A GENERALIZATION OF THE BLOOM-SISASK THEOREM TO ARBITRARY INVERTIBLE COEFFICIENTS

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Abstract. We prove the following conjecture of Shkredov and Solymosi: every subset $A \subset \mathbf{Z}^2$ such that $\sum_{a \in A \setminus \{0\}} \frac{1}{\|a\|^2} = +\infty$ contains the three vertices of an isosceles right triangle. To do this, we adapt the proof of the recent breakthrough by Bloom and Sisask on sets without three-term arithmetic progressions to a more general setting.

Déroulement du stage

Ce document est le rapport du stage fait sous la supervision de Prof. Timothy Gowers (Collège de France) au deuxième semestre de l'année académique 2020-2021. Le stage s'est déroulé exclusivement par vidéoconférence.

Entre février et juin, j'ai lu l'article [2] de Thomas Bloom et Olof Sisask (77 pages pour la première version). Lors de nos réunions hebdomadaires, je présentais les chapitres successifs à Timothy Gowers et nous en discutions.

Entre juillet et début septembre, j'ai pris contact avec Thomas Bloom et Olof Sisask pour leur poser quelques questions. J'ai proposé des modifications pour lesquelles j'ai été remercié dans l'article, dont la deuxième version fait désormais 95 pages. Ils m'ont proposé de travailler à une généralisation de leur résultat, dans le but de démontrer une conjecture de Ilya Shkredov et József Solymosi (voir Section 1).

Ce stage a confirmé mon désir de poursuivre mon parcours en recherche fondamentale par une thèse de doctorat dans un domaine proche de la combinatoire arithmétique.

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1. Introduction

In their 2020 breakthrough paper, Bloom and Sisask [2] improved the best known upper bound on the largest possible size of a subset of $\{1, 2, \dots, n\}$ without three-term arithmetic progression. They showed that, if $A \subset \{1, 2, \dots, n\}$ has no non-trivial three-term arithmetic progression, then

$$|A| \ll \frac{n}{(\log n)^{1+c}}$$

for some absolute constant $c > 0$.

Their result received a lot of attention (see the Quanta Magazine article [3]) as it settled the first interesting case of one of Erdős' favorite conjectures. Erdős conjectured that, if $A \subset \mathbf{N}$ is such that $\sum_{n \in A} \frac{1}{n}$ diverges, then A contains infinitely many k-term arithmetic progressions, for every $k \geq 3$. The result of Bloom and Sisask implies the case $k = 3$. The general case seems to be well beyond the reach of the current techniques.

The theorem of Bloom and Sisask can be applied to the prime numbers to recover a result of Green in analytic number theory. It is an old result of Van der Corput that the set of primes contains infinitely many three-term arithmetic progressions. Much more recently, Green [5] generalized this fact to relatively dense subsets of the primes. The theorem of Bloom and Sisask gives a different proof of this, where Chebyshev's estimate $\pi(x) \gg x/\log x$ is the only fact about the primes that is used.

A three-term arithmetic progression is a solution to the equation $a_1 - 2a_2 + a_3 = 0$. In this report, we adapt the proof of Bloom and Sisask to generalize their theorem to equations of the form $T_1a_1 + T_2a_2 + T_3a_3 = 0$ for an extended class of coefficients T_1, T_2 and T_3. More precisely, we prove the following in Section 4.

Theorem 1.1. *Let G be a finite abelian group and let T_1, T_2, T_3 be automorphisms of G such that $T_1 + T_2 + T_3 = 0$. If A is a subset of G without nontrivial solutions*[1] *to the equation*

$$T_1a_1 + T_2a_2 + T_3a_3 = 0,$$

then

$$|A| \ll \frac{|G|}{(\log |G|)^{1+c}}$$

where $c > 0$ is an absolute constant.[2]

The result [2, Corollary 3.2] of Bloom and Sisask corresponds to the special case $T_1 = T_2 = \mathrm{Id}_G$ and $T_3 = -2\,\mathrm{Id}_G$ of Theorem 1.1.

Remark 1.2 (Assumptions)**.** The condition $T_1 + T_2 + T_3 = 0$ is necessary and ensures that the equation $T_1a_1 + T_2a_2 + T_3a_3 = 0$ is translation invariant. For example, if $G = \mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$, $T_1 = T_2 = T_3 = \mathrm{Id}_G$ and $A = \{1\} \times \mathbf{Z}/m\mathbf{Z}$, then $|A| \asymp |G|$, yet A has no solution to $a_1 + a_2 + a_3 = 0$ as $a_1 + a_2 + a_3 \in \{3\} \times \mathbf{Z}/m\mathbf{Z}$.

It is not known how to remove the invertibility assumption on the T_i's.

In Section 2, we will prove the following corollary, which generalizes [2, Corollary 1.2] to higher dimensions and matrix coefficients.

Corollary 1.3. *Let M_1, M_2, M_3 be nonsingular $d \times d$ matrices with integer coefficients such that $M_1 + M_2 + M_3 = 0$. If $A \subset \mathbf{Z}^d$ satisfies*

$$\sum_{a \in A \setminus \{0\}} \frac{1}{\|a\|^d} = +\infty,$$

then A contains infinitely many nontrivial solutions to the equation $M_1a_1 + M_2a_2 + M_3a_3 = 0$.

Using Corollary 1.3, we are able to solve a conjecture of Shkredov and Solymosi [6, Conjecture 2].

Corollary 1.4. *If $A \subset \mathbf{Z}^2$ is such that $\sum_{a \in A \setminus \{0\}} \frac{1}{\|a\|^2} = +\infty$, then there are infinitely many isosceles right triangles whose vertices are in A.*

It is believed that Corollary 1.4 can be strengthened significantly: a conjecture of Graham states that, under the same hypothesis, A contains infinitely many axes-parallel squares [4, Conjecture 8.4.6].

Proof. This follows from Corollary 1.3 after the observation that $a_1, a_2, a_3 \in \mathbf{R}^2$ satisfy the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} a_1 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} a_2 + \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} a_3 = 0$$

if and only if $a_1 = a_2 = a_3$ or $a_1a_2a_3$ is an isosceles right triangle (oriented counterclockwise and with a_1a_2 being the hypothenuse). □

[1] A solution $(a_1, a_2, a_3) \in A^3$ is trivial if $a_1 = a_2 = a_3$.

[2] In particular, the constant c does not depend on G.

