Many-body localisation for coupled spin chains

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Abstract We base our work off of the article [1] by John Imbrie of the University of Virginia who is a main specialist in the field. Imbrie's paper aims to show the localization of energy in a one-dimensional spin chain with random magnetic field, given an assumption that energy levels be far apart enough. For this he uses a multi-scale method to diagonalize the hamiltonian. We add some simplifying assumptions to his work and try to rewrite some of his proofs more explicitly than in the original article.

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1 Introduction

We use a simple model involving a one-dimensional disordered spin chain with next-neighbor interactions (ie. a spin chain subject to a random magnetic field). A small perturbation in the hamiltonian will allow energy to diffuse throughout the chain, this perturbation will be a spin flip. The hamiltonian for our spin chain is as follows.

$$H = \sum_{-K}^{K'} h_i S_i^z + \sum_{i=-K}^{K'} \gamma S_i^x + \sum_{i=-K-1}^{K'} J_i S_i^z S_{i+1}^z \tag{1}$$

where the h_i and J_i are iid variables, with a probability distribution of finite support.

This hamiltonian operates only on a finite set of spins on the lattice $\Lambda := [-K, K'] \cap \mathbb{Z}$, where we have frozen all the spin variables outside of Λ. This is so that we don't have to deal with any limit effects.

As we've mentioned, the non-perturbative part of the above hamiltonian is diagonal, so the non-perturbed eigenvectors are simply the vectors associated with each site. We will denote by $|X>$ the basis vector $|...010...>$ where the 1 is at site X. The result which we will aim to prove is known as *many-body localisation* (MBL), namely the localisation of energy for small, but nonzero, perturbations (ie. γ sufficiently small). In order to prove this result we first need to make an assumption, known as the *level-spacing assumption*.

Assumption 1. *(Level-spacing assumption) Let us take a hamiltonian H which is defined on a lattice Λ with $|\Lambda| =: n$. Let us call $\mathbb{P}$ the probability distribution of the $(h_i)_i$ and $(J_i)_i$. The level-spacing assumption specifies that there exist ν, C such that the disrtibution of the eigenvalues of H satisfies the following inequality :*

$$\mathbb{P}\left(\min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n \tag{2}$$

for all $\delta > 0$.

Assumption 1 specifies that energy levels are far apart with high probability, so as to avoid resonances between two levels.

We want to prove the localisation of energy for small perturbations. This is embodied by the following inequality :

Definition 1. *(Many-body localisation)*

Take X and Y two sites on the lattice. We call $|X>$ and $|Y>$ the delta functions associated with these respective sites. We say that the system of hamiltonian H is localised if it satisfies the following inequality.

$$\mathbb{E}\left[\sum_{E \text{ eigenvalues}} |<X|E><E|Y>|\right] \leq C exp\left(-|X-Y|/\xi\right) \tag{3}$$

where $|E>$ is the eigenvector of H of eigenvalue E.

More concretely, equation (3) specifies that when we turn on our perturbation, the eigenvectors change by a small amount. Equation (3) also tells us that the spatial scope of the perturbation is

limited, with exponential decay : it does not easily couple two sites that are far from each other. Such is the reason why we say the energy is *localised* in this case.

In order to prove this result we will use a perturbation scheme. We diagonalise the hamiltonian perturbatively, and each step of this diagonalisation generates new resonant and non-resonant regions out of the previous resonant and non-resonant regions (we will define exactly what *resonant* and *non-resonant* mean in this next section). We will be able to prove our bound on the non-resonant regions, and then the key will be to ascertain that the resonant regions do not percolate in the limit, so that the MBL bound "spreads" to the whole lattice.

We have made a few simplifications with respect to Imbrie's article so as to better conceptualize the full scheme, which we will detail below.

1.1 Simplifications

To begin with, once a region becomes resonant, we will neglect anything happening inside it. This means that we will not diagonalise our hamiltonian in resonant regions after they become resonant. Because of this we will only solve part of the problem (namely, controlling the number of resonant regions that are generated out of the non-resonant regions), but doing this already poses significant challenges that we wanted to tackle.

Secondly, we will not take into account the renormalization of energy. This is a component which plays a role in deciding which regions become resonant.

With all of this in mind, let us now sketch our perturbation scheme. To get an idea of it, we will start from the first step (where we will diagonalise the hamiltonian at first order).

2 First step

2.1 Diagonalisation of the hamiltonian to the first order

We use a standard perturbative method to diagonalise our hamiltonian to the first order. For readability purposes, we write $H := H_0 + J$ where H_0 is the diagonal part of the hamiltonian and J is the (small) perturbation. We denote by $(|\sigma^0>)_{\sigma^0}$ the basis of eigenvectors of the unperturbed hamiltonian (which here is nothing but the canonical basis, but we want to keep a general notation). We write $(E^0_\sigma)_{\sigma^0}$ for the associated eigenvalues. We also denote by $(|\sigma^1>)_{\sigma^1}$ the new basis of eigenvectors after the first step in our perturbative process. The eigenvalues associated with these eigenvectors are the same as those of the unperturbed eigenvectors, given that we have neglected the renormalization of energy (and even if we had not, in any case, the lowest-order of energy renormalization is the second order for hamiltonians for which the unperturbed part is diagonal).

With all of these notations in place, we have :

$$\underbrace{<\tau^0|H|\sigma^1>}_{=E^1_\sigma<\tau^0|\sigma^1>+O(||J||^2)} =<\tau^0|H_0|\sigma^1> + <\tau^0|J|\sigma^1> \tag{4}$$

$$E^1_\sigma <\tau^0|\sigma^1> = E^0_\tau <\tau^0|\sigma^1> + <\tau^0|J|\sigma^1> + O(||J||^2) \tag{5}$$

$$<\tau^0|\sigma^1> = \frac{<\tau^0|J|\sigma^1>}{E^1_\sigma - E^0_\tau} + O(||J||^2) \tag{6}$$

with equations (5) and (6) being taken up to the first order in the perturbation J. So the new basis is denoted by

$$|\sigma^1>=\sum_{\tau^0} A^{(1)}_{\tau^0,\sigma^1}|\tau^0> \tag{7}$$

where

$$A^{(1)}_{\tau^0,\sigma^1} := \frac{<\tau^0|J|\sigma^1>}{E^1_\sigma - E^0_\tau} \tag{8}$$

Let us note here that our J is simply a spin flip, so it can only couple two spin chains that are one spin flip apart. Let us note $\sigma^{(i)}$ the spin chain σ where the i^{th} spin has been flipped.

We can then rewrite equation (7) as equations (9) and (10) :

$$A^{(1)} := \sum_{i=-\infty}^{\infty} A^{(1)}(i) \tag{9}$$

$$A^{(1)}(i)_{\sigma,\tau} := \begin{cases} \frac{<\sigma|J|\sigma^{(i)}>}{E_{\sigma^{(i)}}-E_\sigma} & \text{if } \tau = \sigma^{(i)} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

We can now observe that $A^{(1)}$ satisfies the equation :

$$\left[A^{(1)}, H_0\right] = -J \tag{11}$$

Here we have determined that the basis change needed to diagonalise H up to the first order is given by the matrix $1+A$. This, however, poses one problem : $1+A$ is not a unitary matrix, and we want the set of new eigenvalues to remain an orthonormal basis. To fix this, instead of a basis change with matrix $1+A$, we will use the matrix $e^A = \sum_0^\infty \frac{A^n}{n!}$, which is equal to $1+A$ at first order and which has the notable advantage of being a unitary matrix.

With this new basis change, the hamiltonian becomes :

$$H^{(1)} := e^A H e^{-A} = \sum_0^\infty \frac{(ad\, A)^n}{n!} H \tag{12}$$

$$= H_0 + J + \sum_1^\infty \frac{(ad\, A)^n H_0}{n!} + \frac{(ad\, A)^n J}{n!} \tag{13}$$

$$= H_0 + \cancel{J} - \cancel{J} + (ad\, A)J + \sum_2^\infty -\frac{(ad\ \ A)^{(n-1)}J}{n!} + \frac{(ad\ \ A)^n J}{n!} \tag{14}$$

$$= H_0 + \underbrace{(ad\, A)J + \sum_1^\infty n\frac{(ad\ \ A)^n J}{(n+1)!}}_{:=J^{(1)}} \tag{15}$$

(here we have neglected the (1) exponents so as to lighten the notations).

$H^{(1)}$ is therefore the sum of H_0 (which, as a reminder, is diagonal) and a new perturbation $J^{(1)}$, so we find ourselves with a new hamiltonian to apply our perturbation scheme to, iteratively. Our proof of the MBL result will essentially consist in iterating this scheme and making sure we have the proper bounds for it to converge.

Here we notice that all of the terms in the above sum are of the form $[A, ..., [A, J]]$. We can now remark that all of these terms are local :

Proposition 1. *(Locality of step-1 commutators) For any k and any $i_1...i_k$ we have*

$$[A(i_1), ..., [A(i_{k-1}), J(i_k)]] = 0 \tag{16}$$

if and only if there exists l such that $d(i_l, \{i_{l+1}, ..., i_k\}) > 1$.

Proof of locality of the commutators Let us take an example with just one commutator, $[A, J]$.

$$[A(i), J(i')]_{\sigma,\tau} =< \tau|\frac{J(i)}{E_\tau - E_{\sigma^{(i')}}}J(i') - J(i')\frac{J(i)}{E_{\tau^{(i')}} - E_\sigma}|\sigma > \tag{17}$$

$$= \frac{< \tau|J(i)|\sigma^{(i')} >}{E_\tau - E_{\sigma^{(i')}}} - \frac{< \tau^{(i')}|J(i)|\sigma >}{E_{\tau^{(i')}} - E_\sigma} \tag{18}$$

The first term of equation (18) is nonzero if and only if $\tau = \sigma^{(i,i')}$. Likewise, the second term is nonzero if and only is $\tau^{(i')} = \sigma^{(i)}$, which is equivalent to $\tau = \sigma^{(i,i')}$. This has nothing to do with locality yet, but we can rewrite equations (17) and (18) as :

$$[A(i), J(i')]_{\sigma,\sigma^{(i,i')}} = \frac{< \sigma^{(i,i')}|J(i)|\sigma^{(i')} >}{E_{\sigma^{(i,i')}} - E_{\sigma^{(i')}}} - \frac{< \sigma^{(i)}|J(i)|\sigma >}{E_{\sigma^{(i)}} - E_\sigma} \tag{19}$$

and, in the event that $|i' - i| > 1$ (ie. the successive spin flips are not nearest neighbors), we have the equality $E_{\sigma^{(i,i')}} - E_{\sigma^{(i')}} = E_{\sigma^{(i)}} - E_\sigma$. Indeed, when we look at the explicit form of H_0, we notice that it only couples spins that are nearest neighbors, so spin flips on i will not influence the energy difference generated by a spin flip on i' if the two are not nearest neighbors.

Another way of seeing this locality property is by writing $[A, H_0] = -J$ and computing :

$$[A(i), J(k)]|\sigma >= (-A(i)\,[A(k), H_0] + [A(k), H_0]\,A(i))\,|\sigma > \tag{20}$$

$$= (-A(i)A(k)V + A(i)VA(k) + A(k)VA(i) - VA(k)A(i))\,|\sigma > \tag{21}$$

$$= -A(i)A(k)E_\sigma|\sigma > +A(i)A(k)E_{\sigma^{(k)}}|\sigma > +A(k)A(i)E_{\sigma^{(i)}}|\sigma > -A(k)A(i)E_{\sigma^{(k,i)}}|\sigma > \tag{22}$$

$$= A(i)A(k)\,(E_{\sigma^{(k)}} - E_\sigma) - A(k)A(i)\,(E_{\sigma^{(k,i)}} - E_{\sigma^{(i)}}) \tag{23}$$

which cancels out in the event that $|i - k| > 1$. This reasoning generalizes to all of the commutators in the expansion of $H^{(1)}$ (one can understand this more easily with the second method).

