Rapport de stage

by Ethan André

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Présentation du stage

Le stage s'est déroulé au Laboratoire pour la topologie algébrique et les neurosciences au sein de l'EPFL (École polytechnique fédérale de Lausanne) et était dirigé par la mathématicienne Kathryn Hess que j'ai contactée suivant les conseils de mon tuteur Paul Laurain. j'ai ainsi eu l'opportunité de travailler avec le post-doctorant Raphael Reinauer que je tiens à remercier chaleureusement ici.

Déroulement du stage

J'étais donc sous la direction de Raphael Reinauer que je voyais toutes les semaines de 1 à 2 heures en présentiel ou visioconférence. Ce dernier étant Allemand, nous avons travaillé entièrement en anglais, ce qui était une experience nouvelle pour moi est rapidement devenue spontanée et naturelle, et n'a jamais été un frein à notre bonne communication. Avec la petite équipe du laboratoire j'ai pu assister à de nombreux séminaires et exposés autour de divers sujets ratachés à la topologie algébrique hebdomadairement ainsi que rencontrer de nombreux chercheurs et doctorants dans des domaines aussi intéressants que divers. J'ai profité pleinement de la vie de campus pour mon travail et mes loisirs. J'étais en Suisse de mi-février à fin Juin, comment ne pas parler de ses beau paysages d'hiver et de printemps, Le magnifique lac Léman, le début du Jura, les montagnes que j'ai parcourues à ski et à pied avec autant de plaisir. j'en aurai gardé un excellent souvenir si je ne m'étais pas cassé une dent en tombant à vélo la dernière semaine de mon séjour.

Apports

Ce stage était (en omettant les quelques heures passées sur mon TIPE en classes préparatoires) mon premier travail de recherche. Au début le cadre me paraissait vertigineux, toutes ces notions familières à mes collègues m'étaient étrangères et aliennes. Mais rapidement, wikipédia aidant, je me suis familiarisé avec les objets mathématiques manipulés, et bien que mon intuition concernant les suites spectrales avait toujours (et a toujours!) ses limites, mes échanges avec Raphael étaient moins asymétriques, il ne fallait plus tout m'expliquer, et nous alternions en proposant des idées, sans besoin d'expliciter les détails. À la fin des réunions, le tableau finissait couvert d'écritures indifférentiable à du charabia pour le non-initié.

Cadre mathématique

Le sujet du stage était "spectral sequences in persistent homology", deux concepts que j'ai dû comprendre et rapidement assimiler, mon travail s'est principalement concentré autour de 4 ouvrages, [1] pour des résultats fondamentaux sur les suites spectrales, [3] pour l'aspect algorithmique et comprendre comment implementer le calcul de l'homologie persistente . Une fois les notions comprises, nous nous sommes sommes plongés dans les détails en s'appuyant sur deux articles très liés d'Alvaro Torras Casas [2] et de Hee Rhang Yoon [5]

Algebraic Topology in Lausanne

by Ethan André

Abstract

We explore different means of computing persistent homology of simplical complexes using spectral sequences, first using exact couples to describe very explicitely the spectral sequence created. later, we use covers and add together the homology of smaller spaces in order to recover persisstent homology. Finally we discuss algorithms to compute H_0 homology using this principle.

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Definition 1. *a **Simplical complex** X is subspace of $\mathbb{R}^n$ defined by the simplexes it contains (or abstract) (maybe we don't need to define it) a subcomplex Y of X is a subspace of X that is transitive for the $\prec$ relation, i.e. $s \in Y$ and $x \prec s \Rightarrow x \in Y$*

we will write $\sigma^p \in X$ for any $p - \text{simplex}$ appearing in X

Definition 2. ***simplical homology** is the homology of the simplical differential complex, it is known to be isomorphic to our usual homology, we will use the two interchangeably when writing about "homology"*

Definition 3. *a **filtration** on a given simplical complex X is a functor*
$F: (\mathbb{R}, \leqslant) \longrightarrow \text{sub}\, X$ where the morphisms in the category of subspaces of X are inclusions,
*a filtration is called **finite** if there are finitely many $a < b$ such that $F(a) \neq F(b)$.*

Definition 4. *given a point cloud $X = \{x_i\}_{i \in I} \in (\mathbb{R}^m)^I$, it's **Vietoris-Rips complex** is defined as $\forall r \in \mathbb{R}^+ \; X^r := \bigcup_{n \in N} \{[x_1, \ldots, x_n] \in X / \forall (i, j), \|x_i - x_j\| < r\}$ it is a filtration of the space X^D where D is the diameter of X . We note that if X is finite, its Vietoris Rips complex is a finite filtration*

Definition 5. *given a ring R, a $R -$ **persistence module** is a functor $V: (\mathbb{R}, \leqslant) \longrightarrow \text{Vect}_R$, it is called **tame** if $\forall r \in \mathbb{R} \quad \dim(V(r)) < \infty$ and there are only finitely many $t \geq 0$, called critical values, for which the map $V_{t-\varepsilon \leqslant t+\varepsilon} : V_{t-\varepsilon} \rightarrow V_{t+\varepsilon}$ isn't an isomorphism for arbitrarily small $\varepsilon > 0$.*

Theorem 6. ***Structure Theorem** : let $\mathbb{F}$ be a field and $M : \mathbb{R} \rightarrow \text{vect}_{\mathbb{F}}$ be a tame persistence module, then M is isomorphic to a direct sum of interval modules :*

$$M \approx \bigoplus I(a_i, b_i)$$

where $I(a_i, b_i)$ is the persistence module :

$$0 \rightarrow \ldots 0 \rightarrow \mathbb{F} \rightarrow \mathbb{F} \rightarrow \ldots \rightarrow \mathbb{F} \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

and all the morphisms send $1_{\mathbb{F}}$ to $1_{\mathbb{F}}$ and $\forall r \in \mathbb{R} \qquad I(a_i, b_i)(r) = \begin{cases} \mathbb{F} & \text{if } a_i \leqslant r \leqslant b_i \\ 0 & \text{elsewhere} \end{cases}$

we will call this unique decomposition it's barcode moving forward:

