Rapport de stage

Théorie quantique des champs topologique

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Stage de M1 (Février- Juin 2021)

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Présentation générale du stage

J'ai effectué mon stage de M1 à l'université de Copenhague, dans l'équipe de géométrie et topologie. Ce stage a été fait en binôme avec Hugo Panchaud, qui avait demandé à effectuer son stage dans la même équipe ; son maître de stage, Lukas Woike, a proposé de nous encadrer tous les deux afin que nous fassions un travail en tandem. Malheureusement, le Danemark a fermé ses frontières avant que nous puissions partir, et l'université est restée de toute manière fermée tout le semestre ce qui fait que nous avons effectué tout notre stage à distance.
Concernant l'organisation, nous avions une réunion hebdomadaire en visioconférence (parfois toutes les deux semaines) avec notre maître de stage pour décider du travail à faire. Le reste du temps, nous étions libres de nous organiser comme nous voulions ; nous travaillions ensemble à l'ENS en général au moins un ou deux jours par semaine. J'avais beaucoup de temps libre, cela m'a permis de suivre en parallèle le cours avancé de géométrie et relation aux dérivées partielles d'Emmanuel Giroux, dont le thème est est d'ailleurs assez proche du domaine de mon stage.
Concernant le stage en lui-même, l'objectif était de découvrir le domaine de la théorie quantique des champs topologiques (TQFT pour Topological Quantum Field Theory), d'abord en deux dimensions, puis d'étudier un cas particulier de théorie des champs : la théorie de Dijkgraff-Witten. Un bon prétexte pour étudier cette théorie était de prouver la formule de Mednyck qui relie représentations de groupes finis et groupe fondamental d'une surface, qui est le résultat final du mémoire que nous avons rédigé. Lukas nous a ainsi guidé pour réaliser une démonstration de cette formule utilisant la théorie des champs. La formule de Mednyck s'énonce comme suit : Si G est un groupe fini, k un corps algébriquement clos, et S une surface compacte de genre g, alors

$$\sum_{V \in Irrep_k(G)} dim(V)^{\chi(S)} = |G|^{\chi(S)-1} |Hom(\pi_1(S), G)|$$

Où $\chi(S)$ est la caractéristique d'Euler de S et la somme est prise sur les représentations irréductibles de G sur k.

En quelques mots, le stage portait principalement sur les cobordismes et les théories quantiques des champs. Un cobordisme est une variété à bord de dimension n dont le bord est composé d'une variété "de départ" V_0 et d'une variété "d'arrivée" V_1, qui sont de dimension $n-1$. Visuellement, on peut imaginer un cobordisme comme une variété qui "connecte" deux variétés closes (compactes, sans bord). Une théorie quantique des champs topologique est une façon d'assigner à chaque variété de dimension $n-1$ un espace vectoriel, et à chaque cobordisme entre ces variétés une application linéaire entre leurs espaces vectoriels respectifs, satisfaisant certains axiomes de fonctorialité. Dans le langage des catégories, une TQFT est un foncteur monoidal de la catégorie $nCob$ des cobordismes de dimension n vers la catégorie des espaces vectoriels.

Une première partie du stage était consacrée à la lecture/au commentaire d'un ouvrage de Joachim Kock (de l'université de Nice) portant sur les théories des champs en dimension deux. Cela nous a permis de travailler la théorie de Morse,d'approfondir un certain nombre de notions catégoriques, et d'étudier un théorème de classification des théories de champs en dimension 2 qui établit que se donner une $TQFT$ en dimension 2 équivaut à se donner une algèbre de Frobenius, qui est une structure purement algébrique (alors qu'une n-TQFT est a priori un objet lié à la topologie des variétés de dimension n et $n-1$.) Cependant, il n'y avait pas beaucoup de recherche ou de travail de production à fournir dans cette partie du stage, hormis des compte rendus de lecture (pas de théorème à prouver, par exemple, mais plutôt de bien comprendre le livre et de se familiariser avec les objets).
Dans un deuxième temps, Lukas Woike nous a proposé de démontrer la formule de Mednyck en utilisant des méthodes de théorie quantique des champs; entre autres une introduction à la théorie de Dijkgraff-Witten. Par visioconférence, il nous donnait régulièrement une feuille de route ou une présentation de certains nouveaux concepts et nous avions à faire les démonstrations et les rédiger sur un document, que nous lui envoyions périodiquement. Ce travail nous a permis d'écrire un article qui constitue la suite de ce rapport de stage.

Globalement, j'ai assez regretté le fait que le stage soit à distance ; c'était moins motivant et le rythme était parfois un peu relâché, ce qui fait que je n'arrivais pas à me concentrer vraiment sur le travail. Voir mon maître de stage moins d'une fois par semaine, et passer beaucoup de temps à rédiger des preuves et des diagrammes en LaTeXplutôt qu'à parler ou chercher des maths était un peu dommage. Cependant, le sujet m'a quand même

beaucoup intéressé et a confirmé mon attrait pour la géométrie et la topologie. J'ai mis à profit le fait que le stage soit à distance pour suivre divers cours à l'ENS (Théorie de l'information, Relativité, Géométrie et RDP) ce qui a complété mon semestre. J'ai réalisé ainsi un exposé en groupe de travail avec Marcus Nicolas sur le théorème du h-cobordisme, qui établit une condition pour qu'un cobordisme soit un cylindre (c'est-à-dire un cobordisme de la forme $V \times [0,1]$). Le théorème s'énonce comme suit : si W est une variété différentielle simplement connexe à bord, de dimension $n \geq 6$, dont le bord est la réunion de deux variétés V_0 et V_1, et tel que l'inclusion $V_0 \subset W$ est une équivalence d'homotopie, alors W est un cylindre, et V_0 et V_1 sont donc difféomorphes.
La réalisation de cet exposé sur un thème voisin de celui de mon stage était assez motivante car permettait d'étudier certains objets ou théories (cobordismes, fonctions de Morse...) d'un autre point de vue, et m'a aidé à prendre du recul sur le stage en lui-même. La suite de ce rapport est l'article que nous avons écrit avec Hugo sous la supervision de Lukas Woike.

A proof of Mednykh's formula using Dijkgraaf-Witten's 2D Topological Quantum Field Theory

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June 2021

The use of algebraic topology allows us to link topological objects to algebraic ones in some surprising and sometimes amazingly simple formulas. Mednykh's formula is one of these : it connects the genus of a closed surface Σ, a finite-group G and irreductible representations of G. While it is quite difficult to interpret the formula itself, its demonstration is interesting because, as expected, it involves various domains of mathematics. It especially uses a recent notion : Topological Quantum Field Theories (TQFT). Although Quantum Field Theories are obviously physics objects, TQFTs are purely mathematical objects so we won't mention anything about physics in this text. TQFTs are very interesting for many reasons, it provides invariants of manifolds for instance. In our case, we will be focusing on 2D TQFTs because these particular TQFTs admit an extremly simple description in terms of generators and relations (for categories) as stated in [2]. This will allow us to form two 2D TQFTs that seem very different at first, prove that they are in fact equivalent and by identifying the terms we'll obtain the desired formula.

1 An introduction to Topological Quantum Field Theories

In all of this text, a closed manifold will mean a compact manifold with no boundary.

1.1 Introduction and definition of cobordisms

Let's start with the definition of an (unoriented) cobordism. Intuitively, an n-cobordism starts with two closed $(n-1)$-manifolds (remind our convention) which we "join" with an n-manifold. For example, the cylinder $\mathbb{S}^1 \times [0,1]$ is an unoriented cobordism. More formally, we define

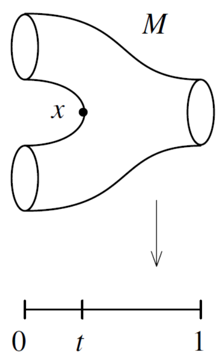
Definition 1. *Let Σ_0, Σ_1 be two closed $(n-1)$-manifolds. A cobordism between Σ_0 and Σ_1 is a compact n-manifold M whose boundary is $\Sigma_0 \sqcup \Sigma_1$.*

Notice that the definition allows us to have empty boundary which means that any compact n-manifold without boundary is an unoriented cobordism between $\emptyset$ and $\emptyset$ (seen as an $(n-1)$-manifold).

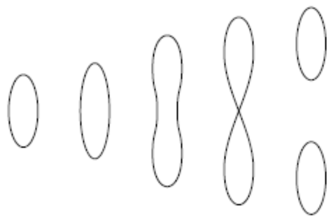
Now let's focus on the "join" meaning. The intuitive idea is that we start from a manifold Σ_0 and by following the cobordism we kind of distort or twist it until we reach Σ_1. First, the notion of a start and an end leads us to orient our manifold. Indeed, defining an orientation creates an in-boundary and out-boundary, like we could do in $\mathbb{R}^3$ with the right-hand rule. In this process we can decide to call the in-boundary the start and the out-boundary the end.

drawing

Now we need to explain the meaning of "following" the manifold. At first sight, if we think of embedded manifolds, we can think of a path $s : [0,1] \to M$ going through our manifold M and we just follow the "projection" of M onto this line (starting at Σ_0 and ending at Σ_1) :



What we actually do is projecting M onto $[0,1]$ with a smooth map f such that $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$ and we look at the preimages $f^{-1}(t)$. This is quite satisfying and also well-known because it is essentially Morse theory. However, let's remind that we manipulate abstract n-manifolds that we don't suppose to be embedded in $\mathbb{R}^n$. So there is no canonical projection and if we think of the "movie" of our transformation, it has many possible representations. For instance, for the last cobordism, one of them (the "natural one") being :



In the end, the use of abstract manifolds rises some problems but it is clear enough to define our main objects, oriented cobordisms and to have an idea of what they represent :

An oriented cobordism from Σ_0 to Σ_1 is an oriented compact n-manifold whose in-boundary is Σ_0 and out-boundary is Σ_1. We will note $\Sigma_0 \longrightarrow \Sigma_1$.

However, this definition doesn't allow us to have a cobordism from Σ to itself. We need to slightly modify it :

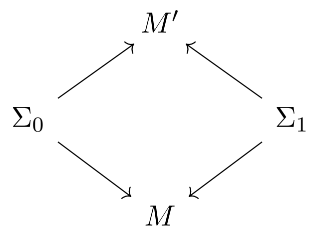
Definition 2. *An oriented cobordism from Σ_0 to Σ_1 is an oriented compact n-manifold and two orientation-preserving applications $\Sigma_0 \to M \leftarrow \Sigma_1$ who diffeomorphically map Σ_0 to the in-boundary of M and Σ_1 to the out-boundary of M.*
We will note $\Sigma_0 \longrightarrow \Sigma_1$.

