Memoire: On the Local Langlands Correspondence of $GL(2)$

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Contents

Introduction

Let F be a nonarchimedean local field, that is to say F is either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_p((t))$. The local class field theory gives us an isomorphism $Art_F : F^\times \longrightarrow W_F^{ab}$. Here W_F is the Weil group of F and W_F^{ab} is its abelianization. The Artin map in the local class field theory will induce a one-to-one correspondence between characters of W_F and characters of $F^\times$. The Langlands correspondence is a higher generalization of class field theory. More precisely in local case, let $\mathcal{G}_n(F)$ be the set of equivalent classes of n-dimensional semisimple Weil-Deligne representations of W_F over $\mathbb{C}$, and $\mathcal{A}_n(F)$ be the set of isomorphism classes of irreducible smooth representations of $GL_n(F)$ over $\mathbb{C}$. Then the local Langlands correspondence gives a bijection between $\mathcal{G}_n(F)$ and $\mathcal{A}_n(F)$. This bijection is characterized by L-factors and ε-factors:

Theorem 0.1. *There is a unique collection of bijections*

$$\sigma_n : \mathcal{A}_n(F) \longrightarrow \mathcal{G}_n(F)$$

satisfying:
(1) When $n = 1$, σ_1 is given by local class field theory.
(2) For any $\pi_1 \in \mathcal{A}_{n_1}(F)$ and $\pi_2 \in \mathcal{A}_{n_2}(F)$, we have

$$L(\pi_1 \times \pi_2, s) = L(\sigma(\pi_1 \otimes \sigma(\pi_2)), s), \qquad \varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\sigma(\pi_1 \otimes \sigma(\pi_2)), s, \psi)$$

(3) For any $\pi \in \mathcal{A}_n(F)$, $\sigma(\pi)^\vee = \sigma(\pi)^\vee$.
(4) If $\pi \in \mathcal{A}_n(F)$ with central quasicharacter ω_π, then $\det \sigma(\pi) = \sigma(\omega_\pi)$.
(5) For any $\pi \in \mathcal{A}_n(F)$ and $\chi \in \mathcal{A}_1(F)$, $\sigma(\pi \otimes (\chi \circ \det)) = \sigma(\pi \otimes \chi)$.

This conclusion has been proved by Harris-Taylor and Henniart using global methods.

In this memoire, we study the local Langlands correspondence of GL_2 mainly following [BH06]. The construction of the correspondence is via theta corrspondence. This proof is purely local, in contrast to Harris-Taylor's and Henniart's proofs of general cases of GL_n.

The organization of this memoire is as follows. In Chapter 1, we give the definitions of Weil-Deligne representations and their L-factors. We also prove the existence of local ε-factors d'après Deligne. In Chapter 2, we study the irreducible smooth representations of $GL_2(F)$, and we define Jacquet-Langlands L-factors. Finally in Chapter 3, we state the local Langlands correspondence of GL_2 precisely, and give the constructions of the correspondence for most of cases, namely so-called the tame Langlands correspondence.

Chapter 1

The Galois Side

Throughout this chapter, F is a nonarchimedean local field of characteristic zero with residue field $k_F = \mathbb{F}_q$, $q = p^d$. That is to say, F is a finite extension of $\mathbb{Q}_p$. π_F is a prime element in O_F, $\mathfrak{p}^n$ be the ideal $\pi_F O_F$, and $U_F^n = 1 + \pi_F^n O_F$.

1.1 The Weil group of Nonarchimedean Local Fields

We first recall the structure of $G_F = Gal(\bar{F}/F)$ and indicate how to construct the Weil group of F. Recall that G_F is defined to be $\varprojlim_K Gal(K/F)$ where the inverse limit is taken over all finite extension K/F, and the topology on G_F is the subspace topology of the product topology on $\prod_K Gal(K/F)$. An algebraic extension E/F is called unramified if πO_E is a prime ideal in O_E. Every unramified extension E/F will give an algebraic extension of residue fields k_E/k_F. It is a fundamental result in local field theory that the set of unramified extensions E/F is bijective to the set of algebraic extensions over $\mathbb{F}_q$. Then we take the maximal unramified extension F^{ur}/F corresponding to the $\overline{\mathbb{F}_q}/\mathbb{F}_q$ on the residue field. Recall that $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ and let the inertia group $I_F = Gal(\bar{F}/F^{ur})$. Then we have the following short exact sequence:

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \hat{\mathbb{Z}} \longrightarrow 1$$

We consider an element $\phi \in G_{k_F}$ defined by $\phi(x) = x^q$. ϕ is called the Frobenius element of k_F . Let $\Phi \in G_F$ be a preimage of ϕ. Now we define the Weil group of F by $W_F := \cup_{n\in\mathbb{Z}} I_F \Phi^n$, and we define the topology on W_F as follows. As a set $W_F = I_F \times \mathbb{Z}$, we equip I_F with the subspace topology of G_F, and $\mathbb{Z}$ with the discrete topology. And then we equip W_F with the product topology. We can deduce the following short exact sequence immediately:

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \mathbb{Z} \longrightarrow 1$$

We use $\varphi_F : W_F \hookrightarrow G_F$ to denote the natural inclusion map, and v_F to denote the map $W_F \longrightarrow \mathbb{Z}$. We define norm $|\cdot|$ on W_F by $|g| = q^{-v_F(g)}$ and the character $\chi_F : W_F \longrightarrow \mathbb{C}^\times$ defined by $\chi_F(g) = |g|$.

We look at the inertia group I_F in more detail. We define $E_n = F^{ur}(\sqrt[n]{\pi_F})$ for any $p \nmid n$,, and the maximal tamely ramified extension of F by $F^{tame} = \cup_{p\nmid n} E_n$. Then we have an isomorphism $t : Gal(F^{tame}/F^{ur}) \xrightarrow{\cong} \prod_{\ell\neq p} \mathbb{Z}_\ell$ given by

$$g(\sqrt[n]{\pi_F}) = \zeta_n^{t(g)} \sqrt[n]{\pi_F}$$

where ζ_n is a primitive n-th root of unity.

We denote the wild inertia group $Gal(\bar{F}/F^{tame})$ by P_F. So we have the following short exact sequence:

$$1 \longrightarrow P_F \longrightarrow I_F \longrightarrow \prod_{\ell \neq p} \mathbb{Z}_\ell \longrightarrow 1$$

Lemma 1.1. *For any $x \in I_F$, and $g \in W_F$, we have*

$$t(gxg^{-1}) = ||g||t(x)$$

The structure of P_F is more complicated. It is a profinite p-group. We do not involve the higher ramification theory here.

Now we recall the local class field theory in terms of Weil groups:

Theorem 1.1 (Local Class Field Theory)**.** *There exists a canonical (topological) group homomorphim*

$$Art_F : F^\times \longrightarrow W_F^{ab}$$

satisfying:
(1) Art_F induces an isomorphism between $F^\times$ and W_F^{ab}.
(2) Let π_F be the prime element in O_F, then we can choose Frobenieus element Φ such that $Art_F(\pi_F) = \Phi^{ab}$, where Φ^{ab} is the image of Φ in W_F^{ab}, and $Art_F(O_F^\times) = I_F^{ab}$.
(3) Suppose E/F is a finite extension, then we have the compatibility

$$\begin{array}{ccc} E^\times & \xrightarrow{Art_E} & W_E^{ab} \\ \downarrow{\scriptstyle Nm} & & \downarrow \\ F^\times & \xrightarrow{Art_F} & W_F^{ab} \end{array}$$

where the first verticle arrow is the norm map and the second verticle arrow is the natural inclusion map.
(4) If $\tau \in Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, then $Art_{\tau K}(\tau x) = \tau Art_K(x)\tau^{-1}$.

Now suppose G is a group and H is a subgroup of finite index in G. Let $G = \cup_{1\leq k\leq n} Hg_k$. For any $g \in G$, we set $\varphi(g) = g_i$ if $g \in Hg_i$.

Definition 1.1. *We define the transfer map $V : G^{ab} \longrightarrow H^{ab}$ by*

$$V(g) = \prod_{1\leq i\leq n} g_i g \varphi(g_i g)^{-1} \mod H^{der}$$

We can check that this map is well-defined and a group homomorphism.

Proposition 1.1. *Let E/F is a finite extension of nonarchimedean local field, then we have the following commutative diagram:*

$$\begin{array}{ccc} E^\times & \xrightarrow{Art_E} & W_E^{ab} \\ \uparrow & & \uparrow{\scriptstyle V} \\ F^\times & \xrightarrow{Art_F} & W_F^{ab} \end{array}$$

where the first verticle arrow upwards is the natural inclusion map and the second verticle arrow upwards is the transfer map $V : W_F^{ab} \longrightarrow W_E^{ab}$.

1.2 Weil-Deligne Representation

We want to study the representations of Weil groups.

Definition 1.2. *A representation V of a locally profinite group G is called smooth representation if for any $v \in V$, v is fixed by a compact open subgroup K in G.*

Remark 1.1. *The definition of smooth representation is independent of the topology of the base field of V.*

Proposition 1.2. *Suppose (ρ, V) is an irreducible smooth representation of W_F, then*
(1) There exists $d \in \mathbb{N}^+$ and $\lambda \in \mathbb{C}$ such that $\rho(\Phi^d) = \lambda Id$. Hence, there exists $s \in \mathbb{C}$ such that the image of $\rho \otimes \chi_F^s(W_F)$ is finite.
(2) The image of $\rho(W_F)$ is finite if and only if ρ factors through $W_F \overset{\varphi_F}{\hookrightarrow} G_F \overset{\widetilde{\rho}}{\longrightarrow} GL(V)$ where $\widetilde{\rho}$ is a smooth Galois representation.

Proof. (1) Since ρ is a smooth representation, there exists an open subgroup I_0 of finite index in I_F such that $\rho|_{I_0}$ is trivial. Then we just take $H = W_F/I_0$ and $F = I_F/I_0$. $H = F \ltimes \mathbb{Z}$ and F is normal in H. Then we take a group isomorphism $\varphi_i : F \to F$ by $\varphi_i(x) = \overline{\Phi}^i x \overline{\Phi}^{-i}$. Since F is a finite group, there exists $d \in \mathbb{N}$ such that $\varphi_d = Id_F$. Then $\overline{\Phi}^d$ is in the center of F. By the Schur's lemma for locally profinite groups, $\rho(\overline{\Phi}^d) = \lambda Id_V$ for some $\lambda \in \mathbb{C}$. Then we take $s = \log \lambda / d \log q$. Hence the image of $\rho \otimes \chi_F^s$ is generated by $\rho \otimes \chi_F^s(I_F)$ and $\rho \otimes \chi_F^s(\Phi)$ of order d hence finite.
(2) "If" part is clear since G_F is a profinite group. We prove the "only if" part. Given such a representation of W_F, then there exists $d \in \mathbb{N}$ such that $\rho(\Phi^d) = 1$. Now for $x = a\Phi^u \in G_F$ where $a \in I_F$ and $u \in \hat{\mathbb{Z}} \cong \prod_\ell \mathbb{Z}_\ell$, we take $\widetilde{\rho}(x) = \rho(a)\rho(\Phi^{\overline{u}})$ where $\overline{u}$ is the image of the quotient map $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}/d\hat{\mathbb{Z}}$. It's easy to verify that $\widetilde{\rho}$ is a group homomorphism extending ρ. □

Now we consider the complex representations.

Lemma 1.2. *Suppose ρ is a finite dimensional complex representation of W_F, then ρ is smooth if and only if ρ is continuous (under the usual topology of complex plane).*

Proof. The "only if" part is clear, we now prove the "if" part. This is a standard "no small subgroups" argument. For $GL_n(\mathbb{C})$, we can choose a enough small neighborhood N of the identity of $GL_n(\mathbb{C})$ such that N does not contain any subgroup expect $\{1\}$. Then we consider $\rho^{-1}(N)$ which is an open neighborhood of the identity in W_F. $\rho^{-1}(N)$ contains an open subgroup I_0 of I_F, and $\rho(I_0)$ is a subgroup contained in N. Thus, $\rho(I_0)$ can only equals 1 and hence ρ is smooth. □

Now we consider continuous ℓ-adic representations of W_F. We give examples to show that, in contrast to the complex case, continuous ℓ-adic representations are not neccessarily smooth.

Example 1.1. *We consider E an elliptic curve over F, and ℓ a prime not equal to p. For any $k \in \mathbb{N}$, let $E[k] = \{x \in E(\bar{F}) | kx = 0\}$. When $\ell \neq p$, $E[\ell^n] \cong \mathbb{Z}/\ell^n\mathbb{Z}$. Then we define $T_\ell(E) = \varprojlim_n E[\ell^n]$ and $T_\ell(E) \cong \mathbb{Z}_\ell^2$. We consider the Galois action on the Tate module $T_\ell(E)$, and we will get a 2-dimensional continuous ℓ-adic representations $\rho_{E,\ell}$ over the space $T_\ell(E) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$. Now suppose E has potential multiplicative reduction over F. This means that there exists a finite extension F'/F, such that the reduction $\bar{E}$ to the residue field $F'/O_{F'}$ has a node. Then we have the following famous conclusion due to Tate.*

Theorem 1.2. *(1) For any $q \in F^\times$ with $|q| < 1$, there is an elliptic curve E_q over F, such that there exists a topological group isomorphism $\phi : \bar{F}^\times / q^{\mathbb{Z}} \longrightarrow E(\bar{K})$ and ϕ is compatible with the action of G_F.*
(2) When E has split multiplicative reduction type over F, then there exists a unique $q \in F^\times$ with $|q| < 1$, such that E is isomorphic to E_q over K.

Then we can in fact get

Proposition 1.3. *When E has potential multiplicative reduction over F, $\rho_{E,\ell}$ is always not smooth.*

For simplicity. we only give a proof when E has split multiplicative reduction type. In fact, we choose a basis of $T_\ell(E)$ as follows. We take

$$e_0 = (\zeta_{\ell^1}, \zeta_{\ell^2}, \zeta_{\ell^3}, \cdots, \zeta_{\ell^n}, \cdots)$$

where ζ_{ℓ^i} are primitive ℓ^i-th roots of unity satisfying $\zeta_{\ell^{i+1}}^\ell = \zeta_{\ell^i}$, and

$$e_1 = (q_1, q_2, \cdots, q_n, \cdots)$$

satisfying $q_n^{\ell^n} = q$ and $q_{n+1}^\ell = q_n$ where $E \cong E_q$ over K. Now since $|q| < 1$, $q_n \notin K^{ur}$ and $q_n \notin K^{ur}(q_{n-1})$ for any $n \in \mathbb{Z}$. Hence $K^{ur}(q_\infty) := \bigcup_i K^{ur}(q_i)$ is an infinite field extension over K^{ur}, and easy to observe that $Gal(K^{ur}(q_\infty)/K^{ur})$ acts faithfully on e_1. Thus, the image of the Galois representation is infinite.

Now we want to study and classify all the continuous representation of W_F over $\overline{\mathbb{Q}}_\ell$. Recall that there is a surjective morphism $t : I_F \to \prod_{r \neq p} \mathbb{Z}_r$. We denote the natural projeciton map $p_\ell : \prod_{r \neq p} \mathbb{Z}_r \longrightarrow \mathbb{Z}_\ell$, and the composition $t_\ell = p_\ell \circ t$.