Corollary 7. *two persistence modules with the same barcode are isomorphic*

Definition 8. *we can now define **persistent homology** :*

Let $X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_N$ be a filtration of a finite complex $X = X_N$. By applying the R-homology functor, we obtain the following sequence of R-modules :

$$PH_\bullet : H_\bullet(X_0, R) \rightarrow H_\bullet(X_1, R) \rightarrow \ldots \rightarrow H_\bullet(X_N, R) \tag{1}$$

which in the case when R is a Field, is a tame persistence module. By the Structure Theorem, (1) decomposes uniquely as $PH_\bullet \approx \bigoplus I(a_i, b_i)$,

we will always be working with $\mathbb{F}_2$ homology coefficients and $\mathbb{F}_2$-persistence modules

Remark 9. Given any short exact sequence of differential complexes, we have a long exact sequence in homology :

$$0 \longrightarrow A \xrightarrow{b} B \xrightarrow{c} C \longrightarrow 0$$

$$\text{gives} \quad H_n(A) \xrightarrow{b^*} H_n(B) \xrightarrow{c^*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

where ∂ is the connecting morphism

1 Computing Persistent Homology

Topological data is extracted through persistent homology in the form of barcodes, the summands $I(a_i, b_i)$ in the decomposition presented above. To compute them directly, one would have to compute homology for every dimension at every step of the filtration which would be very heavy. A more efficient way to proceed is to use a tool that will be the cornerstone of this work : **the spectral sequence** .

1.1 Exact Couples

Definition 10. *: A* ***spectral sequence*** *is a sequence of bigraded modules over a ring R (omitted in notations for clarity),* $\forall r \in \mathbb{N}$ *:*

$E^r_{*,*} = (E^r_{p,q})_{(p,q)\in\mathbb{Z}^2}$ *along with differentials* $d^r : E^r_{p,q} \rightarrow E^r_{p-r,q+r-1}$ *such that* $d^r \circ d^r = 0$

and $E^{r+1}_{p,q} = H(E^r_{p,q}, d^r)$ *i.e.* $E^{r+1}_{p,q} = \dfrac{\ker(d^r : E^r_{p,q} \rightarrow E^r_{p-r,q+r-1})}{\operatorname{Im}(d^r : E^r_{p+r,q-r+1} \rightarrow E^r_{p,q})}$

Definition 11. *an* ***exact couple*** *is a short exact sequence of R-modules*

$$D \xrightarrow{i} D \xrightarrow{j} E \xrightarrow{\partial} D \xrightarrow{i} D \qquad \text{or equivalently} \qquad \begin{array}{ccc} D & \xleftarrow{i^*} & D \\ & \searrow^{j} \quad {}^{\partial}\nearrow & \\ & E & \end{array}$$

let $d = j \circ \partial$ *, we define the derived couple of a given exact couple as*

$E' = \ker(d) / \operatorname{Im}(d)$

$D' = i(D) \qquad i' = i|_D$

next, we define $j'(i(x)) := [j(x)] = j(x) + \operatorname{Im}(d)$

$\partial'([x]) := \partial(x) \in D'$

We verify easily that these morphisms are well defined and form the short exact sequence :

$$D' \xrightarrow{i'} D' \xrightarrow{j'} E' \xrightarrow{\partial'} D' \xrightarrow{i'} D'$$

The rth derived couple is defined by induction couple as

$(D^{(r+1)}, E^{(r+1)}) = \left(D^{(r)\prime}, E^{(r)\prime}\right)$

we see that $j^{(r)}(i^r(x)) := [j(x)]_r = j(x) + \operatorname{im}(d^{(r)})$

$i^{(r)} := i|_{D^{(r-1)}} \qquad D^{(r)} = \operatorname{im}(i^r) \qquad$ *and* $\partial^{(r)}([x]_r) := [\partial(x)]_r$

$$\begin{array}{ccccccccccccc} D & \xrightarrow{i} & & D & \xrightarrow{i} & & \cdots\cdots & D & \xrightarrow{i} & & D & \xrightarrow{i} & D \\ {}_j\searrow & & \nearrow\partial & {}_j\searrow & & \nearrow\partial & & {}_j\searrow & & \nearrow\partial \; {}_j\searrow & & \nearrow\partial & \\ & E & & & E & & \cdots & & E & & & E & \end{array}$$

$i^{(r)}(x) = c \Rightarrow j^{(r)}(x) = [j(c)]_r$

1.2 Spectral Sequences

As described in [1], given any filtration $(X_p)_{p \leqslant N}$ of a topological space X, we construct an exact couple and a spectral sequence on which we can apply theorems to compute persistent homology. Starting with the long exact sequences for the (X_p, X_{p-1}) couples in homology

$$H_n(X_{p-1}) \xrightarrow{i^*} H_n(X_p) \xrightarrow{j^*} H_n(X_p, X_{p-1}) \xrightarrow{\partial} H_{n-1}(X_{p-1}) \xrightarrow{i^*} H_{n-1}(X_p) \xrightarrow{j^*} H_{n-1}(X_p) \to \ldots$$

we "sow" them together in the following complex :

$$\begin{array}{ccccccccc}
 & \downarrow i^* & & & & \downarrow i^* & & & \\
\ldots. & H_n(X_{p-1}) & \xrightarrow{j^*} & H_n(X_{p-1}, X_{p-2}) & \xrightarrow{\partial} & H_{n-1}(X_{p-2}) & \xrightarrow{j^*} & H_{n-1}(X_{p-2}, X_{p-3}) & \ldots. \\
 & \downarrow i^* & & & & \downarrow i^* & & & \\
\ldots. & H_n(X_p) & \xrightarrow{j^*} & H_n(X_p, X_{p-1}) & \xrightarrow{\partial} & H_{n-1}(X_{p-1}) & \xrightarrow{j^*} & H_{n-1}(X_{p-1}, X_{p-2}) & \ldots. \\
 & \downarrow i^* & & & & \downarrow i^* & & & \\
\ldots. & H_n(X_{p+1}) & \xrightarrow{j^*} & H_n(X_{p+1}, X_p) & \xrightarrow{\partial} & H_{n-1}(X_p) & \xrightarrow{j^*} & H_{n-1}(X_p, X_{p-1}) & \ldots. \\
 & \downarrow i^* & & & & \downarrow i^* & & & \\
\ldots. & H_n(X_{p+2}) & \xrightarrow{j^*} & H_n(X_{p+2}, X_{p+1}) & \xrightarrow{\partial} & H_{n-1}(X_{p+1}) & \xrightarrow{j^*} & H_{n-1}(X_{p+1}, X_p) & \ldots. \\
 & \downarrow i^* & & & & \downarrow i^* & & &
\end{array}$$

let $D = \bigoplus_{n,p} H_n(X_p)$ and $E = \bigoplus_{n,p} H_n(X_p, X_{p-1})$
the morphisms are given above : $i = \bigoplus_{n,p} i^*$, $j = \bigoplus_{n,p} j^*$, $\partial = \bigoplus_{n,p} \partial$