Now, let's have a look at a few examples. The most obvious one is the cylinder obtained from an oriented closed $(n-1)$-manifold Σ. Let $M = \Sigma \times [0,1]$ with the standard orientation obtained from the orientation of Σ. Then the canonicals maps $\Sigma \to \Sigma \times \{0\} \to M$ and $\Sigma \to \Sigma \times \{1\} \to M$ are clearly orientation preserving diffeomorphisms. But it seems that we can do better. Instead of using the identity, we can use 2 orientation-preserving diffeomorphisms $\Sigma_0 \simeq si$ (resp. $\Sigma_1 \simeq \Sigma$) and combine them with $\Sigma \to \Sigma \times \{0\} \to M$ (resp. $\Sigma \to \Sigma \times \{1\} \to M$) to obtain another cobordism. So, any diffeomorphic manifolds induce several cobordisms. Indeed, given an orientation-preserving $\psi : \Sigma_0 \to \Sigma_1$, we can chose to take $\Sigma = \Sigma_0$ and consider

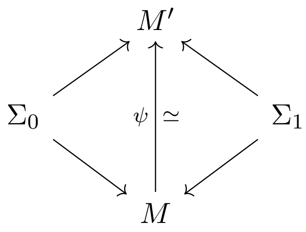
$$\Sigma_0 \xrightarrow{id} \Sigma_0 \to \Sigma_0 \times \{0\} \to \Sigma_0 \times I \leftarrow \Sigma_0 \times \{1\} \leftarrow \Sigma_0 \xleftarrow{\psi} \Sigma_1$$

or the converse with $\Sigma = \Sigma_1$ and ψ^{-1}. It seems natural to say that these two cobordisms are "equivalent". Let's precise that :

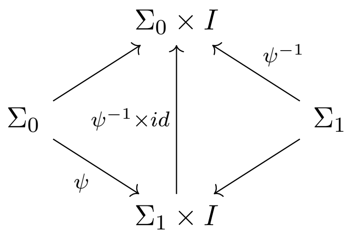
Definition 3. *Given two oriented cobordisms between Σ_0 and Σ_1,*



We say they are equivalent if there is an orientation-preserving diffeomorphism $\psi : M \to M'$ *making this diagram commute :*



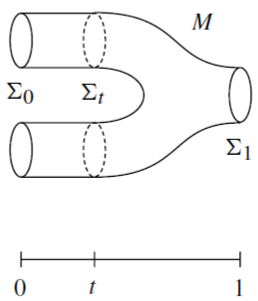
The following easy diagram show that according to our definition, the two cobordisms induced by an orientation-preserving $\psi : \Sigma_0 \to \Sigma_1$ are indeed equivalent :



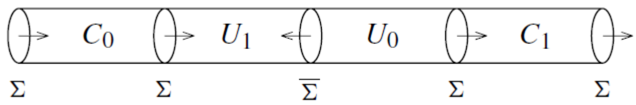
Now that we have laid the foundations for our theory, let's see how we can represent, manipulate our objects and what kind of interesting properties we have.

1.2 Cobordism decomposition

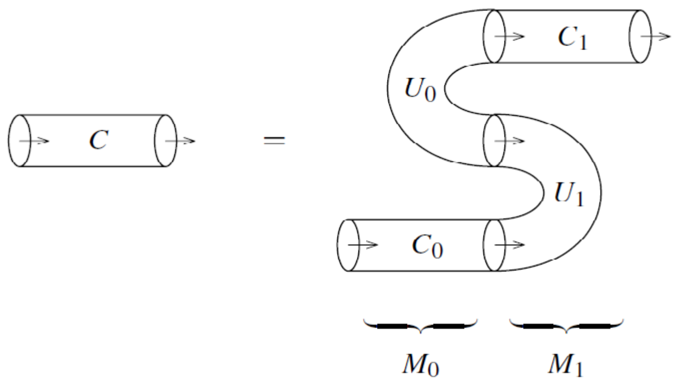
Remember how we explained the idea of "following" our cobordism at the beggining with Morse theory. Just smoothly project our cobordism onto the unit interval $[0, 1]$ and we can look at the preimages $f^{-1}(t)$ at all t to have a "movie" of our cobordism. But we can do more than that. If we consider the part corresponding to $[0, t]$ (so $f^{-1}([0, t])$), we get a "part" of our cobordism. So, this operation actually cuts our cobordisms in two different cobordisms that are "compatible" (we will see later what we mean by that when we will define the converse operation : composition of cobordisms).



One critical application of this construction is the snake decomposition of the cylinder. The idea is to cut our cylinder in half with the right orientation. But we would like to reverse it. This is impossible with our conventions. But we can add one more cut to "compensate" it in the following way :



We will represent it in a more natural way like that :



In this example, we started from a connected cobordism and ended up with two disconnected ones. So, we can see how disconnectedness naturally rises in the study of cobordisms which implies that we have to allow it and use it at our own advantage. For example, it is a common strategy to cut your pieces in a disjoint union of different components which are easier to study.

1.3 The category of cobordism classes

In this section, we are going to form a category of cobordism. We have objects ($(n-1)$-closed manifolds), arrows (oriented n-cobordisms) and an identity (the cylinder) and we even have a natural monoidal operation (disjoint union). We just need to compose in a fine way our cobordism. It is quite natural that we consider cobordism classes instead of cobordisms.

Let's see how we can compose our cobordisms : a reasonable condition is that considering the decomposition M_0, M_1 of a cobordism M like in our last section, we get that $M_0M_1 = M$. Intuitively, considering two cobordisms $M_0 : \Sigma_0 \to \Sigma_1$ and $M_1 : \Sigma_1 \to \Sigma_2$, the composition M_0M_1 should be obtained by gluing M_0 and M_1 along their common boundary Σ_1.

This operation is very intuitive again but if you think about it thoroughly, it is actually well-defined up to diffeomorphism and even not in a canonical way since this diffeomorphism is not unique. That's where the notion of cobordism classes is interesting and necessary to get a well-defined composition.

What we mean by gluing along the common boundary is to consider the topological space $M_0 \bigsqcup_{\Sigma_1} M_1$ endowed with the structure given by the usual charts for points on the outisde of the common boundary and by the gluing of the charts for points on the boundary. It is easy to check that this construction gives a canonical way to have a topological manifold that would fill our requirements for composition. However, this will not happen for smooth manifolds because smoothness is a stronger condition.

Indeed, with only simple examples such as the composition of two closed interval, one can see that there are different ways to produce a smooth structure on the composition of our manifolds and no canonical one. But it appears that any of these structures are actually diffeomorphic relatively to the boundary which will be sufficient if we consider cobordism classes as suggested. So, all we need now is to show this result and prove that there exists a smooth structure for our gluing :

First, we can reduce our problem : if we find a smooth manifold S and homeomorphism from S to $M = M_0M_1$ which restricts to a diffeomorphism on M_0 and M_1, then we can pullback our smooth atlas onto M. Let's consider the "simplest" cobordism : the cylinder over a closed manifold Σ. It's really easy to find a smooth structure on $(\Sigma \times [0,1])(\Sigma \times [1,2]) = \Sigma \times [0,2]$ so we know how to glue two cylinders

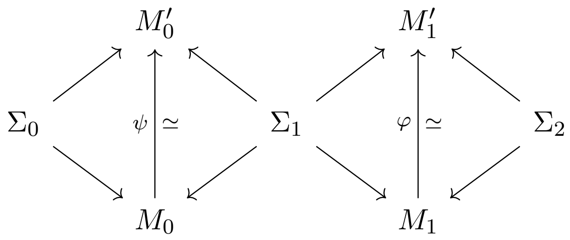
Now let's notice that we can cut any cobordism such that the first (or the second) part is diffeomorphic to a cylinder. Indeed, let's take a Morse function $f : M \to [0,1]$ and $\varepsilon > 0$ such that ε is strictly inferior to the first critical value of f. Then, as in the last paragraphe, we have $M = M_{[0,\varepsilon]}M_{[\varepsilon,1]}$ and by using Hirsch's regular theorem, $M_{[0,\varepsilon]}$ is equivalent to a cylinder over Σ_0.

Thus, to glue 2 cobordisms, we write

$$M_0M_1 = M_{0,[0,t]}M_{0,[t,1]}M_{1,[0,s]}M_{1,[s,1]} \sim M_{0,[0,t]}C_0C_1M_{1,[s,1]}$$

Then, we use the smooth structure above for the area at the boundary of the two cylinders and we keep the old smooth structure for the other points.

In the end, we have defined one smooth structure on $M_0 \bigsqcup_{\Sigma_1} M_1$ but not in canonical way. But, we can prove (see p.42 in [2]) that any of the smooth structures obtained through the above process are diffeomorphic rel. to the boundary. So passing to the diffeomorphism classes makes our structure unique. We just need to verify that the result does not depend on the representatives of the diffeomorphism classes of M_0 and M_1. This is not hard, we have the following diagrams :



We glue ψ and φ in the category of continuous map and we get a homeomorphism from $M_0 M_1$ to $M_0' M_1'$ which is a diffeomorphism χ on M_0 and M_1 by definition. By our precedent remark, we can define a smooth structure on $M_0' M_1'$ which is diffeomorphic through χ to $M_0 M_1$. While it may not be the smooth structure we started with, the two are diffeomorphic (see p.42 in [2]).

Associativity is quite easy to see.

Our identity will naturally be the cylinder class : it intuitively corresponds to doing nothing on Σ. Let's see that : by the same reasoning as for composition, we take $M = M_{[0,\varepsilon]} M_{[\varepsilon,1]}$ where $M_{[0,\varepsilon]} \simeq C'$ where C' is a cylinder, we get $CM = C(M_{[0,\varepsilon]} M_{[\varepsilon,1]}) = (CM_{[0,\varepsilon]}) M_{[\varepsilon,1]} \simeq M_{[0,\varepsilon]} M_{\varepsilon,1]} = M$, where we used that the composition of two cylinders is diffeomorphic to a cylinder. We can prove the same identity for MC in the exact same way.

With all that work done, we have found a category structure **nCob** for our n-cobordism classes :

Our objects are the closed $(n-1)$-manifolds, the arrows are the (oriented) cobordism classes, the identities are the cylinders and the composition is as defined before.

1.4 The monoidal structure of nCob

Monoidal structures arise when there is a natural process of "paralleling", which simply means that given two pair of objects (X, Y) and (X', Y') and arrows $f : X \to Y$ and $g : X' \to Y'$, we can "combine" f and g in a single arrow that acts like f and g simultaneously. We must also specify the neutral object for this operation.

Some of the most natural examples are disjoint union for **Set** ($\emptyset$ is the neutral object) and tensor product for $\mathbf{Vect}_k$. (k is the neutral object).

The structure of a monoidal category C consists in the usual category structure to which we add the "paralleling" functor $\mu : C \times C \to C$ and an identity functor $\eta : 1 \to C$

(where 1 is the category with one object and one morphism) which are subjects to commutativity of diagrams (associativity and neutral element) :

$$\begin{array}{ccccc} & & C\times C\times C & & \\ & \swarrow^{\mu\times id_C} & & \searrow^{id_C\times\mu} & \\ C\times C & & & & C\times C \\ & \searrow_{\mu} & & \swarrow_{\mu} & \\ & & C & & \end{array}$$

$$\begin{array}{ccc} 1\times C & \xrightarrow{\eta\times id_C} & C\times C \\ \downarrow & \swarrow_{\mu} & \\ C & & \end{array}$$

and its symmetric counterpart.

A monoidal category $(C, \Box, I)$ can be endowed with an additional property : symmetry. The idea is to add a "twist" operation τ which interchanges two different factors. This operation is defined for all couple (X, Y) of objects of the category and must respect some axioms : naturality, idempoty and commutativy of diagramms (associativity) :

$$\begin{array}{ccccc} X\Box Y\Box Z & & \xrightarrow{\quad\tau_{X,Y\Box Z}\quad} & & Y\Box Z\Box X \\ & \searrow_{\tau_{X,Y}\Box id_Z} & & \swarrow_{id_Y\Box\tau_{Z,X}} & \\ & & Y\Box X\Box Z & & \end{array}$$

and its symmetric counterpart.

We'll see later that (his twist map can be interpreted as a way to express some sort of commutativity.