Remark 1.5. Theorem 1.1 can be generalized to equations with more than three terms. More precisely, a slight adaptation of the proof shows the following. If $T_1, T_2, \dots, T_k$ are commuting[3] automorphisms of an abelian group G such that $T_1 + T_2 + \dots + T_k = 0$, then any set $A \subset G$ without nontrivial solution to $T_1 a_1 + T_2 a_2 + \dots T_k a_k = 0$ satisfies

$$|A| \ll \frac{|G|}{(\log |G|)^{1+c}}$$

where $c > 0$ is an absolute constant. Corollary 1.3 can also be modified in a similar way. However, for $k \geq 4$, considerably better bounds are available using other methods, which is why we restricted ourselves to the case $k = 3$.

2. Application to Matrix Coefficients

In this section, we show how Corollary 1.3 follows from Theorem 1.1. The proof is standard and involves two steps: first truncating the set A, then embedding this truncation of A inside a finite abelian group.

Proof of Corollary 1.3, assuming Theorem 1.1. Let A be a subset of $\mathbf{Z}^d$ containing only finitely many nontrivial solutions to the equation

$$M_1 a_1 + M_2 a_2 + M_3 a_3 = 0. \tag{1}$$

We want to prove that

$$\sum_{a \in A \setminus \{0\}} \frac{1}{\|a\|^d} < +\infty. \tag{2}$$

After removing a finite number of elements from A, we can assume that A has no nontrivial solution to Eq. (1).

For $T \geq 1$, let A_T be the truncated set

$$A_T := A \cap [-T, T]^d.$$

It is sufficient to prove that, for all $T \geq 2$,

$$|A_T| \ll \frac{T^d}{(\log T)^{1+c}}, \tag{3}$$

where $c > 0$ is the constant from Theorem 1.1.[4] Indeed, we have

$$\sum_{\substack{a \in A \setminus \{0\} \\ \|a\|_\infty \leq M}} \frac{1}{\|a\|^d} = \sum_{N=1}^{M} \sum_{\substack{a \in A \\ \|a\|_\infty = N}} \frac{1}{\|a\|^d}$$
$$\asymp \sum_{N=1}^{M} \frac{1}{N^d} \cdot \#\{a \in A : \|a\|_\infty = N\},$$

and, by partial summation together with Eq. (3), we get

$$\sum_{\substack{a \in A \setminus \{0\} \\ \|a\|_\infty \leq M}} \frac{1}{\|a\|^d} \asymp \frac{|A_M|}{M^d} + \int_1^M \frac{|A_T|}{T^{d+1}} \,\mathrm{d}T \ll 1 + \int_2^M \frac{1}{T(\log T)^{1+c}} \,\mathrm{d}T \ll 1.$$

Taking $M \to +\infty$ proves Eq. (2).

[3] For $k = 3$, this hypothesis is not necessary, see Remark 3.2.

[4] In this proof, the implied constants in the asymptotic notation $\ll$ and $\asymp$ depend only on the dimension d and the matrices M_1, M_2, M_3.

Let $T \geq 1$. Let

$$C = \max\left(\|M_1\|_{\text{op}}, \|M_2\|_{\text{op}}, \|M_3\|_{\text{op}}, |\det M_1|, |\det M_2|, |\det M_3| \right),$$

where $\|M_i\|_{\text{op}}$ is the operator norm of the matrix M_i, viewed as a map $(\mathbf{R}^d, \|\cdot\|_\infty) \to (\mathbf{R}^d, \|\cdot\|_\infty)$. Let p be a prime number between $4CT$ and $8CT$, which exists by Bertrand's postulate.

We embed A_T in the abelian group $(\mathbf{Z}/p\mathbf{Z})^d$. Let $\overline{A_T}$, $\overline{M_1}, \overline{M_2}$ and $\overline{M_3}$ be the reductions of A_T, M_1, M_2 and M_3 modulo p. Clearly, each $\overline{M_i}$ is invertible as its determinant is not divisible by p.

We claim that the map

$$\{(a_1, a_2, a_3) \in (A_T)^3 : M_1a_1+M_2a_2+M_3a_3 = 0\} \to \{(x_1, x_2, x_3) \in (\overline{A_T})^3 : \overline{M_1}x_1+\overline{M_2}x_2+\overline{M_3}x_3 = 0\}$$

given by reduction modulo p is surjective. Indeed, if $(a_1, a_2, a_3) \in (A_T)^3$ is such that

$$M_1a_1 + M_2a_2 + M_3a_3 \equiv 0 \pmod p,$$

then $M_1a_1 + M_2a_2 + M_3a_3 = 0$ in $\mathbf{R}^d$ since we also have

$$\|M_1a_1 + M_2a_2 + M_3a_3\|_\infty \leq 3CT < p.$$

It follows that $\overline{A_T}$ only has trivial solutions to the equation $\overline{M_1}x_1 + \overline{M_2}x_2 + \overline{M_3}x_3 = 0$. By Theorem 1.1, we obtain

$$|A_T| = |\overline{A_T}| \ll \frac{p^d}{(\log p^d)^{1+c}} \asymp \frac{T^d}{(\log T)^{1+c}},$$

which proves Eq. (3) and concludes the proof of Corollary 1.3. □

3. Notations and New Density Increments

We use the same notation as in the paper of Bloom and Sisask [2]. We recall some of it below, but we encourage the reader to familiarize themselves with their article before reading this section.

Notation 3.1. If $X \subset G$, the density of X in G is denoted by $\mu(X) := |X|/|G|$. We write μ_X for the normalized indicator function $\mu_X = \mu(X)^{-1}\mathbf{1}_X$. For $f, g : G \to \mathbf{C}$, we write

$$\langle f, g \rangle := \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)},$$

while for $f, g : \widehat{G} \to \mathbf{C}$,

$$\langle f, g \rangle := \sum_{\gamma \in \widehat{G}} f(\gamma)\overline{g(\gamma)}.$$

Similarly, we normalize the convolutions as $f * g(x) := \frac{1}{|G|} \sum_{y \in G} f(y)g(x - y)$ for $f, g : G \to \mathbf{C}$ and $f * g(x) := \sum_{y \in \widehat{G}} f(y)g(x - y)$ for $f, g : \widehat{G} \to \mathbf{C}$.

In order to suppress logarithmic factors, we use the notation $X \lesssim_\alpha Y$ or $X = \tilde{O}_\alpha(Y)$ to mean that $|X| \leq C_1 \log(2/\alpha)^{C_2} Y$ for some constants $C_1, C_2 > 0$.

Remark 3.2. It suffices to prove Theorem 1.1 when the first automorphism T_1 is the identity, something we will assume from now on. To deduce the case of a general automorphism T_1, simply apply Theorem 1.1 to the set T_1A and the automorphisms $\text{Id}_G, T_2T_1^{-1}, T_3T_1^{-1}$.

