_2(s) \geq \alpha - \nu$  where  $\nu$  does not depend on  $\gamma$ , and  $r$  is coprime to  $s$  so is not a multiple of 2. Therefore there exists  $e' \in \mathbb{Z}$  such that  $e'mr \equiv c_0 \pmod{2^\alpha}$  and  $e'nr \equiv c - c_0 \pmod{2^\alpha}$ . Moreover  $2^{\alpha-\nu}$  is a common divisor of  $2^\alpha$  and  $s$ , so by the above modulo  $2^{\alpha-\nu}$  we have  $e'\delta_i \equiv e'\delta'_i$  equal to the non empty partial sum  $\tilde{\delta}_i$  consisting of the first  $i$  terms of the sum

$$c_0 d_1 + (c - c_0) d_2 + \cdots + c_0 d_{t-2} + (c - c_0) d_{t-1} + 2c_0 d_t + (c - c_0) d_{t-1} + c_0 d_{t-2} + \cdots + (c - c_0) d_2 + c_0 d_1.$$

Similarly  $e'\delta$  is equal modulo  $2^{\alpha-\nu}$  to the whole sum

$$\tilde{\delta} := 2(c_0 d_1 + (c - c_0) d_2 + \cdots + c_0 d_{t-2} + (c - c_0) d_{t-1} + c_0 d_t).$$

By construction  $c_0, c - c_0, d_i$  are positive and do not depend on the choice of  $\gamma$ , so  $0 < c_0 d_1 \leq \tilde{\delta}_i \leq \tilde{\delta}$ , thus for any choice of  $\gamma$  such that  $\alpha > \nu$  and  $\tilde{\delta} < 2^{\alpha-\nu}$  (for instance  $\gamma = \frac{2^u}{2^v+1}$  with  $1 \ll u \ll v$ ), we obtain  $\tilde{\delta}_i, \tilde{\delta} \not\equiv 0 \pmod{2^{\alpha-\nu}}$ , and in consequence  $\delta_i, \delta'_i$  and  $\delta$  are not multiples of  $s$ . Therefore we can now conclude by induction that the actions of  $\rho_i$  and  $\rho'_i$  are the same as that of  $\xi^{\delta_i}$  and  $\xi^{\delta'_i}$ , respectively. Indeed,  $\delta_1$  is not a multiple of  $s$  so  $\rho_1 = \xi^{\delta_1} \eta$  and  $\xi^{\delta_1}$  have the same action on  $E^\oplus$ . If  $\rho_i$  has the same action as  $\xi^{\delta_i}$  on  $E^\oplus$ , then  $\rho'_i = \rho_i \xi^s \eta$  has the same action as  $\xi^{\delta_i} \xi^s \eta = \xi^{\delta'_i} \eta$  on  $E^\oplus$ , and also the same action as  $\xi^{\delta'_i}$  because  $\delta'_i$  is not a multiple of  $s$ . Similarly, if  $\rho'_i$  has the same action as  $\xi^{\delta'_i}$  on  $E^\oplus$ , then  $\rho_{i+1}$  has the

same action as  $\xi^{\delta_{i+1}}\eta$  on  $E^\oplus$ , and also the same action as  $\xi^{\delta_{i+1}}$  because  $\delta_{i+1}$  is not a multiple of  $s$ .

We have thus proved that  $\mu$  has the same action as  $\xi^\delta$  on  $E^\oplus$ , and by symmetry  $\mu^{-1}$  has the same action as  $\xi^{-\delta}$  on  $E^\oplus$ . And  $\delta$  and  $2s - \delta$  are not multiples of  $s$ , so  $\mu\eta\mu^{-1}\xi^{2s}$  and  $\mu^{-1}\xi^{2s}\eta^{-1}\mu$  have the same action as  $\xi^{2s}$  on  $E^\oplus$ , as announced. We shall now observe the action of  $\omega = \mu\eta\mu^{-1}\xi^{2s}\eta^{-1}\mu$  on  $E$ . Let  $M_k$  and  $M_k^j$  denote the monodromy maps of the dessins  $\mathcal{D}_k$  and  $\mathcal{D}_k^j$ .

