Stage de M1

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I - INTRODUCTION

J'ai effectué mon stage de M1 en EDP, auprès du professeur Pierre Raphaël, à l'Université de Cambridge. Je décrirai dans un premier temps mon expérience personnelle et humaine, dans un second temps j'introduirai le domaine de recherche et ce que j'ai précisément étudié. Enfin, mon stage a abouti à l'écriture d'un article qui détaille mes résultats, que je présenterai.

Un fait fondamental : la méthode de mon maître de stage consistait à me laisser en autonomie complète tant que je n'étais pas (trop) bloqué ; je choisissais donc absolument mes horaires et manières et lieux de travail (ainsi j'ai trouvé mes résultats significatifs lors d'un aller-retour en France pour déménager mon appartement). Cette liberté déstabilisante au début m'a -peut-être à la dure- contraint à gagner en autonomie, et à expérimenter l'importance primordiale de passer des semaines sur Google Scholars à feuilleter des articles pour que mûrisse la compréhension des choses dont découle la bonne idée.

De mon expérience, pour quelqu'un de peu sportif et de peu à l'aise avec la rencontre spontanée dans un café, il y a essentiellement deux milieux de sociabilité à l'université de Cambridge. D'une part, la faculté de mathématiques est regroupée dans un bâtiment divisé en pavillons à leur tour divisés en diverses équipes de recherche, bureaux, etc. qui donnent lieu à un riche échange mathématique tant spécifique au domaine d'intérêt (avec les gens du même pavillon au déjeuner ou dans le bureau par exemple) que plus ouvert (avec les évènements à l'échelle du CMS, le centre de mathématiques). Mais hasard des ressources humaines il n'y avait aucun bureau disponible pour moi à mon arrivée : j'ai donc essentiellement dû travailler dans ma chambre ou dans les nombreux coffee shops de Cambridge (les quelques salles de travail du CMS étant bruyantes parce qu'utilisées comme salles de repos / travail en groupe), m'interdisant l'accès à l'échange avec les pairs. D'autre part, à la différence de la France, les étudiants de l'Université sont regroupés par *college*. Les *colleges*, au nombre de 31, sont le coeur du réseau et de la sociabilité cambridgienne : on y mange, on y dort, on y participe aux clubs de sports, d'arts, de musique, on y prends des repas chics en grande tenue aux *formals*, des verres aux bars tenus par les étudiants. Les *colleges* sont un incroyable lieux d'échanges et de rencontres interdisciplinaires, soudant des communautés et amitiés par une tradition quasi millénaire de conversations professeurs / élèves ou autres le long d'immenses tablées. Cela étant, en tant que stagiaire pour quatre mois, je n'étais membre d'aucun *college*. Bref, j'étais environné d'étudiants passant leurs loisirs dans leur *college* et leurs journées de travail dans leur bureau du CMS. J'ai donc dû recourir au troisième moyen secret de l'expérience cambridgienne : l'inépuisable réseau de Français stagiaires / thésards / postdocs issus de l'ENS comme moi (ou parfois de l'école polytechnique), d'évidence très faciles à approcher - mais limitant l'aspect "découverte de l'étranger" du stage.

Au point de vue matériel, j'ai eu la chance non négligeable de me voir financer une chambre dans un internat de la ville par mon maître de stage, Pierre Raphaël, ce qui couplé au coût remarquablement faible de la vie courante m'a assuré de ne pas devoir me soucier de questions financières - avantage dont je ne cesse de mesurer l'étendue en comparant avec ceux partis aux Etats-Unis sans financement ou plus proche ceux qui louaient 800 livres par mois une chambrette dans une collocation douteuse à Cambridge même.

Avant de passer à mon expérience de recherche, *miscellany* sur la vie à Cambridge : le rythme est très perturbant pour un Français, les étudiants enchaînant quelques semaines de cours très intensives suivies d'assez longues vacances de plus d'un mois durant lesquelles ils sont priés de quitter les lieux pour que les *colleges* utilisent leur chambre pour accueillir des conférenciers ou à la saison des locations touristiques : il y a donc finalement peu d'occasion de les rencontrer. En revanche, l'université attire quantité de séminaires et conférences dont les *speakers* à la renommée mondiale sont passionants à écouter : j'ai par exemple eu la chance d'être invité trois jours durant à la conférence Oxbrige, qui dresse un état des lieux de la recherche en EDP chaque année ; c'étaient trois jours entremmêlés de conférences brillantes, de repas / cocktails occasions d'échanger sur mon domaine de recherche avec ceux qui le faisait exister et progresser à l'avant-garde. Enfin, la ville est très petite et entièrement vouée à l'étude ; si l'on souhaite se promener / aller au musée / sortir, aller à Londres, à une heure de train, est un impératif.

En conclusion, ma première expérience de stage longue durée (et ma première expérience seul à l'étranger) a surtout été l'occasion d'apprendre l'autonomie dans la recherche ; j'espère aussi que j'aurai dans de futurs stages l'occasion de me confronter à l'aspect collaboratif de la recherche en mathématiques, que j'ai entraperçu comme essentiel à notre époque sans l'avoir expérimenté. Je suis reconnaissant à mon maître de stage de m'avoir considéré comme suffisamment instruit et capable pour travailler sur des "vrais" sujets, i.e. des questions d'intérêt (bien sûr extrêmement limité), non résolues - j'aurai été déçu de passer mon temps à la bibliothèque à lire des livres d'introduction à des domaines trop compliqués pour être abordés.

II - UN SURVOL DU DOMAINE DE RECHERCHE

En étudiant une Equation aux Dérivées Partielles, parmis les premières questions se pose l'existence et l'unicité d'une fonction solution de l'équation, puis de savoir la régularité de cette solution (est-elle seulement solution faible ? Quelle régularité Sobolev ou autre espaces fonctionnels possède-t-elle ?). Pour une équation dépendant du temps, sur quel intervalle une éventuelle solution existe-elle ? A supposer que l'on sache prouver l'existence et l'unicité locale, on en déduit qu'à condition initiale donnée en $t = 0$ il existe une unique solution sur un intervalle de temps maximal mettons $[0, T)$ avec $T \in (0, +\infty)$. Si dans la plupart des cas on n'a pas de formules explicites pour la solution, on peut s'interroger sur un éventuel comportement asymptotique proche de T : lorsque $T = +\infty$ (solution globale), a-t-on un équivalent de la solution, une estimation de sa "taille" (mesurée avec diverses normes fonctionnelles) ? Lorsque en revanche T est fini, on est souvent dans un cas où la solution "explose" en temps fini, ce qui signifie qu'une norme (par exemple une norme Sobolev) devient arbitrairement grande. C'est ce phénomène d'explosion en temps fini d'une solution, et plus précisément le comportement de la solution au voisinage de T, qui m'a occupé pendant mon stage. Tous les résultats présentés sans référence sont prouvés dans les notes du cours d'EDP donné par Pierre Raphaël à l'université de Cambridge, que j'ai intensément lues avant mon stage

Pour rendre le propos concret, un modèle d'EDP très bien connu est l'équation de Schrödinger non linéaire dans le cas focalisant (NLS) :

$$\begin{cases} i\partial_t u + \Delta u - u|u|^{p-1} = 0 \quad (t,x) \in [0,T) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{2.1}$$

Où u_0 est une condition initiale donnée dans l'espace de Sobolev $H^1(\mathbb{R}^d)$ et $p_{max} > p > 1$ où $p_{max} = +\infty$ lorsque la dimension d est 1 ou 2 et $\frac{d+2}{d-2}$ sinon (cette valeur est essentiellement dictée par les injections de Sobolev : on veut que la régularité Sobolev H^1 assure la régularité $L^{p+1}(\mathbb{R}^d)$ grâce aux injections de Sobolev).

On sait prouver sans trop de difficulté l'existence et l'unicité locale d'une solution à cette équation : on sait donc qu'à u_0 donnée il existe un unique temps maximal $T \in (0, +\infty)$ tel que la solution u existe sur $[0, T)$. Ce qui est alors particulièrement intéressant est la possibilité de décrire très précisément le comportement proche de T de la solution : en posant $s_c := \frac{d}{2} - \frac{2}{p-1}$ (exposant qui sort de calculs et de considérations d'homogénéité), on montre que lorsque $s_c < 0$ les solutions sont globales et bornées dans H^1, et leur comportement asymptotique est explicit. On s'intéressera donc plutôt au cas $s_c \geqslant 0$ dans le but d'étudier les phénomènes d'explosion en temps fini : on a alors un résultat spectaculaire, à savoir que si l'on pose l'énergie

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} \tag{2.2}$$

Alors cette énergie est conservée par l'équation (i.e. en tout temps $E(u(t)) = E(u_0)$) et de plus si $E(u_0) < 0$ alors il y a explosion en temps fini. En outre, les travaux des années 1990 de Franck Merle ont permis une description très explicit du comportement au voisinage de la singularité. Ces résultats reposent sur deux ingrédients essentiels :

D'une part, une équation comme (2.1) présente des *invariance.* En effet, supposons que u est une solution. Alors sont aussi solutions :

$$\begin{cases} (t,x) \to \lambda^{\frac{2}{p-1}} u\left(\lambda^2 t, \lambda x\right), \quad \lambda > 0 \\ (t,x) \to u(t, x + x_0), \quad x_0 \in \mathbb{R}^d \\ (t,x) \to u(t,x)e^{i\gamma} \quad \gamma \in \mathbb{R} \\ (t,x) \to u(t, x - 2\beta t)e^{i\beta\cdot(x-\beta t)}, \quad \beta \in \mathbb{R}^d \end{cases} \tag{2.3}$$

D'autre part, Franck Merle puis Pierre Raphaël ont dégagé l'importance des *solitons.* Ces objets sont des solutions particulières de l'équation qui prennent la forme

$$u(t,x) = e^{i\lambda t} Q(x) \tag{2.4}$$

avec Q une solution de l'équation elliptique

$$-\Delta Q + Q|Q|^{p-1} = \lambda Q \tag{2.5}$$

pour $\lambda \in \mathbb{R}$ un paramètre. Via des changements de fonctions reposant sur les invariances de l'équation, Franck Merle arrive ainsi à démontrer qu'après transformation la solution est asymptotiquement une "somme de solitons".

Suite à ses nombreuses publications en collaboration avec Franck Merle dans les années 2000 sur ces questions, Pierre Raphaël s'est récemment intéressé à leur généralisation à l'équation de l'oscillateur harmonique non linéaire, qui modifie l'équation de Schrödinger non linéaire par l'ajout d'un potentiel harmonique :

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u - u|u|^{p-1} = 0 \quad (t,x) \in [0,T) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{2.6}$$

avec u_0 dans l'espace du viriel $\Sigma := \{f \in H^1(\mathbb{R}^d), xf \in L^2(\mathbb{R}^d)\}$. Contrairement à NLS, le problème reste tout à fait ouvert de savoir si l'on réussi à produire des solutions qui explosent, en temps fini ni même infini (ici, exploser en temps infini signifie que la norme de la solution diverge vers $+\infty$ lors $t \to \infty$).

L'objectif principal de mon stage (et le sujet de mon article) est de construire une solution la plus explicite possible à l'équation *perturbé* qui explose en temps infini : puisqu'on n'arrive pas à construire de solution qui explose à (2.6), on se demande si l'on arrive à construire un potentiel réel lisse $V(t,x)$, $(t,x) \in [0,+\infty) \times \mathbb{R}^d$ tel que $V(t,\cdot)$ ait une forte décroissance lorsque $|x| \to \infty$ (par exemple exponentielle) et soit de limite nulle ainsi que toutes ses dérivées lorsque $t \to \infty$ (typiquement dans toutes les normes Sobolev) ; de sorte que l'équation perturbée

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u - u|u|^{p-1} + V(t,x)u = 0 \quad (t,x) \in [0,T) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{2.7}$$

admette une solution $u(t,x)$ de classe $\mathcal{C}^\infty$ en temps et en espace sur $(0,+\infty)$ dont les normes Sobolev explosent lorsque $t \to \infty$.

Je me suis placé dans le cas le plus simple, i.e. $d = 2$ et $p = 3$ (qui avec la définition ci-haut correspond à $s_c = 0$). Suivant la methode présentée ci-haut pour les résultats d'explosion sur NLS, j'ai consacré le premier mois de mon stage à l'étude des *solitons* de l'équation non perturbée (2.6) ; c'est-à-dire avec la même définiton que j'ai recherché des solutions sous la forme

$$u(t,x) = e^{i\lambda t} Q(x) \tag{2.8}$$

avec

$$-\Delta Q + |x|^2 Q + Q|Q|^2 = \lambda Q \tag{2.9}$$

Cette première phase m'a permis de me familiariser avec des techniques usuelles en EDP : théorie spectrale des opérateurs autoadjoint sur $L^2(\mathbb{R}^2)$, méthode variationnelle, critères de Palais-Smale pour trouver des points critiques de fonctions lisses sur des variétés de Banach via un principe du minimax, etc.

Les plus importants des résultats que j'ai montré sont : (2.9) a une unique solution positive non nulle dans Σ l'espace du viriel pour $\lambda > 2$, et cette solution Q_λ est de classe $\mathcal{C}^\infty$, radiale, strictement positive, de décroissance exponentielle à l'infinie. Par ailleurs, $\lambda \to Q_\lambda$ est lisse sur $(2,+\infty)$ à valeurs dans Σ. En outre, lorsque λ tends vers 2 par valeurs supérieures, Q_λ est de limite nulle (dans les normes Sobolev ainsi qu'en norme infinie), et j'ai calculé un équivalent du type $Q_{2+\varepsilon} \sim C\varepsilon^{\frac{1}{2}} h_0$ où h_0 est la première fonction propre de l'opérateur H dans L^2 lorsque $\varepsilon \to 0$ et C une constante non nulle.

Dans un second temps, en m'inspirant de l'article de Pierre Raphaël et Erwan Faou référencé dans mon article, j'ai étudié les *modulation* que l'on peut appliquer à l'équation (2.6) en conservant sa forme. Plus précisément, en étudiant les *invariances* de NLS présentées ci-haut, on remarque que

$$u(t,x) = e^{-i\lambda t} \frac{1}{L(t)} e^{-i\frac{b(t)|x|^2}{4L(t)^2}} Q\left(\frac{x}{L(t)}\right) \tag{2.10}$$

est une solution de NLH si et seulement si les fonctions (L,b) satisfont un certain système dynamique hamiltonien paramétré par un paramètre réel, l'action a, qui est directement reliée à la norme Sobolev de

u. L'idée est alors de perturber légèrement ce système afin de faire croître l'action, ajoutant un terme dans l'équation que l'on "gère" grâce au potentiel V (que l'on est libre de fixer pourvu qu'il réponde aux contraintes). Dans cet esprit, je prouve la proposition suivante :

Proposition 2.1 (modulated equation)**.** *Let $L(t) > 0$, $\gamma(t)$, $b(t)$ be $\mathcal{C}^1$ functions over $\mathbb{R}_+$. Set*

$$u(t,x) := e^{i\gamma(t)}\frac{1}{L(t)}w(t,\frac{x}{L(t)}) \qquad w(t,y) := e^{-i\frac{b|y|^2}{4}}v(t,y) \qquad \frac{ds}{dt} = \frac{1}{L^2} \tag{2.11}$$

assume $t \to s(t)$ is invertible from $(t_0,+\infty)$ onto itself. Then $u(t,x)$ solves (2.7) *iff $v(s,y)$ solves*

$$i\partial_s v + \Delta v - \gamma_s v + \left(-L^4 + \frac{b_s}{4} - \frac{b^2}{4} - \frac{L_s}{L}\frac{b}{2}\right)|y|^2 v - i\left(\frac{L_s}{L} + b\right)(1+\Lambda)v - v|v|^2 - W(s,y)v = 0 \tag{2.12}$$

where $L_s := \frac{d}{ds}(L(t(s)))$ and similar definitions for b_s, γ_s ; where $\Lambda := y \cdot \nabla$ and finally where

$$W(s,y) = L^2 V(t,x) \tag{2.13}$$

On choisit alors (L,b) solution du système dynamique

$$\begin{cases} L^4 - \frac{b_s}{4} + \frac{b^2}{4} + \frac{L_s}{L}\frac{b}{2} & = 1 + \beta(s) \\ \frac{L_s}{L} + b & = 0 \end{cases} \tag{2.14}$$

où

$$\beta(s) := \frac{\sin(4s)}{s\log s} \qquad s > 1 \tag{2.15}$$

De sorte finalement que $u(t,x)$ est solution de (2.7) si et seulement si $v(s,y)$ est solution de

$$i\partial_s v + \Delta v - |y|^2 v - v|v|^2 - \gamma_s v - W(s,y)v - \beta(s)v = 0 \tag{2.16}$$

C'est alors qu'apparaît l'intérêt de la transformation : en effet, puisque $W(s,y)$ et $\beta(s)$ sont appelés à tendre vers 0 asymptotiquement, on peut écrire informellement qu'en temps grand on cherche une solution de

$$i\partial_s v + \Delta v - |y|^2 v - v|v|^2 - \gamma_s v = 0 \tag{2.17}$$

Pour peu que l'on choisisse alors $\gamma_s = \lambda$ pour un certain $\lambda \in (2,+\infty)$, on dispose d'une solution *stationnaire* de cette équation : le soliton $v(s,y) = Q_\lambda(y)$.