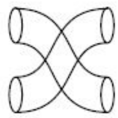
In our preceeding examples, both of our monoidal categories can be made symmetric : in $(\mathbf{Set}, \bigsqcup, \emptyset)$, we use the applications $\tau : (x, y) \mapsto (y, x)$ and in $(\mathbf{Vect}_k, \otimes, k)$ we use $\sigma : v \otimes w \mapsto w \otimes v$.

With our additional structures, we can require from our functors that they respect them, namely :

Definition 4.
- *For a monoidal functor $F : (C, \Box, I) \to (C', \Box', I')$, we require that for all X, Y in C : $F(X\Box Y) = F(X)\Box' F(Y)$.*
- *For a symmetric monoidal functor $F : (C, \Box, I, \tau) \to (C', \Box', I', \tau')$, we require that for all X, Y in C : $F \circ \tau_{X,Y} = \tau'_{F(X),F(Y)}$*

Now, let's come back to **nCob** and see why it has a natural structure of symmetric monoidal category :

What kind of paralleling process can we see ? Well, since our cobordisms mimic an evolution from Σ_0 to Σ_1 through a manifold, we could simply consider to take two disjoints cobordisms and make them act simultaneously. Naturally, this amounts to taking the disjoint union of our cobordism. One can easily check that this operation is well-defined and verify the axioms required, with the empty cobordism $\emptyset_n : \emptyset_{n-1} \to \emptyset_{n-1}$ as the identity element. Moreover we can add a symmetric operation $\tau_{\Sigma,\Sigma'}$ to $(\mathbf{nCob}, \bigsqcup, \emptyset)$. We want it to switch factors, well we have an obvious cobordism to do that :



Notice that this cobordism is different from the union of two cylinders. To see why, simply fix a point in one of the copies of Σ in $\Sigma \bigsqcup \Sigma$ so that we distinguish them and see that the in the parallel cylinders, the marked copies of Σ are in the same connected component conversly to the twist which means they can't be any diffeomorphism between them and they're not equal in **nCob**.

All in all, we showed that **nCob** has a natural structure of monoidal symmetric category. This will help us to easily formulate the definition of a TQFT and to describe **2Cob** and 2D-TQFTs.

1.5 Topological Quantum Field Theories

A n-dimension Topological Quantum Field Theory over a field k was originally defined as a rule sending closed $(n-1)$-manifolds to k-vector spaces and cobordisms to linear maps between these vector spaces with 5 axioms naturally preserving some structures and operations. But the modern formulation of cobordisms allows us to reformulate it as simply a symmetric monoidal functor from the category of **nCob** to the category of k-vector spaces $\mathbf{Vect}_k$. In other words, a TQFT $\mathcal{A}$ assigns to each $(n-1)$-manifold Σ a vector space $\mathcal{A}(V)$ such that

- The empty manifold is sent to the zero vector space.
- The cylinders $\Sigma \times I$ are sent to the identities $id_{\mathcal{A}(\Sigma)}$.
- if W is a cobordism between Σ_1 and Σ_2 then $\mathcal{A}$ is a linear map from $\mathcal{A}(\Sigma_1)$ to $\mathcal{A}(\Sigma_2)$.

- Disjoint union goes to tensor product : $\mathcal{A}(\Sigma \sqcup \Sigma') = \mathcal{A}(\Sigma) \otimes \mathcal{A}(\Sigma')$. This must also hold for cobordisms : if M and M' are cobordism the disjoint union $M \sqcup M'$ must be sent to the maps $\mathcal{A}(M) \otimes \mathcal{A}(M')$.
- The mapping is functorial, i.e. $\mathcal{A}(MM') = \mathcal{A}(M)\mathcal{A}(M')$.

2 A complete description of 2D TQFTs

In order to obtain a complete and most importantly, simple description of **2Cob**, we begin by introducing practical definitions :

2.1 Some useful categorical tools

For a group, a set of generators and relations is a couple (S, R) such that S is a generating set of G and R is a set of relations between elements of S from which we can "deduce" all the relations holding in G. Formally,

Definition 5. *For S a set, we denote $F(S)$ the free group over S. We say that G admits the presentation $\langle S|R\rangle$ if $G \simeq \langle S|R\rangle$ where*

- *$S \subseteq G$ generates G*
- *$R \subseteq F(S)$*
- *$\langle S|R\rangle = F(S)/N$ where N is the smallest normal subgroup of $F(S)$ containing R (to make sense of the quotient)*

One can notice that a group always admit a presentation (by simply taking $S = G$ and $R = \{ghk^{-1}, gh = k\}$) and that it is not unique (starting from any presentation, we can freely add relations for example). Thus, we actually want to find the "simplest" presentation of the group, often meaning having the least possible elements in S and R. Such a presentation often provides a really elemental description of the given group from which one can deduce fine properties.

Example 1. *Let's provide some easy examples : $\mathbb{Z}$ has the presentation $\langle\ x \mid \emptyset\ \rangle$, $\mathbb{Z}^2$ has the presentation $\langle\ x, y \mid xyx^{-1}y^{-1}\ \rangle$ and $\mathbb{Z}/n\mathbb{Z}$ has the presentation $\langle\ x \mid x^n\ \rangle$.*

Naturally, we translate this to category theory. Let C be a category.

Definition 6. *A generating set of arrows is a set S such that any arrow can be written as a composition of arrows in S.*

A relation is an equality of two ways of writing the same arrow.

A set of relations R is complete if every relation can be obtained by combining relations in R.

Just like groups, a presentation for C is a couple $\langle S|R\rangle$ where S is a generating set of arrows and R is a complete set of relations.

Notice that we work with monoidal categories which are endowed with an extra operation on objects and arrows : paralleling. It seems natural to allow its use when generating. For instance, the double cylinder cobordism (the identity between the disjoint union of 2 manifolds) can be obtained by taking the disjoint union of 2 cylinders. So we can really make the presentation of our category simpler. So now, we'll have :

Definition 7. *For a monoidal category, a generating set of arrows is a set S such that any arrow can be written as composition and paralleling of arrows in S.*

Now, notice that we use sets while our categories often have too many objects to form set of objects and arrows. But luckily we can restrict these categories to a much "smaller" but equivalent version of them :

Definition 8. *Let D be a subcategory of C. We say that D is a skeleton of C if D is equivalent to C and D has exactly one object of each isomorphism class in C. An equivalent characterisation is that he inclusion functor $i : D \to C$ is full, faithful and essentially faithful.*

The idea is to restrict the category to one representative of each isomorphism class without losing any "categorical" properties (arrows). It is actually a really common and intuitive process, think about how we think of we often naturally identify 2 objects being isomorphic (groups, vector spaces ...). Now, we do it at the scale of a category with the extra condition of keeping all information in the morphisms.

Example 2. *In $\mathbf{Vect}_k$, the study of k-vector spaces and of finite dimension and linear applications amounts to the study of k^n for $n \in \mathbb{N}$. In* **OrdSet***, We can code all finite ordered sets and increasing functions only using the ordered sets $\{1 \leqslant 2 \leqslant \ldots \leqslant n\}$ etc.*

We can now gives a suitable description of **2Cob** :

2.2 The skeleton of 2Cob

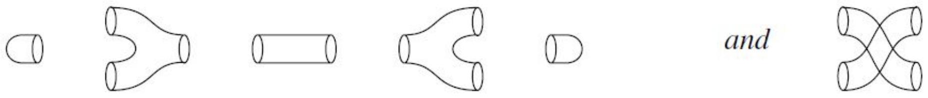
First of all, **2Cob** skeleton is derived from the 2 following results :

- Let Σ be the oriented circle as a 1-manifold. Then any closed 1-manifold (recall that closed means compact with no boundary) is diffeomorphic to the disjoint union of n copies of Σ where n is the number of connected components of Σ. Moreover the result isn't true for $m \neq n$.
- Two closed oriented 1-manifolds Σ_0 and Σ_1 are diffeomorphic if and only if there is an invertible cobordism between them.

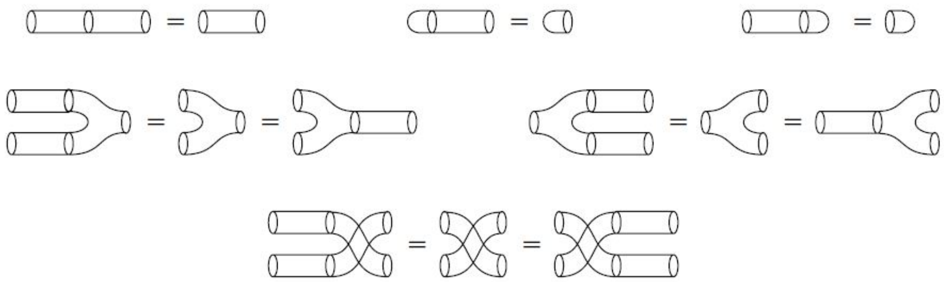
From that, it is clear that isomorphism classes of closed 1-manifold depend exactly on the number of connected components, so we can just chose the empty 1-manifold **0** and the n copies of Σ **n** as representatives (and all cobordisms between them) to obtain a skeleton of **2Cob**.

From now on, **2Cob** will abusively mean its skeleton.

Theorem 1. *The monoidal category* **2Cob** *is generated under composition and disjoint union by the following six cobordisms which are subject to the following relations :*

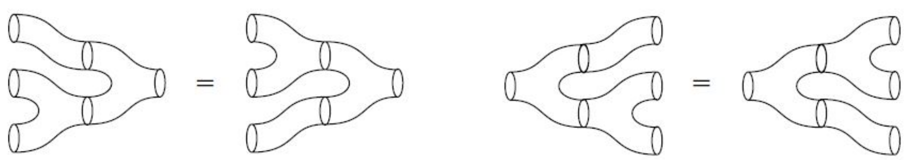


1. Identity relations :



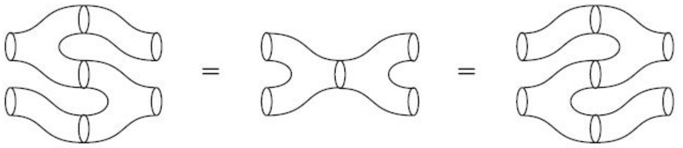
2. Commutativity and Cocommutativity :

3. Associativity and Coassociativity :



4. "Sewing Discs" :

5. The Frobenius relation :



2.3 Frobenius Algebras

In this section, we will briefly define, study Frobenius Algebras and state the main theorem of the first part of this text : 2D TQFTs and Frobenius Algebras are essentially the same thing.

A Frobenius Algebra is an algebra over a field endowed with an additional structure ; It can be defined in many equivalent ways.