Theorem 1.3 (Grothendieck's ℓ-adic monodromy)**.** *Let (ρ, V) be a finite dimensional continuous representation of W_F over $\overline{\mathbb{Q}}_\ell$ for $\ell \neq p$. Then there exists an open subgroup I_0 of I_F and a unique nilpotent element $N_\rho \in End_{\overline{\mathbb{Q}}_\ell}(V)$, such that for any $x \in I_0$*

$$\rho(x) = \exp(t_\ell(x) N_\rho)$$

Proof. The uniqueness of N_ρ is obvious. We prove the existence part. We consider the open subgroup $K_m = 1 + \ell^m M_n(\overline{\mathbb{Z}}_\ell)$ of $GL_n(\overline{\mathbb{Q}}_\ell)$. Then $K_m/K_{m+1} \cong M_n(\overline{\mathbb{Z}}_\ell)/\ell M_n(\overline{\mathbb{Z}}_\ell)$ when $m \geq 1$, hence K_m is a profinite ℓ-group when $m \geq 1$. Let $J_m = \rho^{-1}(K_m) \subset W_F$. Since P_F is a profinite p-group and $\ell \neq p$, $\ker t_\ell / P_F \cong \prod_{r \neq \ell, p} \mathbb{Z}_r$, $J'_m := J_m \cap \ker t_\ell = \{1\}$. Thus, $t_\ell(J_m)$ is a subgroup isomorphic to J_m, and there exists a continuous homomorphism $\varphi : \mathbb{Z}_\ell \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ such that $\rho(x) = \varphi(t_\ell(x))$ for any $x \in J_m$ (Every homomorphism on a subgroup of $\mathbb{Z}_\ell$ can be extended to $\mathbb{Z}_\ell$). Now we we prove that $\rho(x)$ is unipotent for $x \in J_m$. Recall that $t_\ell(gxg^{-1}) = ||g|| t_\ell(x)$ for any $g \in W_F$ and $x \in I_F$. Then

$$(\rho(\Phi x \Phi^{-1}))^q = \rho(x) \qquad x \in J_m$$

This indicates that the set of eigenvalues of $\rho(x)$ is invariant under $\lambda \longmapsto \lambda^q$. Hence the eigenvalues of $\rho(x)$ are roots of unities. Now we take $m = 2$. Suppose α is an eigenvalue of $\rho(h)$, then $(\alpha - 1)/\ell^2$ is an eigenvalue of $(\rho(h) - 1)/\ell^2$. Then $(\alpha - 1)/\ell^2 \in \overline{\mathbb{Z}}_\ell$. We need the following lemma

Lemma 1.3. *If $\alpha \in \overline{\mathbb{Q}}_\ell$ and $(\alpha - 1)/\ell^2 \in \overline{\mathbb{Z}}_\ell$, then $\alpha = 1$.*

We choose some $0 \neq h_0 \in t_\ell(J_m)$. Since $\varphi(h_0)$ is unipotent, we can take $A = \log \varphi(h_0)$ and consider the homorphism $\mathbb{Z}h_0 \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ by $xh_0 \longmapsto \exp(xA)$. Since $\mathbb{Z}h_0$ is dense in $\mathbb{Z}_\ell h_0$, we get $\varphi(x) = \exp(xA)$ for any $x \in \mathbb{Z}_\ell h_0$. Now we take $L_m = t_\ell^{-1}(\mathbb{Z}_\ell h_0)$, and we will get $\rho(x) = \exp(t_\ell(x)A)$ for all $x \in L_m$. □

Remark 1.2. *Such an N_ρ is unique, and ρ is smooth if and only if $N_\rho = 0$.*

For any ρ finite dimensional continuous representation of W_F, since $t_\ell(gxg^{-1}) = ||g||t_\ell(x)$ for any $g \in W_F$ and $x \in I_F$, we fix g and take a new representation of W_F by $\rho_g(x) = \rho(gxg^{-1})$. Then

$$\rho_g(x) = \rho(gxg^{-1}) = \exp(t_\ell(gxg^{-1})N_\rho) = \exp(t_\ell(x)||g||N_\rho)$$

and

$$\rho_g(x) = \rho(g)\exp(t_\ell(x)N_\rho)\rho(g)^{-1} = \exp(t_\ell(x)\rho(g)N_\rho\rho(g)^{-1})$$

in an open subgroup of I_F. By the uniqueness of N_{ρ_g}, we have

$$\rho(g)N_\rho\rho(g)^{-1} = ||g||N_\rho \tag{1.1}$$

for any $g \in W_F$. Now we can define $\rho_\Phi(\Phi^a x) = \rho(\Phi^a x)\exp(-t_\ell(x)N_\rho)$ where $x \in I_F$ and $a \in \mathbb{Z}$. This is a group homomorphism because of the equation 1.1 above. And from the ℓ-adic monodromy theorem ρ_Φ is smooth.

This motivates the following definition:

Definition 1.3. *A Weil-Deligne representaiton of W_F is a pair (ρ, N) where ρ is a finite-dimensional smooth representation of W_F,and $N \in End(V)$ is nilpotent satisfying for any $x \in W_F$*

$$\rho(x)N\rho(x)^{-1} = ||x||N$$

Theorem 1.4. *We have an equivalence of categories*

$$\begin{array}{rcl} Rep_{\overline{\mathbb{Q}}_\ell}(W_F) & \longrightarrow & D - Rep_{\overline{\mathbb{Q}}_\ell}(W_F) \\ \rho & \longmapsto & (\rho_\Phi, N_\rho) \end{array}$$

where $Rep_{\overline{\mathbb{Q}}_\ell}(W_F)$ is the category of finite dimensional continuous representations of W_F over $\bar{\mathbb{Q}}_\ell$, and $D - Rep_{\overline{\mathbb{Q}}_\ell}(W_F)$ is the category of Weil-Deligne representation over $\overline{\mathbb{Q}}_\ell$ Furthermore, the isomorphism class of (ρ_Φ, N_ρ) is independent of the choice of Φ and t.

Remark 1.3. *Since the definition of Weil-Deligne representation does not involve the topology of base field of V, if we fix a field isomorphsm $\tau : \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$, then we will further get an equivalence of category between $Rep_{\overline{\mathbb{Q}}_\ell}(W_F)$ and $D - Rep_{\overline{\mathbb{Q}}_\ell}(W_F)$.*

Definition 1.4. *A Weil-Deligne representation $\rho' = (\rho, N)$ is called semisimple if ρ is semisimple. We use $\mathcal{G}_n(F)$ to denote the set of equivalent classes of n-dimensional semisimple Weil-Deligne representation of W_F over $\mathbb{C}$.*

Proposition 1.4. *A Weil-Deligne representation $\rho' = (\rho, N)$ is semisimple if and only if $\rho(\Phi) \in Aut_{\mathbb{C}}(V)$ is semisimple.*

1.3 Tate's Thesis

In this section, we recall the main conclusions in Tate's thesis and some complementary properties of L-functions and ε-factors of Hecke characters. We denote $\mathbf{F}$ by a number field. $S(\mathbf{F})$ is the set of all places of $\mathbf{F}$, and $S_\infty(\mathbf{F})$ is the set of all archimedean places, $S_f(\mathbf{F})$ is the set of all nonarchimedean places. We first define Adeles and Ideles.

Definition 1.5. *Let $\{X_i\}_{i\in I}$ be a sequence of topological spaces and $Y_i \subset X_i$ be subspaces. Then the restricted product $\prod'_{i\in I} X_i$ of $\{X_i\}_{i\in I}$ with respect to $\{Y_i\}_{i\in I}$ is a topological space defined as follows. $\prod'_{i\in I} X_i = \{(x_i)_{i\in I} | \ x_i \in Y_i$ for all but finitely many $i\}$. The topology on $\prod'_{i\in I} X_i$ is generated by the base $\{\prod_{i\in S_0} U_i \times \prod_{i\notin S_0} Y_i\}$ where S_0 ranges over all finite subsets of I and U_i ranges over all open subsets of X_i.*

Now we define $\mathbb{A}_\mathbf{F} = \prod'_{v\in S(\mathbf{F})} F_v$ with respect to O_{F_v}, and $\mathbb{A}_\mathbf{F}^\times = \prod'_{v\in S(\mathbf{F})} F_v^\times$ with respect to $O_{F_v}^\times$.

Remark 1.4. *The topology on $\mathbb{A}_\mathbf{F}^\times$ is not the same with the subspace topology of $\mathbb{A}_\mathbf{F}$.*

There are natural morphisms $\mathbf{F} \hookrightarrow \mathbb{A}_\mathbf{F}$ and $\mathbf{F}^\times \hookrightarrow \mathbb{A}_\mathbf{F}^\times$.

Definition 1.6. *A Hecke character is a continuous unitary group homomorphism $\chi : \mathbf{F}^\times\backslash\mathbb{A}_\mathbf{F}^\times \to \mathbb{C}^\times$,where unitary means $|\chi(x)| = 1$ for any $x \in \mathbf{F}^\times\backslash\mathbb{A}_\mathbf{F}^\times$.*

For any $v \in S$, there is a natural embedding $F_v^\times \hookrightarrow \mathbf{F}^\times\backslash\mathbb{A}_\mathbf{F}^\times$. Hence we denote the local character $\chi_v : F_v^\times \to \mathbb{C}^\times$. We write $\chi = \prod_v \chi_v$. If v is a nonarchimedean place, we call the local character χ_v is unramified if $\chi_v(O_v^\times) = 1$. We call χ_v is of conductor n or level n if n is the smallest number satisfying $\chi_v|_{U_F^{n+1}} = 1$. Then if χ_v is unramified, then the level of χ_v is 0. We have

Lemma 1.4. *Let χ be a Hecke character, then there is a finite set S of places of $\mathbf{F}$ containing all the archimedean places, such that for any $v \notin S$, χ_v is unramified.*

Now we would like to interpret the Hecke character when $\mathbf{F} = \mathbb{Q}$.

Lemma 1.5. *When $\mathbf{F} = \mathbb{Q}$,*
(1) χ has a unique decomposition $\chi(x) = \chi_1(x)|x|^s$ where χ_1 is an unitary character of finite order.
(2) There is a one-to-one correspondence between the set of Hecke characters of finite order $\chi : \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ and the set of Dirichlet characters.

We need the following lemma:

Lemma 1.6. *Let $S_0 \subset S_f(\mathbf{F})$ be a finite set of nonarchimedean places of $\mathbf{F}$, and $v_0 \in S_f(\mathbf{F})$ be a nonarchimedean place of $\mathbf{F}$ such that $v_0 \neq S_0$. For each $v \in S_0$, we choose an integer $n_v \geq 0$. Then there exists a finite order Hecke character χ of $\mathbf{F}^\times\backslash\mathbb{A}_\mathbf{F}$, satisfying*
(1) χ_{v_0} is unramified.
(2) For every nonarchimedean $v \in S$ such that $v \neq v_0$, the level l_v of χ_v satisfies $l_v \geq n_v$.

For a Hecke character χ, we can define its L-function by the Euler product $L(\chi, s) = \prod_{v\in S} L(\chi_v, s)$, where the local factors $L(\chi_v, s)$ is defined as follows. When $v \in S_f(\mathbf{F})$ is nonarchimedean, we have

$$L(\chi_v, s) = \begin{cases} \frac{1}{1-\chi_v(\pi_v)q^{-s}} & \text{when } \chi_v \text{ is unramified} \\ 1 & \text{when } \chi_v \text{ is ramified} \end{cases}$$

When $v \in S$ is real, we can write the quasicharacter χ_v in the form $\chi_v(x) = (x/|x|_v)^\varepsilon$ for $\varepsilon \in \{0, 1\}$, then we define

$$L(\chi_v, s) = \pi^{-(s+\varepsilon)/2}\Gamma((s+\varepsilon)/2)$$

When $v \in S$ is complex, we can write the quasicharacter χ_v in the form $\chi_v(x) = \chi_{\lambda,k} = |x|_v^\lambda(\frac{x}{\sqrt{|x|_v}})^k$ for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$, then we define

$$L(\chi_v, s) = 2(2\pi^{s+\lambda+\frac{|k|}{2}})\Gamma(s + \lambda + \frac{|k|}{2})$$

When $\mathbf{F} = \mathbb{Q}$, then $L(\chi, s)$ is just the complete Dirichlet L-function. As is the case of Dirichlet L-functions, we wish to prove the following theorem:

Theorem 1.5. *Let χ be a Hecke character $\chi : \boldsymbol{F}^\times\backslash\mathbb{A}_{\boldsymbol{F}} \to \mathbb{C}^\times$, then*
(1) $L(\chi, s)$ is absolutely convergent when $Re(s)$ is sufficiently large, and $L(\chi, s)$ admits a meromorphic continuation to the whole complex plane. Furthermore, it is entire unless $\chi = |x|^\lambda$ for some $\lambda \in \mathbb{C}$ purely imaginary, in which case $L(\chi, s)$ has two poles at $s = -\lambda$ and $s = 1 - \lambda$ when $\boldsymbol{F}$ is a number field.
(2) We have the functional equation

$$L(\chi, s) = \varepsilon(\chi, s)L(\chi^{-1}, 1 - s)$$

where $\varepsilon(\chi, s)$ is an entire function on the complex plane.

We call $\varepsilon(\chi, s)$ the ε-factor of χ. In 1950s, Tate reproved this theorem in his thesis (which we now call Tate's thesis) using the techniques in harmonic analysis. We now recall his arguments here.

We first normalize the Haar measures. We choose the additive Haar measure dx_v on F_v to be the self-dual Haar measure. This means it is normalized in the way $\hat{\hat{f}}(x) = f(-x)$. When $v \in S_f(\mathbf{F})$ is nonarchimedean , $\int_{O_F} 1dx_v = q^{l/2}$ where l is the level of ψ. When $v \in S_\infty(\mathbf{F})$ is archimedean , dx_v is just the usual Lebesgue measure. Now we normalize the multiplicative Haar measure $dx_v^\times$ on $F_v^\times$ by $dx_v^\times = m_v\frac{dx_v}{|x|_v}$ where $m_v = 1$ if v is archimedean and $m_v = (1 - 1/q_v)^{-1}$ if v is nonarchimedean . We take $dx = \prod_v dx_v$ and $d^\times x = \prod_v d^\times x_v$ to be the Haar measure on $\mathbb{A}_F$ and $\mathbb{A}_F^\times$ respectively.

To do analysis on F_v, we need to choose some additive character ψ on F. For additive character ψ, we can also define the level of ψ to be the least number n satisfying $\psi|_{\mathfrak{p}^n} = 1$. We have the following duality for additive characeters:

Proposition 1.5. *Suppose ψ is an additive character on F, then any additive character of F has the form $\psi_a(x) = \psi(ax)$ where $a \in F_v^\times$. If $a \neq 0$, then the level of ψ_a equals the level of ψ minus $v_F(a)$.*

From now on, we fix an additive character $\psi = \prod_v \psi_v$ of $\mathbb{A}_F$. For any $\phi_v \in \mathcal{S}(F_v)$, we define its Fourier transform (with respect to ψ) to be

$$\hat{\phi}(x) = \int_{F_v} \phi_v(y)\psi_v(xy)dy$$

and local zeta functions

$$Z(s, \phi_v, \chi_v) = \int_{\mathbf{F}_v^\times} \phi_v(y)\chi_v(y)|y|^s d^\times y$$

If $\phi = \prod_v \phi_v \in \mathcal{S}(\mathbb{A}_F)$, then we write the global Fourier tranform $\hat{\phi}(x) = \int_{\mathbb{A}_F} \phi(y)\psi(xy)dy$, and the global zeta function $Z(s,\phi,\chi) = \prod_v Z(s,\phi_v,\chi_v) = \int_{\mathbb{A}_F^\times} \phi(y)\chi(y)|y|^s d^\times y$. It's easy to get the following lemma:

Lemma 1.7. *Suppose $v \in S_f(\boldsymbol{F})$ is an nonarchimedean place such that χ_v is unramified and $\phi_v = \mathbf{1}_{O_v}$, then $Z(s,\phi_v,\chi_v) = L(\chi_v,s)$. Therefore, for a Hecke character χ, $Z(s,\phi_v,\chi_v) = L(\chi_v,s)$ for almost all $v \in S_f(\boldsymbol{F})$.*

Proposition 1.6 (Local functional equation)**.** *(1) The function $Z(s,\phi_v,\chi_v)$ is convergent when $Re(s) > 0$, and $Z(s,\phi_v,\chi_v)$ admits a meromorphic continuation to the whole complex plane.*
(2) There exists a meromprphic function $\gamma(s,\chi_v,\psi_v)$ independent of the choice of ϕ_v such that

$$Z(1-s,\hat{\phi}_v,\chi_v^{-1}) = \gamma(s,\chi_v,\psi_v)Z(s,\phi_v,\chi_v)$$

(3) Suppose v is nonarchimedean. Then $Z(s,\phi_v,\chi_v) \in \mathbb{C}[q^s,q^{-s}]$. Let $I_\chi = \{Z(s,\phi_v,\chi_v) \mid \phi_v \in C_c^\infty(F_v)\}$. I_χ is a principal ideal in $\mathbb{C}[q^s,q^{-s}]$ and there exists $p_\chi(x) \in \mathbb{C}[x]$ such that $p_\chi(q^{-s})^{-1}$ is the generator of I_χ. If we normalize p_χ such that $p_\chi(0) = 1$. Then $L(s,\chi) = p_\chi(q^{-s})^{-1}$.