Fix a finite abelian group G and two automorphisms T_2, T_3 such that $\text{Id}_G + T_2 + T_3 = 0$. We count the number of solutions to the equation $a_1 + T_2a_2 + T_3a_3 = 0$ using the inner product

$$T(A_1, A_2, A_3) := \langle \mathbf{1}_{A_1} * \mathbf{1}_{T_2A_2}, \mathbf{1}_{-T_3A_3} \rangle,$$

defined for $A_1, A_2, A_3 \subset G$. Observe that

$$T(A, A, A) = \frac{1}{|G|^2} \cdot \#\{(a_1, a_2, a_3) \in A^3 : a_1 + T_2a_2 + T_3a_3 = 0\}.$$

To make the argument work for general coefficients, we need to change the definition of density increments.

Definition 3.3 (Increments)**.** Let B be a regular Bohr set, and let $B' \subset B$ be a regular Bohr set of rank d. We say that $A \subset B$ of relative density α has an increment of strength $[\delta, d'; C]$ relative to B' if there is a regular Bohr set B'' of the form

$$B'' = B'_\rho \cap \widetilde{B}$$

such that

$$\|\mathbf{1}_A * \mu_{B''}\|_\infty \geq (1 + C^{-1}\delta)\alpha,$$

where $\widetilde{B} = \mathrm{Bohr}(\widetilde{\Gamma}, \widetilde{\nu})$ is a Bohr set of rank $|\widetilde{\Gamma}| \leq Cd'$, and $\rho \in (0, 1]$ satisfies the inequality

$$\left(\frac{\rho}{4}\right)^d \prod_{\gamma \in \widetilde{\Gamma}} \frac{\widetilde{\nu}(\gamma)}{4} \geq (2d(d'+1))^{-C(d+d')}. \tag{4}$$

Remark 3.4. If $A \subset B$ has a density increment of strength $[\delta, d'; C]$ with respect to B' in the sense of Definition 3.3, then A has a density increment of the same strength with respect to B' in the sense of [2, Definition 5.1]. This is because Eq. (4) implies the bound

$$|B''| \geq (2d(d'+1))^{-C(d+d')}|B'|, \tag{5}$$

by a direct application of [2, Lemma 4.4].

The converse is not true in general, but it is true for all the density increments present in [2]. That is, every density increment in [2] is also a density increment in the sense of Definition 3.3, of the same strength. The reason is that, every time the authors show that some set $A \subset B$ has a density increment, they need to prove Eq. (5). To do this, the only tool they use is [2, Lemma 4.4], and thus they prove the stronger Eq. (4).

4. Proof of the Main Theorem

This section is dedicated to the proof of Theorem 1.1. We start by proving an analogue of [2, Lemma 8.2] in our setting. The main difference is that we will have to work with three subsets A_1, A_2, A_3, instead of two. Each statement in this section is an adaptation of a corresponding statement in [2]. To help the reader, we will highlight the changes made to the original statements of [2] in blue.

Lemma 4.1. *There is a constant $c > 0$ such that the following holds. Let $0 < \alpha \leq 1$. Let B be a regular Bohr set of rank d, and B' another regular Bohr set such that $T_3 B' \subset B_\rho$, with $\rho \leq c\alpha/d$. Let $A_1 \subset B$, $A_2 \subset T_2^{-1}B$ and $A_3 \subset B'$, each time with relative density in $[\alpha/2, 2\alpha]$. Then either*

(1) (many solutions) $T(A_1, A_2, A_3) \geq \frac{1}{16}\alpha^3 \mu(B)\mu(B')$, or

(2) (large L^2 mass on a spectrum) there is some $\eta \gg \alpha$ such that

$$\sum_{\gamma \in \Delta_\eta(-T_3 A_3)} |\widehat{\mu_{A/B}}(\gamma)|^2 \gtrsim_\alpha \eta^{-1}\mu(B)^{-1},$$

where A is either A_1 or $T_2 A_2$.

Proof. We have

$$T(A_1, A_2, A_3) = \langle \mathbf{1}_{A_1} * \mathbf{1}_{T_2 A_2}, \mathbf{1}_{-T_3 A_3} \rangle \geq \tfrac{1}{8}\alpha^3 \mu(B)^2 \mu(B') \left\langle \mu_{A_1} * \mu_{T_2 A_2}, \mu_{-T_3 A_3} \right\rangle.$$

Replacing μ_{A_1} and $\mu_{T_2 A_2}$ with their balanced functions $\mu_{A_1/B} = \mu_{A_1} - \mu_B$ and $\mu_{T_2 A_2/B} = \mu_{T_2 A_2} - \mu_B$, we have

$$\langle \mu_{A_1} * \mu_{T_2 A_2}, \mu_{-T_3 A_3} \rangle = \left\langle \mu_{A_1/B} * \mu_{T_2 A_2/B}, \mu_{-T_3 A_3} \right\rangle + E,$$

where

$$E = \langle \mu_{A_1} * \mu_B, \mu_{-T_3A_3}\rangle + \langle \mu_B * \mu_{T_2A_2}, \mu_{-T_3A_3}\rangle - \langle \mu_B * \mu_B, \mu_{-T_3A_3}\rangle .$$

We can estimate E using regularity. Since $-T_3A_3 \subset B_\rho$, we have $\|\mu_{-T_3A_3} * \mu_B - \mu_B\|_1 = O(\rho d)$ by [2, Lemma 4.5]. Moreover, $\|\mu_{A_1}\|_\infty, \|\mu_{T_2A_2}\|_\infty, \|\mu_B\|_\infty \le 2\alpha^{-1}\mu(B)^{-1}$. Therefore

$$\begin{aligned} E &= \langle \mu_{A_1}, \mu_{-T_3A_3} * \mu_B\rangle + \langle \mu_{T_2A_2}, \mu_{-T_3A_3} * \mu_B\rangle - \langle \mu_B, \mu_{-T_3A_3} * \mu_B\rangle \\ &= \langle \mu_{A_1}, \mu_B\rangle + \langle \mu_{T_2A_2}, \mu_B\rangle - \langle \mu_B, \mu_B\rangle + O(\rho d\alpha^{-1}\mu(B)^{-1}) \\ &= \mu(B)^{-1} + \mu(B)^{-1} - \mu(B)^{-1} + O(\rho d\alpha^{-1}\mu(B)^{-1}) \\ &= \mu(B)^{-1} + O(\rho d\alpha^{-1}\mu(B)^{-1}). \end{aligned}$$