En outre, un calcul élémentaire montre que la norme H^1_x (définie par $\|u\|^2_{H^1_x} = \|\nabla u\|^2_{L^2} + \|xu\|^2_{L^2}$ et plus fine que la norme Sobolev H^1) de v est relié à celle de u via un facteur de proportionnalité

$$E(L,b) := \frac{1}{L^2}\left(\frac{b^2}{4} + 1\right) + L^2 \tag{2.18}$$

Où E signifie l'*énergie* du système (2.14) : or, on sait construire [cf article] une solution (L,b) à ce système existant en tout temps tel que l'énergie diverge logarithmiquement :

$$E(L,b) = \log s + O\left(\frac{\log s}{s}\right) \qquad s \to \infty \tag{2.19}$$

Ici réside tout l'intérêt de l'approche modulation / soliton : on va chercher $v(s,y)$ (la fonction après modulation) comme tendant vers Q_λ en temps grand, donc en particulier on connaît asymptotiquement sa norme H^1_x (qui tend vers une constante), et le mécanisme de croissance est alors induit seulement par la modulation, c'est-à-dire par un système dynamique de dimension finie que l'on sait bien mieux étudier.

Le reste du travail (qui prend la majeure partie de mon article) est essentiellement un calcul de perturbation : on rends le raisonnement précédent formel en cherchant

$$v(s,y) = Q_\lambda(y) + w(s,y) \tag{2.20}$$

Où l'on choisit λ proche de 2 et w est une perturbation. J'arrive à prouver que l'on sait construire une solution de cette forme avec w de limite nulle, et qu'en remontant à la solution pré-modulation $u(t,x)$ de l'équation d'intérêt (2.7) induite, on contrôle très précisément le comportement asymptotique de $\|u(t)\|_{H^1_x}$ qui diverge logarithmiquement, ce qui conclue.

On weakly turbulent solutions to the perturbed non linear harmonic oscillator

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I - INTRODUCTION

1.1 Setting of the problem

We introduce the linear operator that is associated with the two dimensional quantum harmonic oscillator, which is on $L^2(\mathbb{R}^2)$

$$H := -\Delta + |x|^2 \tag{1.1}$$

where for $x = (x_1, x_2) \in \mathbb{R}^2$ we denote $|x|^2 = x_1^2 + x_2^2$ and Δ is the Laplace operator. With this operator are associated the Sobolev function spaces defining the domain of H :

$$\forall r \geqslant 0 \qquad H_x^r := \left\{ u \in L^2(\mathbb{R}^2) \quad H^{\frac{r}{2}} u \in L^2 \right\} \tag{1.2}$$

An important problem is to build a solution to the nonlinear Schrödinger equation

$$i\partial_t u = Hu + u|u|^2 \tag{1.3}$$

whose H_x^r norm blows up in infinite time i.e. such that

$$\lim_{t\to\infty} \|u(t)\|_{H_x^r} = +\infty \tag{1.4}$$

The L^2 norm being preserved by the equation, such a phenomenon would be the result of *weakly turbulent* effects i.e. energy transfer between low and high frequencies generating growth of Sobolev norms, thanks to the non linearity (indeed, should we remove the $u|u|^2$ term, the Sobolev norms would be preserved).

We make a step toward this direction by building rather explicitly a solution to the perturbed equation

$$i\partial_t u = Hu + u|u|^2 + V(t,x)u \tag{1.5}$$

where $V(t,x)$ is a real potential, with $\mathcal{C}^\infty$ regularity both in space and in time, moreover such that forall t $V(t,\cdot)$ decays exponentially when $|y| \to \infty$, and such that V vanishes when $t \to \infty$:

$$\forall r \geqslant 0 \qquad \lim_{t\to\infty} \|V(t,\cdot)\|_{H_x^r} = 0 \tag{1.6}$$

and such that however the H_x^1 norm of u has infinite limit when $t \to \infty$. We shall precisely prove the following result :

Theorem 1.1. *There are a $t_0 > 0$, a potential $V(t,x) \in \mathcal{C}^\infty((t_0,+\infty)\times\mathbb{R}^2;\mathbb{R})$ and a solution $u \in \mathcal{C}^\infty((t_0,+\infty)\times\mathbb{R}^2;\mathbb{C})$ such that*

$$\forall (t,x) \in (t_0,+\infty)\times\mathbb{R}^2 \qquad i\partial_t u(t,x) = Hu(t,x) + u(t,x)|u(t,x)|^2 + V(t,x)u(t,x) \tag{1.7}$$

and for all $r \geqslant 0$ and $k \in \mathbb{N}$

$$\lim_{t\to+\infty} \|\partial_t^k V(t,x)\|_{H_x^r} = 0 \tag{1.8}$$

and moreover there are constants $c, C > 0$ such that

$$c\,(\log t)^{\frac{1}{2}} \leqslant \|u(t)\|_{H_x^1} \leqslant C\,(\log t)^{\frac{1}{2}} \qquad t\to\infty \tag{1.9}$$

1.2 Strategy of proof

Following quite closely the approach in [1], we will rely on the study of specific solutions of the unperturbed equation (1.3) that are called *bubbles.* They take the form

$$u(t,x) = e^{-i\lambda t}\frac{1}{L(t)}\exp^{-i\frac{b(t)|x|^2}{4L(t)^2}} Q\left(\frac{x}{L(t)}\right) \tag{1.10}$$

where Q is a *soliton*, that is a solution to the time-independent elliptic equation

$$-\Delta Q + |x|^2 Q + Q|Q|^2 = \lambda Q \tag{1.11}$$

where we will choose carefully a fixed $\lambda \in \mathbb{R}$. Given such a soliton, one can construct an explicit periodic solution to (1.3) by putting $u(t,x) = e^{-i\lambda t}Q(x)$. We also modulate via the *pseudo conformal symetry group* associated to the unperturbed flow, that is u given by (1.10) is a solution to (1.3) iff (L,b) satisfies a quasi hamiltonian dynamical system which has been studied extensively in [1]. The solutions of it are parametrized by one parameter, the *action*, which is preserved by the unperturbed flow. In order to produce growth mecanisms, we will slightly perturbate the dynamical system using the results proven in [1] in order to produce a logarithmic growth of the solution. The main idea is that what produces growth is only the modulation parameters (L,b), i.e. a finite dimensional dynamical system, whereas after modulation the solution will always "look like" a soliton Q. The advantage of this method, somehow hidden in this paper given that we will freely use the results in [1], is that we only have to study a finite dimensional perturbed quasi hamiltonian system, which we can do rather explicitely, rather than deal directly with infinite dimension.

II - PRELIMINARIES

Throughout the rest of this paper, we will only consider radial functions, i.e. all the functions u on $\mathbb{R}^2$ which will appear will satisfy $u(x) = f(|x|)$ for some f. For the purpose of notation we will implicitely suppose that whenever we define a function space. For example, we will denote by $L^2(\mathbb{R}^2)$ the space of *radial* functions over $\mathbb{R}^2$ with integrable squared absolute value.

2.1 Notations

We denote by $L^2(\mathbb{R})$ the real Hilbert space of (radial) functions $\mathbb{R}^2 \to \mathbb{R}$ with squared absolute value integrable. We denote by $L^2(\mathbb{C})$ the space of (radial) functions $\mathbb{R}^2 \to \mathbb{C}$ such that $\int |f|^2 < +\infty$. Unless otherwise stated, L^2 will designate $L^2(\mathbb{C})$. We introduce two scalar products over L^2 : first, the usual hermitian scalar product

$$(f,g) := \int_{\mathbb{R}^2} f\bar{g}$$

So that L^2 may be viewed as a complex Hilbert space. However we can also see $f = f_1 + if_2 \in L^2$ as vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ its two components being in $L^2(\mathbb{R})$, and in that spirit we define

$$\langle f_1 + if_2, g_1 + ig_2\rangle := \int_{\mathbb{R}^2} f_1 g_1 + f_2 g_2$$

So that L^2 is now a real Hilbert space. As $\langle\cdot,\cdot\rangle$ is just the real part of $(\cdot,\cdot)$, they both induce the same norm. For a complex-valued function f, we will always denote $f_1 := Re(f)$ and $f_2 := Im(f)$

2.2 Harmonic oscillator and eigenfunctions

Following for example [2] we know that $H = -\Delta + |x|^2$ is self-adjoint positive over L^2 for $(\cdot,\cdot)$, with compact resolvent, and we have explicitely

Proposition 2.1. *There is a L^2 orthonormal basis of (radial) functions $(h_n)_{n\geqslant 0}$ such that*

$$Hh_n = 4n + 2$$

Moreover, the h_n are real, have $\mathcal{C}^\infty$ regularity, and they decay exponentially when $|x| \to \infty$. Finally, $h_0(x) = C\exp\left(-\frac{|x|^2}{2}\right)$ with C a normalization constant

It is noteworthy that (h_n) is an orthonormal basis of $L^2(\mathbb{R})$ as a real space as well. Moreover, we see that H has simple eigenvalues with fixed frequency gap, which enables the following (for a proof one may look for example [3, p. 123])

Lemma 2.1. *Let E be one of the following function space : $L^2(\mathbb{R}), L^2(\mathbb{C})$ (with hermitian scalar product). Let A be a bounded selfadjoint operator over E with operator norm $\|A\| < 2$. Then $H - 2 + A$ is a selfadjoint operator with discreet spectrum, and there is an orthonormal basis of E $\psi_0, \psi_1, ...$ and a sequence $\lambda_0 < \lambda_1 < ...$ such that*

$$(H - 2 + A)\psi_n = \lambda_n \tag{2.1}$$

Moreoverone there stands the bound :

$$\forall n \qquad |\lambda_n - 4n| \leqslant \|A\| \tag{2.2}$$

2.3 Function spaces

We recall that for $r \geqslant 0$ we define

$$H^r_x := \{u \in L^2, H^{\frac{r}{2}} \in L^2\} \tag{2.3}$$

which is equipped with the norm

$$\|u\|_{H^r_x} := \|H^{\frac{r}{2}} u\|_{L^2} \tag{2.4}$$

More precisely, if $u = \sum_{n \geqslant 0} \alpha_n h_n$ then we have

$$\|u\|^2_{H^r_x} = \sum_{n \geqslant 0} (4n+2)^r |\alpha_n|^2$$

We remark that these norms are non decreasing with r.

We will use some basic properties of the H^r_x norms whose proofs can be found in [1, pp. 6-7]. First, in order to fix ideas, one has the norm equivalence :

$$\exists c_r, C_r > 0 \qquad c_r \|f\|_{H^r_x} \leqslant \|f\|_{H^r} + \| <x>^r f\|_{L^2} \leqslant C_r \|f\|_{H^r_x} \tag{2.5}$$

where $\| \cdot \|_{H^r}$ is the usual Sobolev norm and $<x> := \sqrt{1 + |x|^2}$. Moreover, we have the three lemmas:

Lemma 2.2 (H^r_x is an algebra if $r > 1$)**.** *Let $r > 1$. Then there exists a constant $C_r > 0$ such that*

$$\forall f, g \in H^r_x \qquad \|fg\|_{H^r_x} \leqslant C_r \|f\|_{H^r_x} \|g\|_{H^r_x} \tag{2.6}$$

Lemma 2.3. *For all $r \geqslant 0$ there is a $C_r > 0$ such that for all $f \in H^{r+1}_x$ there stands*

$$\||x|^2 f\|_{H^r_x} \leqslant \|f\|_{H^{r+1}_x} \tag{2.7}$$

Lemma 2.4 (Commutator formula)**.** *For all $r \geqslant 0$ and for all $f \in H^r_x$*

$$\|[H^{\frac{r}{2}}, |x|^2] f\|_{L^2} \leqslant \|f\|_{H^r_x} \tag{2.8}$$

where we denote $[A, B] := AB - BA$ the commutator of A et B.

We prove moreover the following lemma :

Lemma 2.5 (norm estimate)**.** *Given $\varepsilon > 0$ and $r > s$ there exists a $C_{\varepsilon,r,s} > 0$ such that*

$$\forall u \in H^r_x \qquad \|u\|^2_{H^s_x} \leqslant \varepsilon \|u\|^2_{H^r_x} + C_\varepsilon \|u\|^2_{L^2} \tag{2.9}$$

Proof. Write $u = \sum_{n \geqslant 0} \alpha_n h_n$ so that $\|u\|^2_{H^r_x} = \sum_{n \geqslant 0} (4n+2)^t |\alpha_n|^2$ for any $t \geqslant 0$. Find $N \geqslant 0$ so that whenever $n \geqslant N$ there stands $(4n+2)^s \leqslant \varepsilon (4n+2)^r$. One can write

$$\begin{aligned}\|u\|^2_{H^s_x} &= \sum_{n \geqslant N} (4n+2)^s |\alpha_n|^2 + \sum_{n < N} (4n+2)^s |\alpha_n|^2 \\ &\leqslant \varepsilon \|u\|^2_{H^r_x} + (4N+2)^s \|u\|^2_{L^2}\end{aligned}$$

□

2.4 Control of L^∞ norm

We recall the following result over usual Sobolev spaces :

Lemma 2.6. *Take $r > 1$ Suppose $u \in H^r$. Then $u \in L^\infty(\mathbb{R}^2)$ and there stands*

$$\|u\|_{L^\infty} \leqslant C_r \|u\|_{H^r} \tag{2.10}$$

where C_r is a universal constant.