Proposition 12. *(D, E, i, j, ∂) is an exact couple.*

Proof. because of the long exact sequence, $\mathrm{im}(j) = ker(\partial)$ □

It is shown in [1] how to get a spectral sequence from the exact couple above. Letting $E^r_{p,n-p} = E^{(r)}_{p,n}$ and $d^r := d^{(r)}$ makes it a spectral sequence. We can apply the convergence theorem :

Theorem 13. $E^\infty_{p,n-p} = G_p H_n(X)$

Proof. *$E_{*,*}$ is finite, hence there exists $a \in \mathbb{N}$ such that $\forall r \geqslant a$ $d^r_{*,*} = 0$ and $E^\infty_{*,*} = E^a_{*,*}$ using a classical result from [4] chapter 13 , the sequence converges towards $G_p H_n(X)$:*
$$= \frac{\mathrm{Im}(H_n(X_p) \to H_n(X))}{\mathrm{Im}(H_n(X_{p-1}) \to H_n(X))}$$ □

Using this result, we can recover homology :

Corollary 14. $H_n(X) \approx \bigoplus_p G_p H_n(X)$

1.3 Homological Spectral Sequence

Obtaining persistent homology is trickier, it has been shown in [1] that one can get the persistent homology barcodes by computing N spectral sequences, considering how big N can get in the Vietoris Rips complex of a large point cloud, we will provide a simple result in a less general case where the spectral sequence in question directly provides persistent barcodes.

Let us now consider a finite point cloud $X \subset \mathbb{R}^d$ and it's Vietoris-Rips complex $(X^r)_{r>0}$ written as $X^0 \hookrightarrow X^{d_1} \hookrightarrow X^{d_2} \hookrightarrow \ldots X^{d_n}$ we modify it : for all $i \leqslant n$, $X^{d_i} \hookrightarrow X^{d_{i+1}}$ and $X^{d_{i+1}}$ contains finitely many simplexes $\sigma_1, \sigma_2 \ldots . \sigma_n$ that are not in X^{d_i}, we add additional $n-1$ steps to the filtration :

$X^{d_i} \hookrightarrow X^{d_i} \cup \{\sigma_1\} \hookrightarrow X^{d_i} \cup \{\sigma_1\} \cup \{\sigma_2\} \hookrightarrow \ldots . \hookrightarrow X^{d_i} \cup \{\sigma_1\} \cup \ldots \cup \{\sigma_n\} = X^{d_{i+1}}$

after renumerating them, we end up with an **itterative filtration** i.e. $\forall p \geqslant 1$ $X_p = X_{p-1} \bigcup \{\sigma_p\}$ and $X_N = X$

Remark 15. Although we will compute different barcodes because we renumerated the filtration, the original ones can easily be recovered (if $X_p = X^{d_i} \cup \{\sigma_1\}$, replace p by d_i in $I(p,*)$)

We now consider the homological spectral sequence associated to the itterative filtration $(X_p)_{0 \leqslant p \leqslant N}$ on the filtered complex $(C_*(X_p), d)_{0 \leqslant p \leqslant N}$ as defined above

Let us now compute the first page of the Spectral sequence

Proposition 16. $E^1_{p,n-p} = H_n(X_p, X_{p-1}) = \left\{ \begin{array}{ll} <[j(\sigma_p)]> \approx \mathbb{Z}/2\mathbb{Z} & \text{if } n = \dim(\sigma_p) \\ 0 & \text{else} \end{array} \right\}$

Proof. *The group can only be nontrivial when $n = \dim(\sigma_p)$, in that case, $[j(\sigma_p)]$ is a non-trivial class since $j(\sigma_p)$ is homeomorphic to a $p-$sphere. It is the only generator.* □

Since the next pages of the sequence are subquotients of these modules, the $\mathbb{Z}/2\mathbb{Z}$ - modules in the spectral sequence will either be 0 or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the morphisms will all be trivial or the identity.

we have quotient projections $[.]_r : H_{p+q}(X_p, X_{p-1}) \rightarrow E^r_{p,q}$ for $r \geqslant 1$

The only interesting case is when d^r can be non trivial, hence these homology groups are non trivial and generated by $[j(\sigma_p)]_r$ and $[j(\sigma_{p-r})]_r$

Lemma 17. *computing d^r*

Proof. *To compute d^r, we use the exact couple structure and "lift" through i^**

$$\begin{array}{ccccc} & & H_{n-1}(X_{p-r}) & \xrightarrow{j^*} & H_{n-1}(X_{p-r}, X_{p-r-1}) \\ & & \downarrow i^* & & \\ & & H_{n-1}(X_{p-r+1}) & \xrightarrow{j^*} & H_{n-1}(X_{p-r+1}, X_{p-r}) \\ & & \downarrow i^* & & \\ & & \ldots & & \\ & & H_{n-1}(X_{p-2}) & \xrightarrow{j^*} & H_{n-1}(X_{p-2}, X_{p-3}) \\ & & \downarrow i^* & & \\ H_n(X_p, X_{p-1}) & \xrightarrow{\partial} & H_{n-1}(X_{p-1}) & \xrightarrow{j^*} & H_{n-1}(X_{p-1}, X_{p-2}) \end{array}$$

let $s \in H_n(X_p, X_{p-1})$ such that $[s]_r \in E^r$ is non trivial

$s \in \ker(d^{r-2}) \subset \ker(d^1) = \ker(j^* \circ \partial)$, it follows that $\partial(s) \in \ker(j^*) = \operatorname{im}(\ i^*\ )$ hence, since the couples are exact $\exists! c_2 \in H_{n-1}(X_{p-2})$ such that $i^*(c_2) = \partial(s)$ and by definition, $d^2(s) := j^*(c_2) \in \operatorname{Im}(d^1)$.