Definition 9 (Frobenius Algebra)**.** *A Frobenius Algebra over a field k is a pair* (A, ε) *with* A *being a* k*-algebra (with unit) and* $\varepsilon : A \to k$ *is a linear form whose nullspace contains no nontrivial left-ideal. A Frobenius Algebra is called symmetric if* $\varepsilon(ab) = \varepsilon(ba)$ *for all* $a, b \in A$*.*

Here are some equivalent definitions :

Proposition 1. *Let* A *be a Frobenius algebra with form* ε*. We define the bilinear form*

$$\beta : A \ \times A \to k$$

with the formule $\beta(a, b) = \varepsilon(ab)$*.The form* β *is non degenerate and associative, i.e.* $\beta(a, bc) = \beta(ab, c)$*. Conversely, the data of a non-degenerate and associative bilinear form on an algebra* A *amounts to the data of a Frobenius Algebra, by taking* $\varepsilon(x) = \beta(1, x)$*. A Frobenius Algebra is symmetric iff the bilinear form* β *is.*

A convenient fact that we will use later is the existence of a co-pairing to β :

Proposition 2. *There is a unique application* $\gamma : k \to A \otimes A$ *such that*

$$(\beta \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma) = \mathrm{id}_A$$

where β *is seen as an application from* $A \otimes A$ *(because it's bilinear) and we have used the usual canonical identification* $A \simeq A \otimes k \simeq k \otimes A$*.*

Frobenius algebras form a category, whose morphisms are morphisms of algebras that commute with the Frobenius forms ε. The main theorem presented in the book by Joachim Kock is the remarkable fact that there is an equivalence between the category of two-dimensional TQFTs and the category of symmetric Frobenius Algebras. Common examples of Frobenius Algebras are matrix algebras, with ε being the trace operator, and group algebras, that we will study more in detail later. Cohomology rings have also a natural structure of group algebra, but this is not the topic of this text. The following theorem explains how the equivalence works :

Theorem 2. *Let* $\mathcal{F}$ *be a 2D-TQFT. The image of the circle* $\mathcal{F}(\mathbb{S}^1) = A$ *has a natural structure of a symmetric Frobenius Algebra. The multiplication is given by the image under* $\mathcal{F}$ *of the "multiplication" cobordism :*

The form ε is given by the image of the "left open disc" :

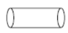
Reciprocally, if A is a symmetric Frobenius algebra, there exists a unique TQFT that sends the circle $\mathbb{S}^1$ to A, and the "multiplication" cobordism to the multiplication of A together with the "left open disc" cobordism to the Frobenius form. These constructions are functorial and lead to an equivalence between the categories of symmetric Frobenius Algebras and of 2D-TQFTs.

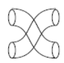
This theorem directly stems from the description by generators and relations of **2Cob**. Indeed, it is very natural, given our building blocks and their relations between them, that 2D TQFTs and Frobenius algebras are essentially the same thing :

$$\mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

$$\mathbf{1} \longmapsto A$$

$$\mathbf{n} \longmapsto A^n$$

 $\longmapsto [\mathrm{id}_A : A \to A]$

 $\longmapsto [\sigma : A^2 \to A^2]$

$$\mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

 $\longmapsto [\eta : \Bbbk \to A]$

 $\longmapsto [\mu : A^2 \to A]$

 $\longmapsto [\varepsilon : A \to \Bbbk]$

 $\longmapsto [\delta : A \to A^2]$

As a remark, one can notice that the existence of a co-pairing γ to the bilinear β is simply the traduction of the snake decomposition we've seen earlier.

3 Dijkgraff-Witten Theory and Integration over Groupoids

3.1 Groupoid cardinality

We begin this section by presenting a very useful concept : the groupoid cardinality.

Definition 10 (Essentially finite groupoid). *Let Γ be a groupoid. It is said to be essentiallyfinite iff its skeleton is a finite groupoid. In an equivalent way, a groupoid is essentially finite if and only if there is only a finite number of isomorphism classes of objects.*

Definition 11 (Covering of groupoid). *A functor $F : \Gamma \to \Omega$ is called a covering if it is surjective on objects and if it has the unique path lifting property : For every morphism $p : y_1 \to y_2$ in Ω and every $x_1 \in \Gamma$ with $F(x_1) = y_1$, there is a unique x_2 and a unique morphism $p' : x_1 \to x_2$ such that $F(p') = p$. If for a covering $F : \Gamma \to \Omega$ all fibers have n elements, we call F n-sheeted.*

Proposition 3 (Groupoid cardinality). *There is a unique assignment of a number $|\Gamma| \in \mathbb{Q}$ to any essentially finite groupoid Γ satisfying the following axioms :*

1. *Normalization : $|*| = 1$, where $*$ is the groupoid with one object and one morphism.*
2. *Homotopy invariance : $\Gamma \simeq \Omega \implies |\Gamma| = |\Omega|$.*
3. *Additivity : $|\Gamma \sqcup \Omega| = |\Gamma| + |\Omega|$.*
4. *Covering property : If $F : \Gamma \to \Omega$ is a n-sheeted covering then $|\Gamma| = n|\Omega|$.*

This assignment is called the groupoid cardinality.

By manipulating this definition it is not difficult to prove this characterization :

Proposition 4 (Groupoid cardinality, bis). *Γ is an essentially finite groupoid, its groupoid cardinality can be defined by*

$$|\Gamma| = \sum_{[x] \in \pi_0(\Gamma)} \frac{1}{|Aut(x)|}$$

We can remark that if Γ and Ω are essentially finite groupoids, $\pi_0(\Gamma \times \Omega)$ is canonically in bijection with $\pi_0(\Gamma) \times \pi_0(\Omega)$ (as for top spaces). Moreover, $\mathrm{Aut}(x, y) \cong \mathrm{Aut}(x) \times \mathrm{Aut}(y)$. Therefore the groupoid cardinality is multiplicative :

$$\begin{aligned} |\Gamma \times \Omega| &= \sum_{([x],[y]) \in \pi_0(\Gamma) \times \pi_0(\Omega)} \frac{1}{|\mathrm{Aut}(x, y)|} \\ &= \sum_{[x] \in \pi_0(\Gamma)} \frac{1}{|\mathrm{Aut}(x)|} \sum_{[y] \in \pi_0(\Omega)} \frac{1}{|\mathrm{Aut}(y)|} = |\Gamma| \times |\Omega|. \end{aligned}$$

We now introduce a notion on groupoid cardinality that will be useful later :

Definition 12. *Let Γ be an essentially finite groupoid and $f : \Gamma \to k$ an invariant function (constant on each isomorphism class). Define the integral of f with respect to groupoid cardinality :*

$$\int f = \sum_{[x] \in \pi_0(\Gamma)} \frac{f(x)}{|Aut(x)|}$$

Let's remark that for the constant function $f = 1$, we have $\int f = |\Gamma|$. Using this integral, we can define several objects :

Definition 13. *Let $F(\Gamma)$ be the k-vector space of k-valued invariant functions on Γ.*

For $f, g \in F(\Gamma)$, let $\langle f, g \rangle = \int fg$.

Proposition 5. *$\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear form.*

Proof. Bilinearity is easily checkable. $F(\Gamma)$ is finite-dimensional (its dimension is the number of isomorphism classes of Γ which is finite by hypothesis). Then, to prove non-degeneracy, we just need to show that $\langle f, g \rangle = 0$ for all g implies $f = 0$.

Let $\delta_{[x]}$ be the function constant equal to 1 on $[x]$ and null everywhere else. Then $\langle f, \delta_{[x]} \rangle = \dfrac{f(x)}{|\mathrm{Aut}(x)|}$. Thus, $\langle f, \delta_{[x]} \rangle = 0 \implies f(x) = 0$ and $\langle f, g \rangle = 0$ for all g implies $f = 0$.

□

Definition 14. *Let $\Phi : \Gamma \to \Omega$ be a functor between essentially finite groupoids. We define $\Phi^* : F(\Omega) \to F(\Gamma)$ by $\Phi^*(f) = f \circ \Phi$.*

We just need to show that our definition makes sense.

Proof. Let $x, y \in \Gamma$ such that $x \sim y$ i.e. $\exists f \in \mathrm{Hom}_\Gamma(x, y)$. Then, $\Phi(f) \in \mathrm{Hom}_\Omega(\Phi(x), \Phi(y))$ and so $\Phi(x) \sim \Phi(y)$. Thus, $\Phi^*(x) = f(\Phi(x)) = f(\Phi(y)) = \Phi^*(y)$ because $f \in F(\Omega)$.

□

Let's have a little interlude on bilinear forms :

Definition 15. *Let $\beta : V \otimes W \to k$ be a non-degenerate bilinear mapping. By definition, we have two isomorphisms $\phi : v \to \langle v, \cdot \rangle$ and $\psi : w \to \langle \cdot, w \rangle$. Let $u : V \to W$ be a linear mapping. We define the adjoint of u with respect to β, $u^\dagger : W \to V$ to be the composition $\psi^{-1} \circ {}^t u \circ \phi$ (where ${}^t u : W^* \to V^*$ is the transpose of u).*

Remark 1. *$u^\dagger$ is linear because it is a composition of linear maps.*

We are now able to compute the adjoint of Φ^* : Φ_*.

Proposition 6.

$$\Phi_*(f)(y) = \sum_{[\Phi(x)] = [y]} f(x) \frac{|Aut(y)|}{|Aut(x)|}$$

Proof. Let $f \in F(\Gamma)$ and $g \in F(\Omega)$. We have,

$$\begin{aligned}
\left({}^t(\Phi^*) \circ \phi(f)\right)(g) &= (\phi(f) \circ \Phi^*)(g) \\
&= \phi(f)(g \circ \Phi) \\
&= \langle f, g \circ \Phi \rangle \\
&= \sum_{[x] \in \pi_0(\Gamma)} \frac{f(x) g(\Phi(x))}{|\mathrm{Aut}(x)|} \\
&= \sum_{[y] \in \pi_0(\Omega)} \sum_{[\Phi(x)]=[y]} \frac{f(x) g(\Phi(x))}{|\mathrm{Aut}(x)|} \\
&= \sum_{[y] \in \pi_0(\Omega)} g(y) \sum_{[\Phi(x)]=[y]} \frac{f(x)}{|\mathrm{Aut}(x)|} \\
&= \sum_{[y] \in \pi_0(\Omega)} \frac{g(y)}{|\mathrm{Aut}(y)|} \sum_{[\Phi(x)]=[y]} f(x) \frac{|\mathrm{Aut}(y)|}{|\mathrm{Aut}(x)|} \\
&= \langle \tilde{f}, g \rangle
\end{aligned}$$

where $\tilde{f}(y) = \sum_{[\Phi(x)]=[y]} f(x) \frac{|\mathrm{Aut}(y)|}{|\mathrm{Aut}(x)|}$.

By definition, we have then $\Phi_*(f) = \tilde{f}$.

□

We end this subsection by stating a definition that will be useful in the next section. Let G be a finite group. We say a set X is a $G-$torsor if it is equipped with a free, transitive left action of G. Let us note that given any point $x_0 \in X$, we have a canonical bijection $f : G \to X$ given by $f(g) = x_0 \cdot g$. This bijection endows X with a group structure which makes the action of G coincide with the tautological action by right multiplication. So, there exists only one G-torsor up to isomorphism.

3.2 The category of principal G-bundles

A G-bundle over a manifold M is a smooth map $\pi : P \to M$ which is a local diffeomorphism together with a smooth right action $P \times G \to P$ such that $\pi(p \cdot g) = \pi(p)$ and such that the action on each fiber is free and transitive. Let $\mathbf{PBun}_G(M)$ be the category of principal G-bundles over M, with morphisms the maps $P \to P'$ which commute with the projection and with the action of G on the fibers.

Proposition 7. *Let M be a finite-dimensional compact manifold and G a finite group. Then the category $\mathbf{PBun}_G(M)$ is a groupoid.*

Proof. Let $\phi : P \to P'$ a morphism of principal G-bundles. It suffices to show that the restriction ϕ_x on the fiber $P_x \to P'_x$ is a bijection. Let us show it is surjective : ϕ

commutes with the action of G, so $\phi(P_x) = \phi(G \cdot P_x) = G \cdot \phi(P_x) = P'_x$ since the action is transitive. Moreover, ϕ_x is injective because if $a \neq b \in P_x$, there exists $g \in G$ s.t. $g \neq e$, $a = g \cdot b$ and then

$$\phi(a) = g \cdot \phi(b) \neq \phi(b).$$

□

The goal of this text is to show that the groupoid $\mathbf{PBun}_G(M)$ is essentially finite (i.e. equivalent to a finite groupoid) when M is compact, and to compute its groupoid cardinality.