Proof. We remind the formula

$$\|u\|^2_{H^r} = \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 < \xi >^{2r} d\xi \tag{2.11}$$

where $\hat{u}$ is the Fourier transform of u. Now, we see that $\|u\|_{L^\infty} \leqslant C\|\hat{u}\|_{L^1}$ for a universal constant C, and we thus can write

$$\begin{aligned}
\|u\|_{L^\infty} &\leqslant C \int_{\mathbb{R}^2} |\hat{u}(\xi)| d\xi \\
&\leqslant C \left(\int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 < \xi >^{2r} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{d\xi}{< \xi >^{2r}} \right)^{\frac{1}{2}} \\
&\leqslant C_r \|u\|_{H^r}
\end{aligned}$$

□

III - SOLITONS

As stated in the introduction, the proof of 1.1 relies crucially on the study of *solitons*, which are solutions of

$$HQ + Q|Q|^2 = \lambda Q$$

We state and prove technical results on solitons in this section

3.1 Existence, uniqueness

We will prove here the following result, using several steps :

Proposition 3.1. *Given $\lambda > 2$ there exists a unique positive solution Q_λ of* (1.11) *in H^1_x. Moreover, $Q \in H^k_x$ for all $k \geqslant 1$*

This proposition ensures in particular that Q has $\mathcal{C}^\infty$ regularity over $\mathbb{R}^2$ ([4]), and $Q(x)$ and all its derivatives decay to zero as $|x| \to \infty$

Let us start with a functional analysis lemma :

Lemma 3.1. *The function space H^1_x is compactly embedded in $L^p(\mathbb{R}^2)$ for all $2 \leqslant p < +\infty$*

Proof. Following for example [4, p.281], we see that H^1_x is continuously embedded into L^p thanks to the Sobolev inequalities (indeed, the H^1_x norm in finer than the H^1 Sobolev norm) ; and the Rellich-Kondrakov theorem [4, P.285] gives compact embedding into L^2_{loc}. We now see that

$$\|u\|^2_{H^1_x} = \|\nabla u\|^2_{L^2} + \|xu\|^2_{L^2} \tag{3.1}$$

Take $u_n \rightharpoonup 0$ in H^1_x (weak convergence). Then we have that (u_n) is bounded in H^1_x. Take $M > 0$ such that $\forall n, \|u_n\|_{H^1_x} \leqslant M$. Take $\delta > 0$. Then for a given $R > 0$ we have

$$\int_{|x| \geqslant R} |u_n|^2 \leqslant R^{-2} \|xu_n\|^2_{L^2} \leqslant M^2 R^{-2} \leqslant \delta$$

should we choose R big enough. Now, as H^1_x is compactly embedded into $L^2(|x| \leqslant R)$, we know that

$$\int_{|x| \leqslant R} |u_n|^2 \to 0$$

This shows that $\limsup \|u_n\|^2_{L^2} \leqslant \delta$, which in turns shows that $u_n \to 0$ in L^2. This shows that H^1_x is compactly embedded into L^2. Now, given $2 \leqslant p < +\infty$, find $p < q < \infty$ and write $p = 2t + q(1-t)$ for some $t \in (0,1)$. Then we have thanks to Hölder's inequality

$$\|u_n\|_{L^p} \leqslant \|u_n\|^t_{L^2} \|u_n\|^{1-t}_{L^q} \to 0$$

Indeed, (u_n) is bounded in L^q. This shows the compactness of the embedding into L^p □

Let us now add a technical lemma :

Lemma 3.2. *Given* $p > 2$ *and* $\varepsilon > 0$ *there exists a* $C_\varepsilon > 0$ *such that for all* $u \in H^1_x$ *one have*

$$\|u\|^2_{L^2} \leqslant C_\varepsilon \|u\|^2_{L^p} + \varepsilon \|u\|^2_{H^1_x} \tag{3.2}$$

Proof. Write first

$$\int_{|x| \geqslant R} |u|^2 \leqslant R^{-2} \|u\|^2_{H^1_x} \leqslant \varepsilon \|u\|^2_{H^1_x}$$

if $R > 0$ is big enough. Now with Hölder inequality

$$\int_{|x| \leqslant R} |u|^p \leqslant |\{|x| \leqslant R\}|^{1-\frac{2}{p}} \|u\|^2_{L^p} = C_\varepsilon \|u\|^2_{L^p}$$

□

We introduce the following function over H^1_x : for $u \in H^1_x$, define

$$J(u) := \frac{1}{2}\|u\|^2_{H^1_x} - \frac{\lambda}{2}\|u\|^2_{L^2} + \frac{1}{4}\|u\|^4_{L^4} = \int \frac{1}{2}\left(|\nabla u|^2 + |xu|^2\right) - \frac{\lambda}{2}|u|^2 + \frac{1}{4}|u|^4 \tag{3.3}$$

which is differentiable, its differential being the continuous linearform over H^1_x

$$J'(u) = -\Delta u + |x|^2 u - \lambda u + u|u|^2 \tag{3.4}$$

where we can define, given $u \in H^1_x$, its distributional laplacian by duality :

$$\forall g \in H^1_x, \qquad \langle -\Delta u, g\rangle_{(H^1_x)' \times H^1_x} := \langle \nabla u, \nabla g\rangle \tag{3.5}$$

Our aim is to show that J realizes its minimum over H^1_x. Let a take a minimizing sequence $(u_n) \subset H^1_x$ such that

$$J(u_n) \to \inf_{u \in H^1_x} J(u) \in [-\infty, +\infty) \tag{3.6}$$

We see by puting $\varepsilon = \frac{1}{2\lambda}$ in lemma (3.2) that there is a $C > 0$ such that

$$J(u) \geqslant \frac{1}{4}\|u\|^2_{H^1_x} - \frac{\lambda C_\varepsilon}{2}\|u\|^2_{L^4} + \frac{1}{4}\|u\|^2_{L^4} \geqslant \frac{1}{4}\|u\|^2_{H^1_x} - C' \tag{3.7}$$

where C' is a constant. Indeed, the last inequality comes from the polynom $\frac{1}{4}x^4 - \frac{\lambda C_\varepsilon}{2}x^2$ being bounded from below over $\mathbb{R}$

Therefore as $J(u_n)$ is bounded from above, (u_n) is bounded in H^1_x. We can assume that u_n converges weakly to a $Q \in H^1_x$. Lemma (3.1) ensures that $u_n \to u_\infty$ strongly both in L^2 and in L^4. Moreover,

$$\|Q\|_{H^1_x} \leqslant \liminf \|u_n\|_{H^1_x} \tag{3.8}$$

We therefore deduce that

$$J(Q) \leqslant \liminf J(u_n) = \inf_{u \in H^1_x} J(u) \tag{3.9}$$

Which ensures J realizes its minimum at Q. We now prove the following convexity result for the Dirichlet integral :

Lemma 3.3 (Convexity of the Dirichlet functional)**.** *Take* $u \in H^1$ *(usual Sobolev space). Then* $|u| \in H^1$ *and there stands*

$$\int_{\mathbb{R}^2} |\nabla |u||^2 \leqslant \int_{\mathbb{R}^2} |\nabla u|^2 \tag{3.10}$$

Proof. By approximation, we can assume that u is regular.

Decompose u in real and imaginary parts $u = f + ig$. Then a.e.

$$\nabla|u| = \nabla\sqrt{f^2+g^2} = \frac{f\nabla f + g\nabla g}{\sqrt{f^2+g^2}}$$

Hence

$$\begin{aligned}\int |\nabla|u||^2 &= \int \frac{|f\nabla f + g\nabla g|^2}{f^2+g^2} \\ &= \int \frac{1}{f^2+g^2}\left[f^2|\nabla f|^2 + g^2|\nabla g|^2 + 2fg\nabla f\cdot\nabla g\right] \\ &= \int |\nabla f|^2 + \int |\nabla g|^2 - \int \frac{|g\nabla f - f\nabla g|^2}{f^2+g^2}\end{aligned}$$

which yields the result. □

This shows that $|Q| \in H^1_x$, and moreover that $J(|Q|) \leqslant J(Q)$, and the equality stands by the minimization property. Therefore we can assume Q to be real and non negative. Moreover, we can prove that Q is non zero by showing that $J(Q) < 0$. Because Q is a minimizer of J it suffices to show that J isn't nonnegative. Let us take

$$h_0(x) := \exp\left(-\frac{1}{2}|x|^2\right)$$

which satisfies $Hh_0 = 2h_0$. There stands :

$$J(th_0) = \frac{1}{2}t^2(2-\lambda)\|h_0\|^2_{L^2} + \frac{1}{4}t^4\|h_0\|^4_{L^4} \tag{3.11}$$

Should we choose $t > 0$ small enough, the right-hand side is negative. This shows that $\inf\limits_{H^1_x} J < 0$ which in turns ensures that $J(Q) < 0$. Let us stress here that we use the hypothesis $\lambda > 2$. Indeed, one can show easily that (1.11) doesn't have a nontrivial solution in H^1_x when $\lambda \leqslant 2$.

We then write that $J'(Q) = 0$ in $(H^1_x)'$ (because of minimization), which reads

$$-\Delta Q + |x|^2Q + Q^3 - \lambda Q = 0 \qquad \text{in } (H^1_x)' \tag{3.12}$$

Now, this ensures that $HQ \in L^2$, which means $Q \in H^2_x$. Lemma (2.6) ensures H^k_x is an algebra for $k \geqslant 2$ so that HQ is in turns in H^2_x, meaning $Q \in H^3_x$. We can iterate and see that

$$Q \in \bigcap_{k\geqslant 1} H^k_x \tag{3.13}$$

Now we thus know with Sobolev injections that Q has $\mathcal{C}^\infty$ regularity over $\mathbb{R}^2$, and that there stands

$$(-\Delta - \lambda)Q = -|x|^2Q - Q^3 \leqslant 0 \tag{3.14}$$

The maximum principle applied to the strictly elliptic operato $-\Delta - \lambda$ thus ensures that Q is positive.

In order to conclude the proof of (3.1) we need to show the unicity of Q. The proof relies on the following lemma :

Lemma 3.4. *Given $f \in H^1_x$ nonnegative, there exists a nonnegative solution $u \in H^1_x$ to the equation*

$$Hu + u|u|^2 = f \tag{3.15}$$

Proof. Given $u \in H^1_x$, let

$$J(u) := \frac{1}{2}\int |\nabla u|^2 + |\cdot|^2|u|^2 + \frac{1}{4}\int |u|^4 - \int fu = \frac{1}{2}\|u\|^2_x + \frac{1}{4}\|u\|^4_4 - \langle f, u\rangle \tag{3.16}$$

We see that J is differentiable over H^1_x, with differential

$$J'(u) = Hu + u|u|^3 - f \in (H^1_x)' \tag{3.17}$$

Moreover, J is bounded from below as there stands

$$J(u) \geqslant \frac{1}{2}\|u\|_{H_x^1}^2 + \frac{1}{4}\|u\|_{L^4}^2 - \|f\|_{L^{\frac{4}{3}}}\|u\|_{L^4} \geqslant \inf_{x\in\mathbb{R}}\left(\frac{1}{4}x^2 - \|f\|_{L^{\frac{4}{3}}}x\right) > -\infty$$

Take $(u_n) \subset H_x^1$ a minimizing sequence for J : we have that (u_n) is bounded in H_x^1 thus, withouth loss of generality, converges weakly towards $u \in H_x^1$. Lemma (3.1) ensures that $u_n \to u$ in L^2 and in L^4. We get using $\|u\|_{H_x^1} \leqslant \liminf \|u_n\|_{H_x^1}$ that

$$J(u) \leqslant \liminf J(u_n) = \inf_{H_x^1} J \tag{3.18}$$

Which shows that u minimizes J over H_x^1. Using lemma (3.3) there stands $J(|u|) \leqslant J(u)$ so we can always assume that $u \geqslant 0$. Write then $J'(u) = 0$ □

Let us now imagine that there are two nontrivial positive solutions u_1 and u_2 to equation (1.11). Let us define $\overline{u} = u_1 + u_2$. We then have

$$H\bar{u} - \lambda\bar{u} + \bar{u}^3 \geqslant H\bar{u} - \lambda\bar{u} + u_1^3 + u_2^3 = 0 \tag{3.19}$$

The inequality having distributional meaning. We set $w_0 = \overline{u}$ then recursively find, using (3.4), $w_{n+1} \in H_x^1$ a nonnegative solution to

$$Hw_{n+1} + w_{n+1}^3 = \lambda w_n \tag{3.20}$$

We show recursively that for all n there stands (in the distributional sense therefore almost everywhere) :

$$u_i \leqslant w_{n+1} \leqslant w_n \leqslant \overline{u} \qquad i = 1, 2 \tag{3.21}$$

Indeed, we can write

$$Hw_1 + w_1^3 = \lambda w_0 \leqslant Hw_0 + w_0^3$$

We test this inequality against the nonnegative H_x^1 function $(w_1 - w_0)^+$ (where $x^+ := \max(0, x)$) :

$$\int |\nabla(w_1 - w_0)^+|^2 + |\cdot|^2|(w_1 - w_0)^+|^2 + |w_1^3 - w_0^3|(w_1 - w_0)^+ \leqslant 0$$

Necessarily there stands $(w_1 - w_0)^+ = 0$ i.e. $w_1 \leqslant w_0$. We can also write

$$Hw_1 + w_1^3 \geqslant \lambda u_i = Hu_i + u_i^3$$

Testing this inequality against the nonnegative H_x^1 function $(u_i - w_1)^+$ yields the other inequality. We can iterate this reasoning because there stands

$$Hw_1 + w_1^3 - \lambda w_1 = \lambda(w_0 - w_1) \geqslant 0 \tag{3.22}$$

Let a subsequence of (w_n) weakly converge to a $w \in H_x^1$: there stands $Hw - \lambda w + w^3 = 0$ in $(H_x^1)'$, and $u_i \leqslant w$ for $i = 1, 2$. As $u_1 \neq u_2$ there stands for example $u_1 \neq w$. We have

$$\langle Hu_1, w\rangle - \lambda\langle u_1, w\rangle + \int u^3 w = 0$$

$$\langle Hw, u_1\rangle - \lambda\langle w, u_1\rangle + \int w^3 u_1 = 0$$

Substracting yields

$$\int u_1 w(u_1^2 - w^2) = 0$$

the integrand being nonpositive and nonzero this is a contradiction. This concludes the proof of the unicity of Q_λ

3.2 Differentiability in λ

Let us study the regularity of the function

$$\lambda \in (2,+\infty) \to Q_\lambda \in H^1_x \tag{3.23}$$

Where Q_λ is the unique positive solution to $HQ_\lambda + Q_\lambda^3 = \lambda Q_\lambda$ given by (3.1)
We will prove the following result

Proposition 3.2. *The function $\lambda \to Q_\lambda$ is differentiable as a function from $(2,+\infty)$ to H^1_x*

It is noteworthy that the proof can easily be iterated, showing the $\mathcal{C}^\infty$ regularity of this function. However, we will not need this result.

Lemma 3.5. *The function $\lambda \to \|Q_\lambda\|_{L^2}$ is increasing over $(2,+\infty)$*

Proof. Given its construction, Q_λ is the unique positive minimizer of

$$J(\lambda) : u \to \|u\|^2_{H^1_x} + \frac{2}{4}\|u\|^4_{L^4} - \lambda\|u\|^2_{L^2}$$

Therefore we can write, given $\mu < \lambda$:

$$\begin{aligned}\|Q_\mu\|^2_{H^1_x} + \frac{2}{4}\|Q_\mu\|^4_{L^4} - \mu\|Q_\mu\|^2_{L^2} &< \|Q_\lambda\|^2_{H^1_x} + \frac{2}{4}\|Q_\lambda\|^4_{L^4} - \mu\|Q_\lambda\|^2_{L^2}\\ &= \|Q_\lambda\|^2_{H^1_x} + \frac{2}{4}\|Q_\lambda\|^4_{L^4} - \lambda\|Q_\lambda\|^2_{L^2} + (\lambda-\mu)\|Q_\lambda\|^2_{L^2}\\ &< \|Q_\mu\|^2_{H^1_x} + \frac{2}{4}\|Q_\mu\|^4_{L^4} - \lambda\|Q_\mu\|^2_{L^2} + (\lambda-\mu)\|Q_\lambda\|^2_{L^2}\end{aligned}$$

Which reads

$$\forall \mu < \lambda \qquad (\lambda-\mu)(\|Q_\lambda\|^2_{L^2} - \|Q_\mu\|^2_{L^2}) > 0$$

□

Thus, as $\|Q_\lambda\|^2_{H^1_x} + \|Q_\lambda\|^4_{L^4} = \lambda\|Q_\lambda\|^2_{L^2}$, we seee that $\lambda \to Q_\lambda$ is locally bounded in H^1_x norm.