In particular, $E \ge \frac{3}{4}\mu(B)^{-1}$, provided ρ is small enough. Thus

$$T(A_1, A_2, A_3) \ge \tfrac{1}{8}\alpha^3\mu(B)^2\mu(B')\Big(\langle \mu_{A_1/B} * \mu_{T_2A_2/B}, \mu_{-T_3A_3}\rangle + \tfrac{3}{4}\mu(B)^{-1}\Big)$$

If the first case of the conclusion doesn't hold, then

$$\langle \mu_{A_1/B} * \mu_{T_2A_2/B}, \mu_{-T_3A_3}\rangle \le -\tfrac{1}{4}\mu(B)^{-1}.$$

By Parseval's identity, followed by the triangle inequality, we deduce that

$$\langle |\widehat{\mu_{A_1/B}}||\widehat{\mu_{T_2A_2/B}}|, |\widehat{\mu_{-T_3A_3}}|\rangle \ge \tfrac{1}{4}\mu(B)^{-1}.$$

Using $xy \le \frac{1}{2}(x^2+y^2)$, we find that

$$\sum_{\gamma\in\widehat{G}} |\widehat{\mu_{A/B}}(\gamma)|^2|\widehat{\mu_{-T_3A_3}}(\gamma)| = \langle |\widehat{\mu_{A/B}}|^2, |\widehat{\mu_{-T_3A_3}}|\rangle \ge \tfrac{1}{4}\mu(B)^{-1},$$

where A is either A_1 or T_2A_2. Since $\|\mu_{A/B}\|_2^2 \le \alpha^{-1}\mu(B)^{-1}$, we can discard the terms of the above sum with $|\widehat{\mu_{-T_3A_3}}| \le \frac{1}{8}\alpha$ to obtain

$$\sum_{\gamma\in\Delta_{\alpha/8}(-T_3A_3)} |\widehat{\mu_{A/B}}(\gamma)|^2|\widehat{\mu_{-T_3A_3}}(\gamma)| \ge \tfrac{1}{8}\mu(B)^{-1}.$$

By the dyadic pigeonhole principle, we conclude that there is some $1 \ge \eta \gg \alpha$ such that

$$\sum_{\gamma\in\Delta_\eta(-T_3A_3)\setminus\Delta_{2\eta}(-T_3A_3)} |\widehat{\mu_{A/B}}(\gamma)|^2|\widehat{\mu_{-T_3A_3}}(\gamma)| \gtrsim_\alpha \mu(B)^{-1}.$$

This concludes the proof since $|\widehat{\mu_{-T_3A_3}}(\gamma)| \asymp \eta$ on the set $\Delta_\eta(-T_3A_3) \setminus \Delta_{2\eta}(-T_3A_3)$. □

The statement of [2, Proposition 8.1] should be modified as follows.

Proposition 4.2. *There is a constant $c > 0$ such that the following holds. Let $k, h, t \ge 20$ be some parameters.*

Let $0 < \alpha \le 1$. Let B be a regular Bohr set of rank d, and B' another regular Bohr set, of rank at most $3d$, such that $B' \subset T_3^{-1}B_\rho$, where $\rho \le c\alpha^2/d$. Let $A_1 \subset B$, $A_2 \subset T_2^{-1}B$ and $A_3 \subset B'$, each time with relative density in $[\alpha/2, 2\alpha]$. Then for either $A = A_1$ or $A = T_2A_2$, one of the following holds

1. *(large density) $\alpha \gg 1/k^2$, or*
2. *(many solutions) $T(A_1, A_2, A_3) \gg \alpha^3\mu(B)\mu(B')$, or*
3. *A has a density increment of strength either*
   - *(a) (small increment) $[1, \alpha^{-1/k}; \tilde{O}_\alpha(h\log t)]$ or*
   - *(b) (large increment) $[\alpha^{-1/k}, \alpha^{-1+1/k}; \tilde{O}_\alpha(h\log t)]$*

   relative to T_3B', or

(4) (non-smoothing large spectrum) there is a set Δ *and three quantities* $\rho_{\text{top}}, \rho_{\text{bottom}}, \rho' \in (0,1)$ *satisfying*

$$\rho_{\text{top}} \gg \alpha^{O(1)}(c/dt)^{O(h)}, \quad \rho_{\text{bottom}} \gg (\alpha/d)^{O(1)}, \quad \text{and} \quad \rho' \gg (\alpha/d)^{O(1)},$$

such that

(a) $\alpha^{-3+O(1/k)} \ll |\Delta| \lesssim_\alpha \alpha^{-3}$,

(b) there exists an additive framework $\tilde{\Gamma}$ *of height* h *and tolerance* t *between*

$$\Gamma_{\text{top}} := \Delta_{1/2}(T_3 B'_{\rho_{\text{top}}}) \quad \text{and} \quad \Gamma_{\text{bottom}} := \Delta_{1/2}(T_3 B'_{\rho_{\text{bottom}}}),$$

(c) Δ *is* $\frac{1}{4}$*-robustly* (τ, k')*-additively non-smoothing relative to* $\tilde{\Gamma}$ *for some* $\alpha^{2-O(1/k)} \gg \tau \gg \alpha^{2+O(1/k)}$ *and* $k \geq k' \gg k$, *and*

(d) if we let $B'' = (T_3 B'_{\rho_{\text{top}}})_{\rho'}$ *then for all* $\gamma \in \Delta + \Gamma_{\text{top}}$

$$|\widehat{\mu_{A/B}}|^2 \circ |\widehat{\mu_{B''}}|^2(\gamma) \gg \alpha^{2+O(1/k)}\mu(B)^{-1},$$

and

(e)

$$\left\| \mathbf{1}_\Delta \circ |\widehat{\mu_{B''}}|^2 \right\|_\infty \leq 2.$$

Few changes have to be made to the proof of [2, Proposition 8.1], so we only give an overview of the modified proof.

Proof sketch. We keep the notation $B^{(0)} := B'$ and $B^{(i+1)} = B^{(i)}_{\rho_i}$ for some ρ_i that are the same as those in the original proof.

By Lemma 4.1, either we are in the second case or there is some $\eta \gg \alpha$ such that

$$\sum_{\gamma \in \Delta_\eta(-T_3 A_3)} |\widehat{\mu_{A/B}}(\gamma)|^2 \gtrsim_\alpha \eta^{-1}\mu(B)^{-1},$$

where A is either A_1 or $T_2 A_2$.