Proposition 1.7 (Global functional equation)**.** *(1) The function $Z(s,\phi,\chi)$ is absolutely convergent when $Re(s) > 1$, and $Z(s,\phi,\chi)$ admits a meromorphic continuation to the whole complex plane. Furthermore, it is entire unless $\chi = |x|^\lambda$ for some $\lambda \in \mathbb{C}$ purely imaginary, in which case $Z(s,\phi,\chi)$ has two poles at $s = \lambda$ and $s = 1-\lambda$ when $\boldsymbol{F}$ is a number field.*
(2) We have the functional equation

$$Z(s,\phi,\chi) = Z(1-s,\hat{\phi},\chi^{-1})$$

Now we compare $Z(s,\phi,\chi)$ and $L(\chi,s)$. We define the local ε-factors

$$\varepsilon(\chi_v,s,\psi_v) := \frac{\gamma(\chi_v,s,\psi_v)L(\chi_v,s)}{L(\chi_v^\vee,1-s)}$$

Remark 1.5. *The local $\varepsilon(\chi_v,s,\psi_v)$ depends on the choices of both the additive character ψ and the additive Haar measure dx_v on F_v.*

We can get immediately from Lemma 1.7 and Proposition 1.7 that

Lemma 1.8. *$\varepsilon(\chi_v,s,\psi_v) = 1$ for almost all $v \in S_f(\boldsymbol{F})$.*

We need the following conclusion:

Proposition 1.8. *For any $v \in S(\boldsymbol{F})$, we have*
(1) For any $s_0 \in \mathbb{C}$, we can choose $\phi_v \in \mathcal{S}(F_v)$ such that $Z(s,\phi_v,\chi_v)$ has neither pole nor zero at s_0.
(2) For any $\phi_v \in \mathcal{S}(F_v)$, $Z(s,\phi_v,\chi_v)/L(\chi_v,s)$ is an entire function.

Thus, we can get the following property of $\varepsilon(\chi,s,\psi)$:

Proposition 1.9. *$\varepsilon(\chi \otimes \chi_F^{s_0},s,\psi) = \varepsilon(\chi,s+s_0,\psi)$.*

Proof. This is immediately from the definition of zeta functions and L-factors. ☐

Proposition 1.10. *$\varepsilon(\chi, s, \psi)$ is an entire function for any $v \in S(\boldsymbol{F})$.*

Proof. By definition $\varepsilon(\chi_v, s, \psi_v) = \frac{Z(1-s,\hat{\phi}_v,\chi_v^{-1})}{L(\chi_v^{-1},1-s)} \cdot \frac{L(\chi_v,s)}{Z(s,\phi_v,\chi_v)}$. For any $s_0 \in \mathbb{C}$, $\frac{Z(1-s,\hat{\phi}_v,\chi_v^{-1})}{L(\chi_v^\vee,1-s)}$ and $\frac{Z(s,\phi_v,\chi_v)}{L(\chi_v,s)}$ are entire from the above proposition, and by Lemma 1.7 we can choose ϕ_v so that $Z(s, \phi_v, \chi_v)$ has no zero at s_0. Thus, $\frac{L(\chi_v,s)}{Z(s,\phi_v,\chi_v)}$ is holomorphic at s_0 and therefore $\varepsilon(\chi_v, s, \psi_v)$ is holomorphic at s_0. Now since $\varepsilon(\chi_v, s, \psi_v)$ is independent of ϕ_v, it is holomorphic at any $s_0 \in \mathbb{C}$. □

Proposition 1.11. *Suppose v is nonarchimedean. Then $\varepsilon(\chi, s, \psi) = aq^{-sn(\chi,\psi)}$ for some $a \in \mathbb{C}$ and $n(\chi, \psi) \in \mathbb{Z}$.*

Proof. We can choose some $\phi \in C^\infty(F)$ such that $Z(s, \phi, \chi) = L(s, \pi)$. Then $\varepsilon(\pi, s, \psi) = \frac{Z(1-s,\hat{\phi},\chi^{-1})}{L(\chi^{-1},1-s)} \in \mathbb{C}[q^s, q^{-s}]$ by Proposition 1.6. We claim that $\varepsilon(\chi, s, \psi)\varepsilon(\chi^{-1}, 1-s, \psi) = \chi(-1)$. With this claim we see that $\varepsilon(\chi, s, \psi)$ is invertible in $\mathbb{C}[q^s, q^{-s}]$. Thus, $\varepsilon(\chi, s, \psi)$ has the form of aq^{ms} for $m \in \mathbb{Z}$. Now we prove our claim. We have $\varepsilon(\chi, s, \psi) = \frac{Z(1-s,\hat{\phi},\chi^{-1})}{L(\chi^{-1},1-s)} \cdot \frac{L(\chi,s)}{Z(s,\phi,\chi)}$ and $\varepsilon(\chi^{-1}, 1-s, \psi) = \frac{L(\chi^{-1},1-s)}{Z(1-s,\hat{\phi},\chi^{-1})} \cdot \frac{Z(s,\hat{\hat{\phi}},\chi)}{L(\chi,s)}$. Therefore, $\varepsilon(\chi, s, \psi)\varepsilon(\chi^{-1}, 1-s, \psi) = \frac{Z(s,\hat{\hat{\phi}},\chi)}{Z(s,\phi,\chi)} = \frac{Z(s,\phi(-x),\chi)}{Z(s,\phi,\chi)} = \chi(-1)$. □

Proof of Theorem 1.5. (1) $Z(s, \phi_v, \chi_v)/L(\chi_v, s)$ is an entire function for any v and equals 1 for almost all v. Hence $Z(s, \phi, \chi)/L(\chi, s)$ is an entire function. So meromorphic continuation and the pole of $L(\chi, s)$ comes from the properties of $Z(s, \phi, \chi)$.
(2) We take $\varepsilon(\chi, s) = \prod_{v \in S(\mathbf{F})} \varepsilon(\chi, \psi, s)$. This is a finite product of entire functions (Lemma 1.8 and Proposition 1.10), hence $\varepsilon(\chi, s)$ is entire. By taking products we get $\varepsilon(\chi, s) = \frac{Z(1-s,\hat{\phi},\chi^{-1})}{L(\chi^\vee,1-s)} \cdot \frac{L(\chi,s)}{Z(s,\phi,\chi)}$. $Z(s, \phi, \chi) = Z(1-s, \hat{\phi}, \chi^{-1})$ by Proposition 1.7. Thus we have the functional equation in (2), and hence $\varepsilon(\chi, s)$ is independent of ψ. □

In the end of this section, we want to do some calculation on the local ε-factors on local field F. We first calculate the relation between different ε-factors when choosing different additive characters.

Proposition 1.12. *For any $a \in F^\times$, we have*

$$\varepsilon(\chi, s, \psi_a) = \chi(a)^{-1}|a|^{s-\frac{1}{2}}\varepsilon(\chi, s, \psi)$$

Proof. We use $\hat{\phi}_a$ to denote the Fourier transform of ϕ under the additive character ψ. We only need to calculate the zeta function $Z(1-s, \hat{\phi}, \chi)$. We have

$$\begin{aligned} Z(1-s, \hat{\phi}_a, \chi) &= \int_{F^\times}\int_F \phi(y)\psi(axy)\chi(x)|x|^{1-s}dyd^\times x \\ &= \chi(a)^{-1}|a|^{s-1}\int_{F^\times}\int_F \phi(y)\psi(ty)\chi(t)|t|^{1-s}dyd^\times t \\ &= \chi(a)^{-1}|a|^{s-1}Z(1-s, \hat{\phi}, \chi) \end{aligned}$$

□

Now we calculate ε-factors explicitly for some additive character.

Proposition 1.13. *When χ is unramified, and the level of ψ has level one. Then*

$$\varepsilon(\chi, s, \psi) = q^{s-1/2}\chi(\pi_F)^{-1}$$

Proof. We choose $\phi = \mathbf{1}_{O_F}$ to be the characteristic function of O_F. Then from Lemma 1.7, $Z(s,\phi,\chi) = L(\chi,s)$, Now we calculate

$$\hat{\phi}(x) = \int_F \phi(y)\psi(xy)dy = q^{1/2}\phi(\pi_F^{-1}x)$$

and hence

$$\begin{aligned} Z(s,\hat{\phi},\chi^{-1}) &= q^{1/2}\int_{F^\times} \phi(\pi_F^{-1}x)\chi(x)^{-1}|x|^s d^\times x \\ &= \chi^{-1}(\pi_F)q^{\frac{1}{2}-s}\int_{F^\times} \phi(x)\chi(x)^{-1}|x|^s d^\times x \\ &= \chi^{-1}(\pi_F)q^{\frac{1}{2}-s}L(\chi^{-1},s) \end{aligned}$$

Thus,

$$\varepsilon(\chi,s,\psi) = \frac{Z(1-s,\hat{\phi}_v,\chi_v^{-1})}{L(\chi_v^{-1},1-s)} = q^{s-\frac{1}{2}}\chi(\pi_F)^{-1}$$

□

Now suppose χ has level $n \geq 1$, we define the Gauss sum of χ with respect to ψ

$$\tau(\chi,\psi) = \sum_{x\in U_F/U_F^{n+1}} \chi^{-1}(cx)\psi(cx)$$

where $c \in F^\times$ is any element satisfying $v_F(c) = -n$. It's easy to observe that $\tau(\chi,\psi)$ is independent of the choice of c.

Proposition 1.14. *Suppose χ has level $n \geq 0$ and χ is ramified, additive character ψ has level one. Then*

$$\varepsilon(\chi,s,\psi) = q^{-ns-\frac{1}{2}}\tau(\chi,\psi)$$

for any $c \in F^\times$ such that $v_F(c) = -n$.

Proof. We choose $\phi = \mathbf{1}_{U_F^{n+1}}$ to be the characteristic function of U_F^{n+1}. The Fourier transform

$$\hat{\phi}(x) = \int_{U_F^{n+1}} \phi(y)\psi(xy)dy = \begin{cases} q^{-n-\frac{1}{2}}\psi(x) & x \in \mathfrak{p}^{-n} \\ 0 & x \notin \mathfrak{p}^{-n} \end{cases}$$

Thus, we have

$$\begin{aligned} Z(s,\hat{\phi},\chi^{-1}) &= q^{-n-\frac{1}{2}}\int_{v_F(y)\geq -n} \psi(y)\chi^{-1}(y)|y|^s d^\times y \\ &= q^{-n-\frac{1}{2}}\sum_{k\geq -n} q^{-ks}\int_{O_F^\times} \psi(\pi_F^k y)\chi^{-1}(\pi_F^k y)d^\times y \end{aligned}$$

When $k > -n$,

$$\begin{aligned}\int_{O_F^\times} \psi(\pi_F^k y)\chi^{-1}(\pi_F^k y) d^\times y &= \sum_{m \in O_F^\times / U_F^{-k}} \int_{U_F^{-k}} \psi(\pi_F^k m r)\chi^{-1}(\pi_F^k m r) d^\times r \\ &= \sum_{m \in O_F^\times / U_F^{-k}} \int_{U_F^{-k}} \psi(\pi_F^k m)\psi(\pi_F^k m(r-1))\chi^{-1}(\pi_F^k m n) d^\times r \\ &= \sum_{m \in O_F^\times / U_F^{-k}} \int_{U_F^{-k}} \psi(\pi_F^k m)\chi^{-1}(\pi_F^k m r) d^\times r \\ &= 0\end{aligned}$$

When $k = -n$,

$$\begin{aligned}\int_{O_F^\times} \psi(\pi_F^n y)\chi^{-1}(\pi_F^{-n} y) d^\times y &= \sum_{m \in O_F^\times / U_F^{n+1}} \int_{U_F^{n+1}} \psi(\pi_F^{-n} m)\psi(\pi_F^{-n} m(r-1))\chi^{-1}(\pi_F^{-n} m r) d^\times r \\ &= \sum_{m \in O_F^\times / U_F^{n+1}} vol^\times(U_F^{n+1})\psi(\pi_F^{-n} m)\chi^{-1}(\pi_F^{-n} m)\end{aligned}$$

Also $Z(s, \phi, \chi) = \int_{F^\times} \phi(y)\chi^y |y|^s d^\times y = vol^\times(U_F^{n+1})$. Thus,

$$\varepsilon(\chi, s, \psi) = \frac{Z(1-s, \hat{\phi}, \chi^{-1})}{Z(s, \phi, \chi)} = q^{-ns-\frac{1}{2}} \sum_{m \in O_F^\times / U_F^{n+1}} \psi(\pi_F^{-n} m)\chi^{-1}(\pi_F^{-n} m)$$

□

We can get the following conclusion immediately:

Corollary 1.1. *Suppose χ has level $n \geq 0$ and χ is ramified, the additive character ψ has level one, then $n(\chi, \psi)$ equals the conductor of χ.*

Proposition 1.15. *Suppose χ has level $n \geq 1$, and the additive character ψ has level one. Let $c \in F$ satisfy*

$$\chi(1+x) = \psi(cx) \qquad x \in \mathfrak{p}^{[n/2]+1}$$

Then

$$\tau(\chi, \psi) = q^{[(n+1)/2]} \sum_{y \in U_F^{[(n+1)/2]} / U_F^{[n/2]+1}} \chi^{-1}(cy)\psi(cy)$$

Proof. From the definition we can get $v_F(c) = -n$. We write

$$\begin{aligned}\tau(\chi, \psi) &= \sum_{y \in U_F / U_F^{[n/2]+1}} \sum_{z \in U_F^{[n/2]+1} / U_F^{n+1}} \chi^{-1}(cy)\chi^{-1}(1+(z-1))\psi(cy)\psi(cy(z-1)) \\ &= \sum_{y \in U_F / U_F^{[n/2]+1}} \chi^{-1}(cy)\psi(cy) \sum_{z \in U_F^{[n/2]+1} / U_F^{n+1}} \psi(c(y-1)(z-1)) \\ &= \sum_{y \in U_F / U_F^{[n/2]+1}} \chi^{-1}(cy)\psi(cy) \sum_{r \in O_F / \mathfrak{p}^{n-[n/2]}} \psi(c(y-1)\pi_F^{[n/2]+1} r)\end{aligned}$$

Let $v_F(y-1)=k$. When $k<[(n+1)/2]$, $\sum_{r\in O_F/\mathfrak{p}^{n-[n/2]}}\psi(c(y-1)\pi_F^{[n/2]+1}r)=0$. When $k<[(n+1)/2]$, $\psi(c(y-1)\pi_F^{[n/2]+1}r)=1$, hence $\sum_{r\in O_F/\mathfrak{p}^{n-[n/2]}}\psi(c(y-1)\pi_F^{[n/2]+1}r)=q^{[n+1]/2}$. Therefore, we can get $\tau(\chi,\psi)=q^{[(n+1)/2]}\sum_{y\in U_F^{[(n+1)/2]}/U_F^{[n/2]+1}}\chi^{-1}(cy)\psi(cy)$.