We can keep going, so long as $[s]_r \in E^r \subset \ker(d^k)$ *i.e.* $k \leqslant r-1$ *,we can lift. This induction provides :*

$\exists\, !c \in H_{n-1}(X_{p-r})$ $i^r(c) = \partial(s)$. *hence :*

$[\, j^*(c)\,]_r = d^r([s]_r)$

□

1.4 linking the Spectral Sequence to Persistent Pairs

We now present our result :

Lemma 18. *let* $(X_p)_{p \leqslant N}$ *be an itterative filtration for a simplical complex* $X = X_N$*, then for all* p*,* σ_p *either creates or kills a barcode in persistent homology.*

Proof. let $\dim(\sigma_p) = n$, since the filtration is iterative and respects the structure, the $(n-1)$ simplexes that form the boundary of σ_p are in X_{p-1}, two cases present:

- either $d(\sigma_p)$ is a boundary in $C_{n-1}(X_{p-1})$ in which case adding σ_p creates a n-cycle
- or $d(\sigma_p)$ is not a boundary in $C_{n-1}(X_{p-1})$, meaning it represents a cycle that is killed by σ_p (contractible) □

Theorem 19. *let* $(X_p)_{p \leqslant N}$ *be an interative filtration for a simplical complex* $X = X_N$*, then each summand* $I(p-r, p)$ *in its persistent homology is in bijectively corresponds to a nontrivial differential* $d^r: E^r_{p,q} \to E^r_{p-r,q+r-1}$ *on the* $r-\text{th}$ *page of the spectral sequence*

Proof. assuming such a summand in the barcode, there exists $c \in H_n(X_{p-r})$ such that :

$$\begin{array}{ccccccc} H_n(X_{p-r-1}) & \xrightarrow{i^*} & H_n(X_{p-r}) & \xrightarrow{i^*} & H_n(X_{p-1}) & \xrightarrow{i^*} & H_n(X_p) \\ & & c & \to & c' & \to & 0 \end{array}$$

where c and c' are non trivial classes and $n = \dim(\sigma_{p-r})$

using the simplical complex structure, we can write $c = [\sigma_{p-r} + \sum_l \sigma_l]$ where σ_l are n-simplexes with $l < p-r$ (if σ_{p-r} didn't appear in the sum, c would be the image of a cycle in X_{p-r-1}, which would go against our hypothesis)

since

$c \in \ker\Big(H_n(X_{p-r}) \xrightarrow{i^*} H_n(X_p)\Big) \backslash \ker\Big(H_n(X_{p-r}) \xrightarrow{i^*} H_n(X_{p-1})\Big)$

there are $(n+1)-\text{simplexes}\, (\sigma_{i_j})_{i_j < p}$ such that $\sigma_p, \sigma_{i_1}, \ldots, \sigma_{i_k}$ fill c, i.e.

$d\big(\sigma_p + \sum_{i_j} \sigma_{i_j}\big) = \sigma_{p-r} + \sum_l \sigma_l = c$

looking at their homological classes, $d(\sigma_p) = c'$ thus $\partial^{(r)}([j(\sigma_p)]_r) := [\partial([j(\sigma_p)])]_r = [c']_r$, and $d^r([j(\sigma_p)]_r) = j^{(r)} \circ \partial^{(r)}([j(\sigma_p)]) = j^{(r)}([c']_r) = j^{(r)}(i^{(r)}([c]_r)) = [j_{p-r}(c)]_r = [j(\sigma_{p-r})]_r$

We just proved that : for any $I(p-r, p)$, $d^r([j(\sigma_p)]_r) = [j(\sigma_{p-r})]_r$. We now prove that both of the classes in question are non trivial, from the Lemma we know that $\forall p \leqslant N$, p appears in a persistence pair.

We now prove that all lower degrees differentials are trivial, that is, if $I(p-r, p)$ is a pair, then $\forall k < r$

$d^k_{p,n-p} = 0$ and $d^k_{p-r,n-p+r} = 0$

$$\begin{array}{ccccccccc} H_n(X_{p-r}) & \to \ldots & H_n(X_{p-k}) & \xrightarrow{j^*} & H_n(X_{p-k}, X_{p-k-1}) & \ldots \xrightarrow{\partial} & H_n(X_p) & \xrightarrow{j^*} & H_n(X_p, X_{p-1}) \\ & & & & \downarrow [.]_k & & & & \downarrow [.]_k \\ & & & & E^k_{p-k,n-p+k} & & \xleftarrow{d^k} & & E^k_{p,n-p} \end{array}$$

for $k < r$, $d^k(\, [j(\sigma_p)]_k\,) = j^{(k)} \circ \partial^{(k)}([j(\sigma_p)]_k) = j^{(k)}([\, c'\,]_k) = j^{(k)}\big(\, i^{(k)}([c]_k)\, \big) = [\, j_{p-k}(c)\,]_k = \, 0\,$ because $\, c\, \in H_n(X_{p-r})$ and $H_n(X_{p-k}) \xrightarrow{j^*} H_n(X_{p-k}, X_{p-k-1})$ sends c to 0 since $X_{p-r} \subset X_{p-k}$

also, $d^k([j(\sigma_{p-r})]_k) = j^{(k)} \circ \partial^{(k)}([j(\sigma_{p-r})]_k) = j^{(k)} \circ \partial^{(k)}(c) = 0$ since c is a boundary.