Graphical interpretation of the expansion We choose to view our problem under a more graphical angle, which will help in deducing certain bounds in the later stages where expansions become more complicated.

The graphical expansion is as follows : as we previously mentioned, each term in the expansion of $H^{(1)}$ will be of the form $[A, ... [A, J]]$. In turn, each of these terms can be decomposed into $[A(i_1), ... [A(i_{n-1}), J(i_n)]]$, where there are no gaps between the $i_1, ...i_n$. Each of these terms will be associated with a graph g that goes through $i_1, .., i_n$. To distinguish the A terms from the J terms, we put an arch over every A site (ie. every site which contributes a denominator to the final commutator). This is illustrated in figure 1.

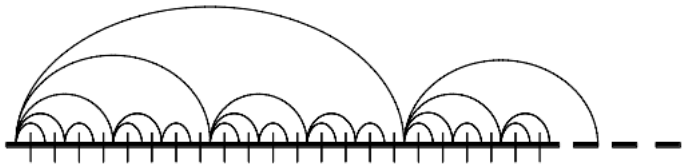

FIGURE 1 – An example of a graph, each denominator is represented by an arch over all of the sites that it encompasses.

Let us denote by g the graph with endpoints $i_1, ..., i_n$ and arches over $i_1, ..., i_{n-1}$. To lighten notations and make things more intuitive, we will denote by $J^{(1)}(g)$ the term in $[A(i_1), ... [A(i_{n-1}), J(i_n)]]$. Therefore we have

$$J^{(1)}_{\sigma,\tau} = \sum_{g:\sigma\to\tau} J^{(1)}(g) \tag{24}$$

At this point, one approximation that we wanted to make was to disregard all of the higher-order terms in the expansion in equation (15), and to keep only the term of order 1. This, however, did not work, because some of the higher-order terms generate terms of relatively low order in the next steps. Keeping only the lowest-order term causes the order of the perturbation to grow as 2^n (where n is the number of iterations of our perturbation scheme, or 'step'), and we found that we could not control the probability of resonant regions for a scheme which converges as fast as that. Therefore, we must a priori take into account all of the terms in the expansion.

2.2 Resonance condition

We will now define what we mean by resonant and non resonant regions. As previously explained, the non-resonant regions will have controlled behavior, and we will be able to prove our MBL bound in these regions. The resonant regions, on the other hand, will cause energy to be displaced, so the goal is to limit how many resonant regions there are (and, ultimately, to check that they do not percolate).

Defining what constitutes a resonant region is a little tricky : we want the bound to be strict enough that there are not too many resonant regions (otherwise they would percolate), but we also want it to be lax enough that there are not too many non-resonant regions (otherwise we would have very little information about the non-resonant regions, and, in particular, we would not be able to show that these regions satisfy the MBL bound).

For this first step, the resonance condition is given by the simple inequality (25).

$$\left|A^{(1)}\right| > \gamma/\epsilon \tag{25}$$

and a step-1 resonant region is defined as any site satisfying equation (25), ie. $\left|A^{(1)}(i)\right| > \gamma/\epsilon$

Assumption 1 yields the almost sure bound $|A(g)| \leq \gamma/\epsilon$ which ensures that almost no regions are resonant in the first step.

Bounds on $J^{(1)}$ We see from the expansion in (15) that

$$\left|J^{(1)}(g)\right| \leq \frac{\gamma(\gamma/\epsilon)^{|g|-1}}{(|g|-1)!} \tag{26}$$

3 Starting point of step n

Here we will anticipate a little to try and write out the starting point for the next steps. Problems begin to arise in step 2, so it will be useful to try to introduce them now in a heuristic way before moving on to the computational details.

Every time we iterate our scheme, we will use the perturbation from our new hamiltonian $H^{(n)} = H_0 + J^{(n-1)}$. We will construct operators $A^{(n)}$ and $J^{(n)}$ at each step. The expansion will therefore be exactly the one from equation (15), only with different superscripts. From here, we can understand that each new perturbation $J^{(n+1)}$ will consist in a sum of commutators of the operators from the scale just below : $\left[A^{(n)}, ... \left[A^{(n)}, J^{(n-1)}\right]\right]$.

At this point, we introduce a length scale $L_k := (1+\beta)^k$, with $\beta < 1$. At step k, we will only consider graphs of length $|g| \in [L_k, L_{k+1}[$ (we have not yet defined the length of a graph g, which we will denote $|g|$, but we will define it shortly). The reason why we do this is because, if our scheme converges too quickly, we will not be able to control the probability of resonant regions. A posteriori, we find that this probability can only be controlled if the length of the graphs evolves within that range.

More explicitly, we decompose as we did in the first step :

$$A^{(n)} =: \sum_{g_{n-1} \text{ step } n-1 \text{ graphs}} A^{(n)}(g_{n-1}) \tag{27}$$

$$J^{(n-1)} =: \sum_{g_{n-1} \text{ step } n-1 \text{ graphs}} J^{(n-1)}(g_{n-1}) \tag{28}$$

and $A^{(n)}$ will be defined as the only operator satisfying equation (29).

$$\left[A^{(n)}, H_0\Big|_{L_{n-1} \leq |g| \leq L_n}\right] := J^{(n-1)} \tag{29}$$

We also denote by $I(g)$ the smallest interval covering all the sites which have been flipped at least once.

Resonant regions at scale n are defined as the connected components of the set

$$\{x \in \mathbb{Z} \mid \exists g \text{ resonant graph st } x \in I(g)\} \tag{30}$$

Namely, resonant regions are the connected sets of points through which there passes at least one resonant graph. Note that we have not yet defined what a scale $n-1$ resonant graph is. This will also be defined shortly.

The definition of $A^{(n)}$ given in (29) is essentially the same as in (11), with one difference : the restriction to $|g| \leq L_n$. This allows us to generate nonzero $A^{(n)}$ for graphs of the desired lengths only.

Let us come back to our remark that $J^{(n)}$ is a sum of commutators of the form $\left[A^{(n)}, ... \left[A^{(n)}, J^{(n-1)}\right]\right]$. By decomposing the latter according to the graphical expansion for graphs of step $n-1$, we get $\left[A^{(n)}(g^1_{n-1}), ... \left[A^{(n)}(g^{k-1}_{n-1}), J^{(n-1)}(g^k_{n-1})\right]\right]$.

We note that $\left[A^{(n)}(g), J^{(n-1)}(h)\right] = 0$ if the graphs g and h are such that $d(I(g), I(h)) > 1$. We will refer to this as the locality property of our commutators. This is analogous to the expression of locality for step-1 operators that we gave in proposition 1. It can be seen inductively, as each operator decomposes into operators of the previous scale.

It follows that all graphs of step n with nonzero contributions to $J^{(n)}$ are concatenations of (potentially many) graphs of step $n-1$, like so :

$$g_n = g^1_{n-1} \vee ... \vee g^k_{n-1} \tag{31}$$

where $g^1_{n-1}, ..., g^k_{n-1}$ are such that the successive commutators in the definition of $J^{(n)}(g)$ are nonzero. This is given by the following condition.

$$d\left(I(g^j_{n-1}), \{I(g^{j+1}_{n-1}), ..., I(g^k_{n-1})\}\right) \leq 1 \tag{32}$$

Now we will attempt to explain the outline of our proof of resonance bounds at step n. Again we are anticipating a little on further problems, but we want to give an intuition for what the resonance condition is and how we will prove our bounds.

The scheme is the following : at step n, we construct our set of A graphs which are concatenations of step $n-1$ graphs that stay in the right length range $[L_n, L_{n+1}[$.

3.1 Long and short graphs

We now split our graphs into two categories :

1. *Long graphs* (or, visually, graphs that "loop a lot"). These are defined as the set of graphs that satisfy the inequality $|g| > \beta^{-1} |I(g)|$.
2. *Short graphs* (or graphs that "do not loop very much"). This is the set that satisfies $|g| \leq \beta^{-1} |I(g)|$.

Since long graphs loop a lot, the associated A operator will decay faster as a function of the number of sites visited (since the length $|g|$ grows fast with the number of sites visited). Heuristically, this faster decay means that we will better be able to control this A operator. Computationally, this is embodied by a change of norm, like so :

$$|g''_n| := |I(g_n)| \vee \beta L_n \tag{33}$$

We will be able to bound the A operator inductively using this new norm. An important thing to note here is that, in order to obtain this bound, we will not need to consider the sub-graph structure. So, heuristically, all long graphs of scale n that cover the same interval $I(g)$ and have the same beginning and end points are equivalent for the inductive procedure. This will allow us to resum all such long graphs. One (more formal) way of seeing this is to set an equivalency relation $\equiv$ defined by $g \equiv h$ iif. g and h are long scale n graphs, $I(g) = I(h)$ and g and h have the same start and end points. We call $[g]_{js}$ a *jump transition*, and an operator over a jump transition is defined as the sum of the operators associated with all of the long graphs in that jump transition, for example :

$$A^{(n)}\left([g]_{js}\right) = \sum_{h\in[g]_{js}} A^{(n)}(h) \tag{34}$$

This way of thinking will make computations simpler.

For short graphs, since the decay as a function of spatial distance is slower, this inductive bound will not suffice. The easy case is when a graph g does not loop at all, because this creates a form of independence (which we will detail later). In this case, we can bound our A operator directly, using a Markov bound.

The tricky part, then, is when some sections of the graph loop, but not enough to make a long graph. To treat these particular graphs, we find that we will have to bound the looping sections inductively and then apply the Markov bound to the remaining sections. This explanation is, of course, completely heuristic : to make this more precise we will need to define what a looping section is and how these bounds come together. Again, for now, we just want to give a general idea of the proof, and computational details will be given later.

We take a moment here to define $|g|$ properly.

A step-n graph g_n is defined as a concatenation of graphs $g^1_{n-1} \vee ... \vee g^k_{n-1}$ on scale $n-1$, and is associated with an operator $J^{(n)}(g_n) = A^{(n)}(g^1_{n-1})...J^{(n-1)}(g^{i_0}_{n-1})...A^{(n)}(g^k_{n-1})$. We define the length $|g_n|$ of g_n inductively, as

$$|g_n| := \left|g^{1''}_{n-1}\right| + ... + \left|g^{i_0}_{n-1}\right| + ... + \left|g^{k''}_{n-1}\right| \tag{35}$$

(the graphs associated with A operators contribute their modified norm to the sum on the right-hand side, instead of their original norm).

Note that a concatenation of two graphs on scale $n-1$ is necessarily a graph of scale n, because it has length $|g_n| = \left|g^1_{n-1}\right| + \left|g^{2''}_{n-1}\right| \geq L_{n-1} + \beta L_{n-1} \geq L_n$. This makes the lower bound of the restriction in equation (29) superfluous.

3.2 Resonance bound

Before tackling the computational aspects of the problem, we first need to define the resonance bound that we want to prove at each step. We have already done this for the first step, in section 2.2, but now we want to extend this to step n.

Concretely, for all step-n graphs g, we want to set a certain ϵ_g such that the probability that $A^{(n)}(g) > \gamma^{|g''|}/\epsilon_g$ is sufficiently small to control the non-percolation of resonant regions. Let us call this probability δ_g. This writes as

$$\mathbb{P}\left(\left|A^{(n)}(g)\right| > \alpha_g\right) \leq \delta_g \tag{36}$$

where δ_g must be such that the resonant regions do not percolate. This non percolation condition will impose an upper bound on δ_g, which will translate to a lower bound on the resonance condition. As mentioned previously, we also need an upper bound on the resonance condition to ensure that we retain enough information on non-resonant regions to be able to prove our MBL result.