Lemma 1. *The following functor is an equivalence of categories :*

$$\begin{aligned} T : \mathbf{PBun}_G(M) \to & \quad [\Pi(M), G\text{–}Tor] \\ P \mapsto & \quad \begin{cases} x \mapsto P_x \\ (x \xrightarrow{\gamma} y) \mapsto (P_x \xrightarrow{T_\gamma} P_y) \end{cases} \end{aligned}$$

Proof. Let us find an inverse functor. If $F : \Pi(M) \to G$–Tor is a morphism of groupoids, we define a space P whose underlying set is $\bigsqcup_{x \in M} F(x)$. The projection π sends every point of $F(x)$ to x. Now we put a topology on P. M is a manifold, so its topology admits a base composed of small open discs (U_i). Let $x \in M$ and U a disc neighborhood of x. For each $p \in F_x$ we take

$$U_p = \{T_{[\gamma]}(p) |\ y \in U, [\gamma] \text{ path class from } x \text{ to } y \text{ in } U\}$$

This collection of sets define a topology on P for which the projection is a local diffeomorphism. This is true because for each point y of U there is a unique homotopy class of path $x \xrightarrow{\gamma} y$ which is in U. Moreover, the constructions carries a right continuous group action $P \times G \to P$ which is free and transitive on each fiber because every $P_x = F(x)$ is a G-torsor. So we have constructed a functor B, and the composition $T \circ B$ is the identity while $B \circ T$ is naturally isomorphic to the identity. □

Remark 2. *The principal G-bundle constructed from a morphism $\Pi(M) \to G - Tor$ can be described more clearly by showing it is isomorphic to the G-bundle $\tilde{M} \times_{\pi_1(M)} G$ with $\tilde{M}$ being the universal cover of M.*

Lemma 2. *If M is connected and compact, then there exists an equivalence of categories*

$$[\Pi(M), G\text{–}Tor] \cong \mathrm{Hom}(\pi_1(M), G) /\!\!/ G$$

where G acts on $\mathrm{Hom}(\pi_1(M), G)$ by conjugation. However, the equivalence is not canonical.

Proof. First of all, let us notice that the groupoid of G-torsors is equivalent to the groupoid with only one G torsor : the group G acting by right multiplication on itself. However, this equivalence is not canonical. This is because there exists only one G-torsor up to isomorphism. The arrows in this groupoid are the right G-equivariant functions on G. Let f be such an automorphism of the G-torsor G. We write $f(1) = g$ and so for all $g' \in G$, we have

$$f(g') = f(1g') = f(1)g' = gg'$$

so we see that $\mathrm{Aut}_{G-Tors}(G) \cong G$ with G acting this time by left mutliplication. Therefore, the category of G-torsors is equivalent to the category with one object and whose automorphism group is G. We name it BG. Moreover, M is connected, so the groupoid $\Pi(M)$ is equivalent to the category $B\pi_1(M)$ with one object and with automorphism group $\pi_1(M)$. Consequently,

$$[\Pi(M), G\text{–Tor}] \cong [B\pi_1(M), BG]$$

natural transformations of this category are identified with endomorphisms of BG i.e. $\mathrm{Aut}(BG) = G$ (so they are all automorphisms). Each of them is represented by an element of G which acts by conjugation on the morphisms $\pi_1(M) \to G$. Thus, this category is equivalent with $Hom(\pi_1, G)/\!\!/G$.

Remark 3. *The proof above uses a strong form of the axiom of choice, and set-theoretic problems can arouse as the category of G-torsors isn't even a set. To avoid this problem, we could just define the category of G-torsors as the category of subsets of a big set E which are endowed with a G-torsor structure. By doing this, $G - Tors$ becomes a set and we can use the axiom of choice without scruples.*

Let's continue with a lemma that will be useful after :

Lemma 3 (Automorphisms of a G-Bundle)**.** *Every element $g \in G$ gives rise to a natural isomorphism of the groupoid $Hom(\pi_1, G)$. If $\phi \in Hom(\pi_1, G)$, g is an automorphism of ϕ if and only $g\phi g^{-}1 = \phi$ (this means that to g corresponds an automorphism of the corresponding G-bundle). This is the case if and only if $g \in C(Im(\phi))$. Therefore the group of automorphisms of a G-bundle E corresponding to ϕ is isomorphic to $C(Im(\phi))$.*

□

Lemma 4. *If M is compact and G is a finite group, then $\mathbf{PBun}_G(M)$ is essentially finite.*

Proof. Let us first restrict to the case when M is connected. If so, the groupoid $\boldsymbol{PBun}_G(M)$ is equivalent to $\mathrm{Hom}(\pi_1(M), G)/\!\!/G$. Then, we use the fact that M is a compact manifold, so it is homeomorphic to a finite simplicial complex and thus the fundamental group of M is finitely generated (see T.Sakai [3]). Therefore $\mathrm{Hom}(\pi_1(M), G)$ is a finite set and the result holds. If M is non-connected, calling $M_1, ..., M_n$ its connected components,

$$\mathbf{PBun}_G(M) \cong \prod_i \mathbf{PBun}_G(M_i) \cong \prod_i \mathrm{Hom}(\pi_1(M_i), G)/\!\!/G.$$

and the result still holds. □

We can now compute the groupoid cardinality of $\mathbf{PBun}_G(M)$.

Corollary 1. *For M a compact manifold and G a finite group,*

$$|\mathbf{PBun}_G(M)| = \prod_{c \in \pi_0(M)} \frac{|\mathrm{Hom}(\pi_1(M_i), G)|}{|G|}$$

Definition 16 (Homotopy Pullback of groupoids)**.** *Let Γ, Ω, Λ be three groupoids. If $\Phi : \Gamma \to \Lambda$ and $\Psi : \Omega \to \Lambda$ are two functors, we define the* homotopy pullback $\Gamma \times_\Lambda \Omega$ *to be the following groupoid :*
Its objects are all the triplets (x, y, η_{xy}) with $(x, y) \in \Gamma \times \Omega$ and $\eta_{xy} : \Phi(x) \xrightarrow{\sim} \Psi(y)$ an isomorphism. A morphism between (x, y, η_{xy}) and $(x', y', \eta_{x'y'}$ is a couple of morphisms $a : x \to x'$ and $b : y \to y'$ such that the following diagram commutes :

$$\begin{array}{ccc} \Phi(x) & \xrightarrow{\eta_{xy}} & \Psi(y) \\ \downarrow{\scriptstyle \Phi(a)} & & \downarrow{\scriptstyle \Psi(b)} \\ \Phi(x') & \xrightarrow{\eta_{x'y'}} & \Psi(y') \end{array}$$

Remark 4. *The homotopy pullback makes the following diagram commutative up to the natural isomorphism η :*

$$\begin{array}{ccc} \Gamma \times_\Lambda \Omega & \longrightarrow & \Gamma \\ \downarrow & \overset{\eta}{\swarrow} & \downarrow{\scriptstyle \Phi} \\ \Omega & \xrightarrow{\Psi} & \Lambda \end{array}$$

We now focus on a special case of this homotopy pullback : when Ω is trivial.

Definition 17 (Homotopy fiber)**.** *Let $\Phi : \Gamma \to \Lambda$ a functor and y an object of Λ. The* homotopy fiber *of y along Φ is defined to be the homotopy pullback described by the diagram above :*

$$\begin{array}{ccc} \Phi^{-1}[y] & \xrightarrow{U_y} & \Gamma \\ \downarrow & & \downarrow{\scriptstyle \Phi} \\ \star & \xrightarrow{y} & \Lambda \end{array}$$

Remark 5. *The homotopy pullback is* not *the subgroupoid $\Gamma_y \subset \Gamma$ of all the elements isomorphic to y. This is because the homotopy fiber considers the couples (x, η_{xy}) and there may be different such isomorphisms η. This means the upper map U_y is not necessarily an isomorphism on Γ_y. In fact it is a covering of groupoids.*

Proposition 8. *The "forgetful functor" U_y defines an $|Aut(y)|$-sheeted covering on Γ_y.*

Proof. The first point to check is that it is surjective on objects. This is straightforward because if $x \in \Gamma_y$ is isomorphic to y via η, the element $(x, \eta) \in \Phi^{-1}[y]$ is sent to x by

U_y. The second point is the unique path lifting property. Let $z \xrightarrow{p} z'$ be an isomorphism between $z, z' \in \Gamma_y$, and $x \in \Phi^{-1}[y]$ such that $U_y(x) = z$. This means $x = (z, \eta_z)$ with $\eta_z : z \xrightarrow{\sim} x$. We are looking for a morphism $(z, \eta_z) \xrightarrow{p} (z', \eta_{z'})$ which makes everything commute. The only thing to determine is $\eta_{z'}$ but it is uniquely determined because the diagram

$$\begin{array}{ccc} \Phi(z) & \xrightarrow{\Phi(p)} & \Phi(z') \\ & {\scriptstyle \eta_z} \searrow \quad \swarrow {\scriptstyle \eta_{z'}} & \\ & y & \end{array}$$

must commute, therefore $\eta_{z'} = \eta_z \circ \Phi(p)^{-1}$ and so is unique. Finally, the fiber at x is in bijection with the set of isomorphisms $x \xrightarrow{\sim} y$ which is of cardinality $|\mathrm{Aut}(x)|$. □

With these definitions we can rewrite the expression of $\Phi_*(f)$ calculated in the previous section :

$$\Phi_*(f) = \sum_{[\Phi(x)]=[y]} f(x) \frac{|\mathrm{Aut}(y)|}{|\mathrm{Aut}(x)|} = |Aut(y)| \int_{\Gamma_y} f.$$

As $\Phi^{-1}[y] \to \Gamma_y$ is an $|\mathrm{Aut}(y)|$-sheeted covering, we have then :

$$\Phi_*(f) = \int_{\Phi^{-1}[y]} U_y^* f$$

with $U_y^* f = f \circ U_y$.

3.3 Construction of Dijkgraff-Witten Theory

Using the previous results, we are going to define an n-dimensional TQFT $Z_G : \mathrm{Cob}(n) \to \mathrm{Vect}_k$.

Definition 18. *Let n be an integer. For Σ an $(n-1)-$dimensional closed manifold, We define*

$$Z_G(\Sigma) := F(\boldsymbol{PBun}_G(\Sigma)) = Map(\pi_0(\boldsymbol{PBun}_G(\Sigma)), k).$$

If $\Sigma_0 \to M \leftarrow \Sigma_1$ is a cobordism, we define the following map between $Z_G(\Sigma_0)$ and $Z_G(\Sigma_1)$: The inclusion $i_0 : \Sigma_0 \to M$ induces a functor

$$r_0^* : F(\boldsymbol{PBun}_G(\Sigma_0)) \to F(\boldsymbol{PBun}_G(M)).$$

We have the same on the other side : $r_1^* : F(\boldsymbol{PBun}_G(\Sigma_1)) \to F(\boldsymbol{PBun}_G(M))$. Let $Z_G(M) := r_{1*} \circ r_0^*$ where r_{1*} is the adjoint map of r_1^*.

