

MÉMOIRE DE STAGE

A BRIEF INTRODUCTION TO l^2 -BETTI NUMBERS

STAGIAIRE
Alex Fernandes

MAÎTRE DE STAGE
Wolfgang Lück



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Introduction et remerciements

Le théorème d'Hurewicz suggère d'étendre les invariants classiques d'une variété riemannienne fermée à son revêtement universel en considérant l'action du groupe fondamental. Un exemple est la "Wall finiteness obstruction", obstruant certains espaces à être homotopiquement équivalent à un CW-complexe fini (pour un traitement plus complet voir [Ros94] chapitre 1 section 7). Les l^2 -nombre de Betti proviennent d'une tentative de généralisation des nombres de Betti dans ce contexte, cependant lorsque le groupe fondamental est infini le revêtement universel est non-compact et ainsi une telle généralisation requiert une certaine finesse lors de la construction.

Les l^2 -nombres de Betti et plus généralement les l^2 -invariants apparaissent dans différentes branches des mathématiques et permettent la démonstration de théorèmes qui de premier abord apparaissent comme distincts de ceux-ci. Ces derniers portent sur des sujets variés : théorie géométrique des groupes, K -théorie, théorie des anneaux, géométrie hyperbolique, théorie spectrale sur le laplacien du revêtement universel d'une variété...

Le texte suivant est une introduction au plus simple des l^2 -invariants : les l^2 -nombres de Betti. Pour des raisons de lisibilité ainsi que de contrainte sur la taille de ce rapport de nombreuses démonstrations ne seront pas explicitées, cependant les (très) grandes lignes seront brièvement abordées.

Je souhaite sincèrement remercier Bruno Dular et Léo Chafloque qui ont toujours pris le temps de creuser chaque question à mes côtés et rassurer mes doutes ainsi que Wolfgang Lück pour son agréable accueil, son accessibilité et surtout sa bienveillance tout au long de ce stage. Plus informellement, un grand merci à Milena et Emilie qui ont rendu la logistique de ce déménagement de quelques mois d'une simplicité salvatrice. Finalement, je souhaite remercier Nicolas Bergeron qui de par son aide et sa disponibilité a rendu tout ceci possible.

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Déroulement du stage

J'ai eu la chance de réaliser mon stage de première année de master à l'université rhénane Frédéric-Guillaume de Bonn en Rhénanie-du-Nord-Westphalie (Allemagne) au sein du groupe de topologie et plus précisément celui de Wolfgang Lück. J'ai pu y étudier sous sa supervision la théorie des l^2 -invariant dont il est un acteur majeur.

Le stage s'est déroulé de la manière suivante : j'assistais à un cours magistral présenté par le professeur Lück deux fois par semaine, de plus nous nous réunissions avec l'équipe de doctorant pour un séminaire toutes les semaines suivies d'une heure où nous pouvions échanger sur les diverses questions que nous avons accumulées au cours de la semaine. Ce cours magistral, bien que destiné aux étudiants en master, était en réalité suivi essentiellement par des doctorants. Les séminaires ont porté sur des sujets très variés allant des interactions entre théorie ergodique et topologie algébrique en passant par la théorie de Hodge et ses relations avec les l^2 -invariants et finalement un groupe de lecture sur les infinies catégories [Lan21]. Le reste du temps était dévoué à l'acquisition des prérequis grandissant avec le temps exigé par le contenu des séminaires, cours ainsi qu'à la complétion du cours qui était sous la forme d'un "survey".

Ayant réalisé ma licence à l'étranger et étant déjà anglophone, ces deux points majeurs d'une expérience à l'étranger m'étaient déjà familiers. J'ai alors pu me concentrer sur une autre opportunité d'un tel stage, une immersion totale dans la culture allemande. Que ce soit en apprenant ex-nihilo la langue (au prix de quelques situations embarrassantes et diverses anecdotes) mais aussi en interagissant ; en-dehors de toute discussion mathématiques ; uniquement dans celle-ci. J'ai pu alors confronter le milieu extrêmement international dont j'avais l'habitude à un cercle très local. Après tout, la ville de Bonn n'est pas si grande.