The resonance condition that will fulfill both of these requirements is given by

$$\left|A^{(n)}(g)\right| > \begin{cases} \gamma^{|g''|} & \text{if } g \text{ is a jump step}\,; \\ (\gamma/\epsilon)^{|g''|}/(g!)^{2/9} & \text{if } g \text{ is a short graph.} \end{cases} \tag{37}$$

where $g!$ is defined recursively, as follows.

Definition 2. *(Factorial of a graph) Let g_n be a scale n graph made up of scale $n-1$ subgraphs $g_{n-1}^1, ..., g_{n-1}^k$. Suppose we have defined $g!$ for all graphs of scales $n-1$ and under. Then we define $g_n!$ as*

$$g_n! := \begin{cases} 1 & \textit{if } g_n \textit{ is a jump step}\,; \\ k! g_{n-1}^1! ... g_{n-1}^k! & \textit{if } g_n \textit{ is a short graph.} \end{cases} \tag{38}$$

We will show that this resonance condition may lead to choosing δ_g as follows, and that this choice of δ_g does indeed guarantee the non-percolation of resonant regions.

$$\delta_g = \begin{cases} \frac{\epsilon^{s|g|}}{(g!)^{2/9}} (C(s))^{|g|} & \text{if } g \text{ is a short graph}\,; \\ \frac{\epsilon^{s|g|}}{g!} (C(s))^{|g|} & \text{if } g \text{ is a long graph.} \end{cases} \tag{39}$$

where $s \in [0, 1[$ is a constant, and $C(s)$ is also a constant.

Three elements have to be proven in order to check the above statement. These are given by theorem 1 below.

Theorem 1. *We have the following three assertions.*

1. *The above resonance condition leads to the non-percolation of resonant regions.*
2. *Equation* (36) *is satified by the above choice of ϵ_g and δ_g.*
3. *The above resonance condition allows us to prove MBL in the non-resonant regions.*

We will start by proving point 1., and point 2. will be proven in the next section. Point 3. will conclude our paper.

Proof of non-percolation of resonant regions The non-percolation condition writes as

$$\mathbb{P}\left(\limsup_n (0 \in Res(n))\right) = 0 \tag{40}$$

where $Res(n)$ denotes all resonant regions up to step n.

Now, this is the part where our first simplification, which stated that we would neglect anything happening in regions that became resonant at steps $n-1$ and before, becomes a little bothersome. Indeed, we have no way of knowing what happens to a region after it becomes resonant, as we just set the perturbation in that region to zero. We therefore lose information on the resonant regions and we cannot be sure that they do not percolate. We will leave this problem aside here, but of course this makes our reasoning sketchy.

In order to obtain equation (40), we compute $\sum_n \mathbb{P}(0 \in Res(n))$.

$$\sum_n \mathbb{P}(0 \in Res(n)) \leq \sum_{g \text{ st } 0 \in Im(g)} \delta_g \tag{41}$$

So the bulk of the problem lies in counting the number of graphs in the right-hand sum and showing that this right-hand sum can be made arbitrarily small so that we can apply Borel-Cantelli lemma 1. We will dedicate subsection 3.2.1 to this.

3.2.1 Combinatorial bound

We will now try to bound the number of terms in the right-hand sum of (41). More specifically we will use the following lemma.

Lemma 1. *Let $c(g)$ be a so-called combinatorial factor such that $\sum_g c(g)^{-1} \leq 1$. Then we can bound any graphical sum $\sum_g f(g)$ by*

$$\left|\sum_g f(g)\right| \leq \sup_g |f(g)|\, c(g) \tag{42}$$

Proof The lemma is relatively intuitive, and its proof quite simple.

$$\sum_g |f(g)| \leq \sum_g |f(g)| \frac{c(g)}{c(g)} \leq \sup_h |f(h)|\, c(h) \sum_g 1/c(g) \tag{43}$$

$$\leq \sup_g |f(g)|\, c(g) \tag{44}$$

To give an intuition of the above lemma, let us think of the case where the sum is over a finite set G, and $c(g) := |G|$. This choice does fulfill the condition $\sum_{g\in G} c(g)^{-1} \leq 1$, and the right-hand side of equation (42) writes as $\left(\sup_{g\in G} |f(g)|\right) |G|$, which is a natural bound of a sum over all the elements in the set.

So, the notion of a combinatoric bound can be seen as a generalization of the bound

$$\left|\sum_{g\in G} f(g)\right| \leq \left(\sup_{g\in G} |f(g)|\right) |G| \tag{45}$$

when such a bound cannot easily be obtained.

Now we move on to the choice of $c(g)$.

We recall that a step n graph g_n is a concatenation of graphs at step $n-1$ that satisfy the locality condition (ie. there are no "gaps" between the scale $n-1$ graphs).

Let us define an equivalency relation : $g \sim_1 h$ iif. g and h are two scale n graphs that have the same set of subgraphs at all scales $1, 2, ..., n-1$. We will choose to define $c(g)$ as an upper bound on the number of graphs in $[g]_1$. This quantity will be computed by induction, and we will then prove that our choice satisfies the condition $\sum_g c(g)^{-1} \leq 1$ of lemma 1.

Naive bound We will start by establishing a first bound on $|[g]|$ which will not be precise enough, but which will help us get an idea of what we are doing. Let us then define $c_k := \prod_{i=1}^{k} L_i^{L_i^{-1}}$ and suppose that $c(g) := c_k^{|g|} g!$ is an upper bound on $|[g]_1|$ for all graphs g on scale $k \leq n-1$. We will prove by induction that this is the case also for graphs on scale n.

Let g_n be a scale n graph made up of subgraphs $g^1_{n-1} \vee ... \vee g^k_{n-1}$. Then we must choose

$$c(g_n) \geq \underbrace{\text{maximum number of rearrangements of the graphs } g^1_{n-1}...g^k_{n-1}}_{=:N(g^1_{n-1},...,g^k_{n-1})} \times \prod_{i=1}^{k} c(g^k_{n-1}) \tag{46}$$

So all that is left is to compute $N(g^1_{n-1}, ..., g^k_{n-1})$ (which we will denote N to lighten notations).

Suppose we have already arranged graphs $g^1_{n-1}, ..., g^l_{n-1}$. We need to arrange graph g^{l+1}_{n-1} in such a way that the commutators are nonzero.
Because of locality, this means that we need $d\left(I(g^{l+1}_{n-1}), \{I(g^1_{n-1}), ..., I(g^l_{n-1})\}\right) \leq 1$.
Since $\left|\{I(g^1_{n-1}), ..., I(g^l_{n-1})\}\right| \leq \sum_{i=1}^{l} \left|g^i_{n-1}\right| \leq lL_{n-1}$, there are at most $(l+1)L_{n-1}$ spins where the rightmost point of graph g^{l+1}_{n-1} may be.

Therefore, $N \leq \prod_{l=1}^{k}(l+1)L_{n-1} \leq L^k_{n-1}k!$.

Now all that is left is to compute

$$|[g_n]_1| := k!L^k_{n-1}\prod_{i=1}^{k} c(g^i_{n-1}) \tag{47}$$

$$= k!L^k_{n-1}\prod_{i=1}^{k} c_{n-1}^{\left|g^i_{n-1}\right|} g^i_{n-1}! \tag{48}$$

$$\leq \left(k!\prod_{i=1}^{k} g^i_{n-1}!\right) c_n^{|g_n|} \tag{49}$$

$$\leq g_n! c_n^{|g_n|} = c(g_n) \tag{50}$$

where (49) is obtained by remarking that $k \leq |g_n| / L_{n-1}$.

This concludes our proof that $|[g]_1| \leq c_n^{|g|} g!$ for all graphs g of scale n. What's more, we observe that the sequence $(c_n)_n$ increases with n and converges with $n \to \infty$, and we write $c_n \to c$.

This allows us to simplify the above definition of $c(g)$ to $c(g) := c^{|g|}g!$

Unfortunately, this combinatorial factor will not suffice for short graphs, and we will only be able to use it in the case of long graphs. To understand why, recall the non-percolation condition (41) as well as equation (39). Sums over long graphs in the right-hand side of (41) write

$$\sum_{g \text{ long graph st } 0 \in Im(g)} \delta_g \leq \sum_{g \text{ long graph}} \frac{\epsilon^{s|g|}}{g!}(C(s))^{|g|} \tag{51}$$

$$\leq \sup_{g \text{ long graph}} c(g)\frac{\epsilon^{s|g|}}{g!}(C(s))^{|g|} \tag{52}$$

$$\leq \sup_{L>0} c^L \epsilon^{sL}(C(s))^L \tag{53}$$

where we have used lemma 1 to get the third line. Therefore, given that we can select ϵ arbitrarily small, we can make the right-hand side arbitrarily small, and therefore ensure that resonant regions will not percolate in the limit.

Since long graphs are resummed to produce jump steps, we can make use of this naïve bound to yield bounds over jump steps. This resummation will allow us to obtain an exponential bound on quantities related to jump steps (the contribution from $g!$ will be "sucked into" the sum), so that equation (39) may become :

$$\delta_g = \begin{cases} \frac{\epsilon^{s|g|}}{(g!)^{2/9}}(C(s))^{|g|} & \text{if } g \text{ is a short graph}\,; \\ (C(s))^{|g|} & \text{if } g \text{ is a jump step.} \end{cases} \tag{54}$$

We will go into detail about this in subsection 4.2.

For sums over short graphs, using the bound from equation (39) yields

$$\sum_n \mathbb{P}(0 \in Res(n)) \leq \sup_g c^{|g|}\epsilon^{s|g|}(C(s))^{|g|}\frac{g!}{(g!)^{2/9}} \tag{55}$$

which is a bound that we cannot control. In order to be able to control the sum graphs, we would therefore like to have $c(g) := c^{|g|}(g!)^{2/9}$ as a combinatorial bound : let us try to establish this.

Precise bound for short graphs Here we define a second equivalence relation that is slightly different from the first : $g \sim_2 h$ iif. g and h *are both short graphs* of scale n that have the same subgraph structure from scale 1 up to scale $n-1$. As in the case of the naive bound, we want to show that $c(g) := c^{|g|}(g!)^{2/9}$ is a good bound on $|[g]_2|$.

In the case of short graphs, the condition $|g| \leq \frac{8}{7}|I(g)|$ imposes that graphs do not loop much. Let's begin the same inductive reasoning as with the naive bound, but this time we'll take this information into account.

Take g_n a *short* scale n graph made up of subgraphs $g^1_{n-1} \vee ... \vee g^k_{n-1}$. Let us compute a new bound for the number of rearrangements of subgraphs $g^1_{n-1}...g^k_{n-1}$ into a step n graph.

We denote by *floating graphs* the set of graphs g^k_{n-1} such that $I(g^k_{n-1}) \subset I(g^1_{n-1}) \cup ... \cup I(g^{k-1}_{n-1})$, and we denote by *pinned graphs* the set of graphs that are not floating graphs.

We will denote by l_p the total length of the pinned graphs, by l_f the total length of the floating graphs, by n_p the number of pinned graphs, and by n_f the number of floating graphs.

The short graphs condition imposes that $l_f \leq \frac{1}{7}l_p$. Moreover, each graph g^i_{n-1} has length $\left|g^i_{n-1}\right| \in [L_{n-1}, L_n[$, with $L_n = \frac{15}{8}L_{n-1} > 2L_{n-1}$. So $n_f \leq l_f/L_{n-1} \leq 2l_f/L_n \leq \frac{2}{7}\frac{l_p}{L_n} \leq \frac{2}{7}n_p$.

This means that at least $7/9$ of the scale $n-1$ subgraphs of g_n are pinned.

Suppose that graphs $g^1_{n-1}...g^l_{n-1}$ have already been arranged, and we want to arrange graph g^{l+1}_{n-1} so as to satisfy the locality condition. If is a floating graph, then there are at most nL_{n-1} spins where we can place the rightmost point of g^{l+1}_{n-1}. If, however, g^{l+1}_{n-1} is a pinned graph, there are only two places where its rightmost point can be.