Suppose first that this is true for some $\eta \geq \frac{1}{2}K^{-1}$. In this case we apply [2, Corollary 7.11] with $T_3 B'$ instead of B, with $T_3 B^{(1)}$ in place of B', the function f chosen to be $\mathbf{1}_{-T_3 A_3}$ and and the weight function ω given by $\omega = |\widehat{\mu_{A/B}}|^2$, restricted to $\Delta_\eta(-T_3 A_3)$. We apply [2, Lemma 7.8] and [2, Lemma 5.7] in the same way as in the original proof, except that we obtain a small density increment for A relative to $T_3 B'$ instead of B'.

The case $\frac{1}{2}K^{-1} \geq \eta \geq K^2\alpha$ is similar. After using [2, Corollary 7.12], [2, Lemma 7.8] and [2, Lemma 5.7], we conclude that A has a large increment relative to $T_3 B'$.

Finally, in the case $\alpha \ll \eta \ll K^2\alpha$, we have

$$\sum_{\gamma \in \tilde{\Delta}} |\widehat{\mu_{A/B}}(\gamma)|^2 \gtrsim_\alpha K^2\alpha^{-1}\mu(B)^{-1},$$

where $\tilde{\Delta} = \Delta_{c\alpha}(-T_3 A_3)$ for some absolute constant $c > 0$. We use [2, Lemma 6.2] to construct an additive framework between $\Gamma_{\text{top}} = \Delta_{1/2}(T_3 B^{(2)})$ and $\Gamma_{\text{bottom}} = \Delta_{1/2}(T_3 B^{(1)})$. Next, we use [2, Lemma 8.5] with A' being replaced by $-T_3 A_3$, B' being replaced by $T_3 B'$, $B^{(1)}$ being replaced by $T_3 B^{(2)}$ and $B^{(2)}$ being replaced by $T_3 B^{(3)}$. This either gives a density increment for A with respect to $T_3 B'$, or else produces a set Δ satisfying most of the conditions of the final case of Proposition 4.2. The rest of the proof is the same, after replacing every occurrence of $2 \cdot A'$ by $-T_3 A_3$ and every occurrence of $2 \cdot B^{(i)}$ by $T_3 B^{(i)}$. □

Next, we reproduce the statement of [2, Lemma 12.1] for three smaller Bohr sets instead of two. The proof of Lemma 4.3 is the same as that of [2, Lemma 12.1], so we shall not repeat it here.

Lemma 4.3. *There is a constant $c > 0$ such that the following holds. Let $\mathcal{B}$ be a regular Bohr set of rank d, let $\mathcal{A} \subset \mathcal{B}$ have relative density α, let $\varepsilon > 0$ and suppose that $B_1, B_2, B_3 \subset \mathcal{B}_\rho$ where $\rho \leq c\alpha\varepsilon/d$. Then either*

(1) ($\mathcal{A}$ has almost full density on B_1, B_2 and B_3) there is an $x \in \mathcal{B}$ such that

$$\mathbf{1}_{\mathcal{A}} * \mu_{B_i}(x) \geq (1-\varepsilon)\alpha$$

for $i = 1, 2, 3$, or

(2) (density increment) $\mathcal{A}$ has a density increment of strength $[\varepsilon, 0; O(1)]$ relative to one of the B_i's.

Proposition 4.4 is the adaptation of [2, Proposition 5.5] to general coefficients.

Proposition 4.4. *There is a constant $C > 0$ such that, for all $k \geq C$, the following holds. Let $\mathcal{B}$ be a regular Bohr set of rank d and suppose that $\mathcal{A} \subset \mathcal{B}$ has density α. Let $\mathcal{B}_1 := \mathcal{B} \cap T_2\mathcal{B} \cap T_3\mathcal{B}$. Either*

(1) $\alpha \geq 2^{-O(k^2)}$,

(2)

$$T(\mathcal{A}, \mathcal{A}, \mathcal{A}) \gg \exp(-\tilde{O}_\alpha(d \log 2d))\mu(\mathcal{B}_1)^2,$$

or

(3) $\mathcal{A}$ has a density increment of one of the following strengths relative to $\mathcal{B}_1$, $\mathcal{B}_2 := T_2^{-1}\mathcal{B}_1$ or $\mathcal{B}_3 := T_3^{-1}\mathcal{B}_1$:

(a) (small increment) $[\alpha^{O(\varepsilon(k))}, \alpha^{-O(\varepsilon(k))}; \tilde{O}_\alpha(1)]$, or

(b) (large increment) $[\alpha^{-1/k}, \alpha^{-1+1/k}; \tilde{O}_\alpha(1)]$,

where $\varepsilon(k) = \frac{\log\log\log k}{\log\log k}$.

Proof. Let $\varepsilon = c_0\alpha^{C_0\frac{\log\log\log k}{\log\log k}}$, for some small constant $0 < c_0 \leq \frac{1}{3}$ and some large constant $C_0 > 0$. We apply Lemma 4.3 with

$$B_1 = (\mathcal{B} \cap T_2\mathcal{B} \cap T_3\mathcal{B})_\rho, \quad B_2 = T_2^{-1}B_1, \quad \text{and} \quad B_3 = (T_3^{-1}B_1)_{\rho'},$$

where $\rho = c\alpha\varepsilon/d$ and $\rho' = c'\alpha^2/d$ (c and c' being small constants, chosen in particular such that B_1 and B_3 are regular.[5] If we are in the second case of Lemma 4.3, then $\mathcal{A}$ has a small increment with respect to B_1, B_2 or B_3. By [2, Lemma 5.2], this translates into a density increment of the same strength with respect to $\mathcal{B}_1$, $\mathcal{B}_2$ or $\mathcal{B}_3$, as required.

Let us assume henceforth that we are in the first case of Lemma 4.3. Let

$$A_1 = (\mathcal{A} - x) \cap B_1, \quad A_2 = (\mathcal{A} - x) \cap B_2 \quad \text{and} \quad A_3 = (\mathcal{A} - x) \cap B_3.$$

If α_i is the density of A_i relative to B_i, for $1 \leq i \leq 3$, then Lemma 4.3 ensures that

$$\alpha_i \in [(1-\varepsilon)\alpha, (1+\varepsilon)\alpha].$$

We now apply Proposition 4.2 with $B = B_1$, $B' = B_3$, $h = \lceil c_1 \log\log k / \log\log\log k \rceil$ and $t = \lceil C_2 \log k \rceil$, for some suitable constants $c_1, C_2 > 0$.

(1) In the first case of the conclusion of Proposition 4.2, $\alpha \gg 1/k^2 \geq 2^{-O(k^2)}$.