De plus, ce stage a été pour moi l'occasion de visiter l'Allemagne. En effet, j'ai pu parcourir la mégapole européenne du dense réseau dans le Rhin-Ruhr aux paysages bucoliques du Rhin-Neckar en passant par l'imposante région de Francfort. Cette zone polarisée entre conflits historiques et profond sentiment européen dispose d'une atmosphère tout particulière et qui je pense pour toute personne amenée à travailler en relation avec différentes cultures européenne est un endroit à visiter. Je ne pourrais jamais suffisamment remercier tous ses amis allemands rencontrés sur les bancs de l'ENS qui ont bien voulu me présenter leurs villes d'origine et de résidence une fois rentrés de leur Erasmus parisien.

Chapter 1

Dimension functions

1.1 Group von Neumann algebra

From now on, unless explicitly mentioned G denotes a countable discrete group. Let H a Hilbert space and denotes $B(H)$ its involutive algebra of bounded endomorphisms.

Definition 1.1.1 – Commutant

The *commutant* of $M \subset B(H)$ denoted M' is defined as operators on H commuting with all operators in M . As expected the *bicommutant* is given by $M'' := (M')'$.

Theorem 1.1.1 – von Neumann Bicommutant Theorem

Let M an unital involutive subalgebra of $B(H)$ then the following are equivalent :

- M is strongly closed
- M is weakly closed
- $M = M''$

Proof. See theorem 2.19 p.18 [Kam19]. □

Remark 1.1.1 An unital involutive subalgebra satisfying these equivalent conditions is called von Neumann algebra .

A specific von Neumann algebra arising from G shall be constructed emphasising the interaction between functional analysis (operator theory) and group theory.

Definition 1.1.2 – Complex group algebra

The *group algebra* $\mathbb{C}G$ is defined as the complex vector space spanned by G with multiplication given on the basis by group composition and linearly extended on $\mathbb{C}G$.

Remark 1.1.2 – Inner product space and induced Hilbert space

One can turn $\mathbb{C}G$ into a inner product space by requiring G to be orthonormal and defining the following inner product : for any two $g, h \in G$ set $\langle g, h \rangle := \bar{g}h$. The Hilbert space completion of $\mathbb{C}G$ is denoted $l^2(G)$ to be the Hilbert space.

Remark 1.1.3 – Right and left regular representations over l^2G

An element $h \in G$ acts on the basis G by $g \mapsto hg$ (resp. $g \mapsto gh^{-1}$) yielding by linear extension the right (resp. left) regular representation. These two turn l^2G into a CG -bimodule.

The right regular representation realise an embedding of involutive algebra $CG \hookrightarrow B(l^2G)$ the involution explicitly acting on CG the following way $\sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \overline{\lambda_g} g^{-1}$.

Definition 1.1.3 – Group von Neumann algebra

The group von Neumann algebra $\mathcal{N}(G)$ is defined as the collection of G -equivariant operators of $B(l^2G)$.

Remark 1.1.4 One can prove using the bicommutant theorem 1.1.1 that $\mathcal{N}(G)$ is the weak/strong closure of CG in $B(H)$ (theorem 2.24 p.19 [Kam19]). Therefore, the notations $\mathcal{R}(G)$ and $\mathcal{L}(G)$ can be found in the literature (e.g [Kam19]) to emphasise on the left or right embedding of CG .

Example 1.1.1 – Amplified group von Neumann algebra

One can consider for any positive integer n the amplified group von Neumann algebra i.e the collection of G -equivariant operators of $B((l^2G)^n)$. It can be identified with $M_n(\mathcal{N}(G))$ see [Kam19] Example 2.28 page 21.

The following von Neumann algebra will play the role of toy model and allows by its use to form conjectures.