For the dessins  $\mathcal{D}_k$  for  $1 \leq k \leq 2p$ , the action of  $\mu$  on  $E^\ominus$  is the transposition  $(A, C)$ , and the action of  $\xi^{2s}$  on  $E^\oplus$  is the transposition  $(B, D)$ , therefore the action of  $\omega$  fixes the set  $E$  on which it induces the permutation

$$M_k(\omega)|_E = (A, C)(B, D) \cdot \prod_{i=0}^{4p-1} (A^{\eta^{2i}}, A^{\eta^{2i+1}}) \cdot (A, C)(B, D).$$

Hence for  $k = 1$ ,

$$\begin{aligned} M_1(\omega)|_E &= (A, A^{\eta^{4p}})(A^\eta, A^{\eta^{4p+1}}) \cdot \prod_{i=0}^{4p-1} (A^{\eta^{2i}}, A^{\eta^{2i+1}}) \cdot (A, A^{\eta^{4p}})(A^\eta, A^{\eta^{4p+1}}) \\ &= \prod_{i=0}^{4p-1} (A^{\eta^{2i}}, A^{\eta^{2i+1}}) \end{aligned}$$

so  $\omega$  and  $\eta^2$  commute on  $E$ . Moreover  $\eta^2$  acts trivially on the complement of  $E$ , so finally  $M_1(\omega)$  and  $M_1(\eta^2)$  commute.

For  $k = 2$ , we observe that  $B^{\omega\eta^2} = D^{\eta^{-1}\eta^2} = D^\eta$  but  $B^{\eta^2\omega} = B^{\eta^2\eta^{-1}} = B^\eta$ . Similarly, for  $3 \leq k \leq 2p$ , we observe that  $A^{\omega\eta^2} = C^{\eta\eta^2} = C^{\eta^3}$  but  $A^{\eta^2\omega} = A^{\eta^2\eta} = A^{\eta^3}$ . Therefore  $M_k(\omega)$  and  $M_k(\eta^2)$  do not commute for  $2 \leq k \leq 2p$ .

For the dessins  $\mathcal{D}_k^j$  for  $1 \leq k \leq 2p$  and  $2 \leq j \leq \frac{p-1}{2}$ ,  $\xi^{2s}$  on  $E^\oplus$  is the transposition  $(B, D)$ , and  $\mu$  acts trivially on  $E^\ominus$ , therefore the action of  $\omega$  fixes the set  $E$  on which it induces the permutation

$$M_k^j(\omega)|_E = (B, D) \cdot \prod_{i=0}^{4p-1} (A^{\eta^{2i}}, A^{\eta^{2i+1}}) \cdot (B, D).$$

Hence we observe that  $B^{\omega\eta^2} = D^{\eta^{-1}\eta^2} = D^\eta$  but  $B^{\eta^2\omega} = B^{\eta^2\eta^{-1}} = B^\eta$ , so  $M_k^j(\omega)$  and  $M_k^j(\eta^2)$  do not commute.

We have thus shown that the actions of  $\omega$  and  $\eta^2$  commute only for  $\mathcal{D}_1$ , this concludes the proof that  $\tilde{\mathcal{D}}$  is a regular dessin with moduli field  $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$ .

### 5.3 Regular dessin with moduli field $\mathbb{Q}(\zeta_3, \sqrt[3]{3})$

Finally, let us exhibit a regular dessin with moduli field  $\mathbb{Q}(\zeta_3, \sqrt[3]{3})$  of smaller degree by choosing a Belyi map that is a rational function instead of a polynomial as was done in the previous subsections. Let

$$\begin{aligned} C: y^2 &= x(x - (1 - \zeta_3))(x - \sqrt[3]{3}), \\ \beta: C &\rightarrow \mathbb{P}^1, (x, y) \mapsto \frac{(x + 3^3)^3}{3^5(x - 3^2)^2}. \end{aligned}$$

The function  $\beta$  is given by the composition of the following maps  $\beta = \beta_1 \circ \beta_0 \circ \pi$ .

1.  $\pi: C \rightarrow \mathbb{P}^1$  is the projection on the coordinate  $x$ , which ramifies over  $\{0, 1 - \zeta_3, \sqrt[3]{3}, \infty\}$ .
2.  $\beta_0 := X^6 \in \mathbb{Q}[X]$ ,  $\text{Crit}(\beta_0) = \{0\}$  so  $\beta_0 \circ \pi$  ramifies over  $\{0, (1 - \zeta_3)^6 = -3^3, 3^2, \infty\}$ .
3.  $\beta_1 := \frac{(X+3^3)^3}{3^3(X-3^2)^2}$ ,  $\text{Crit}(\beta_1) = \{0, 1\}$  so  $\beta = \beta_1 \circ \beta_0 \circ \pi$  ramifies over  $\{0, 1, \infty\}$ .