Let us see that $\lambda \to Q_\lambda$ is continuous. Given $\lambda > 2$, let us take $\lambda_n \to \lambda$. We know that (Q_{λ_n}) is a bounded sequence, therefore it is weakly compact. If $f \in H^1_x$ is a weak limit point, we have

$$-\Delta f + |x|^2 f + f^3 = \lambda f$$

and f is nonnegative. The unicity proved above ensures that f is either equal to Q or 0. However this case cannot happen because of lemma (3.1) : we know indeed that $\|f\|^2_{L^2} = \lim \|Q_{\lambda_n}\|^2_{L^2} \geqslant \|Q_{\lambda-\varepsilon}\|^2_{L^2} > 0$ (we use lemma (3.5)). This ensures that the only limit point of Q_{λ_n} is Q_λ, thus

$$Q_{\lambda_n} \rightharpoonup Q_\lambda \qquad \text{weakly in } H^1_x \tag{3.24}$$

Thanks to lemma (3.1), we have moreover that

$$Q^3_{\lambda_n} - \lambda_n Q_{\lambda_n} \to Q^3_\lambda - \lambda Q_\lambda \qquad \text{strongly in } L^2 \tag{3.25}$$

However we can write

$$Q_{\lambda_n} = H^{-1}(Q^3_{\lambda_n} - \lambda_n Q_{\lambda_n}) \tag{3.26}$$

and using the continuity of H^{-1} from L^2 into H^1_x we obtain that

$$Q_{\lambda_n} \to Q_\lambda \qquad \text{strongly in } H^1_x \tag{3.27}$$

Let us now establish the derivability of $\lambda \to Q_\lambda$.
We will use the following lemma :

Lemma 3.6. *Suppose that* $u \in H^1_x$ *is a solution to*

$$-\Delta u + |x|^2 u + 3Q_\lambda^2 u = \lambda u \qquad \text{in } (H^1_x)' \tag{3.28}$$

Then necessarily $u = 0$

It is rather interesting that with the notations we will define below, for $\lambda = 2 + \varepsilon$ with $\varepsilon > 0$ small enough, this lemma will be a straightforward consequence of the inversibility of the operator $H_+ := H + 3Q_\lambda^2 - \lambda$, which we will prove using spectral theory. The proof here is of a very different nature and doesn't recquire that λ be near 2.

Proof. Take $u \in H^1_x$ a solution. As the equation is the same on the real and imaginary parts of u we can assume u is real-valued. Then we have that $u_+ := \max(0, u) \in H^1_x$ (indeed it is enough to know that $|u| \in H^1_x$, see (3.3)). Moreover, as distributions

$$\Delta(u_+) \geqslant 1_{u>0} \Delta u \tag{3.29}$$

where $1_{u>0} = 0$ if $u \leqslant 0$ and 1 orthérwise. Let us postpone the proof of this inequality. We have

$$\begin{aligned} -\Delta(u_+) &\leqslant 1_{u>0}(-\Delta u) \\ &= 1_{u>0}\left(-|\cdot|^2 u - 3Q_\lambda^2 u + \lambda u\right) \\ &= -|\cdot|^2 u_+ - 3Q_\lambda^2 u_+ + \lambda u_+ \end{aligned}$$

We can apply this distributional inequality to the positive function Q_λ :

$$\begin{aligned} \lambda\langle u_+, Q_\lambda\rangle - 3\langle Q_\lambda^2 u_+, Q_\lambda\rangle &\geqslant \langle H(u_+), Q_\lambda\rangle \\ &= \langle u_+, HQ_\lambda\rangle \\ &= \lambda\langle u_+, Q_\lambda\rangle - \langle u_+, Q_\lambda^3\rangle \end{aligned}$$

Which reads

$$2\int_{\mathbb{R}^2} u_+ Q_\lambda^3 \leqslant 0 \tag{3.30}$$

As Q_λ is positive and u_+ nonnegative, this ensures $u_+ = 0$ i.e.

$$u \leqslant 0 \tag{3.31}$$

We then write

$$\begin{aligned} \lambda\langle u, Q_\lambda\rangle - 3\langle Q_\lambda^2 u, Q_\lambda\rangle &\geqslant \langle H(u), Q_\lambda\rangle \\ &= \langle u, HQ_\lambda\rangle \\ &= \lambda\langle u, Q_\lambda\rangle - \langle u, Q_\lambda^3\rangle \end{aligned}$$

Which reads

$$2\int_{\mathbb{R}^2} u Q_\lambda^3 = 0 \tag{3.32}$$

Thus we finally have $u = 0$ ☐

We now prove the inequality (3.29). Following Kato's inequality [5] we know that

$$\Delta|u| \geqslant sgn(u)\Delta(u) \tag{3.33}$$

Where $sgn(u)$ is the signe of u, i.e. -1 when $u < 0$, 0 when $u = 0$, and 1 when $u > 0$. We can write $u_+ = \frac{1}{2}(u + |u|)$ and conclude using $\frac{1}{2}(1 + sgn(u)) = 1_{u>0}$.

Given $\lambda > 2$, let us consider for h small (for example $h \in]-\varepsilon, \varepsilon[$ with $\lambda - \varepsilon > 2$)

$$\tau(h) := \frac{Q_{\lambda+h} - Q_\lambda}{h} \tag{3.34}$$

which satisfies

$$-\Delta\tau(h)+|x|^2\tau(h)+(Q_{\lambda+h}^2+Q_{\lambda+h}Q_\lambda+Q_\lambda)\tau(h)=\lambda\tau(h)+Q_{\lambda+h} \tag{3.35}$$

Suppose first that $\tau(h)$ is not bounded in H^1_x when $h\to 0$. Then one can find $h_k\to 0$ such that

$$\|\tau(h)\|_{H^1_x}\to\infty$$

. We have

$$\begin{aligned}\|\tau(h_k)\|^2_{H^1_x}&\leqslant\|\tau(h_k)\|^2_{H^1_x}+\int_{\mathbb{R}^2}(Q^2_{\lambda+h_k}+Q_{\lambda+h_k}Q_\lambda+Q_\lambda^2)\tau(h_k)^2\\&=\lambda\|\tau(h_k)\|^2_{L^2}+\int_{\mathbb{R}^2}Q_{\lambda+h_k}\tau(h_k)\\&\leqslant\|\tau(h_k)\|_{L^2}(\lambda\|\tau(h_k)\|_{L^2}+C)\end{aligned}$$

where $C=\sup\|Q_{\lambda+h_k}\|_{L^2}$. This ensures that

$$\|\tau(h_k)\|_{L^2}\to\infty \tag{3.36}$$

and thus one finds a constant C such that

$$\|\tau(h_k)\|_{H^1_x}\leqslant C\|\tau(h)\|_{L^2} \tag{3.37}$$

Now, this ensures that $\frac{\tau(h_k)}{\|\tau(h_k)\|_{L^2}}$ is bounded in H^1_x when $k\to\infty$, therefore one can find a weak limit point $u\in H^1_x$. Using $\frac{Q_{\lambda+h_k}}{\|\tau(h_k)\|_{L^2}}\to 0$ in H^1_x one finds

$$-\Delta u+|x|^2u+3Q_\lambda^2u=\lambda u\qquad\text{in }(H^1_x)' \tag{3.38}$$

Thus using lemma (3.6) we know that $u=0$. However using lemma (3.1) we should have $\|u\|_{L^2}=1$ which is a contradiction. We therefore deduce that $\tau(h)$ is bounded in H^1_x as $h\to 0$, thus is weakly compact in H^1_x
Now, suppose that $h_n\to 0$ is such that

$$\tau(h_n)\rightharpoonup g\qquad\text{weakly in }H^1_x \tag{3.39}$$

We have that g is a distributional solution (thus a solution in $(H^1_x)'$) of

$$Hg+3Q^2_\lambda g=\lambda g+Q_\lambda \tag{3.40}$$

Lemma (3.6) ensures that this equation has a unique solution in H^1_x, thus $\tau(h)$ is weakly compact and have a unique weak limit point in H^1_x when $h\to 0$. This ensures that

$$\tau(h)\rightharpoonup g\qquad\text{weakly in }H^1_x\ h\to 0 \tag{3.41}$$

where g is the unique solution to (3.7). Using now compactness lemma (3.1), the continuity of H from L^2 into H^1_x, and the equation

$$\tau(h)=H^{-1}\left((Q^2_{\lambda+h}+Q_{\lambda+h}Q_\lambda+Q_\lambda)\tau(h)-\lambda\tau(h)-Q_{\lambda+h}\right) \tag{3.42}$$

we can deduce that the convergence $\tau(h)\to g$ happens strongly in H^1_x, thus concluding to the derivability of $\lambda\to Q_\lambda$. We also have the explicit formula

Lemma 3.7 (characterization of $\partial_\lambda Q_\lambda$). *The derivative $\partial_\lambda Q_\lambda$ is the unique solution u in H^1_x to the equation*

$$(H+3Q^2_\lambda-\lambda)u=Q_\lambda \tag{3.43}$$

3.3 Bifurcation branch near $\lambda = 2$

We now study the soliton $Q_{2+\varepsilon}$ when $\varepsilon > 0$ is small.

Proposition 3.3 (Linear approximation near $\lambda = 2$)**.** *Define $h_0(x) = C\exp\left(-\frac{|x|^2}{2}\right)$ with C such that $\|h_0\|_{L^2} = 1$. Then as $\varepsilon \to 0$ there stands in H^1_x*

$$Q_{2+\varepsilon} = \varepsilon^{\frac{1}{2}}\|h_0\|^{-2}_{L^4}h_0 + o(\varepsilon^{\frac{1}{2}}) \tag{3.44}$$

Proof. We first notice that

$$Q_{2+\varepsilon} \to 0 \qquad \text{in } H^1_x \tag{3.45}$$

Indeed, we know that

$$\|Q_{2+\varepsilon}\|^2_{H^1_x} \leqslant (2+\varepsilon)\|Q_{2+\varepsilon}\|^2_{L^2} \tag{3.46}$$

and the right-hand side is bounded as $\varepsilon \to 0$ thank to (3.5). Given u a weak limit point of $Q_{2+\varepsilon}$ in H^1_x when $\varepsilon \to 0$, we see that

$$Hu + u^3 = 2u \tag{3.47}$$

This in turns implies that

$$\langle Hu, u\rangle - 2\|u\|^2_{L^2} = -\int |u|^4 \tag{3.48}$$

As $\langle Hu, u\rangle \geqslant 2\|u\|^2_{L^2}$ with (2.1), this implies $u = 0$. Therefore $Q_{2+\varepsilon}$ converges weakly to 0 in H^1_x and using the same trick as before (compactness of injection and continuity of $H^{-1} : L^2 \to H^1_x$) this convergence happens in H^1_x norm.

Now we study the convergence of $\frac{Q_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}}$. We first have that

$$H\left(\frac{Q_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}}\right) + \frac{Q^3_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}} = (2+\varepsilon)\frac{Q_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}} \tag{3.49}$$

Therefore if $u \in H^1_x$ is a weak limit point to $\frac{Q_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}}$ when $\varepsilon \to 0$ we have that

$$Hu \leqslant 2u \tag{3.50}$$

which impose that u is proportional to h_0. However we have that

$$\left(\frac{\|Q_{2+\varepsilon}\|_{H^1_x}}{\|Q_{2+\varepsilon}\|_{L^2}}\right)^2 + \frac{\|Q_{2+\varepsilon}\|^4_{L^4}}{\|Q_{2+\varepsilon}\|^2_{L^2}} = 2+\varepsilon \tag{3.51}$$

Using moreover that

$$\frac{\|Q_{2+\varepsilon}\|^4_{L^4}}{\|Q_{2+\varepsilon}\|^2_{L^2}} \leqslant C\|Q_{2+\varepsilon}\|^2_{H^1_x} \to 0 \tag{3.52}$$

we now find that

$$\frac{\|Q_{2+\varepsilon}\|_{H^1_x}}{\|Q_{2+\varepsilon}\|_{L^2}} \to \sqrt{2} \qquad \varepsilon \to 0 \tag{3.53}$$

and the compact injection of H^1_x into L^2 (3.1) yields

$$\|u\|_{L^2} = \frac{1}{\sqrt{2}} \tag{3.54}$$

Therefore, $\frac{Q_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}}$ converges weakly to its unique weak limit point $\frac{1}{\sqrt{2}}h_0$ in H^1_x, and this convergence happens in H^1_x norm with again the same trick.

We already know

$$\frac{Q_{2+\varepsilon}}{\|Q_{2+\varepsilon}\|_{H^1_x}} \to \frac{1}{\sqrt{2}}h_0 \qquad \text{in } H^1_x \tag{3.55}$$

Using that $HQ_{2+\varepsilon} + Q^3_{2+\varepsilon} = (2+\varepsilon)Q_{2+\varepsilon}$ and that $Q_{2+\varepsilon} = \|Q_{2+\varepsilon}\|_{L^2}h_0 + o()$ in H^1_x we can write

$$\begin{aligned} &\|\|Q_{2+\varepsilon}\|_{L^2}h_0 + o()\|^2_{H^1_x} + \|\|Q_{2+\varepsilon}\|_{L^2}h_0 + o()\|^4_{L^4} = (2+\varepsilon)\,\|\|Q_{2+\varepsilon}\|_{L^2}h_0 + o()\|^2_{L^2} \\ \text{thus} \quad & 2\|Q_{2+\varepsilon}\|^2_{L^2} + \|Q_{2+\varepsilon}\|^4_{L^2}\|h_0\|^4_{L^4} = (2+\varepsilon)\|Q_{2+\varepsilon}\|^2_{L^2} + o() \\ \text{which reads} \quad & \|Q_{2+\varepsilon}\|_{L^2} = \varepsilon^{\frac{1}{2}}\|h_0\|^{-2}_{L^4} + o(\cdot) \end{aligned}$$

□

3.4 L^∞ bound

We will need to use, at some point, that the multiplication by Q^2 is continuous over the function space H^k_x, and with operator norm arbitrarily small. To this effect, we prove the following statement :

Proposition 3.4. *Take $k \geqslant 0$. There exists a constant $C_k > 0$ such that when $\varepsilon \to 0$ there stands the bound*

$$\|D^kQ\|_{L^\infty} \leqslant C_k\varepsilon^{\frac{1}{2}} \tag{3.56}$$

Let us stress that this bound will typically happen over an open intervall $\varepsilon \in (0, \alpha_k)$ where the upper bound α_k depends of the derivation order k that we consider. However we will always consider only a finite number of derivatives together so we will always be able to have a uniform interval where the bound stands. That is why we will write "when $\varepsilon \to 0$" without specifying the meaning of this assertion : it will mean "there exists a $\alpha > 0$ such that on $(0, \alpha)$ there stands".