Remark 6. *Using the calculations made in the previous section, we can detail the expression of $Z_G(M)$. If $f \in Map(\pi_0(PBun_G(\Sigma_0)), k)$, its image $Z_G(M)(f)$ is the following function $g \in Map(\pi_0(PBun_G(\Sigma_1), k)$: for E_1 a bundle over Σ_1,*

$$g(E_1) = |Aut(E_1)| \sum_{[E] \text{ bundle over } M s.t. E_{|\Sigma_1} \simeq E_1} \frac{f(E_{|\Sigma_0})}{|Aut(E)|}$$

Theorem 3. *Z_G defines an $n-$dimensional TQFT.*

Let's start by showing that $Z_G(\Sigma \times I) = id_{F(\Sigma)}$. In fact, the left inclusion $\Sigma \xrightarrow{i_0} \Sigma \times I$ induces an equivalence of the groupoid of G-bundles. Explicitly, we have a restriction map $r : \mathbf{PBun}_G(\Sigma \times I) \to \mathbf{PBun}_G(\Sigma)$ and a section : $\mathbf{PBun}_G(\Sigma) \to \mathbf{PBun}_G(\Sigma \times I)$ which sends a bundle E to the product bundle $E \times I$. By construction, $ps = id_{\mathbf{PBun}_G(\Sigma)}$ and we have the corresponding isomorphism between sp and the identity : If E is a bundle on $\Sigma \times I$, we write E_0 its restriction to the left boundary Σ. We have an natural isomorphism

$$E_0 \times I \to E$$

$$(x,t) \to \gamma_{t,x} \cdot x$$

with $\gamma_{t,x}$ the natural path that connects $(x,0)$ and (x,t). Therefore, the induced map r_0^* is the identity of $Map(\pi_0(\mathbf{PBun}_G(\Sigma)))$. Similarly, r_{1*} is the identity and therefore $Z_G(Id_\Sigma) = Id_{Z_G(\Sigma)}$. An other point we can see easily is that Z_G is monoidal : In fact

$$\begin{aligned} Z_G(\Sigma \sqcup \Sigma') &= F(PBun_G(\Sigma \sqcup \Sigma')) \\ &\simeq F(PBun_G(\Sigma) \times PBun_G(\Sigma')) \\ &\simeq F(PBun_G(\Sigma) \otimes F(PBun_G(\Sigma)) \end{aligned}$$

because the "Free vector space on a set" functor sends products to tensor products. To go further in the proof, we need some additional results. First, consider $M : \Sigma_0 \to \Sigma_1$ and $M' : \Sigma_1 \to \Sigma_2$ two composable cobordisms. Consider $j_i : \mathbf{PBun}_G(M) \to \mathbf{PBun}_G(\Sigma_i)$ the functors induced by restricting bundles $(i = 0, 1)$.

Proposition 9. *There is a natural equivalence between the homotopy pullback and the bundle groupoid of MM' :*

$$\boldsymbol{PBun}_G(M) \times_{\boldsymbol{PBun}_G(\Sigma_1)} \boldsymbol{PBun}_G(M') \cong \boldsymbol{PBun}_G(MM')$$

.

Proof. We construct functors in both directions that are inverse up to isomorphism. Let $K : \mathbf{PBun}_G(MM') \to \mathbf{PBun}_G(M) \times_{\mathbf{PBun}_G(\Sigma_1)} \mathbf{PBun}_G(M')$ sending a bundle E on MM' to the triplet $(E_{|M}, E_{|M'}, Id_{E_{|\Sigma_1}})$. It is clearly a functorial construction. The quasi-inverse of K is given by a functor L that takes a triplet $T = (E, E', \phi)$ with $\phi : E_{|\Sigma_1} \xrightarrow{\sim} E'_{|\Sigma_1}$, and returns the following bundle over MM':

$$L(T) = (E \sqcup E')/(E_{|\Sigma_1} \xrightarrow{\sim} E'_{|\Sigma_1})$$

With a projection $p'' : L(T) \to MM'$ given by the projection $p : E \to M$ on M and $p' : E' \to M'$, these two coincide on Σ_1 because $p'_{|\Sigma_1} \circ \phi = p_{|\Sigma_1}$ as ϕ is a morphism of bundles.
Moreover, G carries a free and transitive action on the fiber of $L(T)$. Therefore, $L(T)$ is well-defined and is a principal G-bundle over MM'. Now let us show that $L \circ K$ and

$K \circ L$ are both naturally isomorphic to the identities. If E'' is a bundle on the total space MM', we see that

$$K \circ L(E'') = E_{|M} \sqcup_{id} E_{|M'} = E_{|M} \cup E_{|M'} = E''$$

and thus is (isomorphic to) the identity.
In the other direction : if $T = (E, E', \phi)$ is a triplet, we see that

$$KL(T) = ((E \sqcup E'_{|\Sigma_1})/(E'_{|\Sigma_1} \sim E_{|\Sigma_1}), (E_2 \sqcup E_{|\Sigma_1})/(E_{|\Sigma_1} \sim E'_{|\Sigma_1}), id_{(E_{|\Sigma_1} \sqcup E'_{|\Sigma_1})/(E_{|\Sigma_1} \sim E'_{|\Sigma_1})}).$$

We have thus the corresponding isomorphism $T \to KL(T)$:

$$\begin{aligned} &T \to KL(T) \\ &E \xrightarrow{\sim} (E \sqcup E_{|\Sigma_1})/(E_{|\Sigma_1} \sim E'_{|\Sigma_1}) \\ &E' \xrightarrow{\sim} (E' \sqcup E_{|\Sigma_1})/(E_{|\Sigma_1} \sim E'_{|\Sigma_1}) \end{aligned}$$

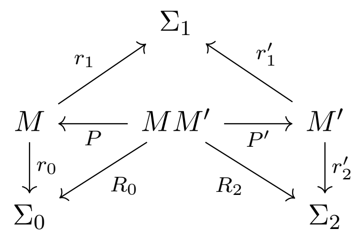
Which is a morphism of triplets because we have the commutativity of

$$\begin{array}{ccc} E_{|\Sigma_1} & \longrightarrow & (E_{|\Sigma_1} \sqcup E'_{|\Sigma_1})/(E_{|\Sigma_1} \overset{\phi}{\sim} E'_{|\Sigma_1}) \\ \downarrow{\scriptstyle \phi_{|\Sigma_1}} & & \downarrow{\scriptstyle id} \\ E'_{|\Sigma_1} & \longrightarrow & (E_{|\Sigma_1} \sqcup E'_{|\Sigma_1})/(E_{|\Sigma_1} \overset{\phi}{\sim} E'_{|\Sigma_1}) \end{array}$$

By definition of a quotient of spaces. □

3.4 Functoriality of the Dijkgraff-Witten theory

The goal of this section is to prove that the Dijkgraff-Witten theory defined above is functorial and thus a TQFT. The situation is the following : $\Sigma_0 \xrightarrow{M} \Sigma_1 \xrightarrow{M'} \Sigma_2$. We define the following restrictions operators on the corresponding principal bundles (the notation is abusive , we write X instead of $PBun_G(X)$ for readability) :



and this diagram commutes. What we want to prove is that $Z_G(\Sigma_0 \to \Sigma_2) = Z_G(\Sigma_1 \to \Sigma_2) \circ Z_G(\Sigma_0 \to \Sigma_1)$ This amounts to saying

$$(R_2)_* R_0^* = (r'_2)_* (r'_1)^* (r_1)_* (r_0)^*$$

But we know $R_2 = r_2'P'$ and $R_0 = r_0 P$ therefore we need to show :

$$(r_2')_*(P')_*(P)^*(r_0)^* = (r_2')_*(r_1')^*(r_1)_*(r_0)^*$$

If we manage to show that

$$(P')_*P^* = (r_1')^*(r_1)_*$$

we will be done. This amounts to show the commutativity of the following diagram :

$$\begin{array}{ccc} F(\mathbf{PBun}_G(MM')) & \xleftarrow[P^*]{} & F(\mathbf{PBun}_G(M)) \\ \big\downarrow{\scriptstyle P'_*} & & \big\downarrow{\scriptstyle (r_1)_*} \\ F(\mathbf{PBun}_G(M')) & \xleftarrow[(r_1')^*]{} & F(\mathbf{PBun}_G(\Sigma_1)) \end{array}$$

We know already that the following diagram (taken without the adjoint maps) is commutative :

$$\begin{array}{ccc} F(\mathbf{PBun}_G(MM')) & \xrightarrow{P_*} & F(\mathbf{PBun}_G(M)) \\ \big\downarrow{\scriptstyle P'_*} & & \big\downarrow{\scriptstyle (r_1)_*} \\ F(\mathbf{PBun}_G(M')) & \xrightarrow{(r_1')_*} & F(\mathbf{PBun}_G(\Sigma_1)) \end{array}$$

Maybe we can obtain the commutativity of the upper diagram by using the other. In fact, using the fact the Proposition 5, we can replace the first diagram by

$$\begin{array}{ccc} F(\mathbf{PBun}_G(M) \times_{\mathbf{PBun}_G(\Sigma_1)} \mathbf{PBun}_G(M')) & \xleftarrow[P^*]{} & F(\mathbf{PBun}_G(M)) \\ \big\downarrow{\scriptstyle P'_*} & & \big\downarrow{\scriptstyle (r_1)_*} \\ F(\mathbf{PBun}_G(M')) & \xleftarrow[(r_1')^*]{} & F(\mathbf{PBun}_G(\Sigma_1)) \end{array}$$

We calculate explicitly both sides and show they are equal.

If $f \in F(\mathbf{PBun}_G(M))$, we have $P^*(f)(E_M, E'_M, \eta_{(E_M)_{|\Sigma_1} \to (E'_M)_{|\Sigma_1}}) = f(E_M)$. Moreover, we get

$$P'_*(P^*(f))(E'_M) = \sum_{\substack{[(E_M, E'_M \eta)] \\ (E_M)_{|\Sigma_1} \xrightarrow{\eta} (E'_M)_{|\Sigma_1}}} \frac{f(E_M)}{|\mathrm{Aut}(E_M, E'_M, \eta)|} |\mathrm{Aut}(E_{M'})|.$$

Where the sum is taken on the isomorphism class of triplets of the form $(E_M, E_{M'}, \eta)$ where an isomorphism of triplets $(E_M, E_{M'}, \eta) \to (\tilde{E}_M, E_{M'}, \tilde{\eta})$ is a couple of isomorphisms $r : E_M \to \tilde{E}_M$ and $s : E_{M'} \to E_{M'}$ such that the diagram on the restriction commutes. Note that as both E_{Σ_1} and $(E_M, E_{M'}, \eta)$ represent principal G-bundles, their automorphism group is G and therefore the sum reduces to

$$P'_*(P^*(f))(E'_M) = \sum_{[(E_M, E'_M \eta)]} f(E_M).$$

Now the other side : If $f \in F(\mathbf{PBun}_G(M))$:

$$(r_1)_*(f)(E_{\Sigma_1}) = \sum_{\substack{[E_M] \\ (E_M)_{|\Sigma_1} \simeq E_{\Sigma_1}}} \frac{f(E_M)}{|\mathrm{Aut}(E_M)|} |\mathrm{Aut}(E_{\Sigma_1})|$$

and then : $\forall E_{M'} \in \mathbf{PBun}_G(M)$,

$$(r_1')^*(r_1)_* f(E_{M'}) = \sum_{\substack{[E_M] \\ (E_M)_{|\Sigma_1} \simeq (E_{M'})_{|\Sigma_1}}} \frac{f(E_M)}{|\mathrm{Aut}(E_M)|} |\mathrm{Aut}((E_{M'})_{|\Sigma_1})|$$

The sum being taken on the isomorphism classes of bundles on M such that their restriction to Σ_1 is isomorphic to that of $E_{M'}$.
(The lemma below is now useless.)