The pair  $(C, \beta)$  is thus a Belyi pair, and we call  $\mathcal{D}$  the dessin corresponding to  $(C, \beta)$ . Similarly as in 5.1,  $\mathcal{D}$  has moduli field  $\mathbb{Q}(\zeta_3, \sqrt[3]{3})$ . We will proceed analogously to show that the regular closure  $\tilde{\mathcal{D}}$  has the same field of moduli. Let us first draw the dessin  $\mathcal{D}_0$  corresponding to the Belyi pair  $(\mathbb{P}^1, \beta_1 \circ \beta_0)$  (see Figure 15), and lift it to the conjugate curves  $C^\sigma$  to obtain the conjugate dessins  $\mathcal{D}^\sigma$  for  $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{3})/\mathbb{Q})$  (see Figure 16).

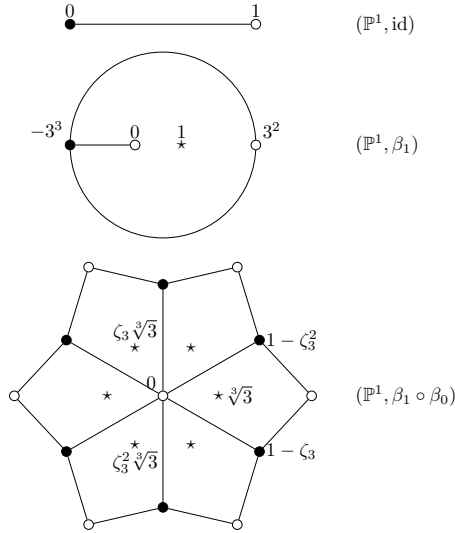
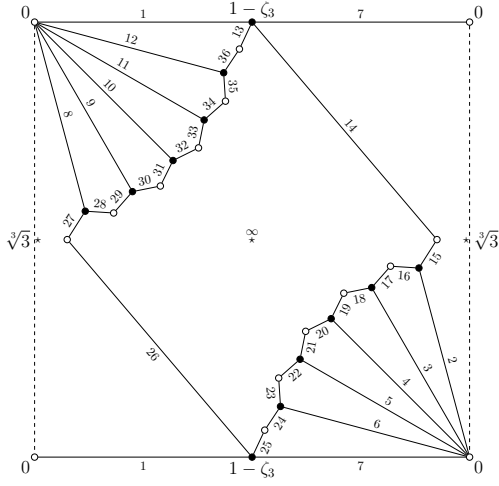
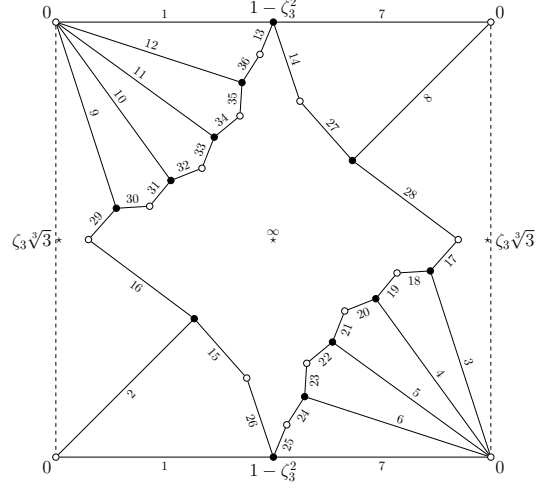


Figure 15: Construction of  $\mathcal{D}_0$ .

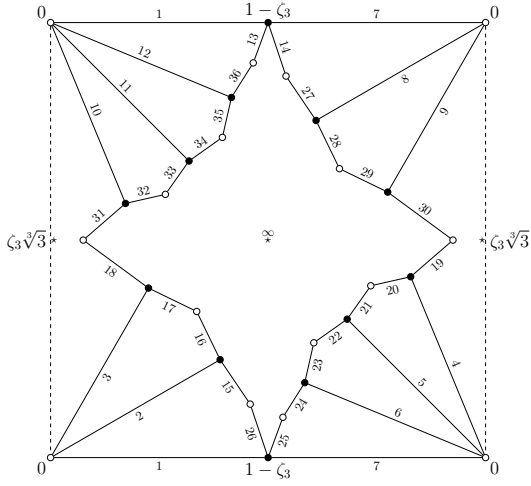
As usual we identify the outermost edges on opposite sides.