Proof. Using lemma (2.6), we see that

$$\|D^kQ\|_{L^\infty} \leqslant C_k\|D^kQ\|_{H^2} \leqslant C_k\|Q\|_{H^{2+k}} \tag{3.57}$$

Thus, using lemma (2.5), we need only prove that $\|Q\|_{H^{k+2}_x} \leqslant C_k\varepsilon^{\frac{1}{2}}$. As however the H^r_x norms are increasing with respect to r, we need only prove that given an even integer $2m$ there stands

$$\|Q\|_{H^{2m}_x} \leqslant C_m\varepsilon^{\frac{1}{2}} \tag{3.58}$$

Thanks to (3.3), we know the statement for $m = 0$. We also see that using (3.1) and (3.3)

$$\begin{aligned} \|Q^3\|_{L^2} &= \|Q\|^3_{L^6} \\ &\leqslant C\|Q\|^3_{H^1_x} \\ &\leqslant C\varepsilon^{\frac{3}{2}} \\ &\leqslant C\varepsilon^{\frac{1}{2}} \end{aligned}$$

provided we suppose ε to be small enough. Thus, we can write

$$\begin{aligned} \|Q\|_{H^2_x} &= \|HQ\|_{L^2} \\ &= \left\|\lambda Q - Q^3\right\|_{L^2} \\ &\leqslant C_2\varepsilon^{\frac{1}{2}} \end{aligned}$$

Now, assume $\|Q\|_{H^{2m}_x} \leqslant C_m\varepsilon^{\frac{1}{2}}$ for a given $m \geqslant 1$. Then we have provided for example $1 > \varepsilon$

$$\begin{aligned} \|Q\|_{H^{2m+2}_x} &= \|HQ\|_{H^{2m}_x} \\ &= \left\|\lambda Q - Q^3\right\|_{H^{2m}_x} \\ &\leqslant 3C_m\varepsilon^{\frac{1}{2}} + C\|Q\|^3_{H^{2m}_x} \\ &\leqslant C_{m+1}\varepsilon^{\frac{1}{2}} \end{aligned}$$

where we use lemma (2.6) to say that H^{2m}_x is an algebra. □

IV - Modulation

4.1 Modulation operators and energy estimates

Given $u \in H^1_x$ and $N > 0$ we set

$$(S_N u)(x) := \frac{1}{N} u\left(\frac{x}{N}\right) \tag{4.1}$$

As we are in dimension 2,

$$\|S_N u\|_{L^2} = \|u\|_{L^2} \tag{4.2}$$

Moreover for $u \in H^1_x$

$$\begin{aligned}
\|x S_N u\|^2_{L^2} &= \int_{\mathbb{R}^2} \frac{|x|^2}{N^2} u\left(\frac{|x|}{N}\right) dx \\
&= N^2 \|xu\|^2_{L^2} \\
\|\nabla S_N u\|^2_{L^2} &= \|\frac{1}{N} S_N\left(\nabla u\right)\|^2_{L^2} \\
&= \frac{1}{N^2} \|\nabla u\|^2_{L^2}
\end{aligned}$$

We also modulate by mutliplication by $e^{im|x|^2}$: this changes neither the L^2 norm neitger $\|xu\|_{L^2}$. However for all $u \in H^1_x$ there stands

$$\|\nabla\left(e^{im|x|^2} u(x)\right)\|^2_{L^2} = \|2imxu(x) + \nabla u(x)\|^2_{L^2} \tag{4.3}$$

which yields

$$\begin{aligned}
\text{When } u \text{ is real-valued } \|\nabla\left(e^{im|x|^2} u(x)\right)\|^2_{L^2} &= 4m^2 \|xu\|^2_{L^2} + \|\nabla u\|^2_{L^2} \\
\text{Otherwise } \|\nabla\left(e^{im|x|^2} u(x)\right)\|^2_{L^2} &\leqslant 4m^2 \|xu\|^2_{L^2} + \|\nabla u\|^2_{L^2}
\end{aligned}$$

Combining these estimates yields the lemma

Lemma 4.1 (energy estimates through modulation)**.** *When u is real-valued there stands*

$$\|S_N e^{im|y|^2} u(y)\|^2_{H^1_x} = \|x S_N e^{im|y|^2} u(y)\|^2_{L^2} + \|\nabla S_N e^{im|y|^2} u(y)\|^2_{L^2} = \left(N^2 + \frac{4m^2}{N^2}\right) \|xu\|^2_{L^2} + \frac{1}{N^2}\|\nabla u\|^2_{L^2} \tag{4.4}$$

Otherwise

$$\|S_N e^{im|y|^2} u(y)\|^2_{H^1_x} \leqslant \left(N^2 + \frac{4m^2+1}{N^2}\right) \|u\|^2_{H^1_x} \tag{4.5}$$

4.2 modulated equation

We study equation (1.7) :

$$i\partial_t u = -\Delta u + |x|^2 u + u|u|^2 + V(t,x)u$$

There stands :

Proposition 4.1 (modulated equation)**.** *Let $L(t) > 0$, $\gamma(t)$, $b(t)$ be $\mathcal{C}^1$ functions over $\mathbb{R}_+$. Set*

$$u := e^{i\gamma(t)} S_L w \qquad w(t,y) := e^{-i\frac{b|y|^2}{4}} v(t,y) \qquad \frac{ds}{dt} = \frac{1}{L^2} \tag{4.6}$$

assume $t \to s(t)$ is invertible from $(t_0, +\infty)$ onto itself. Then $u(t,x)$ solves (1.7) iff $v(s,y)$ solves

$$i\partial_s v + \Delta v - \gamma_s v + \left(-L^4 + \frac{b_s}{4} - \frac{b^2}{4} - \frac{L_s}{L}\frac{b}{2}\right)|y|^2 v - i\left(\frac{L_s}{L} + b\right)(1+\Lambda)v - v|v|^2 - W(s,y)v = 0 \tag{4.7}$$

where $L_s := \frac{d}{ds}(L(t(s)))$ and similar definitions for b_s, γ_s ; where $\Lambda := y \cdot \nabla$ and finally where

$$W(s,y) = L^2 V(t,x) \tag{4.8}$$

Proof. Indeed, let us compute

$$\begin{aligned}
i\partial_t u &= i\partial_t \left(e^{i\gamma(t)} \frac{1}{L(t)} w(t, \frac{x}{L(t)}) \right) \\
&= e^{i\gamma(t)} \left(-\gamma_t \frac{1}{L(t)} w(t, \frac{x}{L(t)}) - i\frac{L_t}{L^2} w + i\frac{1}{L}\partial_t w(t, \frac{x}{L(t)}) - i\frac{L_t}{L^3} x \cdot \nabla w(t, \frac{x}{L(t)}) \right) \\
&= e^{i\gamma} S_L \left(-i\frac{L_t}{L}(1+\Lambda)w + i\partial_t w - \gamma_t w \right) \\
&= \frac{e^{i\gamma}}{L^2} S_L \left(-i\frac{L_s}{L}(1+\Lambda)w + i\partial_s w - \gamma_s w \right)
\end{aligned}$$

where we use only that $\partial_t = \frac{1}{L^2}\partial_s$. This ensures that u solves (1.7) iff

$$\frac{1}{L^2}\left(-i\frac{L_s}{L}(1+\Lambda)w + i\partial_s w - \gamma_s w \right) = -S_L^{-1}\Delta S_L w + S_L^{-1}|x|^2 S_L w + S_L^{-1}\left(S_L w |S_L w|^2 \right) + Vw \tag{4.9}$$

We compute

$$\begin{aligned}
\Delta \left(\frac{1}{L} u \left(\frac{x}{L} \right) \right) &= \frac{1}{L^3} (\Delta u) \left(\frac{x}{L} \right) = S_L \left(\frac{1}{L^2} \Delta u \right) \\
|x|^2 \frac{1}{L} u \left(\frac{x}{L} \right) &= L^2 \frac{1}{L} \left| \frac{x}{L} \right|^2 u \left(\frac{x}{L} \right) = L^2 S_L \left(|y|^2 u(y) \right) \\
\frac{1}{L^3} u \left(\frac{x}{L} \right) \left| u \left(\frac{x}{L} \right) \right|^2 &= \frac{1}{L^2} S_L (u|u|^2)
\end{aligned}$$

Thus u is a solution iff

$$i\partial_s w - i\frac{L_s}{L}(1+\Lambda)w - \gamma_s w = -\Delta w + L^4 |y|^2 w + w|w|^2 + L^2 V w \tag{4.10}$$

Let us here stress an essential algebraic fact : the cubic nonlinearity $u|u|^2$ is preserved through modulation, whereas should we have a nonlinearity of the form $u|u|^{p-1}$ a L^{3-p} would appear. Our proof doesn't therefore apply to other exponents.

Using now that $w(s,y) = e^{-i\frac{b|y|^2}{4}} v(s,y)$ we compute

$$\begin{aligned}
i\partial_s w &= e^{-i\frac{b|y|^2}{4}} \left(\frac{b_s}{4} |y|^2 v + i\partial_s v \right) \\
\Lambda w &= e^{-i\frac{b|y|^2}{4}} y \cdot \left(-i\frac{b}{2} v y + \nabla v \right) = e^{-i\frac{b|y|^2}{4}} \left(-i\frac{b}{2} |y|^2 v + \Lambda v \right) \\
\Delta w &= \nabla \cdot e^{-i\frac{b|y|^2}{4}} \left(-i\frac{b}{2} v y + \nabla v \right) = e^{-i\frac{b|y|^2}{4}} \left(\Delta v - i\frac{b}{2}\Lambda v - ibv + \frac{b^2}{4}|y|^2 v - i\frac{b}{2}\Lambda v \right)
\end{aligned}$$

which yields the proposition ☐

4.3 Hamiltonian structure and resonent trajectory

The following result in proved in [1] very thoroughly. We will not explain the proof here, which relies on carefully lead considerations that rely on hamiltonian dynamical systems.

Proposition 4.2 (resonent trajectory)**.** *Let*

$$\beta(s) := \frac{\sin(4s)}{s \log s} \qquad s > 1 \tag{4.11}$$

There exists a $s_0 > 1$ and a solution with $\mathcal{C}^\infty$ regularity (L, b) over $(s_0, +\infty)$ to the system

$$\begin{cases} L^4 - \frac{b_s}{4} + \frac{b^2}{4} + \frac{L_s}{L}\frac{b}{2} & = 1 + \beta(s) \\ \frac{L_s}{L} + b & = 0 \end{cases} \tag{4.12}$$

Such that the associated hamiltonian

$$E(L,b) := \frac{1}{L^2}\left(\frac{b^2}{4}+1\right)+L^2 = \log s + O\left(\frac{\log s}{s}\right) \qquad s\to\infty \tag{4.13}$$

Moreover there is a $B_0>0$ *such that*

$$\forall s \geqslant s_0 \qquad \frac{1}{B_0\log s} \leqslant L^2 \leqslant B_0\log s \qquad |b(s)| \leqslant B_0(\log s)^3 \tag{4.14}$$

The time defined by $\frac{dt}{ds}=\frac{1}{L^2}>0$ *over* $(s_0,+\infty)$ *satisfies*

$$|t(s)-s| \leqslant B_0(\log s)^2 \tag{4.15}$$

Thus, $t(s)$ *is globally invertible. Finally, if we see* $L(t), b(t)$ *as function of* t*, there stands*

$$\left|\frac{d^kL}{dt^k}(t)\right| + \left|\frac{d^kb}{dt^k}(t)\right| + \left|\frac{d^k}{dt^k}\left(\frac{1}{L}\right)(t)\right| \leqslant B_k\left(\log t\right)^{\alpha_k} \tag{4.16}$$

Thanks to this proposition, after modulation, we need only study the solutions of

$$i\partial_s v = Hv + \beta(s)|y|^2v + \gamma_s v + v|v|^2 + W(s,y)v \tag{4.17}$$

V - SOLITON + PERTURBATION DECOMPOSITION

Let $\lambda = 2+\varepsilon$ with $\varepsilon>0$ small enough. We denote $Q := Q_\lambda$ the corresponding soliton built in section III and we take the explicit choice

$$\gamma_s := -\lambda \tag{5.1}$$

As $\beta(s)$ and $W(s,y)$ should decay to 0 when $s\to\infty$, the equation roughly becomes when $s\to\infty$

$$i\partial_s v = Hv + v|v|^2 - \lambda v \tag{5.2}$$

To which $v(s,y)=Q(y)$ is a stationary solutions. We will therefore try and find a solution to (4.17) of the form

$$v(s,y) = Q(y) + w(s,y) \tag{5.3}$$

where we want $w(s,y)$ to be a perturbation decaying to zero when $s\to\infty$. Indeed, the aim of this section and the following one will be to prove the following statement

Proposition 5.1. *Let* $\beta(s)=\frac{\sin(4s)}{s\log s}$ *et* $W(s,y) := -\alpha\beta(s)Q(y)$ *(where* $\alpha\in\mathbb{R}$ *is a constant to be fixed in the proof). Let* $k\geqslant 1$ *an integer. If we take* $s_0>0$ *large enough, there exists a constant* $B>0$ *depending only of* k *and a solution* $v\in\mathcal{C}((s_0,+\infty),H_x^{2k+1})$ *such that*

$$i\partial_s v = Hv + \beta(s)|y|^2v - \lambda v + v|v|^2 + W(s,y)v \qquad s\in[s_0,+\infty) \tag{5.4}$$

and

$$v(s,y) = Q(y)+w(s,y) \qquad \text{such that} \qquad \forall s\in[s_0,+\infty),\ \|w(s)\|_{H_x^{2k+1}} \leqslant \frac{B}{s\log s} \tag{5.5}$$

5.1 Change of function

As $HQ+Q^3=\lambda Q$ we need to find a solution to

$$i\partial_s w = (H-\lambda)w + 2Q^2w + Q^2\overline{w} + \beta(s)|y|^2w + W(s,y)w + R(s) + N(s) \tag{5.6}$$

where

$$R(s) = \beta(s)|y|^2 Q(y) + W(s,y)Q(y)$$
$$N(s) = 2Q|w|^2 + Qw^2 + w|w|^3$$

The equation is no longer $\mathbb{C}$-linear : we thus write $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and write

$$\partial_s w = \mathcal{L}w + I\left(\beta(s)|y|^2 + W(s,y)\right)w + IR(s) + IN(s) \tag{5.7}$$

where :

$$\mathcal{L} := \begin{pmatrix} 0 & H_- \\ -H_+ & 0 \end{pmatrix}$$
$$H_- := H + Q^2 - \lambda$$
$$H_+ := H + 3Q^2 - \lambda$$
$$I := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

5.2 Spectral theory

We can write $H_{+/-} = H - 2 + A_{+/-}$ with $A_+ := 3Q^2 - \varepsilon$ and $A_- := Q^2 - \varepsilon$. Using lemma (3.4) we have that taking $\varepsilon > 0$ small enough yields that $A_{+/-}$ is a bounded operator over L^2 with operator norm arbitrarily small. Lemma (2.1) yields that $H_{+/-}$ has a discreet spectrum with simple eigenvalues $\lambda_0^{+/-} < \lambda_1^{+/-} < ....$ with

$$|\lambda_n^{+/-} - 4n| < \alpha \tag{5.8}$$

uniformly in n, where $\alpha > 0$ is a given constant that we can choose arbitrarily small with ε. We then observe that

$$H_- Q = 0 \tag{5.9}$$

This follows from the definition of Q. Thus, $\lambda_0^- = 0$. Therefore, H_- is a nonnegative self-adjoint operator with compact resolvent over $L^2(\mathbb{R})$ with kernel $\mathbb{R}Q$. As $H_+ = H_- + 2Q^2$ we infer that $\lambda_0^+ > 0$ which yields that H_+ is positive. Moreover we see straightforwardly that

Lemma 5.1. *Let $k \geqslant 0$. The norm defined by*

$$\|u\|^2_{H_+^k} := \langle H_+^k u, u\rangle \tag{5.10}$$

is equivalent to the H_x^k norms

5.3 Preserved energies

In order to estimate the H_x^k norm of w, we wish to exhibit quantities preserved by the linearized flow $\exp(s\mathcal{L})$ of the equation.