In total, we find a bound on the number of rearrangements of the order of $\mathcal{O}(2^{7k/9}L_{n-1}^{2k/9}n^{2/9}) = \mathcal{O}(L^k_{n-1})(n!)^{2/9}$.

We reiterate the computation of $|[g_n]|$ that we did in equations (47) through (50) to yield :

$$|[g_n]_2| = (k!)^{2/9} L_{n-1}^k \prod_{i=1}^{k} c_{n-1}^{|g_{n-1}^i|} (g_{n-1}^i!)^{2/9} \tag{56}$$

$$\leq \left(k! \prod_{i=1}^{k} g_{n-1}^i! \right)^{2/9} c_n^{|g_n|} \tag{57}$$

$$\leq (g_n!)^{2/9} c_n^{|g_n|} \tag{58}$$

$$\leq (g_n!)^{2/9} c^{|g_n|} = c(g) \tag{59}$$

This new combinatorial factor allows the probability that resonant regions percolate to be bounded by :

$$\sum_n \mathbb{P}(0 \in Res(n)) \leq \sum_L c^L \epsilon^{sL} (C(s))^L \tag{60}$$

which can be made arbitrarily small to ensure that resonant regions do not percolate.

Finally, to check that $c(g)$ is an admissible combinatorial factor, let us check that $\sum_g c(g)^{-1} < 1$.

Conclusion of proof All that is left to show is

$$\sum_g c(g)^{-1} < 1 \tag{61}$$

The left-hand side of the above equation can be expanded into

$$\sum_g c(g)^{-1} = \sum_{A \in G/\sim} \sum_{g \in A} c(g)^{-1} \tag{62}$$

$$= \sum_{A \in G/\sim} |A| \, c(A)^{-1} \tag{63}$$

where G is the set of all graphs, and where we have grouped the graphs by equivalency classes, and defined $c(A)$ as the value of the combinatoric factor over the equivalency class A. The equivalency relation $\sim$ is a new equivalency relation defined by

$$g \sim h \text{ iif.} \begin{cases} \text{g and h are } \textit{short} \text{ graphs of the same scale with identical subgraph structure} \\ \text{g and h are } \textit{long} \text{ graphs of the same scale with identical subgraph structure} \end{cases} \tag{64}$$

Since $c(g)$ is defined as being a bound on the number of graphs per equivalency class, we can bound each term in the right-hand side of (63) by 1. Actually, we can even do better than that : by enlarging the constant c in $c(g) = c^{|g|}(g!)^{2/9}$ (ie. by taking $c > \prod_{k=0}^{\infty} L_k^{L_k^{-1}}$), we can obtain an expression that decays exponentially in $|g|$ for each term in the right-hand sum.

Now all that remains is to compute the number of terms in the sum, ie. the number of equivalency classes for the equivalency relation $\sim$. More specifically, it will suffice to prove that $|G/\sim|$ has exponential growth at most. Indeed, if we can establish this, we can again tweak the constant c to make sum (63) converge to a number smaller than 1.

To give an intuition for the bound over the number of equivalency classes, we will adopt a slightly different point of view : since a graph at scale k is nothing but a concatenation of graphs at scale $k-1$, which themselves are a concatenation of graphs on scale $k-2$,... we can think of a graph g as a tree with one vertex at the top, as represented in figure 2. Each vertex of the tree represents a subgraph of g, with subgraphs of scale k being at height k on the tree (the vertex at the top represents the whole graph g).

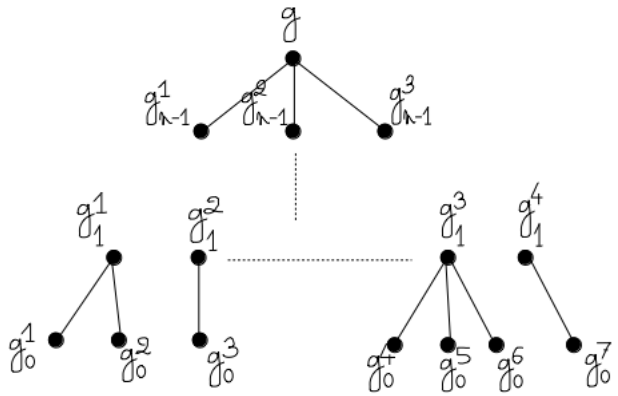

FIGURE 2 – Tree representation of graph g.

This representation does not specify how the subgraphs are arranged together to create graph g. Hence, each tree defines a graph g *up to rearrangement of the subgraphs*, so each equivalency class of $G/\sim$ can be represented by (at least) one such tree.

The converse is not true, however : not all the trees described above represent an equivalency class of $G/\sim$. This is mainly because we have length constraints on the graphs : recall that a graph of scale k must have length $|g| \in [L_k, L_{k+1}[$. So, for instance, if g is a graph of scale n (ie. $|g| \in [L_n, L_{n+1}[$), then the number of vertices at height $n-1$ on the tree is bounded by $|g|/L_{n-1}$. In the same way, the number of vertices at height $n-2$ is also bounded, by $|g|/L_{n-2}$ ... at height $n-k$, we therefore have a bound of $|g|/L_{n-k}$ on the number of vertices of the tree.

Let us call $\mathcal{T}$ the set of trees with this condition. Then we have $|\mathcal{T}| \geq |G/\sim|$. All that is left is then to bound $|\mathcal{T}|$.

Here we are going to tweak the definition of $\mathcal{T}$ a bit : we set L a natural integer, and we impose that trees in $\mathcal{T}$ represent at least one graph of length L. The reason why we do this is because want to control the number of trees *as a function of the length* $|g|$ *of the graph.*

Let us call $\mathcal{T}_n$ the set of trees in $T \in \mathcal{T}$ of height n. Let us also call $\mathcal{T}_n^{T_k}$ the set of trees $T \in \mathcal{T}_n$ that are "frozen" beyond height k in such a way that if we truncate T to keep only the points at heights greater that k, the truncated tree will be T_k.

Our conjecture is that, for all n, k and $T_{k,1}...T_{k,N}$, there exists a constant C such that

$$\left|\mathcal{T}_n^{T_k}\right| \leq \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \tag{65}$$

This looks like a complicated formula, but it will lead to the bound that we are interested in. Indeed,

equation (65) with $k = 0$ yields $\left|\mathcal{T}_n^{T_n}\right| \leq \left(\prod_{i=0}^{n-1} C^{L_i^{-1}}\right)^L$. Since T_n is a tree with height zero, it is nothing but a single vertex, so $\mathcal{T}_n^{T_n} = \mathcal{T}_n$, and therefore (65) leads to

$$|\mathcal{T}_n| \leq \left(\prod_{i=0}^{n-1} C^{L_i^{-1}}\right)^L \tag{66}$$

where the product on the right-hand side converges to $\prod_{i=0}^{\infty} C^{L_i^{-1}}$, and we have the exponential dependence in $|g|$ that we were looking for.

Now, let us prove equation (65) by induction on k. Take n a natural integer. Suppose we have, for a given $k < n$ and for all $T_k \in \mathcal{T}_{n-k}$, $\left|\mathcal{T}_n^{T_k}\right| \leq \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L$We want to show that, if $T_{k+1} \in \mathcal{T}_{n-k-1}$ (ie. we "unfreeze" the k^{th} level), then $\left|\mathcal{T}_n^{T_{k+1}}\right| \leq \left(\prod_{i=0}^{k} C^{L_i^{-1}}\right)^L$

The idea is the following : take a set of points V_k, which will be our vertices at height k. Take another set of points V_{k+1}, these will be our vertices at height $k+1$. Let us call $N := |V_{k+1}|$. We want to compute the number of ways in which we can "group" the points in V_k into groups $V_{k,1}, ..., V_{k,N}$ so that all of the vertices in group $V_{k,j}$ will stem from vertex j of V_{k+1}. This is illustrated in figure 3 below.

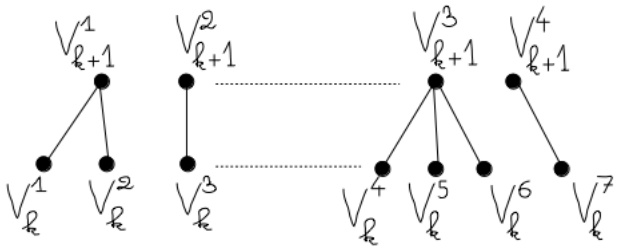

FIGURE 3 – One example of a possible grouping of vertices at height k.

Suppose we have l vertices at height k (so that $|V_k| = l$). Then the number ways in which we can form N groups with these l vertices is $\binom{N+l-1}{l}$, as illustrated in figure 4.

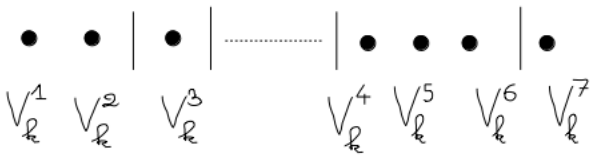

FIGURE 4 – Combinatorics for the number of ways to arrange a set of N elements into l groups : we imagine each group as being separated from the neighboring groups by a "wall", and we have to choose the position of each "wall" among all the elements.

Therefore, we have

$$\left|\{T \in \mathcal{T}_n^{T_{k+1}} \,|\, T \text{ has } l \text{ vertices at height } k\}\right| = \binom{N+l-1}{l} \times \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \tag{67}$$

Of course, we need to get rid of the condition on the number of vertices of our tree at height k. So we write

$$\left|\mathcal{T}_n^{T_{k+1}}\right| \leq \sum_{l=N}^{L/L_k} \binom{N+l-1}{l} \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \tag{68}$$

$$\leq \sum_{l=0}^{L/L_k} \underbrace{\binom{\frac{L}{L_k}+l-1}{l}}_{\leq exp(2ln2L/L_k)} \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \tag{69}$$

$$\leq \frac{L}{L_k} exp\left(\frac{2L}{L_k} ln2\right) \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \tag{70}$$

$$\leq \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \tag{71}$$

where we have recalled that the number of vertices at height k is bounded by L/L_k. The last inequality holds if $C > \sup_i exp(2Lln2ln(L/L_i))$ (the supremum is finite because $L/L_i \longrightarrow_{i\to\infty} 0$).

So, we have proven our bound (65), and therefore equation (66) holds. We therefore have an exponential bound over the number of equivalency classes on our set of graphs for the equivalency relation $\sim$. Coming back to equation (63), this writes as

$$\sum_g c(g)^{-1} \leq \sum_{A\in\{g\}/\sim} \prod_{k=0}^{\infty} \left(L_k^{L_k^{-1}}\right)^{|g|} c^{-|g|} \tag{72}$$

$$\leq \sum_{L=0}^{\infty} \left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \prod_{k=0}^{\infty} \left(L_k^{L_k^{-1}}\right)^L c^{-L} \tag{73}$$

where the sum in equation (73) is over all the possible graphical lengths.

The quantity $\left(\prod_{i=0}^{k-1} C^{L_i^{-1}}\right)^L \prod_{k=0}^{\infty} \left(L_k^{L_k^{-1}}\right)^L c^{-L}$ can be taken to be arbitrarily small by enlarging the constant c, as we alluded to earlier. This concludes our proof of point 1. of theorem 1 : that the resonance condition given by equations (37) and (39) does indeed lead to the non-percolation of resonant regions. □

4 General step

Our next point of interest will be proving assertion 2. of theorem 1, ie. checking that the resonance conditions (37) and (39) satisfy equation (36), which we recall claims that $\mathbb{P}\left(\left|A^{(n)}(g)\right| > \alpha_g\right) \leq \delta_g$. To do this, we will need to run our diagonalisation scheme. Let us start by recalling the starting point of step n.