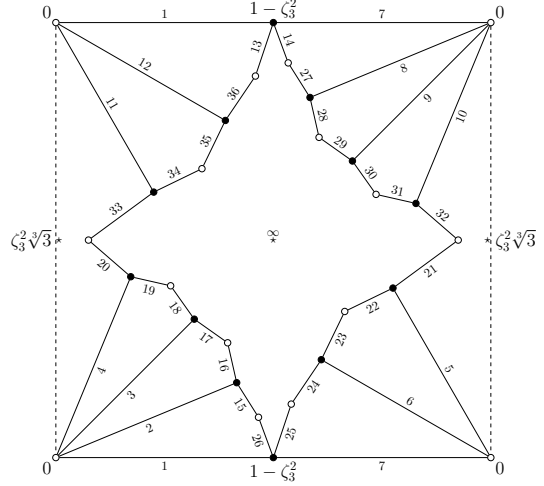
(a)  $\mathcal{D}_1 := \mathcal{D}$



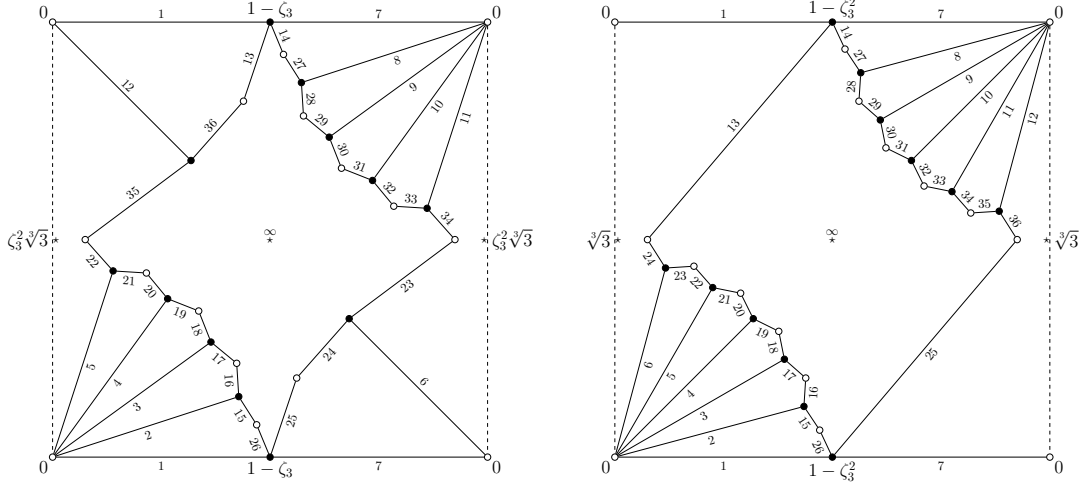
(b)  $\mathcal{D}_2 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3^2, \zeta_3 \sqrt[3]{q})$



(c)  $\mathcal{D}_3 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3, \zeta_3 \sqrt[3]{q})$



(d)  $\mathcal{D}_4 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3^2, \zeta_3^2 \sqrt[3]{q})$



(e)  $\mathcal{D}_5 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3, \zeta_3^2 \sqrt[3]{q})$

(f)  $\mathcal{D}_6 := \mathcal{D}^\sigma, \sigma: (\zeta_3, \sqrt[3]{q}) \mapsto (\zeta_3^2, \sqrt[3]{q})$

Figure 16: Dessins  $\mathcal{D}_1, \dots, \mathcal{D}_6$  in the Galois orbit of  $\mathcal{D}$ .

We can now compute the cartographic groups of the dessins. Let  $M_k$  denote the monodromy map of  $\mathcal{D}_k$ . Then

$$M_k(\xi) = \begin{aligned} & (1,13,14,7,25,26)(2,15,16)(3,17,18)(4,19,20)(5,21,22) \\ & (6,23,24)(8,27,28)(9,29,30)(10,31,32)(11,33,34)(12,35,36) \end{aligned}$$

for all  $1 \leq k \leq 6$ , and

- $M_1(\eta) = (1,2,3,4,5,6,7,8,9,10,11,12)(13,36)(14,15)(16,17)(18,19)(20,21)(22,23)(24,25)(26,27)(28,29)(30,31)(32,33)(34,35)$  ,
- $M_2(\eta) = (1,2,3,4,5,6,7,8,9,10,11,12)(13,36)(14,27)(15,26)(16,29)(17,28)(18,19)(20,21)(22,23)(24,25)(30,31)(32,33)(34,35)$  ,
- $M_3(\eta) = (1,2,3,4,5,6,7,8,9,10,11,12)(13,36)(14,27)(15,26)(16,17)(18,31)(19,30)(20,21)(22,23)(24,25)(28,29)(32,33)(34,35)$  ,
- $M_4(\eta) = (1,2,3,4,5,6,7,8,9,10,11,12)(13,36)(14,27)(15,26)(16,17)(18,19)(20,33)(21,32)(22,23)(24,25)(28,29)(30,31)(34,35)$  ,
- $M_5(\eta) = (1,2,3,4,5,6,7,8,9,10,11,12)(13,36)(14,27)(15,26)(16,17)(18,19)(20,21)(22,35)(23,34)(24,25)(28,29)(30,31)(32,33)$  ,
- $M_6(\eta) = (1,2,3,4,5,6,7,8,9,10,11,12)(13,24)(14,27)(15,26)(16,17)(18,19)(20,21)(22,23)(25,36)(28,29)(30,31)(32,33)(34,35)$  .