Lemma 5.2 (First preserved energy)**.** *Let $\mathcal{H} := \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$. For all $u \in H_x^1$, we set*

$$E(u) := \langle \mathcal{H}u, u\rangle = \langle H_+ u_1, u_1\rangle + \langle H_- u_2, u_2\rangle \tag{5.11}$$

Then the energy E is preserved by the flow $\exp(s\mathcal{L})$

Proof. It is enough to see that

$$\langle \mathcal{H}\mathcal{L}u, u\rangle = 0 \qquad \forall u \tag{5.12}$$

□

Now, as $H_{+/-} = H + A_{+/-}$ with $A_{+/-}$ a bounded operator over $L^2(\mathbb{R})$, we see that in order to bound the H^1_x norm of u, it is enough to bound $E(u)$ and $\|u\|_{L^2}$. However, as H_+ is positive and as the kernel of H_- is $\mathbb{R}Q$ we get that

$$\exists c, C > 0, \forall u \in H^1_x \qquad c\|u\|^2_{H^1_x} \leqslant E(u) + \langle u_2, Q\rangle^2 \leqslant C\|u\|^2_{H^1_x} \tag{5.13}$$

Let us now define

$$\rho := H_+^{-1} Q \in L^2(\mathbb{R}) \tag{5.14}$$

Up to a smaller choice of ε, there stands

$$\langle \rho, Q\rangle > 0 \tag{5.15}$$

Indeed, lemma (3.7) ensures that in fact $\rho = \partial_\lambda Q$ so that

$$\langle \rho, Q\rangle = \frac{1}{2}\partial_\lambda \|Q_\lambda\|^2_{L^2} > 0 \tag{5.16}$$

The inequality follows from $\lambda \to \|Q_\lambda\|_{L^2}$ being increasing : thus its derivative is almost everywhere positive. This enables us to change slightly the previous estimate into :

$$\exists c, C > 0, \forall u \in H^1_x \qquad c\|u\|^2_{H^1_x} \leqslant E(u) + \langle u_2, \rho\rangle^2 \leqslant C\|u\|^2_{H^1_x} \tag{5.17}$$

Now, $E(u)$ reveals insufficient to properly estimate w. Indeed, H^1_x is not an algbera, thus we cannot control the nonlinearity using only H^1_x norm. We therefore introduce another preserved energy

Lemma 5.3 (Higher order preserved energy). *Define* $\mathcal{H} := \begin{pmatrix} H_+H_-H_+ & 0 \\ 0 & H_-H_+H_- \end{pmatrix}$ *and for* $u \in H^3_x$ *set*

$$\mathcal{E}(u) := \langle \mathcal{H}u, u\rangle = \langle H_+H_-H_+u_1, u_1\rangle + \langle H_-H_+H_-u_2, u_2\rangle \tag{5.18}$$

The energy $\mathcal{E}$ *is preserved by the flow* $\exp(s\mathcal{L})$

Proof. It is straightforward from the observation

$$\langle \mathcal{H}\mathcal{L}u, u\rangle = 0 \qquad \forall u \tag{5.19}$$

□

The point is now that there exists $c, C > 0$ such that

$$c\langle H^3 u, u\rangle \leqslant \mathcal{E}(u) + E(u) + \langle u_2, \rho\rangle^2 \leqslant C\langle H^3 u, u\rangle \tag{5.20}$$

Indeed, the right inequality is straightforward, and for the left one we write for example that $H_+H_-H_+ = H^3 + B$ where B is a sum of terms of the form $A_1A_2A_3$ where A_i is one of $H, Q^2 - \lambda, 3Q^2 - \lambda$ with at most 2 of the A_i equals to H. Thus we see that $|\langle Bu, u\rangle| \leqslant C\|u\|^2_{H^2_x}$. Now, lemma (2.5) helps us bound this quantity by $\frac{1}{2}\|u\|^2_{H^3_x} + C\|u\|^2_{L^2}$. Therefore

$$\langle H_+H_-H_+u_1, u_1\rangle \geqslant \frac{1}{2}\|u_1\|^2_{H^3_x} - C\|u_1\|^2_{L^2} \tag{5.21}$$

and we know that $\|u_1\|_{L^2}$ can be bounded using $E(u)$ and $\langle u_2, \rho\rangle$. We bound $\|u_2\|_{H^3_x}$ in the same way.

Now, to bound the H^3_x norm (which is an algebra norm) of a function u, one needs only bound $\mathcal{E}(u)$, $E(u)$, and $\langle u_2, \rho\rangle$

5.4 Explicit computation of $\exp(s\mathcal{L})$

We here give a *formal* computation of $\exp(s\mathcal{L})$. We have that :

$$\begin{aligned}\mathcal{L}^{2n} &= (-1)^n \begin{pmatrix} [H_-H_+]^n & 0 \\ 0 & [H_+H_-]^n \end{pmatrix} \\ \mathcal{L}^{2n+1} &= (-1)^n \begin{pmatrix} 0 & H_-[H_+H_-]^n \\ -H_+[H_-H_+]^n & 0 \end{pmatrix}\end{aligned}$$

Next, we see that H_- and H_+ are nonnegative, so we can properly define over the unbounded operator

$$A := \sqrt{H_+^{\frac{1}{2}} H_- H_+^{\frac{1}{2}}} \tag{5.22}$$

Moreover, as H_+ is invertible, we compute :

$$\begin{aligned}\mathcal{L}^{2n} &= (-1)^n \begin{pmatrix} H_+^{-\frac{1}{2}} A^{2n} H_+^{\frac{1}{2}} & 0 \\ 0 & H_+^{\frac{1}{2}} A^{2n} H_+^{-\frac{1}{2}} \end{pmatrix} \\ \mathcal{L}^{2n+1} &= (-1)^n \begin{pmatrix} 0 & H_+^{-\frac{1}{2}} A^{2n+2} H_+^{-\frac{1}{2}} \\ -H_+^{\frac{1}{2}} A^{2n} H_+^{\frac{1}{2}} & 0 \end{pmatrix}\end{aligned}$$

Which reads

$$\exp(s\mathcal{L}) = \begin{pmatrix} H_+^{-\frac{1}{2}} \cos(sA) H_+^{\frac{1}{2}} & H_+^{-\frac{1}{2}} A \sin(sA) H_+^{-\frac{1}{2}} \\ -H_+^{\frac{1}{2}} s\mathrm{sinc}(sA) H_+^{\frac{1}{2}} & H_+^{\frac{1}{2}} \cos(sA) H_+^{-\frac{1}{2}} \end{pmatrix} \tag{5.23}$$

Where we define $\mathrm{sinc}(x) := \frac{\sin(x)}{x} = \sum_{n \geqslant 0} (-1)^n \frac{x^{2n}}{(2n+1)!}$

Now, this is only formal. We must study A in order to prove that, provided we define $\exp(s\mathcal{L})$ by the expression (5.23), it has a domain large enough.

Lemma 5.4 (Spectrum of A)**.** *Let $\eta > 0$. There exists $\alpha > 0$ such that whenever $0 < \varepsilon < \alpha$, A is a nonnegative operator over L^2, selfadjoint, with compact resolvent. Moreover, its spectra is formed of simple eigenvalues $0 \leqslant \mu_0 < \mu_1 < ...$ with*

$$\forall n \geqslant 0, \qquad |\mu_n - 4n| \leqslant \eta \tag{5.24}$$

Finally, as H_- has a nontrivial kernel, so has A. Therefore, $\mu_0 = 0$

Proof. Take η as in the statement, and $\delta > 0$ assumed small, to be fixed later. Without loss of generality, we assume $\delta < 1$.

We study $A^2 = H_+^{\frac{1}{2}} H_- H_+^{\frac{1}{2}} = H_+^2 - 2H_+^{\frac{1}{2}} Q^2 H_+^{\frac{1}{2}}$. First, let us study the resolvent of A^2 : we must ask ourselves, given a $\zeta \in \mathbb{C}$, when is $A^2 - \zeta$ invertible. To this effect, we introduce $0 < \lambda_0 < \lambda_1 < ...$ the spectrum of H_+, where $\lambda_n = 4n + \varepsilon_n$ and we have that the L^∞ norm of $(\varepsilon_n)_n$ has limit zero when $\varepsilon \to 0$, and we take $\psi_0, \psi_1, ...$ an orthonormal basis of L^2 such that $H_+\psi_n = \lambda_n \psi_n$. We set D_0 the disk of center 0 and radius δ, D_1 the disk of center 16 and radius δ, then D_n the disk of center $16n^2$ and radius $n\delta$. We set Γ_n the boundary circle of D_n positively oriented. We finally set $\Sigma = \bigcup_{n \geqslant 0} D_n$.

Take $\zeta \in \mathbb{C}\backslash\Sigma$. We will show that, provided we take $\varepsilon > 0$ small enough (depending on δ only), $A^2 - \zeta$ is invertible. First, the nth eigenvalue of H_+^2 is $16n^2 + 4n\varepsilon_n + \varepsilon_n^2$. Provided ε is small enough, we have for all $n \geqslant 1$ that

$$|16n^2 - \lambda_n^2| \leqslant \frac{1}{2}\delta n \tag{5.25}$$

and for $n = 0$ that $|\lambda_0^2| \leqslant \frac{1}{2}\delta$. Thus, $H_+ - \zeta$ is invertible and we are able to formally write :

$$(A^2 - \zeta)^{-1} = (H_+^2 - \zeta)^{-1} \left(I - 2H_+^{\frac{1}{2}} Q^2 H_+^{\frac{1}{2}} (H_+^2 - \zeta)^{-1} \right)^{-1} \tag{5.26}$$

where I is the identity over L^2. Our goal is to prove that $2H_+^{\frac{1}{2}}Q^2H_+^{\frac{1}{2}}(H_+^2-\zeta)^{-1}$ is a bounded operator over L^2 with operator norm strictly smaller than 1 which will prove the invertibility of $A^2-\zeta$. To this effect, we first observe that there exists a universal constant $C>0$ such that given any $v\in H_x^1$ there stands

$$\langle H_+(Q^2v),Q^2v\rangle \leqslant C\varepsilon^2\langle H_+v,v\rangle \tag{5.27}$$

Indeed, we can write $H_+=H+B$ where B is a bounded operator over L^2, so

$$\begin{aligned}\langle H_+(Q^2v),Q^2v\rangle &= \langle H(Q^2v),Q^2v\rangle+\langle B(Q^2v),Q^2v\rangle\\ &\leqslant \|\nabla(Q^2v)\|_{L^2}^2+\|xQ^2v\|_{L^2}^2+\|B\|\|Q^2v\|_{L^2}^2\end{aligned}$$

Now, as $\nabla Q^2v=Q^2\nabla v+2Q\nabla Qv$, proposition (3.4) enables us to conclude.

Take now $u=\sum\limits_{n\geqslant 0}\alpha_n\psi_n\in L^2$: there stands

$$\begin{aligned}\left\|2H_+^{\frac{1}{2}}Q^2H_+^{\frac{1}{2}}(H_+^2-\zeta)^{-1}u\right\|_{L^2}^2 &\leqslant C\varepsilon^2\|H_+(H_+^2-\zeta)^{-1}u\|_{L^2}^2\\ &\leqslant C\varepsilon^2\sum_{n\geqslant 0}\frac{\lambda_n}{|\lambda_n^2-\zeta|}|\alpha_n|^2\end{aligned}$$

Given $n\geqslant 1$, as ζ is outside D_n, $|\lambda_n^2-\zeta|\geqslant\frac{1}{2}n\delta$ and thus $\frac{\lambda_n}{|\lambda_n^2-\zeta|}\leqslant\frac{4n+\varepsilon_n}{\frac{1}{2}n\delta}\leqslant\frac{8}{\delta}$. Similarly, we have $\frac{\lambda_0}{|\lambda_0^2-\zeta|}\leqslant\frac{8}{\delta}$. Thus, in terms of operator norm over L^2,

$$\left\|2H_+^{\frac{1}{2}}Q^2H_+^{\frac{1}{2}}(H_+^2-\zeta)^{-1}\right\|\leqslant\frac{C'\varepsilon}{\sqrt{\delta}} \tag{5.28}$$

where C' is a constant. Provided ε is small enough, the right-hand side is stricly smaller than 1 and thus $I-2H_+^{\frac{1}{2}}Q^2H_+^{\frac{1}{2}}(H_+^2-\zeta)^{-1}$ is invertible and formula (5.26) is correct.

First, this ensures that A^2 has compact resolvent (indeed its resolvent is the product of a bounded operator over L^2 and of a compact one, thus is it compact). Therefore we already know that A^2 has a discreet spectrum and that there is an orthonormal basis of L^2 formed of eigenfunctions for A^2. Moreover, as we know that A^2 has no spectrum outside Σ, its eigenvalues are located within the D_n's. We now prove that A^2 has exactly one simple eigenvalue in D_n. Indeed, following Kato [3, p.178] we know that the spectral projector over the eigenfunctions corresponding to the eigenvalues located within D_n can be written as

$$\pi_n^\varepsilon=\frac{1}{2i\pi}\int_{\Gamma_n}(A^2-\zeta)^{-1}d\zeta \tag{5.29}$$

However, using equation (5.28), we see that

$$\left\|\left(I-2H_+^{\frac{1}{2}}Q^2H_+^{\frac{1}{2}}(H_+^2-\zeta)^{-1}\right)^{-1}-I\right\|\leqslant\frac{C\varepsilon}{\sqrt{\delta}} \tag{5.30}$$

for a constant C when ε is small enough, which is an immediate consequence of the formula

$$(I-B)^{-1}-I=\sum_{n\geqslant 1}B^n \tag{5.31}$$

whenever $\|B\|<1$. Thus, if we write

$$\pi_n:=\frac{1}{2i\pi}\int_{\Gamma_n}(H_+^2-\zeta)^{-1}d\zeta \tag{5.32}$$

there stands

$$\begin{aligned}
\|\pi_n^\varepsilon - \pi_n\| &= \frac{1}{2\pi}\left\|\int_{\Gamma_n}(H_+^2-\zeta)^{-1}\left(\left(I-2H_+^{\frac{1}{2}}Q^2H_+^{\frac{1}{2}}(H_+^2-\zeta)^{-1}\right)^{-1}-I\right)d\zeta\right\| \\
&\leqslant \frac{C\varepsilon}{\sqrt{\delta}}\frac{1}{2\pi}\int_{\Gamma_n}\left\|(H_+^2-\zeta)^{-1}\right\|d\zeta \\
&\leqslant \frac{C\varepsilon}{\sqrt{\delta}}\frac{1}{2\pi}\int_{\Gamma_n}\frac{1}{\frac{1}{2}n\delta}d\zeta \\
&\leqslant C\frac{\varepsilon}{\sqrt{\delta}}
\end{aligned}$$

For a constant C which is indepedent of n and of ε. This upper bound can be made arbitrarily small when $\varepsilon \to 0$: should it be smaller than 1 strictly, this ensures that π_n^ε and π_n have the same range (see [3, Lemma 4.1 p.34]) which is equal to 1 as H_+ has simple eigenvalues.

Therefore, provided $\varepsilon > 0$ is small enough, and this depending only of δ, we have proved that A^2 has a discreet spectrum formed of simple eigenvalues $0 \leqslant \mu_0^2 < \mu_1^2 < ....$ with $\mu_n^2 = 16n^2 + \alpha_n$, where $|\alpha_n| \leqslant \delta max(1,n)$ (indeed, $\mu_n^2 \in D_n$).

Thus, A has the discreet spectrum formed of simple eigenvalues $0 \leqslant \mu_0 < \mu_1 < ...$ and we see that $|\mu_0| \leqslant \sqrt{\delta}$, and moreover that for $n \geqslant 1$

$$\begin{aligned}
|\mu_n - 4n| &= \left|\sqrt{16n^2+\alpha_n} - 4n\right| \\
&= 4n\left|\sqrt{1+\frac{\alpha_n}{16n^2}} - 1\right| \\
&\leqslant 4nC\frac{\alpha_n}{16n^2} \\
&\leqslant C\delta
\end{aligned}$$

where C is a constant. Thus, provided we initially choose

$$\max(\sqrt{\delta}, C\delta) < \eta \tag{5.33}$$

we have proved the statement. □

We fix until the end $\psi_0, \psi_1, ...$ an orthonormal basis of eigenfunctions for A

VI - BACKWARD INTEGRATION

Using the a priori bounds on V, we see that the equation (5.6) is locally well-posed in H_x^3, thus we have existence and uniqueness of local solutions for a given initial condition. Thus, given $0 < s_0 < M$, we may set w^M the unique solution to (5.6) defined close to M such that $w^M(M) = 0$. Let us show that for a choice of $s_0, B > 0$ big enough, w^M is defined over $(s_0, M]$ for all $M > s_0$ and there stands

$$\|w^M(s)\|_{H_x^3} \leqslant \frac{B}{s\log s} \qquad s_0 < s < M \tag{6.1}$$

6.1 Another change of function

We set

$$f^M := w^M + \int_s^M \exp((s-\sigma)\mathcal{L})IR(\sigma)d\sigma := w^M + r^M \tag{6.2}$$

f^M is a solution to

$$\partial_s f^M = \mathcal{L}f^M + IK(s)f^M - IK(s)r^M(s) + IN(s) \tag{6.3}$$

where we set $K(s) := \beta(s)(|y|^2 - \alpha Q(y))$. In order to estimate the H_x^3 norm of w^M, we need to estimate f^M and r^M

6.2 Estimate of r^M

We here choose the value of α : we set

$$\alpha := \frac{\langle |y|^2 Q, H_+^{-\frac{1}{2}}\psi_1\rangle}{\langle Q^2, H_+^{-\frac{1}{2}}\psi_1\rangle} \tag{6.4}$$

such that $R(s) = \beta(s)(|y|^2 Q - \alpha Q^2)$ is orthogonal to $H_+^{-\frac{1}{2}}\psi_1$, or to say it differently that $H_+^{-\frac{1}{2}}R(s)$ is orthogonal to ψ_1. We need to justify that α is well defined, that is we need to prove that $\langle Q^2, H_+^{-\frac{1}{2}}\psi_1\rangle \neq 0$ provided $\varepsilon > 0$ is small enough. We compute to this effect that $H_+^{-\frac{1}{2}}\psi_1 \to \frac{1}{2}h_1$ in L^2 when $\varepsilon \to 0$ (this follows from the computations of the proof of lemma (5.4)). Moreover using (3.3) we see that $Q^2 \sim C\varepsilon h_0^2$ when $\varepsilon \to 0$ in L^2 with $C > 0$ Thus, it is enough to know that the scalar product $\langle h_0^2, h_1\rangle$ is nonzero, which is proved in [1].