□

We need the following conclusion later:

Theorem 1.6 (Stability Theorem)**.** *Let θ and χ be characters of $F^\times$ of level $l\geq 0$ and $n\geq 1$ respectively, and $2l<n$. ψ is an additive character of F with $\psi\neq 1$, and $c\in F$ satisfy $\chi(1+x)=\psi(cx)$ for all $x\in\pi_F^{[n/2]+1}O_F$. Then*

$$\varepsilon(\theta\chi,s,\psi)=\theta(c)^{-1}\varepsilon(\chi,s,\psi)$$

Proof. From Proposition 1.12 we only need to calculate the case when ψ has level one. $\theta\chi$ has level n and $\theta(x)=1$ for any $y\in U_F^{[(n+1)/2]}$. By Proposition 1.14 and Proposition 1.15, we have

$$\begin{aligned}\varepsilon(\theta\chi,s,\psi) &= q^{-ns-\frac{1}{2}}\tau(\theta\chi,\psi)\\ &= q^{-ns-\frac{1}{2}}\cdot q^{[(n+1)/2]}\sum_{y\in U_F^{[(n+1)/2]}/U_F^{[n/2]+1}}\theta^{-1}(c)\chi^{-1}(cy)\psi(cy)\\ &= \theta^{-1}(c)q^{-ns-\frac{1}{2}}\tau(\chi,\psi)\\ &= \theta^{-1}(c)\varepsilon(\chi,s,\psi)\end{aligned}$$

□

Corollary 1.2 (Converse Theorem for GL(1))**.** *Let ψ be a nontrivial additive character of F. Suppose χ_1, χ_2 are ramified characters of $F^\times$, satisfying for any character χ of $F^\times$, $\varepsilon(\chi\chi_1,\psi,s)=\varepsilon(\chi\chi_2,\psi,s)$. Then $\chi_1=\chi_2$.*

1.4 Artin L-functions

In this section, we define and study basic properties of Artin L-functions of Galois represnetations.

We recall that F is an nonarchimedean local field. Now suppose ρ is a smooth representation of W_F, then we define

$$L(\rho,s)=det(1-q^{-s}\cdot\rho(\Phi)|_{V^I})^{-1}$$

where $V^I=\{v\in V|\rho(x)v=v\text{for all } x\in I_F\}$. Since I_F is a normal subgroup in W_F, it's obvious that V_I is closed under $\rho(\Phi)$.
Now for Weil-Deligne representation $\rho'=(\rho,N)$ of W_F, we define

$$L(\rho',s)=L(\rho|_{V_N},s)$$

where $V_N=\ker N$, and from the definition $\rho(x)N\rho(x)^{-1}=||x||N$, V_N is a subrepresentation of ρ. We also define Artin L-factors for archimedean local fields. When $F=\mathbb{C}$ we define $W_F=\mathbb{C}^\times$, $\varphi_F:W_F\to G_F$ to be the trivial map, and Art_F to be the identity. Hence the representations of W_F are just the quasicharacters of $\mathbb{C}^\times$ and it can be written in the form $\chi_v=\chi_{\lambda,k}$ as in Section 1.3. Thus, we just define $L(\chi_{\lambda,k},s)$ the same as the local Hecke L-factors. When $F=\mathbb{R}$ we define

$W_F = \mathbb{C}^\times \rtimes \{1, j\}$ where $j^2 = 1$ and $jcj^{-1} = \overline{c}$ for $c \in \mathbb{C}^\times$. And we define $\varphi_F : W_F \to G_F$ to be the projection map to $\{1, j\}$. The Artin map $Art_F : F^\times \to W_F$ is defined to be $Art_F(x) = \sqrt{x}$ when $x > 0$, $Art_F(x) = -j\sqrt{-x}$ when $x < 0$ and $Art_F(0) = 0$.

Lemma 1.9. *Let ρ be an irreducible representation of $W_\mathbb{R}$, then ρ is either of dimension 1 or dimension 2. When $\dim \rho = 2$, we have $\rho = Ind_{\mathbb{C}/\mathbb{R}} \chi_{\lambda,k}$ for some λ and k.*

Thus, we define the local L-factor $L(\rho, s)$ by $L(\chi_\varepsilon)$ when $\dim \rho = 1$ and $L(\chi_{\lambda,k}, s)$ when $\dim \rho = Ind_{\mathbb{C}/\mathbb{R}} \chi_{\lambda,k}$.

Theorem 1.7. *Suppose F is a local field, and then the local Artin L-factors satisfy:*
(1) Let ρ and τ be smooth representations of W_F, then

$$L(\rho \oplus \tau, s) = L(\rho, s)L(\tau, s)$$

(2) Let (ρ, V) a smooth representation of W_E where E/F is a finite field extension, then

$$L(Ind_{E/F}\rho, s) = L(\rho, s)$$

We need the following lemma in representation theory.

Lemma 1.10. *Suppose G is a finite group, and H is a subgroup of G, I is a normal subgroup of G. Let $J = I \cap H$, then for any complex representation V of H, we have an isomorphism*

$$(Ind_H^G V)^I = Ind_{H/J}^{G/I} V^J$$

as a representation of G/I.

Proof of Theorem 1.7. (1) comes immediately from the definition. Now we prove (2). When F is archimedean, then this just comes from the definition. So we assume F is nonarchimedean. Then by the lemma above we have $(Ind_{W_E}^{W_F} V)^{I_F} = Ind_{n\mathbb{Z}}^{\mathbb{Z}} V^{I_E}$ where $n = f(E/F)$. Notice that we also have $\Phi_E = \Phi_F^n$ and $q_E = q_F^n$. So what remains to prove is $\det(1 - \lambda^n A) = \det(1 - \lambda\ Ind_{n\mathbb{Z}}^{\mathbb{Z}} A)$ where $A : W \to W$ is a linear endomorphism on W and $Ind_{n\mathbb{Z}}^{\mathbb{Z}} A : W^{\oplus n} \longrightarrow W^{\oplus n}$ given by $Ind_{n\mathbb{Z}}^{\mathbb{Z}} A(w \otimes e_i) = w \otimes e_{i+1}$ for $1 \le i \le n-1$ and $Ind_{n\mathbb{Z}}^{\mathbb{Z}} A(w \otimes e_n) = A(w) \otimes e_1$. This is an exercise in linear algebra. $\square$

Now we define global Artin L-functions. Let $\mathbf{F}$ be a number field, and S be the set of places of $\mathbf{F}$. Then for each $v \in S$, there is an inclusion $i_{F_v} : G_{\mathbf{F}_v} \hookrightarrow G_{\mathbf{F}}$. Suppose ρ is a representation of $G_{\mathbf{F}}$, then it will induce representations on local Weil groups W_F by $\rho_v := \rho \circ i_{F_v} \circ \varphi_{F_v}$, and we define the Artin L-function to be

$$L(\rho, s) = \prod_{v \in S_{\mathbf{F}}} L(\rho_v, s)$$

Consider the one dimensional Galois representation $\chi : G_{\mathbf{F}} \to \mathbb{C}^\times$. Recall that global class field theory gives the Artin map

$$Art_{\mathbf{F}} : \mathbf{F}^\times \backslash \mathbb{A}_{\mathbf{F}}^\times \longrightarrow Gal(\overline{\mathbf{F}}/\mathbf{F})^{ab}$$

satisfying the local-global compatibility:

$$\begin{array}{ccc} \mathbf{F}^\times \backslash \mathbb{A}_{\mathbf{F}} & \xrightarrow{Art_{\mathbf{F}}} & Gal(\overline{\mathbf{F}}/\mathbf{F})^{abv} \\ \uparrow & & \uparrow \\ F_v^\times & \xrightarrow{Art_{F_v}} & Gal(\overline{F_v}/F_v)^{ab} \end{array}$$

Then $\chi \circ Art_{\mathbf{F}}$ is a Hecke character of $\mathbf{F}$.

We recall the following theorem in the theory of finite group representations:

Theorem 1.8 (Brauer's Induction). *Let G is a finite group, and (ρ, V) is a finite dimensional complex representation. Then there exists finitely many subgroups $\{G_i\}_{i=1}^k$ of G, and chracters χ_i of G_i, such that $[\rho] - n[1_G] = \sum_{i=1}^k n_i Ind_{G_i}^G([\chi_i] - 1_{G_i})$ in the Grothendieck group $K(G)$ for some integers $n_i \in \mathbb{Z}$.*

Corollary 1.3. *Let $\boldsymbol{F}$ be a global field, then*
(1) When χ is a character of $Gal(\overline{\boldsymbol{F}}/\boldsymbol{F})$, we have

$$L(\chi, s) = L(\chi \circ Art_{\boldsymbol{F}}, s)$$

where the L-funciton in the RHS is a Hecke L-function.
(2) Suppose ρ is a finite dimensional representation of $Gal(\overline{\boldsymbol{F}}/\boldsymbol{F})$, then $L(\rho, s)$ admits a meromorphic continuation to the whole complex plane, and it satisfies the functional equation

$$L(\rho, s) = \varepsilon(\rho, s) L(\rho^\vee, 1-s)$$

where $\varepsilon(\rho, s)$ is an entire function. It is called the ε-factor of ρ.
(3) Suppose τ is a finite dimensional representation of $Gal(\overline{\boldsymbol{E}}/\boldsymbol{E})$ where $\boldsymbol{E}/\boldsymbol{F}$ is a finite extension, the global ε-factors satisfy

$$\varepsilon(Ind_{\boldsymbol{E}/\boldsymbol{F}}\tau, s) = \varepsilon(\tau, s)$$

Proof. (1) We only need to prove that for any character $\chi_v : \mathbf{F}_v \to \mathbb{C}^\times$ we have $L(\chi_v, s) = L(\chi_v \circ Art_{F_v}, s)$. Then by the local-global compatibility we will get the global conclusion. When v is nonarchimedean, if χ_v is unramified then $V^{I_v} = 1$ and hence $L(\chi_v \circ Art_{F_v}, s) = (1 - q^{-s}\chi(\pi_F))^{-1} = L(\chi_v, s)$. If χ is ramified then $V^I = 0$ and $L(\chi_v, s) = L(\chi_v \circ Art_{F_v}, s) = 1$. When v is archimedean, this just comes from the definition for $L(\chi_v, s)$.
(2) Since ρ is a smooth representation , then there exists a finite extension $\mathbf{E}/\mathbf{F}$ such that ρ is trivial on $Gal(\overline{\mathbf{E}}/\mathbf{E})$. Then we apply Brauer's induction theorem to $Gal(\overline{\mathbf{F}}/\mathbf{F})/G'$ and we can write

$$L(\rho, s) = \prod_{i=1}^k L(Ind_{\mathbf{K}_i/\mathbf{F}}\chi_i, s)^{n_i}$$

where $\mathbf{E} \supset \mathbf{K}_i \supset \mathbf{F}$ and χ_i are characters on $Gal(\overline{\mathbf{K}_i}/\mathbf{K}_i)$. Then $L(Ind_{\mathbf{K}_i/\mathbf{F}}\chi_i, s) = L(\chi_i, s)$ and $L(\chi_i, s)$ are meromorphic, hence $L(\rho, s)$ is meromorphic, and the functional equation comes from the function equation of Hecke characters.
(3) Since $L(Ind_{\mathbf{E}/\mathbf{F}}\tau, s) = L(\tau, s)$ and $L(Ind_{\mathbf{E}/\mathbf{F}}\tau^\vee, 1-s) = L(\tau^\vee, 1-s)$, $\varepsilon(Ind_{\mathbf{E}/\mathbf{F}}\tau, s) = \varepsilon(\tau, s)$ comes from the function equation in (2). □

1.5 The Existence of Local ε-Factors d'après Deligne

Theorem 1.9. *Suppose ψ is an additive character of F, and for any E/F finite extension, $\psi_E = \psi \circ Tr_{E/F}$ be an additive character of E. Then there is a unique class of function $\varepsilon(\cdot, \psi_E, s)$ for any finite extension E/F*

$$\{\textit{Smooth semisimple representations of } W_E\} \longrightarrow \mathbb{C}[q^s, q^{-s}]$$

satisfying
(1) For any character χ of $E^\times$, we have $\varepsilon(\chi\circ Art_E,\psi_E,s)=\varepsilon(\chi,\psi_E,s)$.
(2) For any two smooth semisimple representations of W_E, $\varepsilon(\rho\oplus\tau,\psi_E,s)=\varepsilon(\rho,\psi_E,s)\varepsilon(\tau,\psi_E,s)$.
(3) Suppose $E\supset K\supset F$, and $\dim\rho=n$, then

$$\frac{\varepsilon(Ind_{E/K}\rho,\psi_K,s)}{\varepsilon(\rho,\psi_E,s)}=\frac{\varepsilon(Ind_{E/K}1_E,\psi_K,s)^n}{\varepsilon(1_E,\psi_E,s)^n}$$

Now for general semisimple Weil-Deligne representation $\rho'=(\rho,N)$ of W_F, we define

$$\varepsilon(\rho',\psi_E,s)=\varepsilon(\rho,\psi_E,s)\cdot\frac{L(\rho^\vee,1-s)}{L(\rho,s)}\cdot\frac{L(\rho_N,s)}{L(\rho^\vee_{N^\vee},s)}$$

By the induction we discussed in the last section, the local ε-factor $\varepsilon(\pi,\psi,s)$ is unique and we can write it as a product of powers of local ε-factors of characters. But the problem is π can be written in different ways in sums of inductions of characters, and we don't know whether the expressions of $\varepsilon(\pi,\psi,s)$ via different ways are coincide. But as is seen in the last section, the global ε-factors are well-defined via Artin L-functions. So Deligne's way to construct local ε-factors is to use global arguments.

We begin with the following lemmas:

Lemma 1.11. *Suppose E/F is a finite Galois extension of nonarchimedean local fields, then there exists finite Galois extension of global fields $\boldsymbol{E}/\boldsymbol{F}$ such that*
(1) There exists a place v_0 of $\boldsymbol{F}$ and a place u_0 of $\boldsymbol{E}$ over $\boldsymbol{F}$ such that $F\cong\boldsymbol{F}_{v_0}$ and $E\cong\boldsymbol{E}_{u_0}$.
(2) The embedding $Gal(E/F)\hookrightarrow Gal(\boldsymbol{E}/\boldsymbol{F})$ is an isomorphism. For any $H\le Gal(E/F)$, let $H=Gal(E/K)$ and $H=Gal(\boldsymbol{E}/\boldsymbol{K})$. Then $K=\boldsymbol{K}_{w_0}$ where w_0 is a place over v_0.
(3) $\boldsymbol{F}$ has no real embedding in $\mathbb{R}$.