4.1 Starting point : review

We will assume that our hamiltonian H has already been transformed into a new hamiltonian $H^{(n)} =: H_0 + J^{(n)}$ which has a perturbative component $J^{(n)}$ of order n, and we will apply our perturbation scheme to this new hamiltonian to create a hamiltonian $H^{(n+1)}$ which will have a perturbative component of order $n+1$.

Let us first set the inductive assumptions from which we start.

1. Operators $A^{(n)}$ and $J^{(n)}$ have been defined. Operator $J^{(n)}$ is a sum of commutators of the form $\left[A^{(n)}, ... \left[A^{(n)}, J^{(n-1)}\right]\right]$, and we expand $J^{(n)}$ according to the graphical expansion, in the following manner :

$$J^{(n)}_{\sigma,\tau} = \sum_{g_n : \sigma \to \tau} J^{(n)}(g_n) \tag{74}$$

2. Graphs g_n are multi-scale graphs made up of graphs g_{n-1} from scale $n-1$. Let us take g_n a graph on scale n such that $g_n := g^1_{n-1} \vee ... \vee g^k_{n-1}$. Then we have

$$J^{(n)}(g_n) = A^{(n)}(g^1_{n-1}) ... J^{(n-1)}(g^{i_0}_{n-1}) ... A^{(n)}(g^k_{n-1}) \tag{75}$$

   The above expansion goes to show that each J graph is constituted of (potentially many) A graphs from the scale just underneath, and only one J graph. This could already be seen from the commutators in the expansion of the new hamiltonian, but once again a graphical approach is a lot more practical.

3. If g_n is a graph on scale n, then we have defined a (modified) norm $|g''_n|$ inductively, using step $n-1$ as follows :

$$|g''_n| := |I(g_n)| \vee \beta L_n \tag{76}$$

   If $g_{n+1} := g^1_n \vee ... \vee g^k_n$ is a graph of scale $n+1$ with i_0 being the position of the $J(n)$ graph in the expansion up to scale n, then the norm $|g_{n+1}|$ is given by $|g_{n+1}| := \left|g^{1''}_n\right| + ... + |g^{i_0}_n| + ... + \left|g^{k''}_n\right|$

4. Graphs are split between long graphs and short graphs, with long graphs satisfying the inequality $|g| \leq \beta |I(g)|$, and short graphs satisfying $|g| \geq \beta |I(g)|$
   We resum all of the quantities associated with all of the long graphs that cover the same intervals $I(g)$, and we call these long graphs *jump transitions*.

5. Any point $x \in \mathbb{Z}$ through which there passes a resonant graph of scale n or below is part of a resonant region.

6. Non-resonant regions satisfy equation (77)

$$\left|A^{(n)}(g_{n-1})\right| \leq \begin{cases} (\gamma/\epsilon)^{|g''_{n-1}|} / \left(g_{n-1}!\right)^{2/9}, & \text{if } g_{n-1} \text{ is not a jump step}; \\ \gamma^{|g''_{n-1}|}, & \text{if } g_{n-1} \text{ is a jump step.} \end{cases} \tag{77}$$

7. The probability of a region being resonant satisfies equation (36) which we recall below :

$$\mathbb{P}\left(\left|A^{(n)}(g)\right| > \alpha_g\right) \leq \delta_g \tag{78}$$

where α_g and δ_g are defined in equations (37) and (39), by

$$\alpha_g = \begin{cases} \gamma^{|g''|} & \text{if } g \text{ is a jump step}\,; \\ (\gamma/\epsilon)^{|g''|}/(g!)^{2/9} & \text{if } g \text{ is a short graph.} \end{cases} \tag{79}$$

$$\delta_g = \begin{cases} \epsilon^{s|g|}/(g!)^{2/9}(C(s))^{|g|} & \text{if } g \text{ is a short graph}\,; \\ (C(s))^{|g|} & \text{if } g \text{ is a jump step.} \end{cases} \tag{80}$$

Point 6. of this assumption is our point 2. of theorem 1, which we will need to prove at step $n+1$ in order to check the consistency of the scheme.

Take g_{n+1} a scale $n+1$ graph, then. We will start with the simpler case : suppose g_{n+1} is a long graph.

4.2 Bound on long graphs

Over long graphs, we will prove the following bounds :

$$\begin{cases} \left|J^{(n)}(g)\right| & \leq \frac{1}{g!}\gamma\left(\frac{\gamma}{\epsilon}\right)^{|g|-1} \\ \left|A^{(n+1)}(g)\right| & \leq \frac{1}{g!}\left(\frac{\gamma}{\epsilon}\right)^{|g''|} \end{cases} \tag{81}$$

Let us first prove that we can deduce bound (79) from this. We will have to use our naïve combinatorial bound to resum long graphs. Let us compute this for the A operator (for J, the computation is analoguous).

Let us take g a scale n long graph. Then, per our naïve combinatorial bound,

$$\left|A^{(n)}([g]_{js})\right| \leq \sup_{h\in[g]_{js}} c(h)\frac{1}{h!}(\gamma/\epsilon)^{|h|} \leq \sup_{h\in[g]_{js}} \tilde{\gamma}^{|h|} \tag{82}$$

where $\tilde{\gamma} := \frac{c}{\epsilon}\gamma$. For convenience, we will simply write $\tilde{\gamma} = \gamma$, as there is simply a multiplicative factor between the two, which does not matter given that we can make γ arbitrarily small.

Since the supremum is taken over graphs of scale n only, we know that $|h| \in [L_n, L_{n+1}[$, and thus

$$\gamma^{L_n} \geq \sup_{h\in[g]_{js}} \gamma^{|h|} \geq \gamma^{(1+\beta)L_n} \tag{83}$$

This means that, for all $\tilde{g} \in [g]_{js}$,

$$\gamma^{|\tilde{g}|} \geq \gamma^{-\beta L_n} \sup_{h\in[g]_{js}} \gamma^{|h|} \tag{84}$$

Therefore,

$$\left|A^{(n)}([g]_{js})\right| \leq \gamma^{|g|}\gamma^{\beta L_n} \tag{85}$$

For long graphs, recall that $|g| \geq \beta^{-1}\,|I(g)|$. Therefore, $|g''| \leq \beta\,|g| \vee \beta L_n \leq \beta\,|g|$. Finally, this yields

$$\left|A^{(n)}([g]_{js})\right| \leq \gamma^{\beta^{-1}|g''|}\gamma^{\beta L_n} \tag{86}$$
$$\leq \gamma^{|g''|}\gamma^{(\beta^{-1}-1)|g''|}\gamma^{\beta L_n} \tag{87}$$
$$\leq \gamma^{|g''|}\gamma^{(1-\beta)L_n}\gamma^{\beta L_n} \tag{88}$$
$$\leq \gamma^{|g''|}\gamma^{L_n} \leq \gamma^{|g''|} \text{ since } \left(\gamma^{L_n}\right)_n \text{ is a decreasing sequence.} \tag{89}$$

□

Let us prove bounds (81) then, as we have said, inductively. Suppose the bound holds for graphs up to scale n. As usual, we set g_{n+1} a long graph at scale $n+1$, made by concatenating graphs $g_n^1, ..., g_n^k$ at scale n.

J graphs at scale $n+1$ are concatenations of (potentially many) scale $(n+1)$ A graphs, and one scale n J graph. This yields the bound

$$\left|J^{(n+1)}(g_{n+1})\right| \leq \left|A^{(n+1)}(g_n^1)\right| ... \left|J^{(n)}(g_n^{i_0})\right| ... \left|A^{(n+1)}(g_n^k)\right| \tag{90}$$
$$\leq \left(\prod_{i=1}^{k} \frac{1}{g_n^i!}\right) \left(\frac{\gamma}{\epsilon}\right)^{\left|g_n^{1\prime\prime}\right|} ...\gamma \left(\frac{\gamma}{\epsilon}\right)^{\left|g_n^{i_0}\right|-1} ... \left(\frac{\gamma}{\epsilon}\right)^{\left|g_n^{k\prime\prime}\right|} \tag{91}$$
$$\leq \frac{1}{g_{n+1}!}\gamma \left(\frac{\gamma}{\epsilon}\right)^{|g_{n+1}|-1} \tag{92}$$

which proves the first bound.

For the second bound, we recall that $A^{(n+2)}_{\sigma,\sigma'}(g_{n+1}) = J^{(n+1)}_{\sigma,\sigma'}(g_{n+1}) / \left(E_{\sigma'} - E_{\sigma}\right)$, where we have denoted by σ and σ' the starting and ending points of g_{n+1}, respectively.
This yields the equality on the norms $\left|A^{(n+2)}_{\sigma,\sigma'}(g_{n+1})\right| = \left|J^{(n+1)}_{\sigma,\sigma'}(g_{n+1})\right| / \epsilon^{|g_{n+1}|}$ a.s.

Now, let us recall that our graph g_{n+1} is a long graph. As such, the condition $|g_{n+1}| \geq \beta^{-1}|I(g_{n+1})|$ is satisfied. Combined with the definition of the new norm $\left|g''_{n+1}\right| := |I(g)_{n+1}| \vee \beta L_{n+1}$, this yields $\left|g''_{n+1}\right| \leq \beta |g_{n+1}| \vee \beta L_{n+1} \leq \beta |g_{n+1}|$. So, we deduce that almost surely,

$$\left|A^{(n+2)}(g_{n+1})\right| \leq \frac{1}{g_{n+1}!}\gamma \left(\frac{\gamma}{\epsilon}\right)^{|g_{n+1}|-1} \left(\frac{1}{\epsilon}\right)^{|g_{n+1}|} \tag{93}$$
$$\leq \frac{1}{g_{n+1}!}\gamma \left(\frac{\gamma}{\epsilon^2}\right)^{|g_{n+1}|-1} \frac{1}{\epsilon} \tag{94}$$
$$\leq \frac{1}{g_{n+1}!} \left(\frac{1}{\epsilon^3}\right) \left(\frac{\gamma}{\epsilon^2}\right)^{\left|g''_{n+1}\right|/\beta} \text{ since } \frac{\gamma}{\epsilon^2} < 1 \tag{95}$$
$$\leq \frac{1}{g_{n+1}!} \left(\frac{1}{\epsilon^3}\right) \left(\frac{\gamma}{\epsilon}\right)^{\left|g''_{n+1}\right|(\beta^{-1}-1)} \left(\frac{\gamma}{\epsilon}\right)^{\left|g''_{n+1}\right|} \tag{96}$$
$$\leq \frac{1}{g_{n+1}!} \left(\frac{\gamma}{\epsilon}\right)^{\left|g''_{n+1}\right|} \text{ for } \gamma \text{ small enough.} \tag{97}$$

where the last inequality holds because $\beta^{-1} - 1 > 0$, and therefore $\left(\frac{\gamma}{\epsilon}\right)^{\left|g''_{n+1}\right|}$ can be made arbitrarily small.

This concludes our proof of equation (81).

□

Now we move on to the short graphs case.

4.3 Bound on short graphs

For short graphs, this inductive reasoning cannot be applied. As is shown above, applying a naïve inductive reasoning yields a denominator of the form $(\epsilon^2)^{|g_{n+1}|}$ instead of the desired $\epsilon^{|g_{n+1}|}$.

So, for short graphs, we will need to use Markov bounds.

Let's start with a very simple case : the loop-free graphs. *Loop-free* graphs are defined as graphs that do not cross over themselves. More formally, we can choose to expand graphs up to scale 0 and view them as a function of a time variable (or "step variable") t, which counts the number of scale 0 operators. Loop-free graphs can then be redefined as injective functions $g : \mathbb{N} \longrightarrow \mathbb{Z}$. Operators on these graphs can be bounded very easily, because each new "step" in the graph creates new independent variables.