We prove by induction $\wp(r) = \forall p \forall n\, d^r_{p,n-p} \neq 0 \Leftrightarrow (p-r, p)$ is a persistence pair

$\wp(1)$ holds since if $(p-1, p)$ is a pair, the results above means $d^1([j(\sigma_p)]_1) = [j(\sigma_{p-1})]_1 \neq 0$ since $H_n(X_p, X_{p-1}) \neq 0$

assume $\wp(r')$ is true for all $r'<r$, consider the $E^r_{*,*}$ page, according to the hypothesis, only the terms appearing in short pairs have been killed by nontrivial differentials, this assures us that if $(p-r,p)$ is a pair, $E^r_{p,n-p}$ and $E^r_{p-r,n-p+r}$ are non trivial, the result above proves $d^r\neq 0$. The rest of the pairs have either already been killed or live longer than r and the result above tells us that in that case $d^r=0$. hence $\wp(r)\square$ □

Acknowledgments. One might expect such a result to provide a new way of computing persistent homology, maybe efficiently. Such was our hope when starting this work. When computing persistent homology of a simplical complex, the input data is given in the form of the position of the points and a matrix for the differential in our case, an adjacency matrix suffices since we are in $\mathbb{F}_2$). From there, the process describes above can be used to compute E^0 and E^1, however it is not obvious how d^2 is computed (Casas's video on the subject shows well how tricky it can be). So far, the only way to compute the differentials in the spectral sequences is by matrix row reduction as in Hatcher, which already provides the persistence pairs, this work is in no way a faster way of computing perstence pairs, for which fast algortihm have been implemented like the one developped in [6]

I justify distributed persistent homology as more than abstract nonsense because it follows from the philosophy of algebraic geometry : transforming topological objects into algebraic ones, here we have done so with the usual homology functor, and in the last section added a "temporal" dimension to it, discovering a more complex algebraic structure, and more properties. here we have a new structure with covers of our space, which, under lax hypotheses add even more properties/information. And we need to use very complex algebraical tools such as the spectral sequence in order to unravel it.

2 Distributed Persistent Homology

We now take a completely different approach, we still consider a point cloud's Vietoris Rips complex with it's itterative filtration, but we add a given 2-cover of this space, assumed to respect the simplical structure.

Definition 20. *let X be a point cloud, with a cover $A\cup B$, for $\varepsilon>0$, let X^*, A^*, B^* denote the associated Vietoris-Rips complexes.*

2.1 Double Complex

for $\varepsilon>0$, the raw topological information we can extract from this are the simplical complexes (with coefficients in $\mathbb{F}_2$:

$$\begin{array}{ll} C_*(X^\varepsilon) & : \quad \ldots\to C_n(X^\varepsilon)\xrightarrow{d_n} C_{n-1}(X^\varepsilon)\to\cdots\to C_0(X^\varepsilon)\\ C_*(A^\varepsilon) & : \quad \ldots\to C_n(A^\varepsilon)\xrightarrow{d_n} C_{n-1}(A^\varepsilon)\to\cdots\to C_0(A^\varepsilon)\\ C_*(B^\varepsilon) & : \quad \ldots\to C_n(B^\varepsilon)\xrightarrow{d_n} C_{n-1}(B^\varepsilon)\to\cdots\to C_0(B^\varepsilon)\\ C_*(A^\varepsilon\cap B^\varepsilon) & : \ldots\to C_n(A^\varepsilon\cap B^\varepsilon)\xrightarrow{d_n} C_{n-1}(A^\varepsilon\cap B^\varepsilon)\to\cdots\to C_0(A^\varepsilon\cap B^\varepsilon)\end{array}$$

additionally, the cover (A,B) comes with a natural simplical structure : the Nerve with the differential

$$\begin{array}{rlll}\delta: & C_*(A^\varepsilon\cap B^\varepsilon) & \to & C_*(A^\varepsilon)\oplus C_n(B^\varepsilon)\\ & x & \mapsto & x\oplus x\end{array}$$

This induces differential morphisms between the simplical complexes :

$$\begin{array}{ccccc} C_n(A^\varepsilon \cap B^\varepsilon) & \xrightarrow{d_n} & C_{n-1}(A^\varepsilon \cap B^\varepsilon) & \to \cdots \to & C_0(A^\varepsilon \cap B^\varepsilon) \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ C_n(A^\varepsilon) \oplus C_n(B^\varepsilon) & \xrightarrow{d_n} & C_{n-1}(A^\varepsilon) \oplus C_{n-1}(B^\varepsilon) & \to \cdots \to & C_0(A^\varepsilon) \oplus C_0(B^\varepsilon) \end{array}$$

where everything commutes. We consider the associated total complex:

$$S_n^{\text{tot},\varepsilon} := C_n(A^\varepsilon) \oplus C_n(B^\varepsilon) \oplus C_{n-1}(A^\varepsilon \cap B^\varepsilon) \text{ and } d^{\text{tot}} := d_n \oplus \delta + d_{n-1}$$

using these notations, [2] proved that using a spectral sequence type argument, one can compute the Homology of this complex and identifed it as :

Theorem 21. $H_n(X^\varepsilon) \approx H_n(S_*^{\text{tot},\varepsilon}) \approx G_1 H_n(S_*^{\text{tot},\varepsilon}) \oplus G_2 H_n(S_*^{\text{tot},\varepsilon}) \approx \mathrm{E}_{0,n}^{\infty,\varepsilon} \oplus \mathrm{E}_{1,n-1}^{\infty,\varepsilon}$

Proof. *proven by Casas in [2]* □

more precisely,

$$\mathrm{E}_{0,n}^{\infty,\varepsilon} := \frac{Z_{0,n}^\infty}{B_{0,n}^\infty} := \frac{\{(c,0) \in S_n^{\text{tot}} / d_n(c) = 0\}}{\{d^{\text{tot}}((x,s)), (x,s) \in S_{n+1}^{\text{tot}} / d_n(s) = 0\}} \approx \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1}) + \delta(\ker(d_n))}$$

$$\approx \frac{H_n(A^\varepsilon) \oplus H_n(B^\varepsilon)}{\mathrm{Im}(\delta^*)} := P_n^\varepsilon$$

$$\mathrm{E}_{1,n-1}^{\infty,\varepsilon} := \frac{Z_{1,n-1}^\infty}{Z_{0,n}^\infty + B_{1,n-1}^\infty} := \frac{\{(x,c) \in S_n^{\text{tot}} / d^{\text{tot}}(x,c) = 0\}}{\ker(d_n) \oplus \{d^{\text{tot}}(x,s), (x,s) \in S_{n+1}^{\text{tot}}\}}$$
$$\approx \{\bar{x} \in H_{n-1}(A^\varepsilon \cap B^\varepsilon) / \delta(x) \in \mathrm{Im}(d_n)\} := Q_n^\varepsilon$$

we will use these isomorphisms to identify classes in $E_{*,*}^{\infty,*}$ *going forward.*