Lemma 1.12. *Suppose E/F is a finite Galois extension of nonarchimedean local fields. There exists an integer $n_{E/F}\ge 1$, such that for any character α of $F^\times$ with level $n\ge n_{E/F}$, for any $E\supset K\supset F$, and any character χ of $K^\times$ satisfying $\chi|_{Nm_{E/K}(E^\times)}=1$ with level l, the level u_K of $\alpha\circ N_{K/F}$ satisfies $u_K>2l$.*

Proof. Suppose $E\supset K\supset F$ is an immediate field. Then $Nm_{E/K}(E^\times)$ is an open subgroup of $K^\times$. Thus, we can find $m(K)\in\mathbb{N}$ such that $U_K^{m(K)}\subset Nm_{E/K}(E^\times)$. If $\chi|_{Nm_{E/K}(E^\times)}=1$, the level of χ is no more than $m(K)$. Now we consider the open subgroup $Nm_{K/F}(U_K^{2m(K)+1})$ and it contains $U_F^{m'(K)}$ for $m'(K)\in\mathbb{Z}$. We take $n_{E/F}=max_K m'(K)$. Then $n_{E/F}$ satisfies the conditions we want. ☐

We go back to the proof of Theorem 1.9. We first prove the following theorem:

Theorem 1.10. *Suppose E/F is a finite Galois representation. Then for any immediate field $E\supset K\supset F$ and finite dimensional smooth representations ρ of $Gal(K/F)$, there exist ε-factors $\varepsilon(\rho,\psi_K,s)$ satisfying the conditions in Theorem 1.9.*

Proof. We take the Galois extension of global fields $\mathbf{E}/\mathbf{F}$ as in Lemma 1.11. From $Gal(\mathbf{E}/\mathbf{F})=Gal(E/F)$, we know that for any $v\in S_f(\mathbf{E})$, there is a unique $u\in S_f(\mathbf{E})$ over v. We take S_0 to be the set of places of $\mathbf{F}$ which is ramified in $\mathbf{E}$. For any $v\in S_0$, we take $n_v=n_{\mathbf{E}_u/\mathbf{F}_v}$ as in Lemma

1.12. Then we can make use of Lemma 1.6 in Section 1.3. to get a finite order Hecke character α of $\mathbf{F}$ satisfying α_{v_0} is unramified and the level of α_v is no less than n_v for any $v \in S_0$. We also choose a nontrivial additive character Ψ of $\mathbb{A}_{\mathbf{F}}/\mathbf{F}$ such that $\Psi_{v_0} = \psi$. Now we fix an immediate field $E \supset K \supset F$ and let $\mathbf{K}_{w_0} = K$. We write $\alpha_{\mathbf{K}} = \alpha \circ Nm_{\mathbf{K}/\mathbf{F}}$ and $\Psi_K = \Psi \circ Tr_{\mathbf{K}/\mathbf{F}}$. Now we define some $\boldsymbol{c} \in \mathbb{A}_{\mathbf{F}}$. When v is archimedean , we just take $c_v = 1$. When v is nonarchimedean , we take $c_v \in \mathbf{F}_v$ such that $\alpha_v(1+x) = \Psi_v(c_v x)$ for all $x \in \mathfrak{p}_v^{[l_v/2]+1}$.

Let χ be a character of $Gal(E/K)$, then it gives rise to $\boldsymbol{\chi}$ a character $Gal(\mathbf{E}/\mathbf{K})$. We use χ and $\boldsymbol{\chi}$ to denote the character induced on $F^\times$ and $\mathbf{F}^\times \backslash \mathbb{A}_{\mathbf{F}}^\times$ respectively. The local-global compatibility will give $\boldsymbol{\chi}_{w_0} = \chi$. Then

$$\varepsilon(\boldsymbol{\chi}\alpha_{K,w}, s, (\Psi_{\mathbf{K}})_w) = \begin{cases} \chi_w^{-1}(c_v)\varepsilon(\alpha_{K,w}, s, (\Psi_E)_w) & w \neq w_0 \\ \varepsilon(\chi, s, \psi_K) & w = w_0 \end{cases}$$

where $c_v \in F_v$ is defined as above. When w is archimedean , since $\mathbf{K}$ has no real embedding, all items in complex case in trivial. When $w \neq w_0$ is nonarchimedean and w is over $u \in S_0$, then this comes from stability Theorem. When $w = w_0$ this comes from the condition $\alpha_{w_0} = 1$ and the local-global compatibility. When $w \neq w_0$ is nonarchimedean and w is over $u \notin S_0$, then $\boldsymbol{\chi}_w$ is always unramified and hence also satisfies the condition of Stability theorem. Therefore

$$\varepsilon(\boldsymbol{\chi}\alpha_K, s) = \varepsilon(\chi, \psi_K, s)\boldsymbol{\chi}^{-1}(c)a(\mathbf{K})$$

where

$$a(\mathbf{K}) = \prod_{w \neq w_0} \varepsilon(\alpha_{K,w}, s, (\Psi_E)_w)$$

is independent of χ. Thus, for any finite dimensional representation ρ of $Gal(K/F)$, we can write

$$\varepsilon(\rho, \psi_K, s) = \frac{\varepsilon(\boldsymbol{\rho}\alpha_K, s)\det \boldsymbol{\rho}(c)}{a(\mathbf{K})^{\dim \rho}}$$

Now we check the conditions in Theorem 1.9. (1) and (2) are obvious, we only need to prove (3). We recall the following lemmas in representation theory:

Lemma 1.13. *Suppose G is a finite group and $H \leq G$. Let ρ be a finite dimensional representation of H and τ be a finite dimensional representation of G. Then there is an isomorphism of representations of G:*

$$Ind_H^G \rho \otimes \tau \cong Ind_H^G(\rho \otimes \tau|_H)$$

Lemma 1.14. *Let G be a group and H be a subgroup of finite index in G. $V : G^{ab} \longrightarrow H^{ab}$ be the transfert map. Let ρ be an n-dimensional representation of H and $x \in G^{ab}$, we have*

$$\det(Ind_H^G([\rho] - n[1_H]))(x) = \det([\rho] - n[1_H])(V(x))$$

Proof. Refer to Proposition 1.2 in [Del73]. □

Thus, $(Ind_{\mathbf{E}/\mathbf{K}}(\boldsymbol{\rho}))\alpha_{\mathbf{E}} = Ind_{\mathbf{E}/\mathbf{K}}(\boldsymbol{\rho}\alpha_{\mathbf{K}})$, hence $\varepsilon((Ind_{\mathbf{E}/\mathbf{K}}(\boldsymbol{\rho})\alpha_{\mathbf{E}}), s = \varepsilon(\boldsymbol{\rho}\alpha_K, s)$. And $\frac{\det(Ind_{\mathbf{E}/\mathbf{K}}(\boldsymbol{\rho}))(c)}{\det \boldsymbol{\rho}(c)} = \det(Ind_{\mathbf{E}/\mathbf{K}}(1))^n(c)$ from Lemma 1.14. $a(\mathbf{K})^{\dim \rho}$ verifies the equation in (3). Hence (3) is verified. □

Proof of Theorem 1.9. Since we know from Brauer's induction theorem that the ε-factors are unique once they exist. Thus, we know that the ε-factors for Galois representations of finite field extensions constructed in Theorem 1.10 are compatible. So we have finished the proof for Galois representations. Recall that in Proposition 1.2, for any smooth representation ρ of W_E, there is $d \in \mathbb{N}^+$ such that $\rho(\Phi_E^d) = \lambda Id$. Now the natural inclusion map $i_{E/F} : W_E \hookrightarrow W_F$ will send $i_{E/F}(\Phi_E) = f(E/F)\Phi_F$. Hence we can find some $s_0 \in \mathbb{C}$ such that $\rho_0 = \rho \otimes \chi_F^{s_0}$ is a representation of W_E factoring through G_E. Now we define

$$\varepsilon(\rho, \psi_E, s) = \varepsilon(\rho_0, \psi_E, s + s_0)$$

Now we check the equation in Theorem 1.9. We need the following lemma:

Lemma 1.15. *Let E/F be a finite extension of nonarchimedean field. Define $\lambda_{E/F}(s, \psi) = \frac{\varepsilon(Ind_{E/F} 1_E, s, \psi_F)}{\varepsilon(1_F, s, \psi_E)}$. Then $\lambda_{E/F}(s, \psi)$ is independent of s.*

(1) comes from Proposition 1.9. (2) is obvious. We now check (3).

$$\begin{aligned}
\frac{\varepsilon(Ind_{E/F}(\rho_0 \otimes \chi_F^{-s_0}), \psi_F, s)}{\varepsilon(\rho_0 \otimes \chi_F^{s_0}, \psi_E, s)} &= \frac{\varepsilon(Ind_{E/F}(\rho_0) \otimes \chi_F^{-s_0}, \psi_F, s)}{\varepsilon(\rho_0 \otimes \chi_F^{s_0}, \psi_E, s)} \\
&= \frac{\varepsilon(Ind_{E/F} 1_E, \psi_F, s - s_0)^n}{\varepsilon(1_E, \psi_E, s - s_0)^n} \\
&= \frac{\varepsilon(Ind_{E/F} 1_E, \psi_F, s)^n}{\varepsilon(1_E, \psi_E, s)^n}
\end{aligned}$$

Hence the proof is finished. □

Chapter 2

The Automorphic Side

2.1 Representation of Totally disconnected Groups

Definition 2.1. *A topological group G is called a totally disconnected group (td group in short) if every neighborhood of the identity contains a compact open subgroup.*

Lemma 2.1. *Let $K_1 \subset K$ be compact open subgroups in G, then K/K_1 is finite. When G is separable, that is to say there is a countable dense subset in G, then G/K is countable.*

Example 2.1. *Let F be a nonarchimedean local field,*
1. The Weil group W_F of F is a td-group.
2. $F^\times$ is a td-group. More generally, for any reductive group G over F, $G(F)$ is a td-group.

Definition 2.2. *A function on G $f : G \longrightarrow \mathbb{C}$ is called smooth if it is locally constant. We use $C^\infty(G)$ to denote the set of all smooth functions on G and $C_c^\infty(G)$ to denote the set of all smooth functions with compact support on G.*

Suppose (π, V) is a representation of G.

Definition 2.3. *(π, V) is called smooth if for any $v \in V$, there exists a compact open subgroup K in G such that v is fixed by K.*

For any representation (π, V), we can define (π_{sm}, V_{sm}) to be $V_{sm} = \bigcup_K V^K$.

Definition 2.4. *(π, V) is called admissible if it is smooth and for any compact open subgroup K in G, $V^K := \{v \in V | \pi(g)v = v \text{ for all } g \in K\}$ is finite dimensional.*

Now suppose (π, V) is a smooth representation, we define the vector space V^* to be the space of linear functionals of V, and the representation (π^*, V^*) is given by $g \cdot \lambda(v) = \lambda(g^{-1}v)$.

Definition 2.5. *For a smooth representation (π, V), its contragredient representation $(\pi^\vee, V^\vee)$ is defined to be $((\pi^*)_{sm}, (V^*)_{sm})$.*

Definition 2.6. *A function $f \in C_c^\infty(G)$ is called a matrix coefficient of (π, V) if there exist $v \in V$ and $v^\vee \in V^\vee$ such that*

$$f(g) = \langle \pi(g)v, v^\vee \rangle$$

Lemma 2.2 (Schur's Lemma for Separable td-group)**.** *For a seperable td-group G, and an irreducible smooth representation (π, V), we have $End_G V = \mathbb{C}$.*

Proof. Refer to Section 2.6 in [BH06]. □

Let Z be the center of G. Then for any $x \in Z$, $\pi(x) \in End_G(V)$. Since π is irreducible, by Schur's Lemma $\pi(x) = \lambda(x) Id_V$ for some $\lambda(x) \in \mathbb{C}$.

Definition 2.7. *The central quasicharacter ω_π of an irreducible smooth representation (π, V) is the quasicharacter on Z defined by $\omega_\pi(x) = \lambda(x)$.*

2.2 Nonarchimedean Representation Theory of $GL(2)$

In this section, we let $G = GL_2(F)$, we define the following algebraic subgroups of G:

$$B = \{\begin{pmatrix} a & b \\ & c \end{pmatrix} \in GL_2(F)\}$$

to be the standard Borel subgroup of G,

$$T = \{\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in GL_2(F)\}$$

to be the standard split maximal torus,

$$N = \{\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in GL_2(F)\}$$

to be the unipotent radical of B, and also

$$Z = \{\begin{pmatrix} a & \\ & a \end{pmatrix} \in GL_2(F)\}$$

to be the center of G.

For the standard Borel group B, we have Levi decomposition $B = T \ltimes N$, and the modular quasicharacter δ_B of B is defined by

$$\delta_B(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}) = ||t_1/t_2||$$

We recall that we have

Proposition 2.1. *(1) Iwasawa decomposition: $G = BK$ where $K = GL_2(O_F)$ is the standard maximal compact subgroup.*
(2) Bruhat decomposition: $G = B \cup BwN$ where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.
(3) For $G = GL_2(F)$, we have Cartan decomposition $G = \bigcup_{a,b \in \mathbb{Z},\ a \leq b} K \begin{pmatrix} \pi_F^a & \\ & \pi_F^b \end{pmatrix} K$

In this section, we study the representation theory of $G = GL_2(F)$. We recall the parabolic induction. Suppose $\chi = \chi_1 \otimes \chi_2$ is a (smooth) character of $T(F)$. Then we define the parabolic induced representation $I(\chi) = Ind_B^G \chi$ by

$$Ind_B^G \chi = \{f : G(F) \longrightarrow V \mid f \text{ is smooth }, f(tng) = \delta_B(t)^{1/2}\chi(t)f(g) \text{ for all } t \in T,\ n \in N\}$$

and

$$(h \cdot f)(g) = f(gh) \qquad \text{for any } h \in G$$

Since any $f \in I(\chi)$ is a smooth function, $I(\chi)$ is a smooth representation.

Lemma 2.3. *$Ind_B^G \chi$ is unitarizable if χ is unitary.*

Now for any smooth representation (π, V) of G, we define $V(N) := \langle v - \pi(n)v \mid v \in V,\ n \in N\rangle$, and $V_N = V/V(N)$. N is normal in B, hence $V(N)$ is closed under B, and N acts trivially on the quotient representation V_N. Thus, we get an action of $T = B/N$ on V_N. $(\pi_N := \pi|_T \otimes \delta_B^{\frac{1}{2}}, V_N)$ is called the Jacquet module of (π, V), and the functor

$$\begin{array}{rcl} Rep(G) & \longrightarrow & Rep(T) \\ (\pi, V) & \longmapsto & (\pi_N, V_N) \end{array}$$

is called Jacquet functor.

Lemma 2.4. *The Jacquet functor is additive and exact.*

Proposition 2.2 (Frobenius Reciprocity Law)**.** *For any smooth represnetation (π, V) of G, and any character χ of T, we have*

$$Hom_G(\pi, I(\chi)) = Hom_T(\pi_N, \chi)$$

Definition 2.8. *An irreducible smooth representation (π, V) of G is called supercuspidal if $V_N = 0$.*

Theorem 2.1. *Let (π, V) be an irreducible smooth representation of G, then it is either a supercuspidal representation itself, or a subrepresentation of $I(\chi)$ where χ is a character of $T(F)$.*

Proof. When V is a subrepresentation of $I(\chi)$. let $\tau : V \hookrightarrow I(\chi)$ be the interwining map. By Frobenius Reciprocity Law $Hom_G(\pi, I(\chi)) = Hom_T(\pi_N, \chi)$ hence $Hom_T(\pi_N, \chi) \neq 0$ and $\pi_N \neq 0$. Therefore, π is not a supercuspidal representation .

Conversely, when π is not supercuspidal, we have to show that $Hom_T(\pi_N, \chi) \neq 0$ for some character χ of T. That is to say V_N has a maximal proper subrepresentation and hence has a nonzero irreducible quotient. We first show that V is a finitely generated module over B. In fact since π is irreducible, we choose a nonzero $v \in V$, V is a finite linear combination of $\pi(g)v$. π is smooth , so suppose v is fixed by $K_0 \subset K$. Lemma 2.1 tells us that K/K_0 is finite. Therefore, $\pi(K)v = \langle v_1, v_2, \cdots v_k\rangle$. And by Iwasawa decomposition $G = BK$ so $v_1, v_2, \cdots v_k$ generates V under B. V is finitely generated over B and V_N is also finitely generated over T. The argument following is standard: let $u_1, u_2, \cdots, u_l$ be a minimal set generating V_N. By Zorn's lemma there is a subrepresentation U which is maximal in the property of $u_l \notin U$. U is proper since the generating set is minimal. $\langle u_1, u_2, \cdots, u_{l-1}\rangle \subset U \subset \langle u_1, u_2, \cdots, u_l\rangle = V$. Thus, U is a maximal proper subrepresentation of V_N.

□

Lemma 2.5. *Suppose σ is an admissible representation of $T(F)$, then $I(\sigma)$ is also admissible.*

Proof. Refer to Proposition 8.2.1 in [GH19]. □

Since χ is smooth character, then $I(\chi)$ is admissible. Now we give the following propositions without proofs.