Example 1.1.2 – Group von Neumann algebra of \mathbf{Z}

Using Fourier transform $l^2\mathbf{Z}$ identifies with $L^2(S^1)$. Under this identification $1 \in \mathbf{Z}$ acts by left multiplication with $z := e^{ix}$. Therefore, $\mathcal{N}(\mathbf{Z})$ is the algebra of operators that are invariant under multiplication Laurent polynomials $\mathbf{C}[z^{\pm 1}]$.

Right multiplication yields $L^\infty(S^1) \hookrightarrow B(L^2(S^1))$, in fact $L^\infty(S^1) \subset \mathcal{N}(\mathbf{Z})$. Continuous functions are dense in $L^2(S^1)$ and Stone Weirstraß theorem allows to write any square integrable function as a L^2 -limit of a sequence of Laurent polynomials. Hence, equivariance and continuity ensure that $T \in \mathcal{N}(\mathbf{Z})$ acts on $L^2(S^1)$ by left multiplication by $T(1)$. Since any $T \in \mathcal{N}(\mathbf{Z})$ is essentially bounded by $\|T\|$, in fact $L^\infty(S^1) = \mathcal{N}(\mathbf{Z})$.

1.2 Hilbert modules

To forecast the next section von Neumann algebra are an object of interest since they admit a trace. Hilbert modules establish a connection in between Hilbert spaces with a G -action and amplified group von Neumann algebras in order to make use of this trace.

Definition 1.2.1 – Finitely generated Hilbert module

A finitely generated Hilbert $\mathcal{N}(G)$ -module (or Hilbert module) is a Hilbert space H with a linear isometric G -action such that there exist an linear G -embedding $H \hookrightarrow (l^2G)^n$ for some $n \in \mathbf{N}$. Together with bounded G -operator, finitely generated Hilbert $\mathcal{N}(G)$ -modules form a category denoted $\text{HMod}_{f.g.}^{\mathcal{N}(G)}$.

Remark 1.2.1 The existence of such an embedding is equivalent to the existence of a surjective map of $\mathcal{N}(G)$ -module $(l^2G)^n \rightarrow H$.

Remark 1.2.2 In a more general setting a Hilbert $\mathcal{N}(G)$ -module is a Hilbert space H with a linear isometric G -action such that there exist a linear G -embedding $H \hookrightarrow V \otimes l^2 G$ for V a Hilbert space. Together with bounded G -operator, Hilbert $\mathcal{N}(G)$ -modules form also a category. Numerous properties that follow can be derived for general Hilbert modules see [Lüco4] section 1.1.2 pages 16.

New Hilbert modules arise from old ones by usual algebraic construction in the following fashion.

Proposition 1.2.1 – Constructing new Hilbert modules

Let H, H_1, H_2 be a finitely generated Hilbert $\mathcal{N}(G)$ -module (respectively $\mathcal{N}(G_0), \mathcal{N}(G_1)$) then :

- [Submodules] If K is a closed G -invariant subspace of H then K turns into a Hilbert module. We say that K is a *Hilbert submodule*.
- [Quotient] If K is a Hilbert submodule of H then H/K turns into a Hilbert module. We say that H/K is a *Hilbert quotient module*.
- [Direct sum] If $G_1 = G_2$ the direct sum $H_1 \oplus H_2$ of Hilbert modules is a Hilbert module.
- [Tensor product] The tensor product $H_1 \otimes H_2$ of Hilbert module is a Hilbert module.

Proof. Let K a closed G -invariant subspace of H . Choose an linear G -embedding $\iota : H \hookrightarrow (l^2 G)^n$, then $\iota|_K$ is a linear G -embedding. Now identify H/K with K^\perp which is a Hilbert submodule. Now choose $\iota_k : H \hookrightarrow (l^2 G_k)^{n_k}$ for $k \in \{1, 2\}$, $\iota_1 \oplus \iota_2$ and $\iota_1 \otimes \iota_2$ yields two desired linear G_1 -embedding and $(G_1 \times G_2)$ -embedding. Indeed $l^2(G_0)^{n_1} \otimes l^2(G_1)^{n_2}$ identifies with $l^2(G_0 \times G_2)^{n_1 n_2}$. \square

The two following definitions yield a glimpse on the interaction in between the lattice of subgroups of G and the various Hilbert modules arising from it.