4.3.1 Markov bound on loop-free graphs

Take a loop-free graph g of scale n. As we said, we can expand g up to elementary steps and view it as a function of time. Let's write $g(0) = i_0$, $g(1) = i_1$, ... $g(N) = i_N$ (where N is the total number of elementary steps in g).

We use the following Markov bound on A

$$\mathbb{P}\left(\left|A^{(n)}(g)\right| \leq \frac{(\gamma/\epsilon)^{|g''|}}{(g!)^{2/9}}\right) \leq \left(\frac{\epsilon}{\gamma}\right)^{s|g|} \frac{(g!)^{2s/9}}{g!}\mathbb{E}\left[\left|A^{(n)}(g)\right|^s\right] \tag{98}$$

$$\leq \frac{\epsilon^{s|g|}}{(g!)^{2/9}}\mathbb{E}\left[\frac{1}{\left|\Delta E_1^{(1)}...\Delta E_1^{(k_1)}\Delta E_2^{(1)}...\Delta E_2^{(k_2)}...\Delta E_n^{(1)}...\Delta E_n^{(k_n)}\right|^s}\right] \tag{99}$$

where we have used $s = 2/7$, and where $\Delta E_i^{(j)}$ denotes the j^{th} denominator of scale i associated with g. Note that, because of the way that our graphs are defined, the number of denominators in the product is exactly the number of elementary steps N taken by the graph.

Once again, we recall that we want to prove the smallness of the resonance probability

$$\mathbb{P}\left(\left|A^{(n)}(g)\right| \leq \frac{(\gamma/\epsilon)^{|g''|}}{(g!)^{2/9}}\right) \leq \frac{\epsilon^{s|g|}}{(g!)^{2/9}}(C(s))^{|g|} \tag{100}$$

So all that is left is to compute the product of denominators in (99). To lighten notations, we will only do this in the case of a graph with two steps, and then give an idea for the general case.

In the case of a graph with two steps i_1 and i_2, we already know that i_1 and i_2 are next neighbors. So, we can set $i_1 = 1$ and $i_2 = 2$ without loss of generality.

If $N = 2$, then, equation (99) writes

$$\mathbb{P}\left(\left|A^{(n)}(g)\right| \leq \frac{(\gamma/\epsilon)^{|g''|}}{(g!)^{2/9}}\right) \leq \frac{\epsilon^{s|g|}}{(g!)^{2/9}}\mathbb{E}\left[\frac{1}{|\Delta E_1 \Delta E_2|^s}\right] \tag{101}$$

where we can explicitly write out ΔE_1 and ΔE_2

$$\begin{cases} \Delta E_1 = 2(h_1\sigma_1 + J_0\sigma_0\sigma_1 + J_1\sigma_1\sigma_2) \\ \Delta E_2 = 2(h_1\sigma_1 + 2h_2\sigma_2 + J_0\sigma_0\sigma_1 + J_2\sigma_2\sigma_3) \end{cases} \tag{102}$$

and therefore

$$\mathbb{E}\left[\frac{1}{|\Delta E_1 \Delta E_2|^s}\right] = \tag{103}$$

$$\int dJ_0 dJ_1 dJ_2 dJ_3 dh_1 dh_2 \frac{1}{2\left|h_1 + J_0\sigma_0 + J_1\sigma_2\right|^s} \frac{1}{2\left|h_1\sigma_1 + 2h_2\sigma_2 + J_0\sigma_0\sigma_1 + J_2\sigma_2\sigma_3\right|^s} \tag{104}$$

where the integral is a shorthand for $\int_{-1}^{1} dJ_0 \int_{-1}^{1} dJ_1 \int_{-1}^{1} dJ_2 \int_{-1}^{1} dJ_3 \int_{-1}^{1} dh_1 \int_{-1}^{1} dh_2$.

Now, we notice that the only terms involving J_2, J_3 and h_2 are in the second denominator, so we can use them to integrate the second denominator first. This is where having a loop-free graph becomes important : it is crucial for this proof to keep the variables independent.

We set

$$C(s) := \max_{b\in\mathbb{R},\sigma_1,\sigma_2\in\{-1,1\}} \int dh dJ_1 dJ_2 \frac{1}{2\left|h + J_1\sigma_1 + J_2\sigma_2 + b\right|} \tag{105}$$

which is well-defined given the boundedness of all of the quantities.

Then, (103) gives

$$\mathbb{E}\left[\frac{1}{|\Delta E_1 \Delta E_2|^s}\right] \tag{106}$$

$$= \int dJ_0 dJ_1 dh_1 \frac{1}{2\left|h_1 + J_0\sigma_0 + J_1\sigma_2\right|^s} \int dJ_2 dJ_3 dh_2 \frac{1}{2\left|h_1\sigma_1 + 2h_2\sigma_2 + J_0\sigma_0\sigma_1 + J_2\sigma_2\sigma_3\right|^s} \tag{107}$$

$$\leq \int dJ_0 dJ_1 dh_1 \frac{1}{2\left|h_1 + J_0\sigma_0 + J_1\sigma_2\right|^s} C(s) \tag{108}$$

$$\leq C(s)^2 \tag{109}$$

In cases where $N > 2$, we proceed analogously, integrating each denominator using the free variables that we have. Each denominator has a free variable because of the structure of our graphs, with each graph being constructed out of a concatenation of one J graph and a certain number of A graphs on the scale below. We call this a *nested structure*, and we recall that this is visually represented on figure 1.

Since there are N denominators in the product in equation (99), we obtain a bound $C(s)^N$ on our probability. It is important to note that, in general, $N \neq |g|$, given that we modify the norm of g at each step in our procedure. However, for loop-free graphs, $|g| = N$ holds : we never effectively modify the norm of a loop-free graph, because $|I(g)| = |g|$ in this particular case.

We then obtain

$$\mathbb{P}\left(\left|A^{(n)}(g)\right| \leq \frac{(\gamma/\epsilon)^{|g''|}}{(g!)^{2/9}}\right) \leq \frac{\epsilon^{s|g|}}{(g!)^{2/9}} C(s)^{|g|} \tag{110}$$

which is exactly the bound that we wanted to prove.

□

4.4 General short graphs case

Obviously, in the general short graphs case, the above reasoning fails. Our goal is still to find an independent integration variable for each denominator, or, equivalently, to attribute a different site to each denominator ΔE and use the integration variables associated with ΔE to integrate ΔE. Two separate problems arise in looking for independent variables when we consider the possible overlapping of subgraphs :

1. If two subgraphs g_k^1 and g_k^2 overlap, then some spin flips are potentially cancelled out (we will call these "mute spin flips"). The question, then, is how to make sure that we have a free spin flip to integrate on.
2. Any overlap potentially creates denominators with no new variables, so that we have no guarantee that we can use a different spin flip to integrate every denominator.

We will start by formulating a procedure to treat problem 1. in the simplified case of graphs where problem 2. does not arise. We will then explain how to generalize this scheme to all graphs.

4.4.1 Simplified case

The first key element we must notice here is that we do not actually need to have an independent integration variable for each denominator : we can weaken this condition. For the purposes of our proof, verifying theorem 2 will suffice.

Theorem 2. *Let g be a graph of scale n. Let h be a subgraph of g on scale $k \leq n$ that contributes an $A^{(k+1)}$ operator to the scale k expansion of g. We write*

$$A^{(k+1)}(h) = \frac{J^{(k)}(h)}{\Delta E_h} \tag{111}$$

We call x_h the site whose associated variables we use to integrate the denominator ΔE_h, and we define $n_x := |\{h \text{ subgraph of } g | x_h = x\}|$ for any $x \in \mathbb{Z}$.

We claim that for all x, $\underline{n_x \leq 4}$.

The above theorem does not quite claim that we can find a free variable to integrate each denominator, but it does claim that no more than 4 denominators will be integrated by the same variable, and this is sufficient. Indeed, using the same reasoning as in (106) yields

$$\mathbb{E}\left[\prod_{h \text{ subgraph of } g} \frac{1}{\Delta E_h}\right] = \prod_{x \in I(g)} \mathbb{E}\left[\prod_{h \text{ st } x_h = x} \frac{1}{\Delta E_h}\right] \tag{112}$$

where, per theorem 2, the products in the expectation on the right-hand side have no more than 4 factors. Therefore we deduce

$$\mathbb{E}\left[\prod_{h \text{ subgraph of } g} \frac{1}{\Delta E_h}\right] \leq \prod_{x \in I(g)} \max_{\substack{b_1,b_2,b_3,b_4 \in \mathbb{R} \\ \sigma_1,\sigma_2 \in \{-1,1\}}} \int dh dJ_1 dJ_2 \prod_{i=1}^{4} \frac{1}{2\,|h + J_1\sigma_1 + J_2\sigma_2 + b_i|} \tag{113}$$

$$\leq \prod_{x \in I(g)} C(s)^4 \leq C(s)^{4|I(g)|} \leq C(s)^{4|g|} \tag{114}$$

So, provided theorem 2 is correct, we would only have to modify the constant $C(s)$ to $C(s)^4$ in order to make our scheme function all the same.

Let us now move on to the proof.

Proof of theorem 2 We first set the terms of our simplified case with three assumptions :

1. If h_1 and h_2 are subgraphs of g (at any scale), then they have at most one site of overlap.

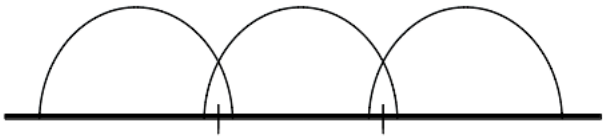

FIGURE 5 – Denominators have at most one site of overlap.

   This is our main assumption here, as we want to first neglect the problem of large loops that generate denominators without free variables.

2. There are no jump steps in the expansion of g.
3. There are no graphs with only mute spin flips.

   Note that this final simplification is not quite equivalent to neglecting energy renormalization. Indeed, even if we do neglect energy renormalization, we should still take into account any J operator without visible (ie. not mute) spin flips, as these can still exist through concatenation of operators that mute each other's spin flips.

Let's call $g_l^1 \vee ... \vee g_l^{N_l}$ the scale l expansion of g. Now, given these assumptions, we will separate our proof into three steps.

Step 1 : Attribution of muted spin variables In this step, we aim to attribute each muted spin flip to 2 denominators. We use the following procedure.

For each mute spin flip x of g, we define

$$\tilde{n}_x := \max\left\{l \in \{0, ..., n\} |\ \exists k \in \{1, ..., N_l\} \text{ st } x \text{ is a visible spin flip for } g_l^k\right\} \tag{115}$$

Take l as in the definition of $\tilde{n}_x$, and take $k_1, k_2 \in \{1, ..., N_l\}$ the indices for which $x \in g_l^{k_i}$ is a visible spin for $g_l^{k_i}$. We know that there are at most two such indices, because assumption 1. forbids any overlap longer than one site, and we know there must be at least two such indices because the visible spin flip becomes mute at the scale above.

We define x to be the integration variable for denominators $g_l^{k_1}$ and $g_l^{k_2}$. We write this as $x_{g_l^{k_1}} := x$ and $x_{g_l^{k_2}} := x$.

Step 2 : Existence of a visible spin flip on the boundary of a hole After step 1, we are left with a new graph $\tilde{g}$, where we will not have to consider the arches that we have integrated in step 1 (we say we have "treated" these). This is equivalent to replacing A operators by J operators for every graph which has been treated in step 1. This changes $A^{(n)}$ into a new operator over g, which we call $\tilde{A}^{(n)}(\tilde{g})$.