Theorem 2.2. *(π, V) is supercuspidal if and only if every matrix coefficient of (π, V) is compactly supported modulo $Z(G)$.*

Theorem 2.3. *Every supercuspidal representation of G is admissible.*

Corollary 2.1. *Any irreducible smooth representation of G is admissible.*

Now we wish to classify the noncuspidal representations of $G = GL_2(F)$.

Example 2.2 (Special representation)**.** *We consider the representation $I(\delta_B^{-\frac{1}{2}})$. Then let $\mathbb{C}$ be the subspace of constant functions in $I(\delta_B^{-\frac{1}{2}})$, and this is a subrepresentation of $I(\delta_B^{-\frac{1}{2}})$. The quotient $I(\delta_B^{-\frac{1}{2}})/\mathbb{C}$ is called the Steinberg representation St_G. More generally, for any character χ_0 of $F^\times$, we have a short exact sequence*

$$0 \longrightarrow \chi_0 \longrightarrow I(\delta_B^{-\frac{1}{2}}\chi) \longrightarrow (\chi_0 \circ \det) \otimes St_G \longrightarrow 0$$

where $\chi = \chi_0 \otimes \chi_0$. We usually write $\chi_0 \cdot St_G$ in short for the twisted Steinberg representation.

We have

Proposition 2.3. *$\chi_0 \cdot St_G$ is irreducible for any character χ_0 of $F^\times$.*

Proposition 2.4. *When $\chi_1\chi_2^{-1} \neq |x|^{\pm 1}$, $I(\chi)$ is irreducible.*

When $I(\chi)$ is irreducible, we call it the principal series representation and we often denote it as $\mathcal{B}(\chi_1, \chi_2)$.

Thus, we can get our classification theorem:

Theorem 2.4 (Classification of representations of $GL_2(F)$)**.** *The following is a complete list of the isomorphism classes of irreducible smooth representations of $G = GL_2(F)$:*
(1) the supercuspidal representations.
(2) $\chi_0 \circ \det$, where χ_0 ranges all characters of $F^\times$.
(3) the special representation $\chi_0 \cdot St_G$, where χ_0 ranges all characters of $F^\times$.
(4) the principal series representation $\mathcal{B}(\chi_1, \chi_2)$, where χ_1 and χ_2 range all characters of $F^\times$ satisfying $\chi_1\chi_2^{-1} \neq |x|^{\pm 1}$.

Proposition 2.5. *Suppose (π, V) is an infinite dimensional irreducible smooth representation of $G = GL_2(F)$, then $\dim V_N = 2$ if π is in the principal series, $\dim V_N = 1$ if π is a special representation, and $\dim V_N = 0$ if π is supercuspidal.*

2.3 Jacquet-Langlands L-function of Representations of $GL(2)$

Similar to the L-functions of local characters, we define local L-factors for irreducible admissible representation π of $GL_2(F)$. Let χ be a character of $F^\times$, we define $\alpha(\chi) = \chi(\pi_F)$ if χ is unramified, and $\alpha(\chi) = 0$ if χ is ramified. Then we define

$$L(\pi, s) = \begin{cases} 1 \qquad \text{when } \pi \text{ is supercuspidal} \\ (1-\alpha(\chi_1)q^{-s})^{-1}(1-\alpha(\chi_2)q^{-s})^{-1} \qquad \text{when } \pi = \mathcal{B}(\chi_1, \chi_2) \text{ is in the principal series} \\ (1-\alpha(\chi_0)q^{-s-\frac{1}{2}})^{-1} \qquad \text{when } \pi = \chi_0 \cdot St_G \text{ is special} \end{cases}$$

More generally, for any character χ of $F^\times$, one can just define $L(\chi \times \pi, s)$ to be $L((\chi \circ \det) \otimes \pi, s)$.

To introduce local ε-factors, we need to introduce Whittaker models and Kirillov models. We fix a nontrivial additive character ψ on F. This will induce a character ψ_N on N defined by $\psi_N(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}) = \psi(b)$.

Definition 2.9. *Let (π, V) be a irreducible admissible representation of $GL_2(F)$, with central character ω. A Whittaker functional $\Lambda : \pi \to \mathbb{C}$ is a functional satisfying*

$$\Lambda(\pi(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})v) = \psi_N(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})v \qquad \text{for all } v \in \pi,\ b \in F$$

Given a nonzero Whittaker functional of π, one can associate $v \in \pi$ to $W_v \in C_c^\infty(G)$ by

$$W_v(g) = \Lambda(\pi(g)v) \qquad g \in GL_2(F)$$

It's easy to observe that $W_{\pi(g)v}(h) = W_v(hg)$. Hence if we take the space $W(\pi) = \{W_v \in C_c^\infty(G) | v \in \pi\}$, then $W(\pi)$ is invariant under the action of G by right translation. Hence the representation $W(\pi)$ under right translation is naturally isomorphic to π, and we call this $W(\pi)$ the Whittaker model of π. For $G = GL_2(F)$, we have the following important conclusion:

Theorem 2.5. *When $\dim \pi$ is infinite, then there exists a nonzero Whittaker functional and it is unique up to scalar.*

Proof. Refer to Section 4.3 and 4.4 in [Bum98]. □

Now we introduce the Kirillov model of π. For $v \in \pi$, we define $\phi_v(a) = W_v(\begin{pmatrix} a & \\ & 1 \end{pmatrix})$, $\phi_v \in C^\infty(F^\times)$. We introduce the Kirillov model $K(\pi) = \{\phi_v \in C^\infty(F^\times) \mid v \in \pi\}$. We check that the space $K(\pi)$ is isomorphic to π. In fact,

Proposition 2.6. *Let (π, V) be a irreducible admissible representation of $GL_2(F)$ with Whittaker functional Λ. Then for any $0 \neq v \in V$, there exists $a \in F^\times$ such that*

$$\Lambda(\pi(\begin{pmatrix} a & \\ & 1 \end{pmatrix})v) \neq 0$$

Proof. Refer to Proposition 4.4.7 in [Bum98]. □

Sometimes we use $\pi_{\mathbf{K}}$ to emphasize the action of G on $K(\pi)$. It's easy to observe that

$$\pi_{\mathbf{K}}(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\phi(x) = \phi(ax)$$

$$\pi_{\mathbf{K}}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})\phi(x) = \psi(bx)\phi(x)$$

$$\pi_{\mathbf{K}}(\begin{pmatrix} u & \\ & u \end{pmatrix})\phi(x) = \omega_\pi(u)\phi(x)$$

Here ω_π is the central quasicharacter of π. So the action of B on the Kirrilov model is determined. Recall that in Proposition 2.1, we have Bruhat decomposition $G = B \cup BwB$. Hence the representation π is uniquely determined by $\pi_{\mathbf{K}}(w)$ on $K(\pi)$ where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Now we determine the space $K(\pi)$.

Theorem 2.6. *Suppose π is an infinite dimensional irreducible smooth representation, then $K(\pi)(N) = C_c^\infty(F^\times)$.*

Proof. We first prove that $K(\pi)(N) \subset C_c^\infty(F^\times)$. In fact, since π is smooth, v is fixed by $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ when $b \in \mathfrak{p}^k$ for some $k \in \mathbb{N}^+$. Then this means $\phi(y) = \psi(xy)\phi(y)$ when $x \in \mathfrak{p}^k$. We can choose t such that $\psi(t) \neq 0$, then $\phi(y) = 0$ for any $y \in t\mathfrak{p}^{-k}$. Hence $\phi \in C_c^\infty(F)$. On the other hand, $K(\pi)(N)$ is generated by functions of the form $(\psi(xy) - 1)\phi(y)$. $\psi(xy) = 1$ when $y \in \mathfrak{p}^{k'}$ for some $k' \in \mathbb{N}^+$. Therefore, $(\psi(xy) - 1)\phi(y) \in C_c^\infty(F^\times)$.

Then we prove $C_c^\infty(F^\times)$ is irreducible under B. Since $K(\pi)(N)$ is a nonzero subspace of $C_c^\infty(F^\times)$, this will induce the conclusion. We only need to check for any $\phi \in C_c^\infty(F^\times)$, and $a \in F^\times$, we have $\mathbf{1}_{U_F^n \cdot a} \in \pi_K(B)\phi$ for n sufficiently large. Since $\pi_{\mathbf{K}}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})\phi(x) = \psi(bx)\phi(x)$, we can assume $a = 1$ and $\phi(1) \neq 0$ without loss of generality. Let $f \in C_c^\infty(F)$, we consider

$$\begin{aligned} \phi'(y) &= \int_F f(b)\pi_{\mathbf{K}}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})\phi(y)db \\ &= \int_F f(b)\psi(by)\phi(y)db \\ &= \hat{f}(y)\phi(y) \end{aligned}$$

f is a finite sum of characteristic functions and $\pi_{\mathbf{K}}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})\phi$ is locally constant, hence $\phi' \in \pi_K(N)\phi$. Now we take $\hat{f}(y) = \mathbf{1}_{U_F^n}(y)/\phi(y)$ when n is sufficiently large, and $f(y) = \hat{\hat{f}}(-y)$. Hence $\int_F f(b)\pi_{\mathbf{K}}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})\phi(y)db = \mathbf{1}_{U_F^n}(y)$ and the theorem is proved. □

Then we can get immediately:

Corollary 2.2. *Suppose (π, V) is an infinite dimensional irreducible smooth representation, the space $V(N)$ is irreducible over B.*

Corollary 2.3. *When π is a supercuspidal representation, then $K(\pi) = C_c^\infty(F^\times)$.*

We give following conclusions without proofs:

Proposition 2.7. *Let $(\pi, V) = \pi(\chi_1, \chi_2)$ be an irreducible principal series representation . Then (1) When $\chi_1 \neq \chi_2$, $K(\pi)$ consists of the functions $\phi \in C^\infty(F^\times)$ satisfying $\phi(t)$ vanishes when $|t|$ is large and when $|t|$ is small $\phi(t) = C_1|t|^{1/2}\chi_1(t) + C_2|t|^{1/2}\chi_2(t)$ for some constants C_1 and C_2.*
(2) When $\chi_1 = \chi_2$, $K(\pi)$ consists of the functions $\phi \in C^\infty(F^\times)$ satisfying $\phi(t)$ vanishes when $|t|$ is large and when $|t|$ is small $\phi(t) = C_1|t|^{1/2}\chi_1(t) + C_2 v_F(t)|t|^{1/2}\chi_2(t)$ for some constants C_1 and C_2.

Proposition 2.8. *Let $(\pi, V) = \chi_0 \cdot St_G$ be a special representation. Then $K(\pi)$ consists of the functions $\phi \in C^\infty(F^\times)$ satisfying $\phi(t)$ vanishes when $|t|$ is large and when $|t|$ is small $\phi(t) = C|t|\chi_0(t)$ for some constant C.*

We can define zeta functions similar to Tate's thesis:

Definition 2.10. *For any $\phi \in K(\pi)$, we define $Z(s, \phi, \chi) = \int_{F^\times} \phi(y)\chi(y)|y|^{s-\frac{1}{2}} d^\times y$, where $d^\times y$ is a multiplicative Haar measure on $F^\times$.*

Proposition 2.9. *(1) $Z(s, \phi, \chi)$ is convergent if $Re(s)$ is sufficiently large, and it admits a meromorphic continuation to the whole complex plane.*
(2) $Z(s, \phi, \chi) \in \mathbb{C}(q^s)$. Let $I_\chi = \{Z(s, \phi, \chi) \mid \phi_v \in K(\pi)\}$. $I_{\pi,\chi}$ is a fractional principal ideal in $\mathbb{C}[q^s, q^{-s}]$ and there exists $p_{\pi,\chi}(x) \in \mathbb{C}[x]$ such that $p_{\pi,\chi}(q^{-s})^{-1}$ is the generator of $I_{\pi,\chi}$. If we normalize $p_{\pi,\chi}$ such that $p_{\pi,\chi}(0) = 1$. Then $L(s, \chi \times \pi) = p_{\pi,\chi}(q^{-s})^{-1}$.

Proof. (1) Since we know from the proof of Theorem 2.6, $\phi \in C_c^\infty(F)$. Then $Z(s, \phi, \chi)$ is exactly the same with the zeta functions in Tate's thesis, hence the proofs coincide. (2) comes from the description of □

Theorem 2.7 (Local functional equation)**.** *In the context above, there exists a meromorphic function $\gamma(\chi \times \pi, s, \psi)$ only depend on χ, π and ψ, such that for any $\phi \in K(\pi)$, we have*

$$Z(1 - s, \pi(w)\phi, \omega_\pi^{-1}\chi^{-1}) = \gamma(\chi \times \pi, s, \psi)Z(s, \phi, \chi)$$

We need the following lemma in the proof.

Lemma 2.6. *Given π an irreducible smooth representation of G and quasicharacter χ of $F^\times$. Let $\Lambda_{\chi,s}$ be the space of linear functionals L satisfying*

$$L(\pi(\begin{pmatrix} a & \\ & 1 \end{pmatrix})v) = \chi(a)|a|^s L(v)$$

Then for any fixed χ, there are at most two s modulo $2\pi i/\log q$ such that $\dim \Lambda_{\chi,s} \geq 2$.

Proof of Theorem 2.7. We check that $L_1 = Z(s, \phi, \chi)$ and $L_2 = Z(1-s, \pi(w)\phi, \omega_\pi^{-1}\chi^{-1})$ is contained in $\Lambda_{\chi',s}$ where $\chi'(x) = \chi^{-1}(x)|x|^{-1/2}$. We first check $L_1 \in \Lambda_{\chi',s}$. Recall that $\pi_{\mathbf{K}}(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\phi(x) = \phi(ax)$. So
$Z(s, \phi(ax), \chi) = \int_{F^\times} \phi(ax)\chi(x)|x|^{s-1/2} d^\times x = \int_{F^\times} \phi(t)\chi(a^{-1}t)|a^{-1}t|^{s-1/2} d^\times t = \chi(a)^{-1}|a|^{-s+1/2} Z(s, \phi, \chi)$.

Now we check $L_2 \in \Lambda_{\chi',s}$. Let $\phi' = \pi(w)\phi$. Then $\pi(w)\pi(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\phi = \pi(\begin{pmatrix} a & \\ & a \end{pmatrix})\pi(\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix})\phi'$. We have

$$\begin{aligned} Z(1-s, \pi(w)\pi(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\phi, \omega_\pi^{-1}\chi^{-1}) &= \int_{F^\times} \omega_\pi(a)\phi'(a^{-1}x)|x|^{-s+1/2}\omega_\pi^{-1}(x)\chi^{-1}(x)d^\times x \\ &= \chi(a)^{-1}|a|^{-s+1/2}\int_{F^\times} \omega_\pi(a)\phi'(t)|t|^{-s+1/2}\omega_\pi^{-1}(at)\chi^{-1}(t)d^\times t \\ &= \chi(a)^{-1}|a|^{-s+1/2}Z(1-s, \pi(w)\phi, \omega_\pi^{-1}\chi^{-1}) \end{aligned}$$

Since we have all most all $s_0 \in \mathbb{C}$, $\dim \Lambda_s = 1$, then there exists $\gamma(\chi \times \pi, s_0, \psi) \in \mathbb{C}$ independent of ϕ satisfying $Z(1-s, \pi(w)\phi, \omega_\pi^{-1}\chi^{-1}) = \gamma(\chi \times \pi, s, \psi)Z(s, \phi, \chi)$. $\gamma(\chi \times \pi, s, \psi)$ is meromorphic because both of zeta functions are meromorphic. □

Now we can define the local ε-factors as

$$\varepsilon(\chi \times \pi, s, \psi) := \frac{\gamma(\chi \times \pi, s, \psi)L(\chi \times \pi, s)}{L(\chi^\vee \times \pi^\vee, 1-s)}$$

We have a description for ε similar to $GL(1)$ case in Proposition 1.11 and Corollary 1.1:

Proposition 2.10. *(1) $\varepsilon(\pi, s, \psi) = aq^{-sn(\pi,\psi)}$ for some $a \in \mathbb{C}$ and $n(\pi, \psi) \in \mathbb{Z}$.*
(2) We put $K_1(N) = \{g \in GL_2(O_F) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{p}^N\}$. Suppose the additive character ψ has level one, then $n(\pi, \psi)$ equals the least nonnegative integer N such that $V^{K_1(N)} \neq 0$.