Definition 1.2.2 – Restriction functor

Let $G_0 \leq G$ a subgroup of finite index. We define the *restriction functor*

$$\text{res}_{G_0}^G : \text{HMod}_{f.g}^{\mathcal{N}(G)} \rightarrow \text{HMod}_{f.g}^{\mathcal{N}(G_0)}$$

by restricting the action from G to G_0 .

Remark 1.2.3 The functor yields finitely generated Hilbert modules, indeed a system of representatives for left cosets ensures that $\text{res}_{G_0}^G(l^2 G)$ identifies with $(l^2 G)^{|G:G_0|}$.

Definition 1.2.3 – Induction functor

Let $\iota : H \hookrightarrow G$ a group embedding, it yields a *induction functor* :

$$\iota_* : \text{HMod}_{f.g}^{\mathcal{N}(H)} \rightarrow \text{HMod}_{f.g}^{\mathcal{N}(G)}.$$

For M a Hilbert $\mathcal{N}(H)$ -module define $\iota_* M$ as the Hilbert completion of $\text{CG} \otimes_{\text{CH}} M \cong M^{\oplus |G:H|}$.

Remark 1.2.4 The induction functor is compatible with direct sums and send $l^2 H$ to $l^2 G$.

A more detailed approach on the induction is treated in [Lüco4] section 1.1.5 pages 29-30. The mention of finitely generated in definition 1.2.1 indicate that there is a more general definition of Hilbert $\mathcal{N}(G)$ -module.

1.3 Dimension function for Hilbert Modules

Von Neumann algebras admit a trace that extends naturally to amplified group von Neumann algebra. This trace can be extended to finitely generated Hilbert $\mathcal{N}(G)$ -modules and yields a candidate object to define an algebraic invariant.

Definition 1.3.1 – von Neumann trace

The *von Neumann trace* on $\mathcal{N}(G)$ is defined by $\text{tr}_{\mathcal{N}(G)} : T \mapsto \langle Te, e \rangle$. It satisfies, as expected, the *trace property* i.e for any $S, T \in \mathcal{N}(G)$ one can check that $\text{tr}_{\mathcal{N}(G)}(ST) = \text{tr}_{\mathcal{N}(G)}(TS)$.

Remark 1.3.1 This trace extend to G -equivariants bounded endorphism of the amplified von Neumann algebra considering first the usual trace on matrices.

Example 1.3.1 – von Neumann trace on $\mathcal{N}(\mathbf{Z})$

Under the identification $l^2\mathbf{Z} = L^2(S^1)$ and $\mathcal{N}(\mathbf{Z}) = L^\infty(S^1)$, the trace is $\text{tr}_{\mathcal{N}(\mathbf{Z})} : f \mapsto \frac{1}{2\pi} \int_{S^1} f(x) dx$.

Proposition 1.3.1 – von Neumann trace of Hilbert modules

Let H be a finitely generated Hilbert $\mathcal{N}(G)$ -module and f a G -equivariant bounded endomorphism then $\text{tr}_{\mathcal{N}(G)}(\iota f \pi_H)$ is invariant under any choice of linear G -embedding of H .