For $\varepsilon < \varepsilon'$ *, the inclusion* $X^\epsilon \subset X^{\varepsilon'}$ *induces inclusion morphisms for all of the modules above, where everything commutes as well :*

Let $(\varepsilon_i)_{1 \leqslant i \leqslant N}$ *be the (finitely many) instants when the Vietoris rips complex is changed, we have naturally induced morphisms :*

$$C_*(X^{\varepsilon_1}) \xrightarrow{h_1} C_*(X^{\varepsilon_2}) \longrightarrow \cdots \xrightarrow{h_N} C_*(X^{\varepsilon_N})$$
$$C_*(A^{\varepsilon_1}) \xrightarrow{h_1} C_*(A^{\varepsilon_2}) \longrightarrow \cdots \xrightarrow{h_N} C_*(A^{\varepsilon_N})$$
$$C_*(B^{\varepsilon_1}) \xrightarrow{h_1} C_*(B^{\varepsilon_2}) \longrightarrow \cdots \xrightarrow{h_N} C_*(B^{\varepsilon_N})$$
$$C_*(A^{\varepsilon_1} \cap B^{\varepsilon_1}) \xrightarrow{h_1} C_*(A^{\varepsilon_2} \cap B^{\varepsilon_2}) \longrightarrow \cdots \xrightarrow{h_N} C_*(A^{\varepsilon_N} \cap B^{\varepsilon_N})$$

These morphisms commute with the differentials, since S_*^{tot} *is constructed by summing these modules, these morhisms go to homology* $PH_n(S_*^{\text{tot}}) := (H_n(S_*^{\text{tot},\varepsilon_i}), h_i^*)_{1 \leqslant i \leqslant N}$ *creating a persistence module structure. In fact this persistent module is isomorphic to* $PH_n(X)$ *:*

Theorem 22. $PH_n(S_*^{\text{tot}}) \approx PH_n(X)$ *as persistence morphisms.*

Proof. *Yoon proved it in [5]* □

2.2 recovering the persistence modules

$$\forall i \;\; P_n^\varepsilon = \frac{H_n(A^\varepsilon) \oplus H_n(B^\varepsilon)}{\mathrm{Im}(\delta^*)} \;\; Q_n^\varepsilon = \ker(\,\delta^* \colon H_{n-1}(A^\varepsilon \cap B^\varepsilon) \to H_n(A^\varepsilon) \oplus H_n(B^\varepsilon))$$

In [5], Yoon the same modules are expressed using cosheaf homology theory, we will not need to use it.

Consider :

$$\begin{array}{ccccccc}
H_n(X^i) & \longrightarrow & H_n(X^{i+1}) & \longrightarrow & \ldots \rightarrow & H_n(X^N) \\
\downarrow & & \downarrow & & & \downarrow \\
H_n(S_*^{\text{tot},i}) & \rightarrow & H_n(S_*^{\text{tot},i+1}) & \longrightarrow & \ldots \rightarrow & H_n(S_*^{\text{tot},N}) \\
\downarrow & & \downarrow & & & \downarrow \\
P_n^i \oplus Q_n^i & & P_n^{i+1} \oplus Q_n^{i+1} & & \ldots & P_n^N \oplus Q_n^N
\end{array}$$

with commutating diagramms above. Our goal is to complete this diagram with arrows between the P, Qs so that we have an isomorphism between persistent modules.

We already start with morphisms provided by the natural inclusion morphisms $C_n(X^{\varepsilon}) \longrightarrow C_n(X^{\varepsilon'})$. They respect the simplical differentials and the δ_n's , hence the induce moprphisms in homology :

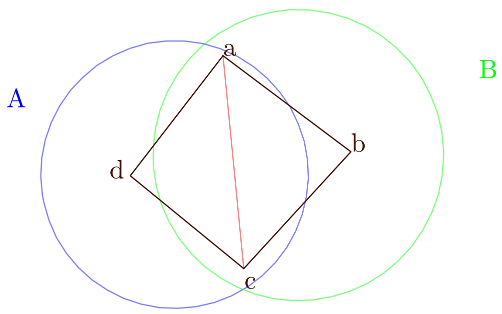
$$\begin{array}{ccccccc}
P_n^i & \xrightarrow{\Phi_i} & P_n^{i+1} & \xrightarrow{\Phi_{i+1}} & \ldots & \xrightarrow{\Phi_{N-1}} & P_n^N \\
\oplus & & \oplus & & & & \oplus \\
Q_n^i & \xrightarrow{\Phi_i} & Q_n^{i+1} & \xrightarrow{\Phi_{i+1}} & \ldots & \xrightarrow{\Phi_{N-1}} & Q_n^N
\end{array}$$

However, the diagram

$$\begin{array}{ccc}
H_n(S_*^{\text{tot},i}) & \xrightarrow{i^*} & H_n(S_*^{\text{tot},i+1}) \\
\downarrow & & \downarrow \\
P_n^i \oplus Q_n^i & \xrightarrow{\Phi_i \oplus \Phi_i} & P_n^{i+1} \oplus Q_n^{i+1}
\end{array}$$

doesn't always commute, we provide an example of such a case :

Example 23. Let us consider the following point cloud $X = \{a, b, c, d\}$ where $l = \text{ac} < \text{db}$



let $e < l$ and $s = [\text{ab} + \text{bc} + \text{cd} + \text{da}]$, then

$$\begin{array}{ccc} H_1(X^e) & \longrightarrow & H_1(X^l) \\ \downarrow \pi^e & & \downarrow \pi^l \\ P_1^e \oplus Q_1^e & \xrightarrow{\Phi_e \oplus \Phi_e} & P_1^l \oplus Q_1^l \end{array}$$