Proof. (1) Similar to $GL(1)$ case, we can choose some $\phi \in K(\pi)$ such that $Z(s, \phi, \chi) = L(s, \pi)$. Then $\varepsilon(\pi, s, \psi) = \frac{Z(1-s,\pi(w)\phi,\omega_\pi^{-1}}{L(\pi^\vee,1-s)} \in \mathbb{C}[q^s, q^{-s}]$ by Proposition 2.9. From functional equation we have $\varepsilon(\omega_\pi^{-1}\chi^{-1}\pi, 1-s, \psi)\varepsilon(\chi\pi, s, \psi) = -1$. Therefore, $\varepsilon(\pi, s, \psi)$ is invertible and hence has the form of aq^{-ns}.
(2) Refer to [Mat13]. □

We also do some explict calculation on local ε-factors as we did in $GL(1)$ case. Let χ be a character of $F^\times$ and $k \in \mathbb{Z}$, we take function $\xi_{\chi,k} \in C_c^\infty(F^\times)$ defined by

$$\xi_{\chi,k} = \begin{cases} \chi(x) & v_F(x) = k \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.8. *Let π be a supercuspidal representation of G with central character ω_π, then*

$$\pi_k(w)(\xi_{\chi,k}) = \varepsilon(\chi^{-1}\pi, \frac{1}{2}, \psi)\ \xi_{\chi^{-1}\omega_\pi, -n(\chi^{-1}\pi,\psi)-k}$$

where $n(\chi, \psi) \in \mathbb{Z}$ is defined as in Proposition 2.10.

Proof. We first prove that if $\phi_1, \phi_2 \in K(\pi)$ satisfies $Z(s, \phi_1, \chi) = Z(s, \phi_2, \chi)$ for any $s \in \mathbb{C}$ and characters χ of $F^\times$, then $\phi_1 = \phi_2$. It's sufficient to show that $\phi_1(1) = \phi_2(1)$. In fact for any $a \in F^\times$, let $\phi_i'(x) = \phi_i(ax)$. $Z(s, \phi_i', \chi) = \chi(a)^{-1}|a|^{-s+1/2}Z(s, \phi_i, \chi)$, hence $Z(s, \phi_1' - \phi_2', \chi)$ for any s and χ. And $\phi_i'(1) = \phi_i(a)$ so $\phi_1(a) = \phi_2(a)$.

We recall the following version of Fourier transform:

Lemma 2.7. *Let M be a compact abelian td-group and Haar measure dm on M is normalized such that $vol(M) = 1$. $f \in C^\infty(M)$. Then*

$$f(1) = \sum_{\chi \in \hat{M}} \int_M f(m)\chi(m)dm$$

where $\hat{M}$ is the group of all characters of M.

Thus, if we put $Z_{\chi,n} = \int_{|y|=q^{-n}}(\phi_1(y) - \phi_2(y))\chi(y)d^\times y$, then $\phi_1(1) - \phi_2(1) = \sum_{\chi \in \hat{O_F^\times}} Z_{\chi,1}$. Therefore, it's enough to show that $Z_{\chi,n} = 0$ for any χ and n. Now we can write $Z(s, \phi_1 - \phi_2, \chi) = \sum_{n\in\mathbb{Z}} Z_{\chi,n}(q^{-s})^n$. This is a finite sum since $\phi_1, \phi_2 \in C_c^\infty(F^\times)$. Since this power series vanish for all q^{-s}, we can get all coefficients $Z_{\chi,n} = 0$. Hence, our claim is proved.

In view of this, we only need to check the RHS satisfies the functional equation in Theorem 2.7 for any χ and s. In fact

$$Z(s, \xi_{\chi,k}, \chi_0) = \int_{F^\times} \xi_{\chi,k}(x)\chi_0(x)|x|^{s-1/2}d^\times x = \begin{cases} aq^{-k(s-1/2)} & \chi_0 = a\chi^{-1} \text{ for some constant } a \text{ on } \{v_F(x) = k\} \\ 0 & \text{otherwise} \end{cases}$$

and similarly

$$Z(1-s, \xi_{\chi^{-1}\omega_\pi, -n(\chi^{-1}\pi,\psi)-k}, \omega_\pi^{-1}\chi_0^{-1}) = \begin{cases} aq^{-(n(\chi^{-1}\pi,\psi)+k)(s-1/2)} & \chi_0 = a\chi^{-1} \text{ for some constant } a \text{ on } \{v_F(x) = k\} \\ 0 & \chi_0 \neq \chi^{-1} \end{cases}$$

Recall that $n(\pi, \psi)$ is defined by $\varepsilon(\pi, s, \psi) = q^{(1/2-s)n(\pi,\psi)}\varepsilon(\pi, \frac{1}{2}, \psi)$ (Proposition 2.10). So zeta functions satisfy exactly $Z(1 - s, \varepsilon(\chi^{-1}\pi, \frac{1}{2}, \psi)\xi_{\chi^{-1}\omega_\pi, -n(\chi^{-1}\pi,\psi)-k}, \omega_\pi^{-1}\chi_0^{-1}) = \varepsilon(\chi_0\pi, s, \psi)Z(s, \xi_{\chi,k}, \chi_0)$ for any χ and s. The theorem is proved.

□

Since $\{\xi_{\chi,k}\}$ is a basis for $C_c^\infty(F^\times)$, and π is uniquely determined by the central quascharacter and $\pi_{\mathbf{K}}(w)$ on $C_c^\infty(F^\times)$, we can get:

Corollary 2.4. *If π_1 and π_2 are supercuspidal representations, and for any character χ of $F^\times$, $\varepsilon(\chi \times \pi_1, s, \psi) = \varepsilon(\chi \times \pi_2, s, \psi)$. Then $\pi_1 \cong \pi_2$.*

Proof. We only need to prove the central quasicharacters $\omega_{\pi_1} \cong \omega_{\pi_2}$. For any $0 \neq b \in F$, we have

$$\begin{pmatrix} 1 & -b \\ & 1 \end{pmatrix}\begin{pmatrix} b^2 & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} -b & \\ & -b \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & -b^{-1} \\ & 1 \end{pmatrix}$$

in $GL_2(F)$. Let A denote the matrix in the LHS, and B denote the matrix in the RHS. Claim: $(\pi_1)_{\mathbf{K}}(A) = (\pi_2)_{\mathbf{K}}(A)$. In fact we write

$$\psi(y) = \sum_{k\in\mathbb{Z}} \sum_j a_{j,k}\xi_{\chi_j,k}$$

Then let $A' = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$,

$$\begin{aligned} A'(\xi_{\chi,k})(y) &= \varepsilon(\chi^{-1}\pi_i, \frac{1}{2}, \psi)\pi_{\mathbf{K}}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})(\psi(by)\xi_{\chi^{-1}\omega_{\pi_i}, -n(\chi^{-1}\pi_i,\psi)-k}(y)) \\ &= \varepsilon(\chi^{-1}\pi_i, \frac{1}{2}, \psi)\pi_{\mathbf{K}}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})(\sum_j a_{j,-n(\chi^{-1}\pi_i,\psi)-k+v_F(b)}\chi_j(b)\xi_{\chi_j^{-1}\chi^{-1}\omega_{\pi_i}, -n(\chi^{-1}\pi_i,\psi)-k}(y)) \\ &= \varepsilon(\chi^{-1}\pi_i, \frac{1}{2}, \psi)\sum_j a_{j,-n(\cdot)-k+v_F(b)}\chi_j(b)\varepsilon(\chi_j\chi\omega_{\pi_i}^{-1}\pi_i, \frac{1}{2}, \psi)\xi_{\chi_j\chi, k+n(\cdot)-n(\cdot)} \end{aligned}$$

Also, $A(\xi_{\chi,k})(y) = \psi(-by)A'(\xi_{\chi,k})(b^2y)$. Since by the proof of Proposition 2.10,

$$\varepsilon(\omega_\pi^{-1}\chi_j\chi\pi, s, \psi) = -\varepsilon(\chi_j^{-1}\chi^{-1}\pi, 1-s, \psi)^{-1}$$

$n(\cdot)$ only depends on ε-factors, the claim is proved. Hence we have $(\pi_1)_{\mathbf{K}}(B) = (\pi_2)_{\mathbf{K}}(B)$. While

$$\begin{aligned} (\pi_i)_{\mathbf{K}}(B)(\xi_{\chi,k})(y) &= \omega_{\pi_i}(-b)(\pi_i)_{\mathbf{K}}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})(\psi(-b^{-1}y)\xi_{\chi,k}(y)) \\ &= \omega_{\pi_i}(-b)(\pi_i)_{\mathbf{K}}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})(\sum_j a_{j,k}\chi_j(-b)^{-1}\xi_{\chi\chi_j,k}(y)) \\ &= \omega_{\pi_i}(-b)\sum_j a_{j,k}\chi_j(-b)^{-1}\varepsilon(\chi^{-1}\chi_j^{-1}\pi, \frac{1}{2}, \psi)\xi_{\chi^{-1}\chi_j^{-1}\omega_{\pi_i}, -n(\cdot)-k} \\ &= \omega_{\pi_i}(-by)\sum_j a_{j,k}\chi_j(-b)^{-1}\varepsilon(\chi^{-1}\chi_j^{-1}\pi, \frac{1}{2}, \psi)\xi_{\chi^{-1}\chi_j^{-1}, -n(\cdot)-k} \end{aligned}$$

The sum over j are independent of central quasicharacters . Hence $\omega_{\pi_1}(-by) = \omega_{\pi_2}(-by)$ when then sum over j is nonzero. Now since for any $x \in F^\times$, we can choose proper $f \in C_c^\infty(F^\times)$ such that $(\pi_i)_{\mathbf{K}}(B)(x) \neq 0$, we have $\omega_{\pi_1}(x) = \omega_{\pi_2}(x)$ for any $x \in F^\times$. □

For noncuspidal representations , we also have the following:

Lemma 2.8. *If $\pi \cong \pi(\chi_1, \chi_2)$ is in the principal series, then we have $\varepsilon(\chi \times \pi, s, \psi) = \varepsilon(\chi \times \chi_1, s, \psi)\varepsilon(\chi \times \chi_2, s, \psi)$. If $\pi \cong \chi_0 \cdot St_G$ is the special representation, then $\varepsilon(\chi \times \pi, s, \psi) = -\varepsilon(\chi \times \chi_0, s, \psi)$.*

Proof. Refer to Section 26 in [BH06]. □

Combining these two lemmas, we get immediately:

Theorem 2.9 (Converse theorem for $GL(2)$)**.** *Suppose π and π' are irreducible smooth representations of $GL_2(F)$, and*

$$\begin{aligned} L(\chi \times \pi, s) &= L(\chi \times \pi', s) \\ \varepsilon(\chi \times \pi, s, \psi) &= \varepsilon(\chi \times \pi', s, \psi) \end{aligned}$$

for all characters χ of $F^\times$. Then $\pi \cong \pi'$.

Chapter 3

Local Langlands Correspondence of $GL(2)$

3.1 Statement of the Correspondence

Let $\mathcal{G}_n(F)$ denote the set of equivalent classes of n-dimensional semisimple Weil-Deligne representation of W_F over $\mathbb{C}$, and $\mathcal{A}_n(F)$ denote the the set of isomorphism classes of irreducible smooth representation of $GL_n(F)$ over $\mathbb{C}$.

Theorem 3.1 (Local Langlands Correspondence of $GL(2)$)**.** *Let ψ be a additive character on F and $\psi \neq 1$. There is a unique bijective map*

$$\pi : \mathcal{G}_2(F) \longrightarrow \mathcal{A}_2(F)$$

such that

$$\begin{aligned} L(\chi\pi(\rho), s) &= L(\chi \otimes \rho, s) \\ \varepsilon(\chi\pi(\rho), s, \psi) &= \varepsilon(\chi \otimes \rho, s, \psi) \end{aligned}$$

for all $\rho \in \mathcal{G}_2(F)$ and for all characters χ of $F^\times$. Furthermore, π is independent of the choice of ψ.

Notice that the uniqueness of the correspondence comes immediately from the converse theorem. We will prove the existence part for most of cases in the following sections.

Suppose $\tau \in Aut(\mathbb{C})$, then for complex vector space V, it will induce morphism $\tau_V : GL(V) \longrightarrow GL(V)$. If $\pi : G \longrightarrow GL(V)$ is a complex representation of a group G, we take ${}^\tau\pi = \tau_V \circ \pi$. Now we see that τ acts on both $\mathcal{G}_n(F)$ and $\mathcal{A}_n(F)$ by the way we described above. Then we can see that the Langlands correspondence in Theorem 3.1 is not compatible with the action of τ. In fact we have the following modification:

Definition 3.1. *For $\rho \in \mathcal{G}_2(F)$. we define $\widetilde{\rho} \in \mathcal{G}_2(F)$ by*

$$\widetilde{\rho}(x) = \rho(x)||x||^{\frac{1}{2}}, \qquad x \in W_F$$

Theorem 3.2. *If we define the map*

$$\begin{aligned} \pi_{\mathbb{C}} : \mathcal{G}_2(F) &\longrightarrow \mathcal{A}_2(F) \\ \rho &\longmapsto \pi(\widetilde{\rho}) \end{aligned}$$

Then it satisfies

$$\pi_{\mathbb{C}}({}^{\tau}\rho) = {}^{\tau}\pi_{\mathbb{C}}(\rho)$$

Moreover, $\pi_{\mathbb{C}}$ is the unique map satisfying

$$\begin{aligned} L(\chi\pi_{\mathbb{C}}(\rho), s) &= L(\chi \otimes \rho, s - \frac{1}{2}) \\ \varepsilon(\chi\pi_{\mathbb{C}}(\rho), s, \psi) &= \varepsilon(\chi \otimes \rho, s - \frac{1}{2}, \psi) \end{aligned}$$

Now let $\mathcal{G}_2(F, \overline{\mathbb{Q}}_\ell)$ denote the set of equivalent classes of 2-dimensional semisimple Weil-Deligne representation of W_F over $\overline{\mathbb{Q}}_\ell$, and $\mathcal{A}_2(F, \overline{\mathbb{Q}}_\ell)$ denote the the set of isomorphism classes of irreducible smooth representation of $GL_2(F)$ over $\overline{\mathbb{Q}}_\ell$.

Theorem 3.3 (ℓ-adic Local Langlands Correspondence of $GL(2)$)**.** *There is a unique bijection*

$$\pi_\ell : \mathcal{G}_2(F, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathcal{A}_2(F, \overline{\mathbb{Q}}_\ell)$$

such that

$$\pi_\ell({}^{\tau}\rho) = {}^{\tau}\pi_{\mathbb{C}}(\rho)$$

for all $\rho \in \mathcal{G}_2(F)$ and for all field isomorphisms $\tau : \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{Q}}_\ell$.

3.2 The Tame Langlands Correspondence

Let $\mathcal{G}_n^0(F) \subseteq \mathcal{G}_n(F)$ (resp. $\mathcal{G}_n^1(F)$) be the subset of Weil-Deligne representations (ρ, N) for which ρ is irreducible (resp. reducible). We also let $\mathcal{A}_n^0(F) \subseteq \mathcal{A}_n(F)$ (resp. $\mathcal{A}_n^1(F)$) be the subset of supercuspidal (non-supercuspidal) representations, Then we first give the correspondence from $\mathcal{G}_n^1(F)$ to $\mathcal{A}_n^1(F)$:

Theorem 3.4. *There exists a bijective correspondence*

$$\pi^1 : \mathcal{G}_2^1(F) \longrightarrow \mathcal{A}_2^1(F)$$

satisfying the conditions in Theorem 3.1.