Lemma 5. *Let Γ, Ω, Λ be three groupoids. If $\Phi : \Gamma \to \Lambda$ and $\Psi : \Omega \to \Lambda$ are two functors, and $(x, y, \eta) \in \Gamma \times_\Lambda \Omega$ an triplet in the homotopy pullback, then we have an equality :*

$$|Aut(x, y, \eta) \times Aut(\Phi(x))| = |Aut(x) \times Aut(y)|)$$

.

Proof. Let

$$F : \mathrm{Aut}(x, y, \eta) \times \mathrm{Aut}(\Phi(x)) \to \mathrm{Aut}(x) \times \mathrm{Aut}(y)$$
$$(r, s), u \mapsto (ru^{-1}, s)$$

and

$$G : \mathrm{Aut}(x) \times \mathrm{Aut}(y) \to \mathrm{Aut}(x, y, \eta) \times \mathrm{Aut}(\Phi(x))$$
$$(r, s) \mapsto (ru, s), u \text{ with } u = r^{-1}\eta^{-1}s\eta.$$

A straightforward calculation shows that F and G and inverse of each other. □

Therefore, for each triplet $(E_M, E_{M'}, \eta)$ we have the following :

$$\frac{|\mathrm{Aut}(E_M)|}{|\mathrm{Aut}((E_{M'})_{|\Sigma_1}} = \frac{|\mathrm{Aut}(E'_M)|}{|\mathrm{Aut}(E_M, E_{M'}, \eta)|}.$$

Now we want to identify the two terms $P'_*(P^*(f))(E'_M)$ and $(r_1')^*(r_1)_* f(E_{M'})$. Using the results of the previous part, we have

$$P'_*(P^*(f))(E'_M) = \int_{P'^{-1}[E_{M'}]} f(E_M)$$

and

$$(r_1')^*(r_1)_* f = \int_{r_1{}^{-1}[E_{M'|\Sigma_1}]} f(E_M)$$

We have expressed the two terms as integrals of f over two groupoids. The abusive notation $f(E_M)$ stands for U_*f with U the forgetful functor that are $G_1 = P'^{-1}[E_{M'}]$ and $G_2 = {r_1}^{-1}[E_{M'|\Sigma_1}]$. It suffices then to show that G_1 and G_2 are equivalent (and that the equivalence is compatible with f) to conclude. To prove it, let's describe what are these groupoids.

G_1 is the groupoid of pairs $((E_M, F_{M'}, \eta), \varepsilon)$ with $\eta : E_{M|\Sigma_1} \xrightarrow{\sim} F_{M'|\Sigma_1}$ and $\varepsilon : F_{M'} \xrightarrow{\sim} E_{M'}$. A morphism between $((E_M, F_{M'}, \eta), \varepsilon)$ and $((\tilde{E}_M, \tilde{F}_{M'}, \tilde{\eta}), \tilde{\varepsilon})$ is a pair (r, s) of morphisms with $r : E_{M|\Sigma_1} \xrightarrow{\sim} \tilde{E}_{M|\Sigma_1}$ and $s : F_{M'|\Sigma_1} \xrightarrow{\sim} \tilde{F}_{M'|\Sigma_1}$ such that the following diagrams commute :

$$\begin{array}{ccc} (E_M)_{|\Sigma_1} & \xrightarrow{\eta} & (F_{M'})_{|\Sigma_1} \\ \downarrow{\scriptstyle r_{|\Sigma_1}} & & \downarrow{\scriptstyle s_{|\Sigma_1}} \\ (\tilde{E}_M)_{|\Sigma_1} & \xrightarrow{\tilde{\eta}} & (\tilde{F}_{M'})_{|\Sigma_1} \end{array}$$

and

$$\begin{array}{ccc} F_{M'} & \xrightarrow{\quad s \quad} & \tilde{F}_{M'} \\ & {\scriptstyle \varepsilon}\searrow \quad \swarrow{\scriptstyle \tilde{\varepsilon}} & \\ & E_{M'} & \end{array}$$

G_2 is the groupoid of pairs (E_M, η) with $\eta : E_{M|\Sigma_1} \xrightarrow{\sim} E_{M'|\Sigma_1}$. We have two natural functors between G_1 and G_2 :

$$\begin{aligned} \Pi : & G_1 \to G_2 \\ & ((E_M, F_{M'}, \eta), \varepsilon) \mapsto (E_M, \varepsilon_{|\Sigma_1} \circ \eta) \end{aligned}$$

which is functorial because if (r, s) is a morphism in G_1, we define $\Pi(r, s) = r$ and it is a morphism in G_2 because the following diagram is commutative :

$$\begin{array}{ccccc} (E_M)_{|\Sigma_1} & \xrightarrow{\eta} & (F_{M'})_{|\Sigma_1} & \xrightarrow{\varepsilon_{|\Sigma_1}} & E_{M'|\Sigma_1} \\ \downarrow{\scriptstyle r_{|\Sigma_1}} & & \downarrow{\scriptstyle s_{|\Sigma_1}} & \nearrow{\scriptstyle \tilde{\varepsilon}_{|\Sigma_1}} & \\ (\tilde{E}_M)_{|\Sigma_1} & \xrightarrow{\tilde{\eta}} & (\tilde{F}_{M'})_{|\Sigma_1} & & \end{array}$$

$$\begin{aligned} K : & G_2 \to G_1 \\ & (E_M, \eta) \mapsto ((E_M, E_{M'}, \eta), id_{F_{M'}}) \end{aligned}$$

K is functorial because if η is a morphism in G_1, we can take $K(\eta) = (\eta, id_{E_{M'}})$. Moreover, $\Pi K(E_M, \eta) = (E_M, \eta)$ and we have $K\Pi((E_M, F_{M'}, \eta), \varepsilon) = ((E_M, E_{M'}, \varepsilon_{\Sigma_1} \circ$

$\eta), id_{E_{M'}})$. We have then the following isomorphism from the identity functor to $K\Pi$:

$$((E_M, F_{M'}, \eta), \varepsilon) \xrightarrow{(id_{E_M}, \varepsilon)} ((E_M, E_{M'}, \varepsilon_{|\Sigma_1} \circ \eta), id_{E_{M'}})$$

Therefore G_1 and G_2 are equivalent and the two integral terms are equal. Conclusion : We have shown that Z_G is a monoidal functor $Cob(n) \to Vect_k$ and so an n-dimensional $TQFT$.

3.5 Group algebras and 2D-TQFTs

In this section, we define another two-dimensional TQFT using Froeenius algebras and group representation theory. While it is yet unclear why, we will see that this TQFT is equivalent to the one defined in the last section.

Definition 19. *Let G be a finite group and k an algebraically closed field of characteristic 0. We define the* group algebra $k[G]$ *to be the vector space k^G with a multiplication given by*

$$\forall a, b \in k^G, a \star b(g) = \sum_{h,h' \in G, hh'=g} a(h)b(h').$$

This multiplication endows k^G with a $k-$algebra structure. Define A_G to be the center of $k[G]$ i.e. the subset of central functions on G. We see that A_G is a commutative subalgebra of $k[G]$.

We highlight some interesting and easy points that will be used later in this section :

- A_G can also be seen as the functions of k^G that are constant on conjugacy classes of G.
- $k[G]$ is equivalently the algebra of formal sums $\sum_{g \in G} k_g \cdot g$ with the multiplication induced by the group multiplication.

Now, let's see how we can give A_G a structure of Frobenius algebra :

Definition 20. *Let define $\varepsilon : A_G \to k$ by $\varepsilon(a) = \frac{a(1)}{|G|}$ and let $\delta_{[g]}$ be the element of A_G which takes the value 1 on $[g]$, the conjugacy class of g, and 0 elsewhere.*

It is clear that the $\delta_{[g]}$ with g spamming G form a basis of A_G. Another less trivial fact is :

Theorem 4. *(A_G, ε) is a commutative Frobenius algebra.*

Proof. We have already seen that A_G is commutative. It suffices to show that $\ker(\varepsilon)$ contains no nontrivial left ideal. Take $a \in \ker(\varepsilon)$ which means $a(1) = 0$. If $a \neq 0$, we know there exists g such that $a(g) \neq 0$. Take such a g Consider $b = \delta_{[g^{-1}]} \star a$. We compute its image under ε :

$$b(1) = \sum_{xx'=1} a(x)\delta_{[g^{-1}]}(x') = \sum_{x\in G} a(x)\delta_{[g^{-1}]}(x^{-1})$$
$$= \sum_{x\in[g]} a(x) = \#[g]a(g) \neq 0$$

where the last term isn't null because $\mathrm{char}(k) \neq 0$.

Therefore $\ker(\varepsilon)$ cannot contain a nontrivial left ideal and (A_G, ε) is a Frobenius algebra. □

Now the goal is to compute the $TQFT$ associated to this Frobenius structure. What we want is to know what are the images of the generators of $Cob(2)$. We already have the unit ($k \ni 1 \mapsto \delta_1 \in A_G$), the counit ε and the multiplication $\star$ that we will write as $\mu : A_G \otimes A_G \to A_G$ for clarity. Thus we have to compute the comultiplication δ.

Definition 21. *We write $\beta = \varepsilon\mu$ the natural pairing on A_G. More explicitly*

$$\beta(a, b) = \frac{1}{|G|} \sum_{g\in G} a(g)b(g^{-1})$$

. Since A_G is a Froebenius algebra, β admits a unique copairing that we will call γ defined by :

$$(id_{A_G} \otimes \beta) \circ (\gamma \otimes id_{A_G}) = id_{A_G}$$

To compute our operations, we will use a particular basis involving characters. But first, we need some basic results about representation theory of finite groups from [1].

Let's state some notations and definitions :

Definition 22. *If V is a k-vector space of finite dimension, $[V : k]$ is the dimension of V over k.*

By a $k[G]$-module, we mean here a left $k[G]$-module which has a finite k-basis so that the data of a $k[G]$-module is equivalent to a representation of G (by choosing a base $\{m_1, \dots, m_n\}$ and writing $gm_i = \sum_{j=1}^{n} t_{ji}(x)m_j$) (as stated in [1] (30.3)).

We will use the following theorems from [1] :

Lemma 6. *Let $Z_1, \dots, Z_s$ be a full set of pairwise irreducible $k[G]$-modules. Then*

i $k[G] \cong a_1Z_1 \oplus \dots a_sZ_s$

ii s *is the number of conjugacy classes in* G

iii *for all* i, $a_i = [Z_i : k]$

Proof. These are classical proofs from representation theory. In [1], see (27.20) for i and iii, (27.22) for ii.

□

Now, we can use these facts to build a basis that will be much more handy than $(\delta_{[g]})_{g\in G}$:

Lemma 7. *Starting with the decomposition $k[G] \cong a_1 Z_1 \oplus \dots a_s Z_s$ in Lemma 6, we know we can associate to any $k[G]$-module a representation of G and thus a character (see Definition 22). Let $\zeta^{(i)}$ be the character afforded by Z_i. Then, $\left(\zeta^{(1)}, \dots, \zeta^{(s)}\right)$ are linearly independent over k.*

Proof. Again, a classical proof that can be found in (30.12) from [1] since $\mathrm{char}(k) \neq 0$.

□

Now, let us remark that $[A_G : k] = s$. Indeed, the $(\delta_{[g]})$ form a basis of A_G.

Using, this and Lemmas 6 and 7, we see that :

Theorem 5. $\mathcal{B} = (\zeta^{(1)}, \dots, \zeta^{(s)})$ *is a basis of* A_G.