Proof. For sake of simplicity H is identified with its image. By consideration of the form $\tilde{i} = i \oplus 0$, one can suppose w.l.o.g $i, j : H \rightarrow (l^2G)^n$ two linear G -embeddings. Writing $(l^2G)^n$ as $H \oplus H^\perp$ we can extend the inverse i^{-1} to $i' : (l^2G)^n \rightarrow H$ by $i' = i^{-1} \oplus 0$ and define $u = ji'$ a $\text{HMod}_{f,g}^{\mathcal{N}(G)}$ partial-isometry. Observing that $j = ui$ one can conclude using the trace property. A more detailed proof can be found in 2.33 p.25 [Kam19]. \square

Definition 1.3.2 – Trace of $\text{HMod}_{f,g}^{\mathcal{N}(G)}$ -endomorphisms

Let H be a finitely generated Hilbert $\mathcal{N}(G)$ -module and f a G -equivariant bounded endomorphism the *von Neumann trace* of f is define by $\text{tr}_{\mathcal{N}(G)}(f) = \text{tr}_{\mathcal{N}(G)}(\iota f \pi_H)$ for ι any G -embedding of H .

Proposition 1.3.2 – Some properties of the extended trace

For H a Hilbert module the trace $\text{tr}_{\mathcal{N}(G)}$ is as expected \mathbb{C} -linear weakly continuous and admit the trace property. Moreover it admit the following properties :

- For $t \in B(H)^G$, $\text{tr}_{\mathcal{N}(G)}(t^*t) = 0$ if and only if $t = 0$
- For $t, s \in B(H)^G$ such that $s \leq t$ then $\text{tr}_{\mathcal{N}(G)}(s) \leq \text{tr}_{\mathcal{N}(G)}(t)$
- If $r \in B(H)^G$, $s \in B(K)^G$, $t \in B(L)^G$ induce an endomorphism of the short exact sequence

$$0 \rightarrow H \rightarrow K \rightarrow L \rightarrow 0$$

then $\text{tr}_{\mathcal{N}(G)}(s) = \text{tr}_{\mathcal{N}(G)}(r) + \text{tr}_{\mathcal{N}(G)}(t)$.

- For H_i a Hilbert $\mathcal{N}(G_i)$ -module and $t_i \in B(H_i)^{G_i}$ with $i \in \{0, 1\}$ then

$$\text{tr}_{\mathcal{N}(G_1 \times G_2)}(t_1 \otimes t_2) = \text{tr}_{\mathcal{N}(G_1)}(t_1) \text{tr}_{\mathcal{N}(G_2)}(t_2)$$

Proof. A detailed proof can be found in [Kam19] p.26-28 theorem 2.35, 2.36. Essentially the proof relies on elementary properties following from the trace definition, the fact that any short exact sequence of Hilbert module splits and the existence of a polar decomposition preserving G -equivariance. \square

One can explicit the interaction of the preceding trace with the induction and restriction functor and therefore establish a link between the von Neumann trace and the lattice of subgroups of G .

Proposition 1.3.3 – Interaction between trace and induction/restriction functor

Let $G_0 \leq G$ and H a Hilbert $\mathcal{N}(G)$ -module.

- For $s \in B(H)^G$ and G_0 of finite index we have $\text{tr}_{\mathcal{N}(G_0)}(\text{res}_{G_0}^G(s)) = \text{tr}_{\mathcal{N}(G)}(s) |G : G_0|$.
- For $i : \mathcal{N}(G_0) \rightarrow \mathcal{N}(G)$ the ring homomorphism induced by the induction functor we obtain $\text{tr}_{\mathcal{N}(G_0)} = \text{tr}_{\mathcal{N}(G)} \circ i : \mathcal{N}(G_0) \rightarrow \mathbf{C}$.