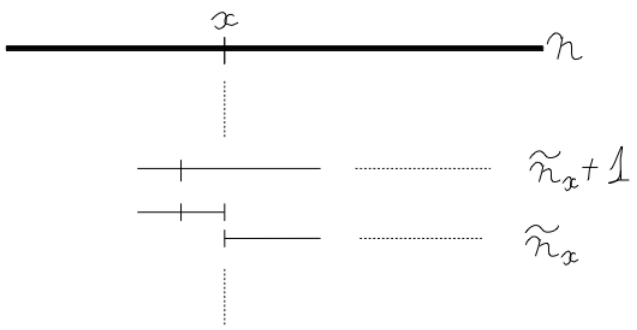

FIGURE 6 – Illustration of the definition of $\tilde{n}_x$, the first scale at which a visible spin flip appears

In step 2, we aim to prove that there exists a visible spin flip on the boundaries of certain specific regions of $\tilde{g}$, defined below.

Definition 3. *(Hole)*
Take h_{l+1} a graph of scale $l+1$, made up from scale k subgraphs $h_l^1 \vee ... \vee h_l^k$. We denote

$$\tilde{A}^{(l+1)}(g_{l+1}) = \frac{\tilde{O}^{(l)}(h_l^1)...\tilde{O}^{(l)}(h_l^k)}{\Delta E_{h_{l+1}}} \tag{116}$$

where $\tilde{O} = \tilde{A}$ or $\tilde{J}$.
Now let us consider our original graph $\tilde{g}$. At each scale k, we denote by

$$I_k := \bigcup_{\substack{i\in\{1,...,N_k\} \\ \tilde{O}^{(i)}(\tilde{g}_i^k)=\tilde{J}^{(i)}(\tilde{g}_i^k)}} I(\tilde{g}_i^k) \tag{117}$$

the union of all of the intervals covered by J graphs at scale k. We also denote by

$$I := \bigcap_{k\geq 1} I_k \tag{118}$$

the intersection of all such intervals.
We call <u>hole</u> any connected component of I.

We want to show lemma 2.

Lemma 2. *There exists a hole $T \in I$ of $\tilde{g}$ such that $\partial T \cup \partial\left(T^c\right)$ contains a visible spin flip of $\tilde{g}$.*

Proof of lemma 2 First of all, if all of the arches of g of scale $k \geq 1$ have been erased in the first step (except, of course, for the arch covering all of g), we use a spin flip at the boundary of $I(g)$ or neighboring the boundary (ie. in $\partial I(g) \cup \partial\left(I(g)^c\right)$). One of these spin flips must be visible so that the operator $A^{(n)}(g)$ may have a nonzero commutator with its neighbors.

Now that this special case has been set aside, we define

$$k_1 := \max\left\{1 \leq j < n \mid \exists l \text{ st } \tilde{O}^{(j)}(\tilde{g}_j^l) = \tilde{A}^{(j)}(\tilde{g}_j^l)\right\} \tag{119}$$

the largest scale at which we can still see an arch, excluding scale 0.

We begin by proving the following lemma :

Lemma 3. *There exists a visible spin flip x such that :*

$$\exists l' \ st \ \begin{cases} d\left(x, \partial I(\tilde{g}^{l'}_{k_1})\right) \leq 1 \\ d\left(x, \partial I(\tilde{g}^{l'+1}_{k_1})\right) \leq 1 \end{cases} \tag{120}$$

$$x \in \overset{\circ}{\widehat{I(\tilde{g}^{i}_{k_1+1})}} \tag{121}$$

$$\begin{cases} \tilde{O}^{(k_1)}(\tilde{g}^{l'}_{k_1}) = \tilde{A}^{(k_1)}(\tilde{g}^{l'}_{k_1}) \ and \ \tilde{O}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) = \tilde{J}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) \\ or \\ \tilde{O}^{(k_1)}(\tilde{g}^{l'}_{k_1}) = \tilde{J}^{(k_1)}(\tilde{g}^{l'}_{k_1}) \ and \ \tilde{O}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) = \tilde{A}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) \end{cases} \tag{122}$$

Proof of lemma 3 We consider graph $\tilde{g}$ at scales $k_1 + 1$ and k_1.
We take each graph $\tilde{g}^{i}_{k_1+1}$ and expand it to scale k_1 : $\tilde{g}^{i}_{k_1+1} = \tilde{g}^{j_i}_{k_1} \vee ... \vee \tilde{g}^{j_{i+1}-1}_{k_1}$. By definition of k_1,

$$\exists i \in \{1, ..., N_{k_1+1}\}, \exists l \in \{j_i, ..., j_{i+1} - 1\} \ \text{st} \ \tilde{O}^{(k_1)}(\tilde{g}^{l}_{k_1}) = \tilde{A}^{(k_1)}(\tilde{g}^{l}_{k_1}) \tag{123}$$

We have

$$\tilde{A}^{(k_1+1)}(\tilde{g}^{i}_{k_1+1}) = \tilde{O}^{(k_1)}(\tilde{g}^{j_i}_{k_1})...\tilde{O}^{(k_1)}(\tilde{g}^{j_{i+1}-1}_{k_1}) \tag{124}$$

where we know that at least one of the $\tilde{O}$ is a $\tilde{J}$ (and, again, per equation (123), at least one of the $\tilde{O}$ is an $\tilde{A}$). Therefore, we can choose a site x that is on the boundary between an $\tilde{A}$ operator and a $\tilde{J}$ operator. This writes as

$$\exists x \in \overset{\circ}{\widehat{I(\tilde{g}^{i}_{k_1+1})}}, \ \exists l' \ \text{st} \ \begin{cases} d\left(x, \partial I(\tilde{g}^{l'}_{k_1})\right) \leq 1 \\ d\left(x, \partial I(\tilde{g}^{l'+1}_{k_1})\right) \leq 1 \end{cases} \tag{125}$$

$$\text{with} \ \begin{cases} \tilde{O}^{(k_1)}(\tilde{g}^{l'}_{k_1}) = \tilde{A}^{(k_1)}(\tilde{g}^{l'}_{k_1}) \text{ and } \tilde{O}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) = \tilde{J}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) \\ \text{or} \\ \tilde{O}^{(k_1)}(\tilde{g}^{l'}_{k_1}) = \tilde{J}^{(k_1)}(\tilde{g}^{l'}_{k_1}) \text{ and } \tilde{O}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) = \tilde{A}^{(k_1)}(\tilde{g}^{l'+1}_{k_1}) \end{cases} \tag{126}$$

We claim the following :

Lemma 4. *Among all the sites x satisfying equation* (125) *above, there exists a visible spin flip for $\tilde{g}$ (and for g).*

Proof of lemma 4 Let us take l' satisfying equation (125). There are at most 2 values of x that can satisfy equation (125) with this value of l', and we claim that one of these is the site of a visible spin flip.

Indeed, there needs to be at least one visible spin flip at the boundary between graphs so that these can be concatenated with a nonzero commutator (because of locality).

Because of assumption 1., we know that overlaps between arches are of length at most 1. We also know that $x \in \overset{\circ}{\widehat{I(\tilde{g}^i_{k_1+1})}}$. We can even make a stronger statement than this : $d\left(x, I(\tilde{g}^i_{k_1+1})^c\right) > 1$. This is because $k_1 \geq 1$, so graphs of scale k_1 are of length 2 at least, and since these are not jump steps per assumption 2., they all have spatial length at least 2. The condition written in (125) then makes it apparent that $d\left(x, I(\tilde{g}^i_{k_1+1})^c\right) > 1$.

Therefore, site x is not in an overlap with any graph of scale $k > k_1$. Since x is a visible spin flip at scale k_1, it is therefore also a visible spin flip at scale n.

□

Let us now conclude the proof of lemma 2.

We set aside this visible spin flip we've found as our first candidate, and we consider either graph $\tilde{g}^{l'}_{k_1}$ or $\tilde{g}^{l'+1}_{k_1}$ (whichever of the two graphs bears a $\tilde{J}$ operator... let's say $\tilde{g}^{l'}_{k_1}$). If all of the arches of $\tilde{J}^{(k_1)}(\tilde{g}^{l'}_{k_1})$ at scales $1 \leq k < k_1$ have been treated in step 1, we stop our procedure here. If not, we reiterate the same procedure we used to prove lemma 3, but this time we use graph $\tilde{g}^{l'}_{k_1}$ instead of $\tilde{g}$.

We end up with a new candidate at a new, smaller scale k_2. We reiterate this procedure until we end up with a graph with no arches on scales greater than 1. Let us call m the number of times we have to run this procedure before achieving the stopping condition.

We claim that the last candidate that this procedure gives us (which we still call x) is the visible spin flip in lemma 2.

We already know, because of how we defined our procedure, that x is a visible spin flip. All that remains to be proven is that x is on the boundary of a hole, or next to the boundary of a hole.

We already know that $x \in \partial I_{k_\alpha} \cup \partial\left(I^c_{k_\alpha}\right)$ where $\alpha \in \{1, ..., m\}$ is any step in our procedure. This is explicitly stated in lemma 3, which states that our chosen site is at the boundary of a J graph, or neighbors the boundary of a J graph.

Next, take a scale $j \in \{k_{\alpha+1}+1, ..., k_\alpha - 1\}$. Let us call g_{k_α} the graph that we have obtained after α steps in our procedure. Per definition, scale $k_{\alpha+1}$ is the largest scale at which we observe an arch on g_{k_α} (which is a graph on scale k_α). Our candidate x satisfies $x \in I(g_{k_\alpha}) \cup \partial\left(I(g_{k_\alpha})^c\right)$, so x is either in a J graph of scale j, or x neighbors the boundary of a J graph of scale j. Therefore, $x \in I_j \cup \partial\left(I^c_j\right)$.

Ultimately, we find that $x \in I \cup \partial I^c$.

Moreover, because of the way that our procedure works, $x \in \partial I(g_{k_m}) \cup \partial\left(I(g_{k_m})^c\right)$. Therefore, finally, $x \in \partial I \cup \partial\left(I^c\right)$.

□

Step 3 : Conclusion of proof. Let us rename x_g the spin flip selected for graph g by lemma 2.

We run the same procedure used to prove lemma 2 on all of the subgraphs $\tilde{g}^l_k$ of $\tilde{g}$ of scale $k \geq 2$ that bear an arch (ie. $\tilde{O}^{(k)}(\tilde{g}^l_k) = \tilde{A}^{(k)}(\tilde{g}^l_k)$). This way, we find a spin flip $x_{g^l_k}$ that we will use to integrate each arch.

For a given subgraph h of g with scale greater than or equal to 1, we may be in one of two cases :

1. Subgraph h has a subgraph h' on scale greater than or equal to 1 with an arch. In this case, the spin flip chosen to integrate h is not in $\partial I(h) \cup \partial\left(I(h)^c\right)$.
2. Subgraph h does not have any subgraphs on scales greater than or equal to 1 without arches. In this case, the chosen spin flip is in $\partial I(h) \cup \partial\left(I(h)^c\right)$.

For subgraphs h on scale 0, since the lengths of these is 1, we will take x_h to be the only element of $I(h)$.

Now, to conclude, let us try to bound $n_x = |\{h \mid x_h = x\}|$ for all x. We have :

$$n_x = \underbrace{|\{h \text{ in case 1.} \mid x_h = x\}|}_{\text{term } T1} + \underbrace{|\{h \text{ in case 2.} \mid x_h = x\}|}_{\text{term } T2} + \underbrace{|\{h \text{ on scale 0} \mid x_h = x\}|}_{\text{term } T3} \tag{127}$$

It is apparent that, if h_1 and h_2 are in case 1., then $x_{h_1} \neq x_{h_2}$. Therefore $T_1 \leq 1$.

For term T_3, assumption 2. (according to which there are no jump steps in our decomposition) makes it impossible for two graphs on scale 0 to share the same site. So $T_3 \leq 1$.