We prove the following lemma :

Lemma 6.1. *There is a constant B_0 independent of s and M such that*

$$\|r^M(s)\|_{H^3_x} \leqslant \frac{B_0}{s\log s} \tag{6.5}$$

The choice of α enables us to write

$$\begin{aligned} r^M(s) &= \int_s^M \frac{\sin(4\sigma)}{\sigma\log\sigma} \exp((s-\sigma)\mathcal{L}) \begin{pmatrix} 0 \\ -(|y|^2Q - \alpha Q^2) \end{pmatrix} d\sigma \\ &= -\begin{pmatrix} H_+^{-\frac{1}{2}} A \int_s^M \frac{\sin(4\sigma)}{\sigma\log\sigma} \cos((s-\sigma)A) H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2) d\sigma) \\ H_+^{\frac{1}{2}} \int_s^M \frac{\sin(4\sigma)}{\sigma\log\sigma} \sin((s-\sigma)A) H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2) d\sigma \end{pmatrix} \end{aligned}$$

However

(6.6)
$$\sin(4\sigma)\cos((s-\sigma)A)H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2) = \frac{1}{2}\left(\sin(sA - \sigma(A-4)) + \sin(sA - \sigma(A+4))\right) H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2)$$

And we have that

$$\sin(sA - \sigma(A-4))H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2) = \sum_{n\neq 1} \sin(s\mu_n - \sigma(\mu_n - 4))\langle H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2), \psi_n\rangle\psi_n \tag{6.7}$$

where we are able to exclude $n = 1$ because of the definition of α.

Now, if we denote $\phi(y) := H_+^{-\frac{1}{2}}(|y|^2Q - \alpha Q^2)$ there stands :

$$\int_s^M \frac{\sin(sA - \sigma(A-4))}{\sigma\log\sigma}\phi(y)d\sigma = \sum_{n\neq 1}\left(\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n - 4))}{\sigma\log\sigma}d\sigma\right)\langle\phi(y), \psi_n\rangle\psi_n \tag{6.8}$$

However :

$$\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n-4))}{\sigma\log\sigma}d\sigma = \left[\frac{\cos(s\mu_n - \sigma(\mu_n - 4))}{(\mu_n - 4)\sigma\log\sigma}\right]_s^M + \int_s^M \frac{\cos(s\mu_n - \sigma(\mu_n-4))}{\mu_n - 4}\frac{\log\sigma + 1}{\sigma^2(\log\sigma)^2}d\sigma \tag{6.9}$$

As $n \neq 1$ we see that $|\mu_n - 4| \geqslant c > 0$ with c a constant. Thus, we have the bound :

$$\left|\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n - 4))}{\sigma\log\sigma}d\sigma\right| \leqslant \frac{B_0}{s\log s} \tag{6.10}$$

where B_0 does not depend on M nor on n.

The same kind of reasoning gives :

$$\int_s^M \frac{\sin(sA - \sigma(A+4))}{\sigma \log \sigma}\phi(y)d\sigma = \sum_{n\neq 1}\left(\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n+4))}{\sigma\log\sigma}d\sigma\right)\langle \phi(y), \psi_n\rangle\psi_n \tag{6.11}$$

where

$$\left|\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n+4))}{\sigma\log\sigma}d\sigma\right| \leqslant \frac{B_0}{s\log s} \tag{6.12}$$

Let us now see that writing

$$r_1^M(s) = H_+^{-\frac{1}{2}}A\sum_{n\neq 1}\alpha_n(s)\langle\phi(y),\psi_n\rangle\psi_n \tag{6.13}$$

with $|\alpha_n(s)| \leqslant \frac{B_0}{slogs}$ yields :

$$\begin{aligned}
\|r_1^M(s)\|_{H_x^3}^2 &\leqslant \|r_1^M(s)\|_{H_+^3}^2 \\
&= \left\| H_+^{-\frac{1}{2}}A\sum_{n\neq 1}\alpha_n(s)\langle\phi(y),\psi_n\rangle\psi_n\right\|_{H_+^3}^2 \\
&= \left\| A\sum_{n\neq 1}\alpha_n(s)\langle\phi(y),\psi_n\rangle\psi_n\right\|_{H_+^2}^2
\end{aligned}$$

Now, given a $u \in H_x^2$, there stands :

$$\begin{aligned}
\|u\|_{H_+^2}^2 &= \langle H_+^2 u, u\rangle \\
&= \langle A^2 u, u\rangle + 2\langle H_+^{\frac{1}{2}}Q^2 H_+^{\frac{1}{2}}u, u\rangle \\
&\leqslant \langle A^2 u, u\rangle + \varepsilon\|u\|_{H_+^1}^2 \\
&\leqslant \langle A^2 u, u\rangle + \frac{1}{2}\|u\|_{H_+^2}^2 + C\|u\|_{L^2}^2
\end{aligned}$$

where C is a constant independent of u. Thus, we see that

$$\|u\|_{H_+^2}^2 \leqslant C\left(\langle A^2 u, u\rangle + \|u\|_{L^2}^2\right) \tag{6.14}$$

From which we obtain :

$$\begin{aligned}
\|r_1^M(s)\|_{H_x^3}^2 &\leqslant C\langle A^2 A\sum_{n\neq 1}\alpha_n(s)\langle\phi(y),\psi_n\rangle\psi_n, A\sum_{n\neq 1}\alpha_n(s)\langle\phi(y),\psi_n\rangle\psi_n\rangle + C\left\|A\sum_{n\neq 1}\alpha_n(s)\langle\phi(y),\psi_n\rangle\psi_n\right\|_{L^2}^2 \\
&= \sum_{n\neq 1}(\mu_n^4 + \mu_n^2)\alpha_n(s)^2\left|\langle\phi(y),\psi_n\rangle\psi_n\right|^2 \\
&\leqslant \frac{B_0^2}{s^2(\log s)^2}\sum_{n\neq 1}(\mu_n^4+\mu_n^2)\left|\langle\phi(y),\psi_n\rangle\psi_n\right|^2 \\
&\leqslant \frac{B_0^2}{s^2(\log s)^2}C\|\phi(y)\|_{H_x^2}
\end{aligned}$$

Wich we can write as :

$$\|r_1^M(s)\|_{H_x^3} \leqslant \frac{B_0}{s\log s} \tag{6.15}$$

with B_0 another constant without importance.

Now, we similarly show that

$$r_2^M(s) = H_+^{\frac{1}{2}} \sum_{n \neq 1} \beta_n(s) \langle \phi(y), \psi_n \rangle \psi_n \tag{6.16}$$

with $|\beta_n(s)| \leqslant \frac{B_0}{s \log s}$ uniformly in n. Thus, we are able to write :

$$\begin{aligned} \|r_2^M(s)\|_{H_x^3}^2 &\leqslant C \|r_2^M(s)\|_{H_+^3}^2 \\ &= \left\| \sum_{n \neq 1} \beta_n(s) \langle \phi(y), \psi_n \rangle \psi_n \right\|_{H_+^4}^2 \end{aligned}$$

A few computations show that

$$\|u\|_{H_+^4}^2 \leqslant C \left(\langle A^4 u, u \rangle + \|u\|_{L^2}^2 \right) \tag{6.17}$$

Thus, there stands

$$\|r_2^M(s)\|_{H_x^3}^2 \leqslant C \langle A^4 \sum_{n \neq 1} \beta_n(s) \langle \phi(y), \psi_n \rangle \psi_n, \sum_{n \neq 1} \beta_n(s) \langle \phi(y), \psi_n \rangle \psi_n \rangle + C \left\| \sum_{n \neq 1} \beta_n(s) \langle \phi(y), \psi_n \rangle \psi_n \right\|_{L^2}^2$$

and can conclude in the same way that

$$\|r_2^M(s)\|_{H_x^3} \leqslant \frac{B_0}{s \log s} \tag{6.18}$$

6.3 An estimate

We will need the following technical result :

Lemma 6.2. *There is a constant B_0 independent of M such that*

$$\left| \int_s^M \langle r_1^M(\sigma), Q \rangle \right| \leqslant \frac{B_0}{s \log s} \tag{6.19}$$

This lemma is essentially a consequence of the oscillatory nature of r_1^M. Indeed, we set

$$H_+^{-\frac{1}{2}} (|y|^2 Q - \alpha Q^2) = \sum_{n \neq 1} \alpha_n \psi_n \tag{6.20}$$

thus,

$$\begin{aligned} r_1^M(s) &= H_+^{-\frac{1}{2}} A \int_s^M \frac{\sin(4\sigma)}{\sigma \log \sigma} \cos((s-\sigma)A) H_+^{-\frac{1}{2}} (|y|^2 Q - \alpha Q^2) d\sigma \\ &= \frac{1}{2} H_+^{-\frac{1}{2}} A \sum_{n \geqslant 2} \alpha_n \left(\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n - 4)) + \sin(s\mu_n - \sigma(\mu_n + 4))}{\sigma \log \sigma} d\sigma \right) \psi_n \end{aligned}$$

where we can remove both $n = 0$ and $n = 1$ because there is a factor A before the sum. Now,

$$\int_s^M \langle r_1^M(\sigma), Q \rangle = \langle \sum_{n \geqslant 2} \int_s^M \left(\int_\sigma^M \frac{\sin(\sigma\mu_n - \tau(\mu_n - 4)) + \sin(\sigma\mu_n - \tau(\mu_n + 4))}{\tau \log \tau} d\tau \right) d\sigma \alpha_n \psi_n, A H_+^{-\frac{1}{2}} Q \rangle \tag{6.21}$$

We then compute

$$\begin{aligned}\int_s^M \frac{\sin(s\mu_n - \sigma(\mu_n-4))}{\sigma\log\sigma}d\sigma &= \left[\frac{\cos(s\mu_n - \sigma(\mu_n-4))}{(\mu_n-4)\sigma\log\sigma}\right]_s^M + \int_s^M \frac{\cos(s\mu_n-\sigma(\mu_n-4))}{(\mu_n-4)}\frac{\log\sigma+1}{\sigma^2(\log\sigma)^2}d\sigma \\ &= \frac{\cos(s\mu_n - M(\mu_n-4))}{(\mu_n-4)M\log M} - \frac{\cos(4s)}{(\mu_n-4)s\log s} + \int_s^M \frac{\cos(s\mu_n-\sigma(\mu_n-4))}{(\mu_n-4)}\frac{\log\sigma+1}{\sigma^2(\log\sigma)^2}d\sigma\end{aligned}$$

First there stands

$$\begin{aligned}\int_s^M \frac{\cos(\sigma\mu_n - M(\mu_n-4))}{(\mu_n-4)M\log M}d\sigma &= \frac{\sin(4M)}{\mu_n(\mu_n-4)M\log M} - \frac{\sin(s\mu_n - M(\mu_n-4))}{\mu_n(\mu_n-4)M\log M} \\ \int_s^M \frac{\cos(4\sigma)}{(\mu_n-4)\sigma\log\sigma}d\sigma &= \left[\frac{\sin(4\sigma)}{4(\mu_n-4)\sigma\log\sigma}\right]_s^M + \int_s^M \frac{\sin(4\sigma)}{4(\mu_n-4)}\frac{1+\log\sigma}{\sigma^2(\log\sigma)^2}d\sigma\end{aligned}$$

with all the right-hand terms bounded by $\frac{B_0}{s\log s}$ (because $n \geqslant 2$)
Moreover,

$$\begin{aligned}\int_s^M \frac{\cos(s\mu_n - \sigma(\mu_n-4))}{(\mu_n-4)}\frac{\log\sigma+1}{\sigma^2(\log\sigma)^2}d\sigma &= \left[\frac{\sin(s\mu_n-\sigma(\mu_n-4))}{(\mu_n-4)^2}\frac{\log\sigma+1}{\sigma^2(\log\sigma)^2}\right]_s^M \\ &+ \int_s^M \frac{\sin(s\mu_n-\sigma(\mu_n-4))}{(\mu_n-4)^2}\left(\frac{1}{\sigma^3(\log\sigma)^2} - (1+\log\sigma)\frac{2\sigma(\log\sigma)^2+2\sigma\log\sigma}{\sigma^4(\log\sigma)^4}\right)d\sigma\end{aligned}$$

the bracket and the integral are both bounded by $\frac{B_0}{s^2\log s}$. Using now that

$$\int_s^M \frac{d\sigma}{\sigma^2\log\sigma} \leqslant \frac{C}{\sigma\log\sigma} \tag{6.22}$$

with C independent of M, we are able to conclude.
The computations are exactly the same for the other terme $\int_s^M \frac{\sin(s\mu_n-\sigma(\mu_n+4))}{\sigma\log\sigma}d\sigma$

6.4 estimate of f^M

We now want to bound the H^3_x norm of f^M. For that purpose, (5.20) tells us that we need only bound $\mathcal{E}(f^M)$, $E(f^M)$, and $\langle f_2^M, \rho\rangle$. We set given $B > 0$:

$$T_{[s_0,M]}(B) := \inf\{s \in [s_0, M], \forall s < \sigma < M, \mathcal{E}(f^M(\sigma)) + E(f^M(\sigma)) + \langle f_2^M(\sigma), \rho\rangle^2 \leqslant \frac{B^2}{\sigma^2(\log\sigma)^2}\} \tag{6.23}$$

We will prove that provided s_0, B are big enough, for any $M > s_0$ there stands $T_{[s_0,M]}(B) = s_0$.
Let us take $T_{[s_0,M]}(B) < s < M$. There stands :

$$\partial_s \begin{pmatrix} f_1^M \\ f_2^M \end{pmatrix} = \begin{pmatrix} H_- f_2^M + \beta(s)(|y|^2 - \alpha Q(y))f_2^M - \beta(s)(|y|^2-\alpha Q(y))r_2^M \\ -H_+ f_1^M - \beta(s)(|y|^2-\alpha Q(y))f_1^M + \beta(s)(|y|^2-\alpha Q(y))r_1^M \end{pmatrix} + IN(s) \tag{6.24}$$