Proof. For the restriction let $m = |G : G_0|$ and $\{g_i\}_{i=1}^m$ a set of representatives for G/G_0 . The unitary $u : (h_i)_{i=1}^m \mapsto \sum_i h_i g_i$ yields an embedding $u^* : l^2 G \hookrightarrow (l^2 G_0)^m$. For $s \in \mathcal{N}(G)$ define $\bar{s} = u^* s \pi_{l^2 G_0}$ and seeking simplicity let us denote also by s the image of s by the restriction functor. The relation $\bar{s}_{ij}(e) = \pi_{l^2 G_0}(g_j s(e) g_i)$ and $1 \leq i, j \leq m$ ensures that $\text{tr}_{\mathcal{N}(G_0)}(\bar{s}_{ii}) = \sum_i \text{tr}_{\mathcal{N}(G_0)}(\bar{s}_{ii}) = \sum_i \text{tr}_{\mathcal{N}(G)}(s) = m \text{tr}_{\mathcal{N}(G)}(s)$. For a general $s \in B(H)^G$ and a general embedding $\iota : H \hookrightarrow (l^2 G)^n$ we use a similar argument with the composition $(u^*)^{\oplus n} \circ \iota$. The induction functor case is a direct consequence of the definition of i . \square

Inspired by the formula linking the usual trace and dimension in linear algebra, i.e $\dim_K = \text{tr}(\text{id})$ for K a field, one can define a dimension analog in the von Neumann algebra framework.

Definition 1.3.3 – Von Neumann dimension

The *von Neumann dimension* of a finitely generated Hilbert $\mathcal{N}(G)$ -module H is defined as

$$\dim_{\mathcal{N}(G)}(H) = \text{tr}_{\mathcal{N}(G)}(\text{id}_H).$$

Remark 1.3.2 *The von Neumann dimension extends the classical dimension for G trivial.*

The von Neumann dimension of a finitely generated Hilbert $\mathcal{N}(G)$ -module as one major difference with the classical v can take any real positive value.

Example 1.3.2 – Arbitrary von Neumann dimension

If $G = \mathbf{Z}$, take A of Lebesgue measure λ the characteristic function $\chi_A \in L^\infty(\mathbf{S}^1)$ is a projection such that $\text{tr}_{\mathcal{N}(\mathbf{Z})}(\chi_A) = \lambda/2\pi$. Therefore, with the properties of proposition 1.3.4 one can show that $\text{im}(\chi_A)^n$ is a $\mathcal{N}(\mathbf{Z})$ -module of dimension $\frac{n\lambda}{2\pi}$ for any $n \in \mathbf{N}$.

Proposition 1.3.4 – Properties von Neumann dimension

Let H a Hilbert $\mathcal{N}(G)$ -modules, the von Neumann dimension admits the following properties :

- $\dim_{\mathcal{N}(G)}(l^2 G) = 1$
- For H a Hilbert module $\dim_{\mathcal{N}(G)}(H) = 0$ if and only if H is trivial.

- If $0 \rightarrow H \rightarrow K \rightarrow L \rightarrow 0$ is an exact sequence of $\mathcal{N}(G)$ -modules then

$$\dim_{\mathcal{N}(G)}(K) = \dim_{\mathcal{N}(G)}(L) + \dim_{\mathcal{N}(G)}(H).$$

- If H_i is a Hilbert $\mathcal{N}(G_i)$ -module for $i \in \{1, 2\}$ then

$$\dim_{\mathcal{N}(G_1 \times G_2)}(H_1 \otimes H_2) = \dim_{\mathcal{N}(G_1)}(H_1) \dim_{\mathcal{N}(G_2)}(H_2)$$

- If $\{H_i\}_{i \in I}$ is a \subset -directed system of $\mathcal{N}(G)$ -submodules of H then

$$\dim_{\mathcal{N}(G)} \overline{\bigcup_{i \in I} H_i} = \sup_{i \in I} \dim_{\mathcal{N}(G)} H_i$$

- If $\{H_i\}_{i \in I}$ is a \supset -directed system of $\mathcal{N}(G)$ -submodules of H then

$$\dim_{\mathcal{N}(G)} \bigcap_{i \in I} H_i = \inf_{i \in I} \dim_{\mathcal{N}(G)} H_i$$