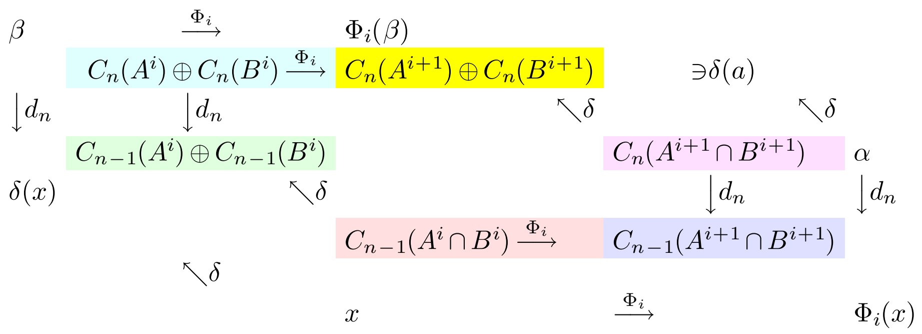
- $H_1(X^e) \approx \mathbb{F}_2$ is generated by s
- $H_1(X^l) \approx \mathbb{F}_2^2$ is generated by [ab + bc + ca] and [ac + cd + da]
- the top arrow sends s to a nonzero class.
- $P_1^e = 0 \qquad Q_1^e = \langle [a + c] \rangle$
- $P_1^l = \mathbb{F}_2^2 \qquad Q_1^l = 0$
- $\Phi_e(\pi^e(s)) = 0$ and π^l is supposed to be an isomorphism, hence the diagram doesn't commute

Remark 24. This in fact the only type of problem that this rightful intuition raises, we solve it by looking at classes in $\ker\Big(Q_n^* \xrightarrow{\Phi_e} Q_n^{*+1}\Big)$ and correcting by sending them to a class in P_n^{*+1} . In the example, that would be

$Q_1^e \ni \pi^e(s) \longrightarrow [\mathrm{ab} + \mathrm{bc} + \mathrm{ca}] \oplus [\mathrm{ac} + \mathrm{cd} + \mathrm{da}] \in P_1^l$

2.3 Constructing the morphisms

Let $K_n^i = \ker(Q_n^i \to Q_n^{i+1})$ $f\colon K_n^i \to P_n^{i+1}$, there are multiple ways of lifting these classes, a choice must be made and once it is, it has consequences later.



Let $[x] \in K_n^i \subset Q_n^i$, we know $x \in C_{n-1}(A^i \cap B^i)$
$\Phi_i([x]) = 0$ in $Q_n^{i+1} \subset H_{n-1}(A^{i+1} \cap B^{i+1})$ hence $\exists\ \alpha \in C_n(A^{i+1} \cap B^{i+1})$ s.t. $d_n(\alpha) = \Phi_i(x)$
$[x] \in \ker(\delta^*)$ hence $\exists\ \beta \in C_n(A^i) \oplus C_n(B^i)$ s.t. $d_n(\beta) = \delta(x)$

Let $f([x]) := [\ \Phi_i(\beta) + \delta(\alpha)\ ]$ be our expression of choice, this doesn't depend on the choice for α, but it yields different results depending on the β lift chosen. To make f into a morphism , we will need to chose a basis (b_*) at each point of the modules and corresponding lifts (β_*).

Following in the context of the previous example, $x = a + c$, $\Phi_e(x) = a + c$ choices for α and β are $\alpha = [\mathrm{ca}]$ $\beta = [\mathrm{ab} + \mathrm{bc}] \oplus [\mathrm{cd} + \mathrm{da}]$, we have $[\Phi_e(\beta) + \delta(\alpha)] = [\mathrm{ab} + \mathrm{bc} + \mathrm{ca}] \oplus [\mathrm{cd} + \mathrm{da} + \mathrm{ca}] \in H_1(A^l) \oplus H_1(B^l)$

we “only” need to lift $K_n^i \subset Q_n^i$ but it impacts the rest :

$$\begin{array}{lllll} P_n^i \xrightarrow{\Phi_i} & P_n^{i+1} \xrightarrow{\Phi_{i+1}} & \ldots & \xrightarrow{\Phi_{N-1}} P_n^N \\ \oplus f^i \nearrow & \oplus \quad f^{i+1} \nearrow & & f^{N-1} \nearrow \oplus \\ Q_n^i \xrightarrow{\Phi_i} & Q_n^{i+1} \xrightarrow{\Phi_{i+1}} & & \xrightarrow{\Phi_{N-1}} Q_n^N \end{array}$$

$$\begin{array}{llll} P_n^i \xrightarrow{\Phi_i} & P_n^{i+1} \xrightarrow{\Phi_{i+1}} & P_n^{i+2} \xrightarrow{\Phi_{i+2}} & \\ \oplus \quad f^i \nearrow & \oplus \quad f^{i+1} \nearrow & \oplus \quad f^{i+2} \nearrow & \ldots \\ K_n^i \xrightarrow{\Phi_i = 0} & K_n^{i+1} \xrightarrow{\Phi_{i+1} = 0} & K_n^{i+2} \xrightarrow{\Phi_{i+2} = 0} & \end{array}$$

$Q_n^1 = K_n^1 \oplus V$ let (k_*, v_*) be a basis, we have (β_*) arbitrarily chosen , then $(\Phi^1(v_*))$ are linearly independant, and can be completed as a basis $((\Phi^1(v_*)), \gamma_*)$ of Q_n^2 . We define β on this basis :
$\beta(\Phi^1(v)) := \Phi^1(\beta(v))$
$\beta(\gamma) := c \quad s.t. \quad d(c) = \delta(\gamma)$ (chose any)

Theorem 25. *The persistence module created by this process is isomorphic to $PH_n(X)$*