We finally claim that $T_2 \leq 2$. Indeed, suppose that $T_2 \geq 3$. This would mean that there exist $h_1, h_2, h_3 \in \{h \text{ in case 2.} \mid x_h = x\}$, on scales k_1, k_2 and k_3 respectively. Without loss of generality, suppose $k_1 > k_2$

h_2 cannot be a subgraph of h_1, as then h_1 would not be in case 2. So h_1 and h_2 must not have more than one site of overlap. The same reasoning can be applied with h_1 and h_3, and with h_2 and h_3. This is impossible if all three graphs share a site. Therefore $T_2 \leq 2$.

So we finally come to the conclusion that $n_x \leq 4$, which concludes our proof of theorem 2.

□

4.4.2 General short graphs case

We will not go into heavy computational detail in this section, as this can get quite heavy notation-wise. We will simply sketch the general idea of how to deal with graphs that loop over themselves (ie. that pass several times over the same site).

This involves what we will call an *erasure procedure*. After we run this erasure procedure, we will bound erased denominators inductively. We will then use our Markov bound on the rest of the arches. One then has to modify the algorithm we just ran in section 4.4.1 to account for all of the erased denominators, which will free us of the problems that arise when we drop the assumptions we used in the previous section.

This erasure procedure will have to be chosen wisely, because erased denominators will not contribute any smallness to our estimate, which means that in order to maintain an exponential decay, we cannot erase too many denominators.

Let us now give an idea of what this erasure procedure is.

Erasure procedure Everywhere the graph loops onto itself, potential problems arise, in that we may not be able to find visible spins to integrate arches in these regions. The point of the erasure procedure is to erase these arches for which we may not easily find integration spins.

Definition 4. *(Looping segment)*
For all $x \in \mathbb{Z}$, consider the set

$$N_x := \left| \left(\bigcup_{\substack{h \text{ subgraph of } g \\ h \text{ scale } 0}} \mathbb{1}_{h(0)=x} \right) \cup \left(\bigcup_{\substack{h \text{ subgraph of } g \\ h \text{ long graph}}} \mathbb{1}_{x \in I(h)} \right) \right| \tag{128}$$

(this can be seen as the number of times that graph g passes through point x, where we have reduced jump steps to the interval they cover).

Now consider the set $\{x \in \mathbb{Z} \mid N_x \geq 2\}$. Its connected components $(T_\alpha)_\alpha$ are called looping segments. The length of a looping segment $|T_\alpha|$ is defined as

$$|T_\alpha| := \sum_{x \in T_\alpha} (N_x - 1) \tag{129}$$

Clearly, arches with both ends inside of looping segments pose problems, because the spins under these arches may be mute. So we need to erase any arches that have both ends in looping segments. However, some arches which have only one endpoint in a looping segment can also pose problems, as is illustrated figure 7. We will now attempt to give a bound on the number of problematic arches.

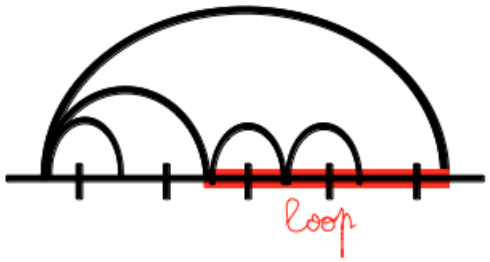

FIGURE 7 – An example of a problem section. In the looping segment, we do not know which spins have been muted. If all of the spins in the looping segment are mute, we may not be able to find a visible spin flip to integrate the big arch, so we need to erase some arches to free up spins.

Let $l := 2 \max_\alpha T_\alpha$.

Take k such that $l \in [L_{k-1}, L_k[$. We claim that arches on scales k and above do not pose any looping problems.

Indeed, for any looping segment T_α, $2|T_\alpha| \geq \sum_{x \in T_\alpha} N_x$. Since $2|T_\alpha| \leq l < L_k$, we deduce $\sum_{x \in T_\alpha} N_x < L_k$.

Now, endpoints of arches on scales k and above are separated by graphs h on scales k or above. Take h a subgraph of g on scale k or above, we have $|h| = \sum_{x \in I(h)} N_x \geq L_k$, so the endpoints of two arches on scales k or above cannot be in the same looping segment. The problem presented in figure 7 does not arise.

Now, suppose the erasure procedure has been run for all scales greater than or equal to $i+1 < k$. We now want to run it on scale i. We apply the same reasoning as above to deduce that arches of scale i only pose problems for looping segments of whose lengths satisfy $2|T_\alpha| \geq L_i$. For these, we have to erase any arches on scale i that have an endpoint in T_α.

Ultimately, for any looping segment $T_\alpha = \{x, ..., y\}$ with $|T_\alpha| \in [L_i, L_{i+1}[$, we define an extended looping segment $\overline{T_\alpha}$ as the union of T_α and of all of the points at "distance" less than L_{i+1} from T_α :

$$\overline{T_\alpha} := T_\alpha \cup \left(\bigcup_{\substack{z<x \\ \sum_{j=z}^{x} N_j < L_{i+1}}} \{z\} \right) \cup \left(\bigcup_{\substack{z>y \\ \sum_{j=y}^{z} N_j < L_{i+1}}} \{z\} \right) \tag{130}$$

The erasure procedure erases any arches with both endpoints in an extended looping segment.

Conclusion of the short graphs case Now all that is left is to bound the portion of arches that we have erased through this procedure.

Because our graph is short, it cannot have too many loops. Computationally, this is given by the short graphs condition $|g| \geq \beta^{-1} |I(g)|$. We have

$$\sum_\alpha |T_\alpha| = \sum_\alpha \sum_{x \in T_\alpha} (N_x - 1) = \sum_x (N_x - 1) \tag{131}$$

$$\leq |g| - |I(g)| \leq \frac{1}{8} |g| \tag{132}$$

Therefore,

$$\sum_\alpha \left|\overline{T_\alpha}\right| := \sum_{x \in \overline{T_\alpha}} N_x \tag{133}$$

$$\leq \underbrace{\sum_\alpha 2 |T_\alpha|}_{\text{looping segment}} + 2 \underbrace{\sum_\alpha (1+\beta) |T_\alpha|}_{\text{extension}} \tag{134}$$

$$\leq \sum_\alpha 6 |T_\alpha| \tag{135}$$

$$\leq \frac{3}{4} |g| \tag{136}$$

This essentially means at least one quarter of the arches remains unerased. This will be enough in order to obtain our resonance bound (100).

The idea is to bound inductively all of the erased subgraphs (which we will call $g_{\alpha_1}^{\beta_1}$,...,$g_{\alpha_l}^{\beta_l}$). Since the erased graphs are, at most, only a proportion of the total length of the graph, this will leave enough arches intact to use our Markov bound. We also have to bound subgraphs that are jump steps with our inductive bound (we call these $g_{i_1}, ..., g_{i_m}$).

The actual computation is quite heavy notation-wise (we attempt to illustrate it below), but the idea is quite simple : we bound jump steps and erased subgraphs with an inductive bound, and then use a Markov bound on the rest.

We have neglected to write the factorials, as this would have made the notations even heavier.

$$\mathbb{P}\left(\left|A^{(k+1)}(g)\right| > \left(\frac{\gamma}{\epsilon}\right)^{|g''|}\right) = \mathbb{P}\left(\left|A^{(k+1)}(g)\right| > \left(\frac{\gamma}{\epsilon}\right)^{|g|}\right) \tag{137}$$

$$= \mathbb{P}\left(\frac{A^{(k)}(g_1)...J^{(k-1)}(g_{i_0})...A^{(k)}(g_k)}{\Delta E_g} > \left(\frac{\gamma}{\epsilon}\right)^{|g|}\right) \tag{138}$$

$$\leq \mathbb{P}\left(\gamma^{\sum_{g_i \text{ jump step}}|g_i|}\frac{A^{(k)}(g_{\tilde{i}_1})...J^{(k-1)}(g_{\tilde{i}_{i'_0}})...A^{(k)}(g_{\tilde{i}_m})}{\Delta E_g} > \left(\frac{\gamma}{\epsilon}\right)^{|g|}\right) \tag{139}$$

$$\leq \mathbb{P}\left(\frac{A^{(k)}(g_{i_1})...J^{(k-1)}(g_{i_I})...A^{(k)}(g_{i_m})}{\Delta E_g} > \frac{\gamma^{\sum_{j=1}^m |g_{i_j}|}}{\epsilon^{|g|}}\right) \tag{140}$$

$$\leq \mathbb{P}\left(\left(\frac{\gamma}{\epsilon}\right)^{\sum_{j=1}^l \left|g_{\alpha_j}^{\beta_j}\right|}\frac{\gamma^{\sum_{j=1}^m |g_{i_j}|+\sum_{j=1}^l \left|g_{\alpha_j}^{\beta_j}\right|}}{\prod_{\substack{h \text{ subgraph of } g \\ h\neq g_{i_j}\ ,\ h\neq g_{\alpha_j}^{\beta_j}}}\Delta E_h} > \frac{\gamma^{\sum_{j=1}^m |g_{i_j}|}}{\epsilon^{|g|}}\right) \tag{141}$$

$$\leq \mathbb{P}\left(\frac{1}{\prod_{\substack{h \text{ subgraph of } g \\ h\neq g_{i_j}\ ,\ h\neq g_{\alpha_j}^{\beta_j}}}\Delta E_h} > \frac{1}{\epsilon^{|g|-\sum_{j=1}^l \left|g_{\alpha_j}^{\beta_j}\right|}}\right) \tag{142}$$

$$\leq \mathbb{P}\left(\frac{1}{\prod_{\substack{h \text{ subgraph of } g \\ h\neq g_{i_j}\ ,\ h\neq g_{\alpha_j}^{\beta_j}}}\Delta E_h} > \frac{1}{\epsilon^{|g|/4}}\right) \tag{143}$$

$$\leq \epsilon^{s|g|/4}\mathbb{E}\left[\frac{1}{\prod_{\substack{h \text{ subgraph of } g \\ h\neq g_{i_j}\ ,\ h\neq g_{\alpha_j}^{\beta_j}}}\Delta E_h^s}\right] \tag{144}$$

$$\leq \left(\epsilon^s C(s)^4\right)^{|g|/4} \tag{145}$$

5 Conclusion of proof

We now conclude by proving point 3. of theorem 1.

We now consider graphs in non-resonant regions only. We use our combinatoric factor again to bound the change of basis that we had to do to diagonalise our hamiltonian. At scale n, we have

$$\left|A_{\sigma,\tau}^{(n)}\right| \leq \sup_{\substack{g \\ g(0)=\sigma\ ,\ g(N)=\tau}} c(g)\left|A_{\sigma,\tau}^{(n)}(g)\right| \tag{146}$$

$$\leq c^{L_{n+1}}\left(\epsilon^s C(s)^4\right)^{L_{n+1}} \tag{147}$$

Keep in mind that we can make this arbitrarily small by changing ϵ.

We have diagonalized our hamiltonian H perturbatively, so we need to sum over all of these perturbative basis changes to ensure that we can still control the total basis change. We therefore need to compute

$$\sum_{n=0}^{\infty} c^{L_{n+1}} \left(\epsilon^s C(s)^4\right)^{L_{n+1}} \tag{148}$$

This converges. Again, we can make the sum arbitrarily small by changing ϵ, so that the total basis change is arbitrarily small.

Moreover, for any two points x and y on the lattice, the shortest graph g linking the two has length $|g| = d(x, y)$. If $d(x, y) \in [L_k, L_{k+1}[$, this means that any graph linking x and y is on scale k or above, which goes to show that there is a spatial decay in correlations (the sum in equation (148) starts at $n = k$). We are not going to prove in detail why this decay is expontential in the distance (which would finish our proof of our MBL result in non-resonant regions), but we just wanted to give an idea of this.

The reasoning that has allowed us to obtain the MBL result is still incomplete, as we have made simplifying assumptions in order to get it. In particular, we are currently looking to reintroduce the renormalization of energy into the problem. However, we hope to have addressed some of the problems posed by the multi-scale method through this work.

References

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