Which yields

$$\begin{aligned}\partial_s\langle f_2^M, \rho\rangle &= -\langle f_1^M, Q\rangle - \beta(s)\langle f_1^M, |y|^2Q - \alpha Q^2\rangle + \beta(s)\langle r_1^M(s), |y|^2Q-\alpha Q^2\rangle + \langle N(s)_1, Q\rangle \\ \partial_s E(f) &= \langle \mathcal{H}IK(s)f, f\rangle - \langle \mathcal{H}IK(s)r^M, f\rangle + \langle \mathcal{H}f, IN(s)\rangle \\ \partial_s\mathcal{E}(f) &= -\langle H_+H_-H_+K(s)f_2, f_1\rangle + \langle H_-H_+H_-K(s)f_1, f_2\rangle - \langle \mathcal{H}_3IK(s)r^M(s), f\rangle + \langle \mathcal{H}_3, f\rangle IN(s)\end{aligned}$$

a. We first recall lemma (2.7) :

$$\||y|^2 u\|_{H^r_x} \leqslant C_r \|u\|_{H^{r+1}_x} \tag{6.25}$$

This enables us to estimate via the second equation

$$|\partial_s E(f)| \leqslant \frac{C}{s\log s}\|f\|_{H^3_x}^2 + \frac{C}{s^2(\log s)^2}\|f\|_{H^3_x} + \|f\|_{H^3_x}\|N(s)\|_{L^2} \tag{6.26}$$

Moreover there stands $\|N(s)\|_{L^2} \leqslant \|N(s)\|_{H^3_x}$. Now, H^3_x is an algebra thanks to (2.6) and $N(s)$ is quadratic in w^M, so we have the bound

$$\|N(s)\|_{L^2} \leqslant C\|w^m\|_{H^3_x} \tag{6.27}$$

Using now $w^M = f^M - r^M$ and the hypothesis $\|f^M\|_{H^3_x} \leqslant \frac{B}{s\log s}$ and (**??**) we have by choosing $B \geqslant B_0$ that

$$\|N(s)\|_2 \leqslant \frac{CB^2}{s^2(\log s)^2} \tag{6.28}$$

This ensures that

$$|\partial_s E(f^M)| \leqslant \frac{CB^3}{s^3(\log s)^3} \tag{6.29}$$

Integrating between s and M (as $f^M(M) = 0$) yields

$$E(f^M) \leqslant \frac{CB^3}{s^2(\log s)^3} \leqslant \frac{1}{6}\frac{B^2}{s^2(\log s)^2} \tag{6.30}$$

as long as $CB \leqslant \frac{1}{6}\log s_0$

b. Now, let us study the first equation : we use the following algebraic fact which is fundamental : $v^M := w^M + Q$ solves

$$\begin{cases} i\partial_s v^M & = (H-\lambda)v^M + \beta(s)(|y|^2 - \alpha Q)v^M + v^M|v^M|^2 \\ v^M(M) & = Q \end{cases} \tag{6.31}$$

This equation preserves the L^2 norm. We thus see that

$$\|w^M(s)\|_{L^2}^2 + 2\langle w_1^M(s), Q\rangle + \|Q\|_{L^2}^2 = \|w^M(s) + Q\|_{L^2}^2 = \|Q\|_{L^2}^2 \tag{6.32}$$

Thus, $\langle w_1^M, Q\rangle$ is *quadratic* rather than linear as could be expected :

$$|\langle w_1^M(s), Q\rangle| = \frac{1}{2}\|w^M(s)\|_{L^2}^2 \leqslant \frac{CB^2}{s^2(\log s)^2} \tag{6.33}$$

as long as $T_{[s_0,M]}(B) < s < M$. Now, using $f_1^M = w_1^M - r_1^M$, there stands

(6.34)
$$\langle f_2^M(s), \rho\rangle = -\int_s^M \langle r_1^M(\sigma), Q\rangle d\sigma + \int_s^M \left(\langle w_1^M(\sigma), Q\rangle + \beta(\sigma)\langle f_1^M(\sigma), |y|^2 Q - \alpha Q^2\rangle + \beta(\sigma)\langle r_1^M(\sigma), |y|^2 Q - \alpha Q^2\rangle + \langle N(\sigma)_1, Q\rangle\right.$$

Using lemma (6.2), the first integral is bounded by $\frac{B_0}{s\log s}$ with B_0 universal, and the second integral's integrand can be bounded by $\frac{CB^2}{\sigma^2(\log\sigma)^2}$ with C universal, thus we can bound the second integral by $\frac{CB^2}{\sigma(\log\sigma)^2}$ with C a constant. We finally get

$$\left|\langle f_2^M(s), \rho\rangle\right| \leqslant \frac{CB^4}{s^2(\log s)^3} + \frac{B_0^2}{s^2(\log s)^2} \leqslant \frac{1}{6}\frac{B^2}{s^2(\log s)^2} \tag{6.35}$$

provided we choose B big enough in regard to B_0 then s_0 big enough.

c. We finally need to bound $\mathcal{E}(f^M)$. As we examine the right-hand side of the third equation, we can decompose it as $\beta(s)\left(-\langle H^3|y|^2 f_2, f_1\rangle + \langle H^3|y|^2 f_1, f_2\rangle\right) +$ (remainder) where the remainder can be bounded using that H^3_x is an algbera ((2.6)), lemma (2.7), lemma (6.1) and proposition (3.4), and the a priori bound over $\|f^M\|_{H^3_x}$, by $\frac{CB^3}{s^3(\log s)^3}$ with C a universal constant. We do not detail the proof which is very similar to what we have done so far.

We then use lemma (2.8)

$$|-\langle H^3|y|^2 f_2, f_1\rangle + \langle H^3|y|^2 f_1, f_2\rangle| \leqslant C\|f\|^2_{H^3_x} \tag{6.36}$$

Indeed we can write

$$-\langle H^3|y|^2 f_2, f_1\rangle + \langle H^3|y|^2 f_1, f_2\rangle = \langle H^{\frac{3}{2}} f_1, \left[H^{\frac{3}{2}}, |y|^2\right] f_2\rangle + \langle H^{\frac{3}{2}} f_2, \left[H^{\frac{3}{2}}, |y|^2\right] f_1\rangle \tag{6.37}$$

Thus, by choosing B big enough with regards to universal constants, there strands

$$|\partial_s \mathcal{E} f^M| \leqslant \frac{CB^3}{s^3(\log s)^3} \tag{6.38}$$

Hence

$$\mathcal{E}(f^M(s)) \leqslant \frac{CB^3}{s^2(\log s)^3} \tag{6.39}$$

with C a universal constant. Thus, by choosing s_0 big enough, there stands

$$\mathcal{E}(f^M(s)) \leqslant \frac{1}{6}\frac{B^2}{s^2(\log s)^2} \tag{6.40}$$

d. We have proved that there exists a suitable choice of B and s_0 such that while $T_{[s_0,M]}(B) \leqslant s \leqslant M$ there stands

$$\mathcal{E}(f^M(\sigma)) + E(f^M(\sigma)) + \langle f_2^M(\sigma), \rho\rangle^2 \leqslant \frac{1}{2}\frac{B^2}{\sigma^2(\log \sigma)^2} \tag{6.41}$$

Which ensures that $T_{[s_0,M]}(B) = s_0$ for all $M \geqslant s_0$, which is the proclaimed result.

6.5 Cauchy sequence

Let us now show that (w^M) is a Cauchy sequence in $\mathcal{C}([s_0, T], H^3_x)$ when $M \to \infty$ for all T. We use the notations introduced so far : let s_0, B as built before, and for $s_0 < N < M$ let us see that

$$\partial_s(w^M - w^N) = \mathcal{L}(w^M - w^N) + IK(s)(w^M - w^N) + I(N^M(s) - N^N(s)) \tag{6.42}$$

This ensures that

$$\begin{gathered}
\partial_s\langle (w_2^M - w_2^N)(s), \rho\rangle = -\langle (w_1^M - w_1^N)(s), Q\rangle - \beta(s)\langle w_1^M - w_1^N, |y|^2 Q - \alpha Q^2\rangle + \langle N_1^M(s) - N_1^N(s), Q\rangle \\
\partial_s E(w^M - w^N) = \langle \mathcal{H} IK(s)(w^M - w^N), w^M - w^N\rangle + \langle \mathcal{H}(w^M - w^N), I(N^M(s) - N^N(s))\rangle \\
\partial_s \mathcal{E}(w^M - w^N) = \langle \mathcal{H}_3 IK(s)(w^M - w^N), w^M - w^N\rangle + \langle \mathcal{H}_3(w^M - w^N), I(N^M - N^N)\rangle
\end{gathered}$$

Using (2.6) we thus have up to a bigger s_0 that

$$\|N^M(s) - N^N(s)\|_{H^3_x} \leqslant C\|w^M(s) - w^N(s)\|_{H^3_x}(\|w^M(s)\|_{H^3_x} + \|w^N(s)\|_{H^3_x}) \leqslant \frac{CB}{s\log s}\|w^M(s) - w^N(s)\|_{H^3_x} \tag{6.43}$$

With the same techniques as before, there stands

$$\begin{aligned}
|\partial_s E(w^M - w^N)| &\leqslant \frac{CB}{s\log s}\|w^M - w^N\|^2_{H^3_x}\\
|\partial_s \mathcal{E}(w^M - w^N)| &\leqslant \frac{CB}{s\log s}\|w^M - w^N\|^2_{H^3_x}
\end{aligned}$$

Moreover using $\langle w_1^M, Q\rangle = -\frac{1}{2}\|w^M\|^2_{L^2}$ we find

$$\begin{aligned}
\left|\langle w_1^M(s) - w_1^N(s), Q\rangle\right| &= \frac{1}{2}\left|\|w^M(s)\|^2_{L^2} - \|w^N(s)\|^2_{L^2}\right|\\
&\leqslant C\|w^M(s) - w^N(s)\|_{H^3_x}(\|w^M(s)\|_{H^3_x} + \|w^N(s)\|_{H^3_x})\\
&\leqslant \frac{CB}{s\log s}\|w^M(s) - w^N(s)\|_{{}^3_x}
\end{aligned}$$

From which we deduce

$$|\partial_s\langle w_2^M(s) - w_2^N(s), \rho\rangle| \leqslant \frac{CB}{s\log s}\|w^M(s) - w^N(s)\|_{H^3_x} \tag{6.44}$$

and

$$|\partial_s\left(\langle w_2^M(s) - w_2^N(s), \rho\rangle^2\right)| = 2|\langle w_2^M(s) - w_2^N(s), \rho\rangle||\partial_s\langle w_2^M(s) - w_2^N(s), \rho\rangle| \leqslant \frac{CB}{s\log s}\|w^M - w^N\|^2_{H^3_x} \tag{6.45}$$

Integrating between s and N, and using $w^N(N) = 0$, yields

$$\begin{aligned}
|\langle w_2^M(s) - w_2^N(s), \rho\rangle^2| &\leqslant |\langle w_2^M(N), \rho\rangle|^2 + \int_s^N \frac{CB}{\sigma\log\sigma}\|w^M(\sigma) - w^N(\sigma)\|^2_{H^3_x}d\sigma\\
E(w^M(s) - w^N(s)) &\leqslant E(w^M(N)) + \int_s^N \frac{CB}{\sigma\log\sigma}\|w^M(\sigma) - w^N(\sigma)\|^2_{H^3_x}d\sigma\\
\mathcal{E}(w^M(s) - w^N(s)) &\leqslant \mathcal{E}(w^M(N)) + \int_s^N \frac{CB}{\sigma\log\sigma}\|w^M(\sigma) - w^N(\sigma)\|^2_{H^3_x}d\sigma
\end{aligned}$$

Thus there exists a universal constant C such that

$$\forall s_0 \leqslant s \leqslant N\ \|w^M(s) - w^N(s)\|^2_{H^3_x} \leqslant C\|w^M(N)\|^2_{H^3_x} + \int_s^N \frac{C}{\sigma\log\sigma}\|w^M(\sigma) - w^N(\sigma)\|^2_{H^3_x}d\sigma \tag{6.46}$$

The Gronwall lemma in its version proven in [1], and the fact $\|w^M(N)\|^2_{H^3_x} \leqslant \frac{C}{N^2(\log N)^2}$ yield

$$\begin{aligned}
\|w^M(s) - w^N(s)\|^2_{H^3_x} &\leqslant \frac{C}{N^2(\log N)^2} + \int_s^N \frac{C}{N^2(\log N)^2\sigma\log\sigma}\exp\left(\int_s^N \frac{C}{\sigma\log\sigma}\right)\\
&\leqslant \frac{C}{N^2(\log N)^2}\left(1 + \int_s^N \frac{(\log\sigma)^{C-1}}{\sigma}d\sigma\right)\\
&\leqslant \frac{C'}{N}
\end{aligned}$$

where C' is a constant.

Thus, (w^M) is a Cauchy sequence in $\mathcal{C}((s_0, T), H^3_x)$ which converges uniformly to a $w(s)$. This is a solution of (5.6) over $(s_0, +\infty)$ with the bound

$$\|w(s)\|_{H^3_x} \leqslant \frac{B}{s\log s} \tag{6.47}$$

VII - PROOF OF THEOREM (1.1)

Let us choose s_0 such as built in the previous section, $t(s_0) = s_0$ and $\frac{dt}{ds} = \frac{1}{L^2(s)}$ as in proposition (4.2). We have built

$$u(t,x) = e^{-i\lambda s(t)} S_{L(t)} e^{-i\frac{b(t)|y|^2}{4}} v(s(t),y) \tag{7.1}$$

a solution to

$$i\partial_t u(t,x) = Hu(t,x) + u(t,x)|u(t,x)|^2 + V(t,x)u(t,x) \tag{7.2}$$

with the potential

$$V(t,x) = -\alpha \frac{1}{L^2(t)} \beta(s(t)) Q\left(\frac{x}{L(t)}\right) \tag{7.3}$$

Using proposition (4.2) there stands for all k, r that

$$\lim_{t\to\infty} \|\partial_t^k V(t,x)\|_{H^r_x} = 0 \tag{7.4}$$

Moreover there stands

$$u(t,x) = u_0(t,x) + u_1(t,x) \tag{7.5}$$

where

$$\|u_0(t)\|^2_{H^1_x} \geqslant C \log t \qquad t \to \infty \tag{7.6}$$

with C a constant, and there exists $C' > 0$ such that

$$\|u_1(t,x)\|^2_{H^1_x} \leqslant C' \frac{1}{t^2 \log t} \tag{7.7}$$

We have indeed showed that $v(s,y) = Q(y) + w(s,y)$ with $\|w(s,y)\|_{H^1_x} \leqslant \frac{B}{s \log s}$ so if we set

$$u_0 = e^{-i\lambda s(t)} S_{L(t)} e^{-\frac{ib(t)|y|^2}{4}} Q(y) \tag{7.8}$$

then lemma (4.1) and proposition (4.2) yield

$$\begin{aligned}
\|u_0\|^2_{H^1_x} &= \left(L^2 + \frac{b^2}{4L^2}\right) \|xQ\|^2_{L^2} + \frac{1}{L^2} \|\nabla Q\|^2_{L^2} \\
&\geqslant E(L,b) \min(\|xQ\|^2_{L^2}, \|\nabla u\|^2_{L^2}) \\
&\geqslant C \log(t)
\end{aligned}$$

Moreover

$$\begin{aligned}
\|u_0\|^2_{H^1_x} &= \left(L^2 + \frac{b^2}{4L^2}\right) \|xQ\|^2_{L^2} + \frac{1}{L^2} \|\nabla Q\|^2_{L^2} \\
&\leqslant E(L,b) \max(\|xQ\|^2_{L^2}, \|\nabla u\|^2_{L^2}) \\
&\leqslant C' \log(t)
\end{aligned}$$

Finally with proposition (4.2) there stands

$$\begin{aligned}
\|S_{L(t)} e^{-\frac{ib(t)|y|^2}{4}} w(s,y)\|^2_{H^1_x} &\leqslant E(L,b) \|w(s,y)\|^2_{H^1_x} \\
&\leqslant C' \log(t) \frac{1}{t^2 (\log t)^2}
\end{aligned}$$

which nearly concludes the proof to theorem 1. It remains to be shown that u has $\mathcal{C}^\infty$ regularity. However that is a straightforward consequence of local unicity and the fact that Q, L, b have $\mathcal{C}^\infty$ regularity.

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