Proof. *This result is proven by Yoon in [5]* □

Remark 26. All of the work discussed above holds on the condition that the cover “respects the simplical structure” in order to pass to homology, for this to be true, A^ε, B^ε, $A^\varepsilon \cap B^\varepsilon$ must be subcomplexes of X^ε, this means that if $\sigma \in A^\varepsilon$ and $\sigma \succ s \in B^\varepsilon$ then $s \in A^\varepsilon \cap B^\varepsilon$ and can only be the case for ε small enough, this limit ε^* is computed by Yoon [5]. In the following section, we will assume that X^* reach connectedness before ε^*

3 Persistent $0-$Homology

This case illustrates some of the mechanisms at play in the constructions above : according to the Theorem, we have an isomorphism of persistence modules :

$$\begin{array}{ccccc} H_0(X^1) & \xrightarrow{i^*} & H_0(X^2) & \rightarrow \cdots \rightarrow & H_0(X^N) \\ \downarrow & & \downarrow & & \downarrow \\ P_0^1 & \xrightarrow{f^1} & P_0^2 & \rightarrow \cdots \rightarrow & P_0^N \end{array}$$

where $P_0^i = \dfrac{H_0(A^i) \oplus H_0(B^i)}{\mathrm{Im}(\delta\colon H_0(A^i \cap B^i) \rightarrow H_0(A^i) \oplus H_0(B^i))}$

We understand what $PH_0(X)$ is : $H_0(X^0)$ is generated by the connected components of X^0 (ie the points) and at each length l where an edge appears in X^l, if it connects two different connected components a and b of X^{l^-}, the corresponding classes in $H_0(X^{l-})$ are sent to the new class $i(a) = i(b)$ in $H_0(X^l)$.

$$\begin{array}{ccc} H_0(X^{l-}) & \rightarrow & H_0(X^l) \\ <a> & \rightarrow & <i(a)> \\ \oplus & \nearrow & \\ <b> & & \end{array}$$

We recognize Kruskal's algorithm for computing the minimal spanning tree of a weighed connected graph with the Euclidian distance. This is sound since we assumed X^{ε^*} to be connected.

Remark 27. Drawing the persistence module $PH_0(X)$ and $T(X)$ side by side, one will see that they are the same as "graphs" if one considers arrows in PH_0 as edges and classes in $H_0(X^*)$ as vertices, although I wasn't able to write this as a fonctorial equivalence of some sort.

Definition 28. *A Minimal spanning tree (mst) of a weighed connected graph G is a spanning subgraph T of G that minimises* $\sum_{e \in T} \text{length}(e)$*. It happens to be a tree*

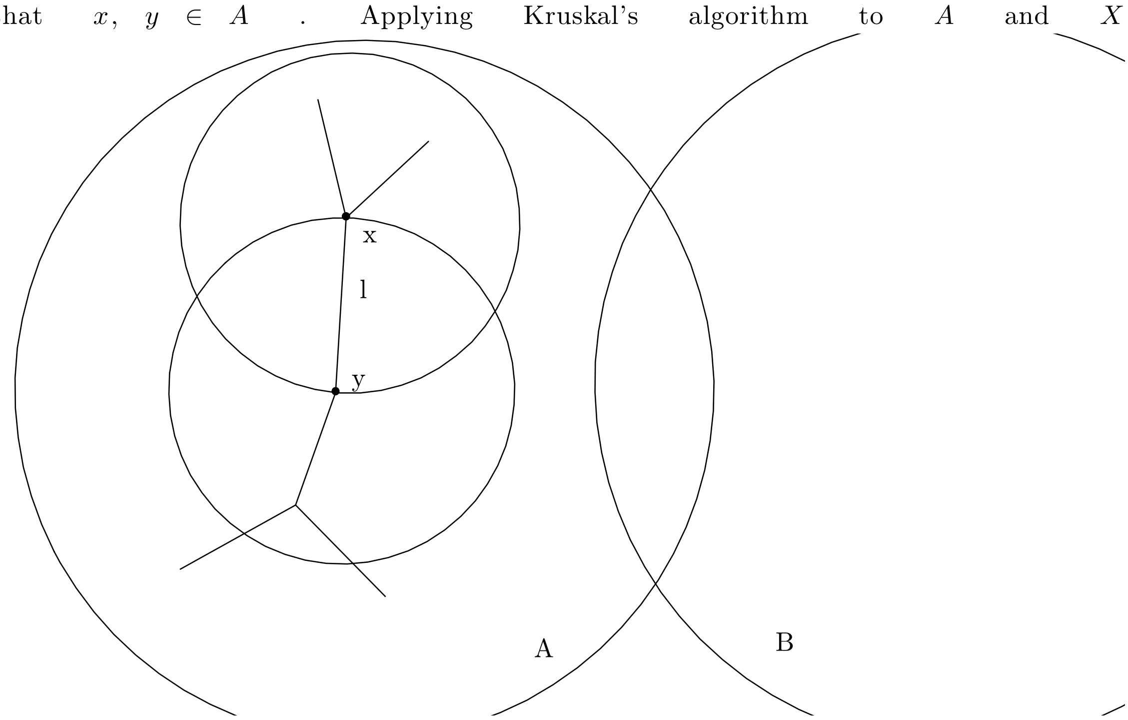
To compute $PH_0(X^*)$ from $PH_0(A^*)$ and $PH_0(B^*)$, we propose the following algorithm using mst's

Algorithm 1

- compute $T(A)$ and $T(B)$, and merge them (i.e. points are connected by an edge if they are connected in $T(A)$ or $T(B)$
- delete the longest edges that appear in cycles

Lemma 29. *$T(X) \subset T(A) \cup T(B)$.*

Proof. Let $x\, y \in T(X)$ be of lengh l ,we can assume that $x, y \in A$. Applying Kruskal's algorithm to A and X



l is the smallest radius of a sphere where the connected components of x and y touch in X , so it must be the same in A (since these connected components can only be smaller than they are in X) : $x\,y \in T(A)$. □

Theorem 30. *The Algorithm finishes*

Proof. *We know the result to be spanning since it must at least contain $T(X)$ according the Lemma, deleting all the longest edges in cycles of $T(A) \cup T(B)$ will make it acyclic and minimal by unicity* □

Note 31. In fact, we have an isomerty between edges in the minimal spanning tree and barcodes in the persistence module.

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