This base will be very convenient to computer the elements of our TQFT. But before seeing why, let's compute $\delta_{[1]} = 1_{A_G}$ in $\mathcal{B}$ as an example :

Lemma 8. *Let M be a $k[G]$-module. If M is isomorphic to a direct sum $M_1 + \dots + M_k$ of $k[G]$-modules, then if μ is the character afforded by M and $\mu_1, \dots, m_k$ are the characters afforded by $M_1, \dots, M_k$, $\mu = \mu_1 + \dots m_k$.*

Proof. See (30.7) in [1].

□

A very easy calculation show that 1_{A_G} is the character afforded by $k[G]$ as a $k[G]$-module. Then, using i from Lemma 6 and the Lemma 8, we prove that

Example 3. *In $\mathcal{B}$,* $1_{A_G} = \dfrac{1}{|G|}\left(a_1\zeta^{(1)} + \dots + a_s\zeta^{(s)}\right)$.

Now, the reason this base is convenient for our calculation is that it achieves a diagonalization of the multiplication $\star$:

Lemma 9. $\zeta^{(i)} \star \zeta^{(j)} = \zeta^{(i)} b_i \delta_{i,j}$ *where* $b_i = \dfrac{|G|}{a_i}$.

Proof. It is a result of orthogonality. See (31.6) in [1].

□

Moreover, we have

Remark 7. $\beta\left(\zeta^{(i)}, \zeta^{(j)}\right) = \varepsilon\left(\zeta^{(i)} \star \zeta^{(j)}\right) = \zeta^{(i)}(1)\dfrac{b_i}{|G|}\delta_{i,j}$.

Since $\zeta^{(i)}(1) = [Z_i : k] = a_i$ *(from Lemma 6), we have* $\beta\left(\zeta^{(i)}, \zeta^{(j)}\right) = \delta_{i,j}$*, i.e.* $\mathcal{B}$ *is an orthonormal basis for* A_G*.*

In this convenient base, we can compute all of the elements of our Frobenius algebra :

Theorem 6. *Recall that we defined* γ *as the copairing of* β *and* δ *as the comultiplication in* A_G*. Then :*

- $\gamma(1) = \sum_{i=1}^{s} \zeta^{(i)} \otimes \zeta^{(i)}$
- $\delta\left(\zeta^{(i)}\right) = b_i \zeta^{(i)} \otimes \zeta^{(i)}$

Proof. Let $\gamma(1) = \sum_{i,j} \gamma_{i,j}\ \zeta^{(i)} \otimes \zeta^{(j)}$. By definition of γ (see Definition 21), using the identity on $\zeta^{(k)}$, we have :

$$\begin{aligned}
\zeta^{(k)} &= (id_{A_G} \otimes \beta)\left((\gamma \otimes id_{A_G})\left(1 \otimes \zeta^{(k)}\right)\right) \\
&= (id_{A_G} \otimes \beta)\left(\sum_{i,j} \gamma_{i,j}\ \zeta^{(i)} \otimes \zeta^{(j)} \otimes \zeta^{(k)}\right) \\
&= \sum_{i,j} \gamma_{i,j}\ \zeta^{(i)} \otimes \beta\left(\zeta^{(j)}, \zeta^{(k)}\right) \\
&= \sum_{i,j} \gamma_{i,j} \zeta^{(i)} \delta_{j,k} \\
&= \sum_{i} \gamma_{i,k}\ \zeta^{(i)}
\end{aligned}$$

By identification, $\gamma_{i,j} = \delta_{i,j}$.

Likewise, by definition of δ,

$$\begin{aligned}
\delta\left(\zeta^{(k)}\right) &= \sum_{i} \zeta^{(i)} \otimes \left(\zeta^{(i)} \star \zeta^{(k)}\right) \\
&= \sum_{i} \zeta^{(i)} \otimes b_i \delta_{i,k}\ \zeta^{(i)} \\
&= b_k\ \zeta^{(k)} \otimes \zeta^{(k)}
\end{aligned}$$

□

We can then compute the handle element $\omega = \mu\delta$ (where μ is the function multiplication $\star$) :

Corollary 2. *$\omega\left(\zeta^{(i)}\right) = b_i^2\zeta^{(i)}$ i.e. ω is the multiplication by $\left(b_1^2\zeta^{(1)} + \dots b_s^2\zeta^{(s)}\right)$.*

Proof.

$$\begin{aligned}\mu\left(\delta\left(\zeta^{(i)}\right)\right) &= \mu\left(b_i\zeta^{(i)} \otimes \zeta^{(i)}\right)\\ &= b_i(b_i\zeta^{(i)})\\ &= b_i^2\zeta^{(i)}\end{aligned}$$

□

Recalling that given ω we can compute $Z(\Sigma_g)$ the image of the closed surface of genus g, we can finally state :

Theorem 7.

$$Z(\Sigma_g) = \sum_{i=1}^{s}\frac{[Z_i:k]^{2-2g}}{|G|^{2-2g}} = \sum_{V\in \mathrm{irr}(G)}\frac{\dim(V)^{\chi(\Sigma_g)}}{|G|^{\chi(\Sigma_g)}}$$

where in the right-hand side of the equation, we sum on the irreducible (non-isomorphic) representations of G.

Proof. We know, by the equivalence of categories of [2] (see (3.3.1)), that $Z(\Sigma_g) = \varepsilon\left(\omega^g\left(1_{A_G}\right)\right)$.

We have,

$$\begin{aligned}\omega^g\left(1_{A_G}\right) &= \omega^g\left(\frac{1}{|G|}\sum_{i=1}^{s}a_i\zeta^{(i)}\right) && \text{(see Example 3)}\\ &= \frac{1}{|G|}\sum_{i=1}^{s}a_ib_i^{2g}\zeta^{(i)} && \text{(see Corollary 2)}\end{aligned}$$

Then,

$$\begin{aligned}\varepsilon\left(\frac{1}{|G|}\sum_{i=1}^{s}a_ib_i^{2g}\zeta^{(i)}\right) &= \frac{1}{|G|^2}\sum_{i=1}^{s}a_ib_i^{2g}\zeta^{(i)}(1) && \text{by definition}\\ &= \frac{1}{|G|^2}\sum_{i=1}^{s}a_ib_i^{2g}a_i && \text{because } \zeta^{(i)}(1) = [Z_i:k] = a_i\\ &= \sum_{i=1}^{s}b_i^{2g-2} && \text{by definition of } b_i\\ &= \sum_{i=1}^{s}\frac{[Z_i:k]^{2-2g}}{|G|^{2-2g}} && \text{by defintion of } b_i\end{aligned}$$

□

4 A proof of Mednyck's formula

Now we show that the 2-dimensional Dijkgraff-Witten theory defines a $TQFT$ equivalent to the one defined with the Frobenius algebra A_G. The proof is inspired by [4].
The image under Z_G of the circle S^1 (which is the only 1-dimensional manifold up to isomorphism) is the vector space $Map(\pi_0(\mathbf{PBun}_G(S^1)), k)$. We know that the groupoid $\mathbf{PBun}_G(S^1)$ is equivalent to $Hom(\pi_1(S^1), G)//G \simeq G//G$ with G acting by conjugation on itself. Therefore, $Z_G(S^1)$ is naturally in bijection with the maps from conjugacy classes of G to k. So $Z_G(S^1) \simeq A_G$ as vector spaces. Now all we have to do is to show that the multiplication and the counit are the same as the one on A_G. This will imply that the two Frobenius algebras are isomorphic.

- Let $\varepsilon' = Z_G(D^2)$ with D^2 being considered as the cobordism from S^1 to $\emptyset$. Using the remark 10, we can write :

$$\begin{aligned} \varepsilon' : Z_G(S^1) = Map(C(G), k) \to & \qquad k = Z_G(\emptyset) \\ f \mapsto & \sum_{[E] \text{ bundle on } D^1} \frac{f(E_{|S^1})}{Aut(E)} \end{aligned}$$

  There is only one principal G-bundle over D^2 because this space is simply connected. It is the trivial bundle $D^2 \times G$ corresponding to the unique morphism of groups from 1 to G . Thus, denoting E_0 the trivial bundle on S^1, which corresponds to the zero morphism $\mathbb{Z} \to G$, we have $\varepsilon'(f) = \frac{f(E_0)}{|G|}$. Modulo the identification with A_G, $\varepsilon' = \varepsilon$.

- Now we show the multiplication is the same. Let us write M for the natural cobordism from $S^1 \sqcup S^1$ to S^1, and $\mu = Z_G(M)$. If $f \in Z_G(S^1 \sqcup S^1)$ and E_2 is a bundle over S^1 (the target), to which corresponds an element $g_2 \in G$.

$$\mu f(E_2) = |\mathrm{Aut}(E_2)| \sum_{[E] \text{ on } M,\, E_{|S^1} \simeq E_2} \frac{f(E_{S^1 \sqcup S^1})}{|Aut(E)|}$$

  But $\pi_1(M) = F_2$ the free group on two elements, generated by the paths along the two left circles. Under the identification with $Hom(\pi_1(M), G)$ we see that a bundle on M amounts to a choice of a couple $(g_0, g_1) \in G$ and that its restriction to the right circle corresponds to the element $g_0 g_1$. Modulo the identifications, we can then rewrite the action of μ:

$$\mu f(g_2) = \int_\Gamma f(g_0, g_1)$$

  where Γ is the homotopy fiber groupoid, that is the groupoid of triplets (g_0, g_1, h) with $g_0 g_1 = hgh^{-1}$, a morphism $(g_0, g_1, h) \to (g_0', g_1', h')$ is an (unique) element u such that $h'u = h$ by abstract nonsense. We see that each object has only the identity as automorphism. Let's write Ω for the totally disconnected groupoid composed of the couples (g_0, g_1) such that $g_0 g_1 = g$. the map $\Gamma \to \Omega$ given by

$$(g_0, g_1, h) \mapsto (hg_0h^{-1}, hg_1h^-1)$$

is functorial, sending every morphism to the identity (because $h'u = h$ with the previous notations). It is faithful and essentially surjective. Finally we have :

$$\mu f(g_2) = \sum_{(g_0,g_1)\in G, g_0g_1=g_2} f(g_0, g_1)$$

and modulo the identification $F(S^1 \sqcup S^1) \simeq F(S^1) \otimes F(S^1)$, we get the following multiplication :

$$\forall (f_0, f_1) \in F(S^1)^2,\ \mu(f_1, f_2)(g) = \sum_{(g_0,g_1)\in G, g_0g_1=g_2} f_0(g_0)f_1(g_1)$$

and so identifies with the multiplication on A_G.

Conclusively, Z_G and A_G define the same Frobenius algebra and thus the same $TQFT$. What remains is to calculate the image of a closed surface of genus g under Z_G. Let us denote such a surface by Σ_g. The image $Z_G(\Sigma_g)$ is identified with a scalar. Let u denote the left empty restriction functor $PBun_G(\Sigma_g) \to PBun_G(\emptyset) = \{\emptyset\}$. We have

$$Z_G(\Sigma_g)(1) = u_* \circ u^* = \sum_{[E] \text{ bundle on } \Sigma_g} \frac{1}{|\mathrm{Aut}(E)|}$$

which is the groupoid cardinality of $PBun_G(\Sigma_g) \simeq Hom(\pi_1(G))//G$. We have consequently

$$Z_G(\Sigma_g) = A_G(\Sigma_g)$$

and therefore the following result [4] :

Theorem 8 (Mednyck's formula)**.** *Let k be an algebraically closed field, G a finite group and Σ a closed oriented surface of genus g. Then we have*

$$\sum_{V \in Irrep_k(G)} (\dim V)^{\chi(\Sigma)} = |G|^{\chi(\Sigma)-1} Hom(\pi_1(\Sigma), G).$$

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