Proof. All the properties not involving directed systems are consequences of proposition 1.3.2. By consideration of $\{H_i^\perp\}_{i \in I}$ we only need to prove one of the two other statements. The proof relies on the weak continuity of the trace and the following fact :

$$\lim_{i \in I} \operatorname{tr}_{\mathcal{N}(G)}(\pi_{H_i}) = \sup_{i \in I} \dim_{\mathcal{N}(G)}(H_i).$$

A detailed proof is provided in [Lüc04] theorem 1.12 pages 21-23. □

Proposition 1.3.5 – Interaction von Neumann dimension and restriction/induction functor
Let $G_0 \leq G$ a subgroup with H is a Hilbert $\mathcal{N}(G)$ -module and M is a Hilbert $\mathcal{N}(G_0)$ -module :

- If G_0 has finite index then $\dim_{\mathcal{N}(G_0)} \operatorname{res}_{G_0}^G(H) = |G : G_0| \dim_{\mathcal{N}(G)}(H)$.
- The induction yields $\dim_{\mathcal{N}(G_0)}(H) = \dim_{\mathcal{N}(G)}(i_* M)$.

Proof. Elementary consequence of 1.3.3. □

Example 1.3.3 – Vector space dimension and von Neumann dimension

If G is finite setting G_0 as the trivial group we obtain by 1.3.5 that $|G| \dim_{\mathcal{N}(G)} = \dim_{\mathbb{C}}$.

1.4 Extended dimension function

The preceding von Neumann dimension function was defined for *finitely generated* Hilbert $\mathcal{N}(G)$ -modules. To forecast the next sections one can now able to derive a notion of Betti-number for G -spaces yielding Hilbert $\mathcal{N}(G)$ -module that are finitely generated. The category of those spaces happen to be a little too restrictive to make fully use of the notion of l^2 -Betti numbers. In particular if one wants to define l^2 -Betti numbers of discrete groups using classifying spaces, one needs to extend the von Neumann dimension to a larger class of Hilbert module.

This section is (heavily) based on [Lüc98] in particular the section 2 and 3. Therefore, the careful

reader may consider the reading of those.

In order to understand the point of view hereby presented one must acknowledge the two following facts : the \mathbf{C} -category of finitely generated Hilbert $\mathcal{N}(G)$ -module is equivalent to the \mathbf{C} -category of finitely generated projective $\mathcal{N}(G)$ -modules, $\mathcal{N}(G)$ is semihereditary and therefore the category of finitely presented $\mathcal{N}(G)$ -module is abelian. A summary on abelian categories and their classical properties can be found in [Gro57] chapter I (in French), [Buc55] (in English) and in order to learn more about enriched categories one can read [Kel82].

The extension of a dimension function will be first treated for an abstract notion of dimension in the category of R -modules. Some notations, let R be a unitary ring and for any M a R -module denote M^* the *dual* of M i.e $\text{Hom}_R(M, R)$.

Definition 1.4.1 – Closure of a submodule

Let M be a submodule of a R -module denoted N ; one can define the *closure* of M in N as

$$\overline{M} := \bigcap_{\substack{f \in N^* \\ M \subset \ker(f)}} \ker(f)$$

and also the *torsion part* \mathbf{TM} and *projective part* \mathbf{PM} as follows :

$$\mathbf{TM} = \bigcap_{f \in M^*} \ker(f), \quad \mathbf{PM} = M/\mathbf{TM}.$$

Remark 1.4.1 One can compare the preceding definition of closure with the topological closures characterisation by geometric form of Hahn-Banach's theorem.

Definition 1.4.2 – Weak exact sequence

The sequence of R -modules $L \xrightarrow{i} M \xrightarrow{q} N$ is called *weakly exact* if $\overline{\text{im}(i)} = \ker(q)$.