Proof. Let $\rho' = (\rho, N) \in \mathcal{G}_2^1(F)$. Since ρ is reducible, $\rho = \chi_1 \oplus \chi_2$. When $\chi_1\chi_2^{-1}(x) \neq ||x||^{\pm 1}$, $\pi(\chi_1, \chi_2)$ is in the principal series. We claim that in this case, $N = 0$. In fact, suppose $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the relation formula of Weil-Deligne representation can be writen as

$$\begin{pmatrix} \chi_1(x) & \\ & \chi_2(x) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \chi_1^{-1}(x) & \\ & \chi_2^{-1}(x) \end{pmatrix} = \begin{pmatrix} a & \chi_1\chi_2^{-1}(x)b \\ \chi_1^{-1}\chi_2(x)c & d \end{pmatrix} = ||x|| \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then we can deduce $a = d = 0$. And neither $\chi_1\chi_2^{-1}(x)$ nor $\chi_1^{-1}\chi_2(x)$ equals $||x||$, $b = c = 0$. Hence $N = 0$. Thus, we define

$$\pi^1((\chi_1 \oplus \chi_2)) = \pi(\chi_1, \chi_2)$$

in this case. $L(\chi\pi_1(\rho'), s) = L(\chi \otimes \rho', s)$ by definition and $\varepsilon(\chi\pi_1(\rho'), s, \psi) = \varepsilon(\chi \otimes \rho', s, \psi)$ by Lemma 2.4.

When $\chi_1\chi_2^{-1}(x) = ||x||$ or $||x||^{-1}$, WLOG we assume $\chi_1 = \phi(x)||x||^{1/2}$ and $\chi_2 = \phi(x)||x||^{-1/2}$, then $N = 0$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ from our calculations above. Now we put $\pi^1((\rho, 0)) = \phi \cdot \det$ and $\pi^1((\rho, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})) = \phi \circ St_G$ in this case. Then the relation of L-factors come from definitions and the relation of ε-factors come from Lemma 2.4. $\square$

Now we consider the bijection from $\mathcal{G}_n^0(F)$ to $\mathcal{A}_n^0(F)$. We do not give the full part of the correspondence here, but give the "tame Langlands correspondence" when $n = 2$. We begin with the following conclusion in the Galois side:

Proposition 3.1. *Suppose $p \nmid n$, then for any $(\rho, V) \in \mathcal{G}_n^0(F)$, there exists a seperable extenstion E/F of degree n and a character θ of $E^\times$, such that $\rho \cong Ind_{E/F}\theta$.*

Proof. We recall the following elementary conclusion in the representation theory of finite group: Suppose G is a finite group, and V is an irreducible (complex) representation of G, then $\dim V$ divides $|G|$. Since P_F is a profinite p-group, and ρ is a smooth representation, so there is an open subgroup $K_0 \subset P_F$ such that $\rho|_{K_0} = 1$, and P_F/K_0 is a finite p-group. we conclude that $\rho|_{P_F} = \bigoplus_{1\le i\le m} \bigoplus_i n_i\chi_i$, where χ_i are distinct characters of P_F. We denote $V_i = n_i\chi_i$.

Since P_F is normal in W_F, for any $g \in W_F$, we take $\rho_g(x) = \rho(gxg^{-1})$. ρ_g is also a representation of P_F. Now we consider $\rho(g) : V \longrightarrow V$:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \downarrow{\scriptstyle \rho(g)} & & \downarrow{\scriptstyle \rho(g)} \\ V & \xrightarrow{\rho} & V \end{array}$$

Hence $\rho(g)$ is a morphism of representations between ρ and $\rho(g)$. We also write $(\chi_i)_g(x) = \chi_i(gxg^{-1})$ as a character on P_F. Then $\rho_g \cong \bigoplus_i n_i(\chi_i)_g$. Since χ_i are distinct characters and $(\chi_i)_g$ are also distinct characters , there is a permutation ι on $\{1, 2, \cdots m\}$ such that $\rho(g)V_i \subset V_{\iota(j)}$. Since $\rho(g)$ is an isomorphism of vector spaces, we have $n_i = n_{\iota(j)}$. Now since V is irreducible, for any $i \neq j$, there exists $g \in W_F$ such that $\rho(g)V_i \subset V_{\iota(j)}$. Thus, we have $n_i = n_j$ for any i and j.

Let $H = \{g \in W_F \mid gV_1 \subset V_1\}$. Then $V = Ind_H^{W_F} V_1$ follows from the following lemma in representation theory.

Lemma 3.1. *Suppose G is a finite group and $H \le G$ is a subgroup. V is an irreducible representation of G. Then $V = Ind_H^G W$ for some subspace W of V if and only if there exists a decomposition $V = \oplus_r W_r$ such that for any $g \in G$ and $1 \le i \le r$, $gW_i = W_j$ for some j, and $H\{g \in G \mid gW_1 \subset W_1\}$. In this case $V = Ind_H^G W_1$.*

Now we analyse the representation V_1 of H. We denote it by τ_1. When $\dim V_1 = 1$ then we are done. So we assume $\dim V_1 \geq 2$ form now on. We need the following lemma in representation theory:

Lemma 3.2. *Suppose G is a finite group and H is a normal subgroup in G, and G/H is abelian. Suppose there is a representation π of G such that $\pi|_H = \chi Id_V$ for some character of H. Then χ can be extended to G.*

Since I_F/P_F and W_F/I_F are both abelian, we can extend χ_1 to χ_1' on W_F. (Since χ_1 is smooth so we only involve essntially finite groups here.) And $\chi_1'^{-1}\tau$ is trivial on P_F. Since $(H \cap I_F)/P_F$ is abelian, $\tau_1 = \bigoplus_{1 \leq j \leq n_1} \widetilde{\chi}_j$ on $H \cap I_F$. We claim that $\widetilde{\chi}_{j_1} \neq \widetilde{\chi}_{j_2}$ for any $j_1 \neq j_2$. In fact suppose so, then $V_1 = \oplus r\widetilde{\chi}_j$ for some $r \geq 2$ by the arguments similar to above. Now we take $V_{1,1} = r\widetilde{\chi}_1$ and $H_{1,1} = \{g \in H | \ gV_{1,1} \subset V_{1,1}\}$. Then also by the arguments similar to above, $V_1 = Ind_{H_{1,1}}^{H} V_{1,1}$. But $H_{1,1}/(H \cap I_F)$ is abelian, and $\dim H_{1,1} \geq 2$. So $V_{1,1}$ is reducible on $H_{1,1}$ which contradicts to the condition that V is irreducible. □

Let $\mathcal{G}_n^{in}(F)$ be the set of irreducible Weil-Deligne representations which is induced by a single character, then the "tame Langlands correspondence" just gives the correspondence from $\mathcal{G}_n^{in}(F)$ to a subset of $\mathcal{A}_n^0(F)$. And the above proposition tells us that when $p \nmid n$, $\mathcal{G}_n^{in}(F) = \mathcal{G}_n^0(F)$. Hence in this case the "tame Langlands correspondence" will give us the full part of Langlands correspondence.

To give the "tame Langlands correspondence", it's enough to construct supercuspidal representations from a character $\chi : E^\times \to \mathbb{C}^\times$ where E/F is a field extension of degree n, and then verifying the conditions in Theorem 3.1. In the next section, we will do this for the case when $n = 2$ and $p \nmid 2$ by making use of Weil representation.

3.3 Constructing Supercuspidal Representations from Weil Representation

We first recall on the classical Weil representation. Suppose (V, q) is an *even* dimensional vector space with quadratic form q, and ψ is an additive character on F. Then the Weil represnetaiton of $SL_2(F) \times O(V)$ on $C_c^\infty(V)$ can be defined as:

- $\omega_\psi(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix})f(x) = \chi_{(V,q)}(a)|a|^{\dim V/2} f(ax), \quad a \in F^\times,$
- $\omega_\psi(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})f(x) = \psi(bq(x))f(x), \quad b \in F,$
- $\omega_\psi(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})f(x) = \gamma(V, q)\hat{f}(x),$
- $\omega_\psi(h)f(x) = f(h^{-1}x), \quad h \in O(V)$

we should explain the symbols appearing here. the character $\chi_{(V,q)}$ is defined as

$$\chi_{(V,q)}(a) = (a, (-1)^{\dim V/2} det(V))_F$$

where $(\ , \ )_F$ is the Hilbert symbol, and $det(V) \in F^\times/(F^\times)^2$ is the determinant of the matrix ,$\gamma(V, q)$ is the Weil index, and the fourier transform is defined by

$$\hat{f}(x) = \int_{V(F)} f(y)\psi(\langle x, y \rangle)dy$$

We can extend this representation to a larger subgroup in $GL_2(F) \times GO(V)$. We define $\tilde{G} = \{(g, h) \in GL_2(F) \times GO(V) \mid \det(g) = \det(h)^{-1}\}$.

We consider the special case when $V = (E, N_E)$, where E/F is a seperated quadratic extension viewed as a 2-dimensional vectore space over F and N_E is the norm map on E which is a quadratic form on E. Then $E^\times \hookrightarrow GO(E)$ by multiplication maps. Thus, we get a natural representation of the group $\mathcal{G} = \{(g,h) \in GL_2(F) \times E^\times \mid det(g) = N_E(h)^{-1}\}$ on the space $C_c^\infty(E)$ by restricting the Weil representation of $\widetilde{G}$ on $\mathcal{G}$. We denote this representation by ρ_ψ. We first choose generators of $\mathcal{G}$ by $\omega \in GL_2(F)$ and $E^1 \subset E^\times$ and

$$\begin{aligned} n(b) &= (\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, 1) \quad b \in F \\ c(u) &= (\begin{pmatrix} u & \\ & u \end{pmatrix}, u^{-1}) \quad u \in F^\times \\ u(a) &= (\begin{pmatrix} N_{E/F}(a) & \\ & 1 \end{pmatrix}, (a^\tau)^{-1}) \quad a \in E^\times \end{aligned}$$

Then we can write ρ_ψ explicitly:

- $\rho_\psi(n(b))f(y) = \psi(bN_{E/F}(y))f(y)$
- $\rho_\psi(c(u))f(y) = \varkappa(u)f(y)$
- $\rho_\psi(u(a))f(y) = |a|_E^{1/2}f(ay)$
- $\rho_\psi(\omega)f(y) = \lambda_{E/F}(\psi)\hat{f}(y^\tau)$
- $\rho_\psi(h)f(y) = f(h^{-1}x), \quad h \in E^1$

We observe that when we restrict ρ_ψ to the subgroup $SL_2(F) \times E^1$ where $E^1 = \ker Nm_{E/F} \subset E^\times$, then $\rho_\psi|_{SL_2(F)\times E^1} \cong \bigoplus_i (\pi_i \boxtimes \chi_i)$. Inspired by this, we can "seperate" supercuspidal representations of $GL_2(F)$ from ρ_ψ. Let Θ be a character of $E^\times$ such that $\theta = \Theta|_{E^1} \neq 1$. We call this kind of Θ the regular character of $E^\times$ (with respect to $F^\times$).

Lemma 3.3. *$\theta = \Theta|_{E^1} \neq 1$ if and only if $\Theta \neq {}^\tau\Theta$ where $Gal(E/F) = \{1, \tau\}$.*

We construct a representaiton of $GL_2(F)$ as follows. We take $C_c^\infty(E,\theta) := \{f \in C_c^\infty(E) \mid f(xy) = \theta(x)f(y)$ for any $x \in E^1,\ y \in E\ \}$ be the θ component of $C_c^\infty(E)$ with respect to E^1. Since $f(x)$ is constant when $|x| < a$ for some $a \in \mathbb{R}$. Then we take $|y| < a$ and $x \in E^1$ such that $\theta x \neq$. Then $f(0) = f(xy) = \theta(x)f(y) = \theta(x)f(0)$. Thus, $f(0) = 0$ and $C_c^\infty(E,\theta) \subset C_c^\infty(E^\times)$.

Now we take $\xi(\Theta,\psi) = (\Theta^{-1}\rho_\psi, C_c^\infty(E,\theta))$. Then $\xi(\Theta,\psi)|_{E^1} = 1$ and we can regard $\xi(\Theta,\psi)$ as a representation of the group $\widetilde{G}/E^1 \cong G_\varkappa$ where $G_\varkappa = \{g \in G \mid \det g \in Nm(E^\times/F^\times)\}$. We can also consider the function $\Theta^{-1}f$. Then $\Theta^{-1}f(xy) = f(y)$ for $x \in E^1$. This will induces $f_0 \in C_c^\infty(F_\varkappa^\times)$ such that $\Theta^{-1}f(y) = f_0(\overline{y})$ where $F_\varkappa^\times = E^\times/E^1 \cong \{x \in F^\times \mid x \in Nm(E^\times/F^\times)\}$. And this map

$$\begin{aligned} C_c^\infty(E,\theta) &\longrightarrow C_c^\infty(F^\times\varkappa) \\ f &\longmapsto f_0 \end{aligned}$$

is an isomorphism. Now we define $\pi(\Theta,\psi) = Ind_{G_\varkappa}^G \xi(\Theta,\psi)$. Deriving from the definition of Weil representation , we can write the representation $\xi(\Theta,\psi)$ of group $G_\varkappa$ on $C_c^\infty(F_\varkappa^\times)$ explicitely:

- $\xi(\Theta,\psi)(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})f(y) = \psi(by)f(y) \quad b \in F$
- $\xi(\Theta,\psi)(\begin{pmatrix} u & \\ & u \end{pmatrix})f(y) = \varkappa(u)\Theta(u)f(y) \quad u \in F^\times$
- $\xi(\Theta,\psi)(\begin{pmatrix} a & \\ & 1 \end{pmatrix})f(y) = |x|^{1/2}\Theta(a)f(ay) \quad a \in F_\varkappa^\times$

Proposition 3.2. *$\pi(\Theta, \psi)$ is irreducible and supercuspidal.*

Proof. For convenience in our proof, we use V to denote $C_c^{\infty}(F_{\varkappa}^{\times})$ and W to denote $Ind_{G_{\varkappa}}^{G} C_c^{\infty}(F_{\varkappa}^{\times})$. Since $N \leq G_{\varkappa} \leq G$, then $W(N) = Ind_{B_{\varkappa}}^{B} V(N)$. And $B/B_{\varkappa} = G/G_{\varkappa}$, so $W = W(N)$ if and only if $V = V(N)$. So we check $V = V(N)$ next. We have $C_c^{\infty}(F_{\varkappa}^{\times})$ is irreducible under $B_{\varkappa}$ (similar to the proofs in), and hence $\xi(\Theta, \psi)$ is irreducible. Now for irreducible smooth representation $\xi(\Theta, \psi)$, $V(N)$ is irreducible under $B_{\varkappa}$ (similar to the proofs in). Thus, we see $V(N) = C_c^{\infty}(F_{\varkappa})$. □

We compare the local ε-factors of $\pi(\Theta, \psi)$ and Θ :

Theorem 3.5. *Suppose ψ is a nontrivial additive chracter of F, and Θ is a character of $E^{\times}$. Let $\rho = Ind_{E/F}\Theta$ and $\pi = \pi(\Theta, \psi)$. Then we have*

$$\varepsilon(\chi\pi, s, \psi) = \varepsilon(\chi\rho, s, \psi)$$

for any chracter χ of $F^{\times}$.

Corollary 3.1. *In the context of Theorem 3.5, we can give the correspondence map*

$$\begin{array}{rcl} \mathcal{G}_2^0(F) & \longrightarrow & \mathcal{A}_2^0(F) \\ \rho = Ind_{E/F} & \longmapsto & \pi = \pi(\Theta, \psi) \end{array}$$

satisfying the conditions of 3.1.

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