Using the computer algebra system SageMath [12], we determined that

$$|\langle M_1(\xi), M_1(\eta) \rangle| = 42467328 = 2^{19} \cdot 3^4.$$

Moreover,  $M_1(\xi)$ ,  $M_1(\eta)$  and  $M_1(\xi\eta)$  respectively have orders 6, 12 and 12, so the Euler characteristic of the underlying surface of  $\widehat{\mathcal{D}}_1$  is  $\chi = |\langle M_1(\xi), M_1(\eta) \rangle| \cdot \left( \frac{1}{\text{ord } M_1(\xi)} + \frac{1}{\text{ord } M_1(\eta)} + \frac{1}{\text{ord } M_1(\xi\eta)} - 1 \right) = -28311552 = -2^{20} \cdot 3^3$ , and its genus is  $g = 1 - \frac{\chi}{2} = 14155777$ .

We will now show that  $\widetilde{\mathcal{D}}_1$  is not isomorphic to  $\widetilde{\mathcal{D}}_2, \dots, \widetilde{\mathcal{D}}_6$ . We claim that  $\omega := [\xi^{-1}\eta^2\xi, \xi\eta] \in \ker M_1 \setminus \bigcup_{2 \leq k \leq 6} \ker M_k$ , thus concluding the proof. Indeed, we obtain:

- $M_1(w) = \text{id}$ ;
- $M_2(w) = (13, 25)(15, 27)(21, 33)(23, 35)$ ;
- $M_3(w) = (17, 29)(21, 33)$ ;
- $M_4(w) = (13, 25)(15, 27)(19, 31)(21, 33)$ ;
- $M_5(w) = (13, 25)(17, 29)$ ;
- $M_6(w) = (13, 25)(19, 31)(21, 33)(23, 35)$ .

We have thus constructed a regular dessin  $\widetilde{\mathcal{D}}$  of degree  $2^{19} \cdot 3^4$  and genus 14155777 with moduli field  $\mathbb{Q}(\zeta_3, \sqrt[3]{3})$ .

## References

- [1] Marston D. E. Conder, Gareth A. Jones, Manfred Streit, and Jürgen Wolfart. Galois actions on regular dessins of small genera. *Rev. Mat. Iberoam.*, 29(1):163–181, 2013.
- [2] Clifford J. Earle. On the moduli of closed Riemann surfaces with symmetries. In *Advances in the theory of riemann surfaces (Proc. Conf., Stony Brook, N. Y., 1969)*, pages 119–130. Ann. of Math. Studies, No. 66, 1971.
- [3] Ernesto Girondo and Gabino González-Diez. *Introduction to compact Riemann surfaces and dessins d’enfants*, volume 79 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2012.
- [4] Gabino González-Diez and Andrei Jaikin-Zapirain. The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces. *Proc. Lond. Math. Soc. (3)*, 111(4):775–796, 2015.
- [5] Alexandre Grothendieck. Esquisse d’un programme. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 5–48. Cambridge Univ. Press, Cambridge, 1997. With an English translation on pp. 243–283.
- [6] Pierre Guillot. An elementary approach to dessins d’enfants and the Grothendieck-Teichmüller group. *Enseign. Math.*, 60(3-4):293–375, 2014.
- [7] Moises Herradón Cueto. The field of moduli and fields of definition of dessins d’enfants. 09 2014.
- [8] Ruben A. Hidalgo and Saul Quispe. Regular dessins d’enfants with field of moduli  $\mathbb{Q}(\sqrt[4]{2})$ . *Ars Math. Contemp.*, 13(2):323–330, 2017.
- [9] Gareth A. Jones and Jürgen Wolfart. *Dessins d’enfants on Riemann surfaces*. Springer Monographs in Mathematics. Springer, Cham, 2016.
- [10] Bernhard Köck. Belyi’s theorem revisited. *Beiträge Algebra Geom.*, 45(1):253–265, 2004.
- [11] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

- [12] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.0)*, 2020-01-01. <https://www.sagemath.org>.
- [13] J. Wolfart. *ABC* for polynomials, dessins d'enfants and uniformization—a survey. In *Elementare und analytische Zahlentheorie*, volume 20 of *Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main*, pages 313–345. Franz Steiner Verlag Stuttgart, Stuttgart, 2006.