(2) In the second case,

$$T(A_1, A_2, A_3) \gg \alpha^3\mu(B_1)\mu(B_3) \gg \exp(-\tilde{O}_\alpha(d \log 2d))\mu(\mathcal{B} \cap T_2\mathcal{B} \cap T_3\mathcal{B})^2$$

Since the A_i's are subsets of the same translate of $\mathcal{A}$ and the equation $a_1 + T_2a_2 + T_3a_3 = 0$ is translation-invariant, any lower bound for $T(A_1, A_2, A_3)$ is a lower bound for $T(\mathcal{A}, \mathcal{A}, \mathcal{A})$.

[5] Note that the regularity of B_2 follows immediately from that of B_1.

(3) In the third case, either $A_1 \subset B_1$ or $T_2A_2 \subset B_1$ has a density increment of strength
$$[1, \alpha^{-1/k}; \tilde{O}_\alpha(h \log t)] \quad \text{or} \quad [\alpha^{-1/k}, \alpha^{-1+1/k}; \tilde{O}_\alpha(h \log t)]$$
with respect to $T_3B_3 = (B_1)_{\rho'}$. Note that $h \log t = \tilde{O}_\alpha(1)$, or else we have the first case of the conclusion. Observe also that, for $i = 1, 2$,
$$(1 + \tilde{O}_\alpha(1))\alpha_i \geq (1 + \tilde{O}_\alpha(1))\alpha \quad \text{and} \quad (1 + \tilde{O}_\alpha(\alpha^{-1/k}))\alpha_i \geq (1 + \tilde{O}_\alpha(\alpha^{-1/k}))\alpha$$
by our choice of ε. This implies that $\mathcal{A}$ has a density increment of strength $[1, \alpha^{-1/k}; \tilde{O}_\alpha(1)]$ or $[\alpha^{-1/k}, \alpha^{-1+1/k}; \tilde{O}_\alpha(1)]$ relative to either $(B_1)_{\rho'}$ or $T_2^{-1}(B_1)_{\rho'} = (B_2)_{\rho'}$. By [2, Lemma 5.2], this implies that $\mathcal{A}$ has an increment of the same strength relative to $\mathcal{B}_1$ or $\mathcal{B}_2$.

(4) Finally, suppose that the last case of the conclusion of Proposition 4.2 holds. Then we may apply [2, Proposition 11.8] with $B = B_1$, $B' = (T_3B_3)_{\rho_{\text{top}}}$ and $B'' = (T_3B_3)_{\rho_{\text{top}}\rho'}$. The hypotheses of [2, Proposition 11.8] exactly match the last case of Proposition 4.2 for some $K = \alpha^{-O(1/k)}$.

The number M in [2, Proposition 11.8] satisfies $M = \alpha^{-O(\varepsilon(k))}$, or else $\alpha \geq 2^{-O(k^2)}$ and we are in the first case of our conclusion. Taking C large enough in the statement of Proposition 4.4, we see that the first case of [2, Proposition 11.8] cannot hold. In the other two cases, either $A_1 \subset B_1$ or $T_2A_2 \subset B_1$ has a density increment of strength
$$[\alpha^{O(\varepsilon(k))}, \alpha^{-O(\varepsilon(k))}; \tilde{O}_\alpha(1)] \quad \text{or} \quad [\alpha^{-1/k}, \alpha^{-1+1/k}; \tilde{O}_\alpha(1)]$$
with respect to B''. As in the previous case, we conclude that $\mathcal{A}$ has a density increment of the same strength with respect to $\mathcal{B}_1$ or $\mathcal{B}_2$ after noting that
$$(1 + \tilde{O}_\alpha(\alpha^{O(\varepsilon(k))}))\alpha_i \geq (1 + \tilde{O}_\alpha(\alpha^{O(\varepsilon(k))}))(1-\varepsilon)\alpha \geq (1 + \tilde{O}_\alpha(\alpha^{O(\varepsilon(k))}))\alpha,$$
which holds because we chose $\varepsilon = c_0\alpha^{C_0 \frac{\log\log\log k}{\log\log k}}$. □

The prove Theorem 1.1, we will need the following weaker bound as a black box. It is the same statement as [2, Theorem 5.4], but with our different definition of $T(A, A, A)$. This bound can be deduced from [1, Theorem 2.3].

Theorem 4.5. *Let B be a regular Bohr set of rank d and suppose that $A \subset B$ has density α. Then*
$$T(A, A, A) \gg \exp\left(-\tilde{O}_\alpha(d + \alpha^{-1}) \log 2d\right) \mu(B)^2.$$

We are now ready to prove Theorem 1.1. The proof consists in iterating Proposition 4.4.

Proof of Theorem 1.1. Let $A \subset G$ of density α. In this proof, α will always denote the density of this initial A.

Let $1 \leq C_1 = O(1)$ be some fixed constant, chosen in particular larger than the implied constants in the exponents of the small increment case of Proposition 4.4. Let k be some constant large enough such that Proposition 4.4 holds and
$$C_1\varepsilon(k) \leq \frac{1}{20}.$$
Let $1 \leq C_2 = \tilde{O}_\alpha(1)$ be some quantity depending only on α, chosen in particular larger than the implicit constants of Proposition 4.4 hidden in the $\gg$, $O(\cdot)$ and $\tilde{O}_\alpha(1)$ notation. Note that we may assume $\alpha \leq 1/(2C_2^2)$, or else we are done by an application of Theorem 4.5 with $B = G$. We may similarly suppose that $\alpha \leq 2^{-C_2k^2}$ and $\log(1/\alpha) \leq \alpha^{-C_1\varepsilon(k)}$.

By induction, we construct sequences $(A^{(i)})_{0\leq i\leq l}$, $(B^{(i)})_{0\leq i\leq l}$, $(\alpha^{(i)})_{0\leq i\leq l}$, $(d^{(i)})_{0\leq i\leq l}$, $(D^{(i)})_{0\leq i\leq l-1}$, $(T^{(i)})_{1\leq i\leq l}$, $(\rho^{(i)})_{1\leq i\leq l}$ and $(\widetilde{B}^{(i)})_{1\leq i\leq l}$, such that

- for $0 \leq i \leq l$, $B^{(i)}$ is a regular Bohr set of rank $d^{(i)}$;
- for $0 \leq i \leq l$, $A^{(i)}$ is a subset of $B^{(i)}$ of relative density $\alpha^{(i)}$;

- for $1 \leq i \leq l$, $A^{(i)}$ is a subset of a translate of $A^{(i-1)}$;
- for $0 \leq i \leq l$,

$$D^{(i)} = B^{(i)} \cap T_2 B^{(i)} \cap T_3 B^{(i)}; \tag{6}$$

- for $1 \leq i \leq l$,

$$B^{(i)} = \left(T^{(i)} D^{(i-1)}\right)_{\rho^{(i)}} \cap \widetilde{B}^{(i)}, \tag{7}$$

where $T^{(i)} \in \{\mathrm{Id}_G, T_2^{-1}, T_3^{-1}\}$;
- for $1 \leq i \leq l$,

$$\alpha^{(i)} \geq \left(1 + C_2^{-1} \alpha^{C_1 \varepsilon(k)}\right) \alpha^{(i-1)};$$

- for $1 \leq i \leq l$, $\widetilde{B}^{(i)} = \mathrm{Bohr}(\widetilde{\Gamma}^{(i)}, \widetilde{\nu}^{(i)})$ where $|\widetilde{\Gamma}^{(i)}| \leq C_2 \alpha^{-C_1 \varepsilon(k)}$ and

$$\frac{\rho^{(i)}}{4} \prod_{\gamma \in \widetilde{\Gamma}^{(i)}} \frac{\widetilde{\nu}^{(i)}(\gamma)}{4} \geq \exp\left(-C_2\, 3d^{(i-1)} \log(3d^{(i-1)})\, \alpha^{-2C_1 \varepsilon(k)}\right). \tag{8}$$

Inductive construction. We let $A^{(0)} = A$ and $B^{(0)} = G$, which we regard as a Bohr set of rank $d^{(0)} = 1$. Also, let $\alpha^{(0)} = \alpha$ and $D^{(0)} = B^{(0)} \cap T_2 B^{(0)} \cap T_3 B^{(0)}$. Assume that we have constructed $(A^{(i)})_{0 \leq i \leq n}$, $(B^{(i)})_{0 \leq i \leq n}$, $(\alpha^{(i)})_{0 \leq i \leq n}$, $(d^{(i)})_{0 \leq i \leq n}$, $(D^{(i)})_{0 \leq i \leq n}$, $(T^{(i)})_{1 \leq i \leq n}$, $(\rho^{(i)})_{1 \leq i \leq n}$ and $(\widetilde{B}^{(i)})_{1 \leq i \leq n}$ for some $n \geq 0$.

We apply Proposition 4.4 to the sets $\mathcal{A} = A^{(n)}$ and $\mathcal{B} = B^{(n)}$. If case (3)(a) does not occur, then we stop the construction of the sequences, so that $l = n$. Otherwise, we are in the case (3)(a) and we have a small increment for $A^{(n)}$. By Proposition 4.4, for some $T^{(n+1)} \in \{\mathrm{Id}_G, T_2^{-1}, T_3^{-1}\}$, we know that $A^{(n)}$ has a density increment of strength $[\alpha^{C_1 \varepsilon(k)}, \alpha^{-C_1 \varepsilon(k)}; C_2]$ relative to $T^{(n+1)} D^{(n)}$. By Definition 3.3, this means that there is a regular Bohr set $B^{(n+1)}$ of the form

$$B^{(n+1)} = \left(T^{(n+1)} D^{(n)}\right)_{\rho^{(n+1)}} \cap \widetilde{B}^{(n+1)},$$

for some $\rho^{(n+1)}$ and some $\widetilde{B}^{(n+1)} = \mathrm{Bohr}(\widetilde{\Gamma}^{(n+1)}, \nu^{(n+1)})$ where $|\widetilde{\Gamma}^{(n+1)}| \leq C_2 \alpha^{-C_1 \varepsilon(k)}$ and

$$\left(\frac{\rho^{(n+1)}}{4}\right)^{\mathrm{rk}(D^{(n)})} \prod_{\gamma \in \widetilde{\Gamma}^{(n+1)}} \frac{\widetilde{\nu}^{(n+1)}(\gamma)}{4} \geq \left(2\, \mathrm{rk}(D^{(n)})(\alpha^{-C_1 \varepsilon(k)} + 1)\right)^{-C_2(\mathrm{rk}(D^{(n)}) + \alpha^{-C_1 \varepsilon(k)})}.$$

Since $\mathrm{rk}(D^{(n)}) \leq 3d^{(n)}$, this implies the inequality

$$\begin{aligned} \frac{\rho^{(n+1)}}{4} \prod_{\gamma \in \widetilde{\Gamma}^{(n+1)}} \frac{\widetilde{\nu}^{(n+1)}(\gamma)}{4} &\geq \left(12 d^{(n)} \alpha^{-C_1 \varepsilon(k)}\right)^{-C_2(3d^{(n)} + \alpha^{-C_1 \varepsilon(k)})} \\ &\geq \exp\left(-C_2\, 3d^{(n)} \log(3d^{(n)})\, \alpha^{-2C_1 \varepsilon(k)}\right), \end{aligned}$$

as required. Finally, let $D^{(n+1)} = B^{(n+1)} \cap T_2 B^{(n+1)} \cap T_3 B^{(n+1)}$ and $\alpha^{(n+1)}$ be the relative density of $A^{(n+1)}$ in $B^{(n+1)}$.

Upper bound for l. Since $\alpha^{(l)}$ is a relative density, and thus cannot exceed 1, we have the inequality

$$1 \geq \alpha^{(l)} \geq \left(1 + C_2^{-1} \alpha^{C_1 \varepsilon(k)}\right)^l \alpha,$$

from which we deduce that

$$l = \tilde{O}_\alpha(\alpha^{-2C_1 \varepsilon(k)}).$$

Upper bound for $d^{(l)}$. Iterating the naive bound

$$\mathrm{rk}(B^{(i)}) \leq \mathrm{rk}(D^{(i)}) + \mathrm{rk}(\widetilde{B}^{(i)}) \leq 3\,\mathrm{rk}(B^{(i-1)}) + \mathrm{rk}(\widetilde{B}^{(i)})$$

does not give an acceptable upper bound for $d^{(l)}$. The reason why this bound is too weak is that the frequency sets of the Bohr sets $B^{(i-1)}$, $T_2B^{(i-1)}$ and $T_3B^{(i-1)}$ are largely intersecting.

Let W_i be the set of all automorphisms obtained by composing i elements of $\{\mathrm{Id}_G, T_2, T_3, T_2^{-1}, T_3^{-1}\}$. Since $T_3 = -\mathrm{Id}_G - T_2$, these automorphisms commute, which implies that $|W_i| \leq (2i+1)^2$.

If we write $\widetilde{\Gamma}^{(0)} = \{0\}$, then an immediate induction using Eqs. (6) and (7) implies that the frequency set of $B^{(i)}$ is contained in

$$\bigcup_{T\in W_i} \bigcup_{0\leq j\leq i} T\big(\widetilde{\Gamma}^{(j)}\big).$$

Therefore,

$$d^{(i)} \leq d^{(l)} \leq (2l+1)^2 \cdot (l+1) \cdot C_2\alpha^{-C_1\varepsilon(k)} = \tilde{O}_\alpha(\alpha^{-7C_1\varepsilon(k)})$$

for $1 \leq i \leq l$.

Lower bound for $\mu(B^{(l)})$ and $\mu(D^{(l)})$. Using our bound for $d^{(i)}$, Eq. (8) becomes

$$\frac{\rho^{(i)}}{4} \prod_{\gamma\in\widetilde{\Gamma}^{(i)}} \frac{\widetilde{\nu}^{(i)}(\gamma)}{4} \geq \exp\left(-\tilde{O}_\alpha(\alpha^{-10C_1\varepsilon(k)})\right).$$

To bound the size of $D^{(l)}$, we observe the inclusion

$$D^{(l)} \supset \left(\bigcap_{T\in W_{l+1}} \bigcap_{0\leq i\leq l} \widetilde{B}^{(i)} \right)_{\rho^{(1)}\rho^{(2)}\dots\rho^{(l)}},$$

which is easily proven by induction using Eqs. (6) and (7). Here we wrote $\widetilde{B}^{(0)}$ for $B^{(0)}$. The size of the right-hand side can be bounded using [2, Lemma 4.4]. We obtain

$$\begin{aligned}
\mu(D^{(l)}) &\geq \prod_{1\leq i\leq l} \left(\frac{\rho^{(i)}}{4}\right)^{(2l+3)^2\cdot(l+1)\cdot C_2\alpha^{-C_1\varepsilon(k)}} \frac{1}{4}\left(\prod_{1\leq i\leq l} \prod_{\gamma\in\widetilde{\Gamma}^{(i)}} \frac{\widetilde{\nu}^{(i)}(\gamma)}{4} \right)^{(2l+3)^2} \\
&\geq \frac{1}{4} \prod_{1\leq i\leq l} \left(\frac{\rho^{(i)}}{4} \prod_{\gamma\in\widetilde{\Gamma}^{(i)}} \frac{\widetilde{\nu}^{(i)}(\gamma)}{4} \right)^{\tilde{O}_\alpha(\alpha^{-7C_1\varepsilon(k)})} \\
&\geq \exp\left(-\tilde{O}_\alpha(\alpha^{-19C_1\varepsilon(k)})\right).
\end{aligned}$$

In particular, $\mu(B^{(l)}) \geq \exp\left(-\tilde{O}_\alpha(\alpha^{-19C_1\varepsilon(k)})\right)$ as well.

Concluding the proof. We now apply Proposition 4.4 to $\mathcal{A} = A^{(l)}$ and $\mathcal{B} = B^{(l)}$. The small increment case cannot occur, by construction of the sequence $(A^{(i)})$.

- If we are in the first case of the conclusion of Proposition 4.4, then $\alpha^{(l)} \geq 2^{-C_2k^2}$. In this case we apply Theorem 4.5 for the bound

$$T(A,A,A) \geq T(A^{(l)},A^{(l)},A^{(l)}) \gg \exp\left(-\tilde{O}_\alpha\left(\alpha^{-7C_1\varepsilon(k)} + 2^{C_2k^2}\right)\right)\mu(B^{(l)})^2.$$

- In the second case, we directly obtain

$$T(A,A,A) \geq T(A^{(l)},A^{(l)},A^{(l)}) \gg \exp\left(-\widetilde{O}_\alpha\left(\alpha^{-7C_1\varepsilon(k)}\right)\right)\mu(D^{(l)})^2.$$

- Finally, in the large increment case, there are some $\rho > 0$, $T'' \in \{\mathrm{Id}_G, T_2^{-1}, T_3^{-1}\}$ and $\widetilde{B} = \mathrm{Bohr}(\widetilde{\Gamma}, \widetilde{\nu})$ such that $|\widetilde{\Gamma}| \leq C_2 \alpha^{-1+1/k}$ and, if
$$B'' := (T'' D^{(l)})_\rho \cap \widetilde{B},$$
then $\|\mathbf{1}_A * \mu_{B''}\| \gtrsim_\alpha \alpha^{1-1/k}$ and
$$\left(\frac{\rho}{4}\right)^{\mathrm{rk}(T'' D^{(l)})} \prod_{\gamma \in \widetilde{\Gamma}} \frac{\widetilde{\nu}(\gamma)}{4} \geq \exp\left(-\tilde{O}_\alpha\left(\alpha^{-7C_1\varepsilon(k)} + \alpha^{-1+1/k}\right)\right). \tag{9}$$
Equation (9) implies, by [2, Lemma 4.4], that
$$\mu(B'') \geq \exp\left(-\tilde{O}_\alpha\left(\alpha^{-7C_1\varepsilon(k)} + \alpha^{-1+1/k}\right)\right) \mu(D^{(l)}).$$
We now apply Theorem 4.5 with a suitable subset of a translate of A and the Bohr set B'' to find that

$$T(A,A,A) \geq \exp\left(-\tilde{O}_\alpha\left(\alpha^{-7C_1\varepsilon(k)} + \alpha^{-1+1/k}\right)\right) \exp\left(-\tilde{O}_\alpha\left(\alpha^{-7C_1\varepsilon(k)} + \alpha^{-1+1/k}\right)\right) \mu(D^{(l)})^2$$

Since
$$\mu(B^{(l)}) \geq \mu(D^{(l)}) \geq \exp\left(-\tilde{O}_\alpha\left(\alpha^{-19C_1\varepsilon(k)}\right)\right),$$
we obtain, in all three cases, the lower bound
$$T(A,A,A) \geq \exp\left(-\tilde{O}_\alpha\left(\alpha^{-19C_1\varepsilon(k)} + \alpha^{-1+1/k} + 2^{C_2k^2}\right)\right).$$
Choosing $c = 1/(2k)$, say, we obtain
$$T(A,A,A) \geq \exp\left(-O(\alpha^{-1+c})\right).$$
On the other hand, since A contains only trivial solutions to $a_1 + T_2a_2 + T_3a_3 = 0$, we have
$$T(A,A,A) = \frac{\alpha}{|G|} \leq \frac{1}{|G|}.$$
Therefore, $|G| \leq \exp\left(O(\alpha^{-1+c})\right)$, which can be rewritten as
$$|A| \ll \frac{|G|}{(\log|G|)^{1+c'}},$$
where $c' = \frac{1}{1-c} - 1$. This finishes the proof of Theorem 1.1. □

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