Remark 1.4.2 – Properties of \mathbf{TM} and \mathbf{PM}

The following properties follow from the definition of \mathbf{TM} and \mathbf{PM} :

- $\mathbf{TM} = \overline{\{0\}} = \ker(M \rightarrow M^{**})$
- $\mathbf{PPM} = \mathbf{PM}$
- $\mathbf{PM} = 0$ iff. $M^* = 0$
- $\mathbf{TPM} = 0$
- $M^* = (\mathbf{PM})^*$

Definition 1.4.3 – Extended dimension function

Suppose that \dim is a function that assigns to any finitely generated projective R -module an element of \mathbf{R}_+ such that :

- If P, Q are two finitely generated projective R -modules :

$$P \cong Q \Rightarrow \dim(P) = \dim Q, \quad \dim(P \oplus Q) = \dim(P) + \dim(Q).$$

- If Q is a finitely generated projective R -module and K a Q -submodule then \overline{K} is a direct summand and

$$\dim(\overline{K}) = \sup\{\dim(P) \mid P \text{ is a finitely generated projective } K\text{-submodule}\}.$$

For M a R -module, the *extended dimension* is defined by :

$$\text{Dim}(M) = \sup\{\dim(P) \mid P \text{ is a finitely generated projective } M\text{-submodule}\}.$$

Proposition 1.4.1 – Projectivity under existence of an extended dimension function

Suppose that (R, \dim) is a pair satisfying the hypothesis of 1.4.5, then :

- (Semiheredity) Finitely generated R -submodules of projective modules are projective.
- (Projective submodules) If K is a submodule of the finitely generated R -module M then M/\overline{K} is finitely generated projective.
- (Projective quotients) If M is finitely generated then $\mathbf{P}M$ is finitely generated projective with $M \cong \mathbf{P}M \oplus \mathbf{T}M$.

Proof. Standard commutative algebra, see [Lüco4] theorem 6.7 p.240. □

Proposition 1.4.2 – Properties of the extended dimension function

Suppose that (R, \dim) is a pair satisfying the hypothesis of 1.4.5, then the extended dimension function admit the following properties :

- (Extension) If M is a finitely generated projective R -module then $\dim(M) = \text{Dim}(M)$.
- (Additivity) If $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ is a short exact sequence of R -modules then $\text{Dim}(N) = \text{Dim}(M) + \text{Dim}(P)$.
- (Cofinality) If $\{M_i\}_{i \in I}$ is a directed system of submodule such that $\bigcup_{i \in I} M_i = M$ and for any two $i, k \in I$ there is $j \in I$ with $M_i, M_k \subset M_j$ then $\text{Dim}(M) = \sup\{\text{Dim}(M_i) \mid i \in I\}$.
- (Continuity) If K is a submodule of M a finitely generated R -module then $\text{Dim}(K) = \text{Dim}(\overline{K})$.
- (Projective/Torsion) If M is finitely generated then $\text{Dim}(M) = \text{Dim}(\mathbf{P}M)$ and $\text{Dim}(\mathbf{T}M) = 0$.
- (Uniqueness) The extension, additivity, cofinality and continuity properties uniquely determine the extended dimension function.

Proof. The proof involves the definition of the extended dimension function, closure and standard commutative algebra see [Lüco4] theorem 6.7 pages 239-242. □

Remark 1.4.3 *The notation \dim instead of Dim for the extended dimension function is now unambiguous.*

Now that any dimension function satisfying some assumptions can be extended beyond

Remark 1.4.4 *We denote by $\text{Mod}_{f.g.p}^{\mathcal{N}(G)}$ the category of finitely generated projective $\mathcal{N}(G)$ -modules. Both $\text{Mod}_{f.g.p}^{\mathcal{N}(G)}$ and $\text{HMod}_{f.g}^{\mathcal{N}(G)}$ can be seen as \mathbf{C} -categories, i.e categories enriched over the complex vector spaces category see [Kel82].*

Definition 1.4.4 – Dimension for finitely generated projective $\mathcal{N}(G)$ -modules

One can define a dimension \dim over $\text{Mod}_{f.g.p}^{\mathcal{N}(G)}$ the following way : $\dim(P) = \sum_{i=1}^n \text{tr}_{\mathcal{N}(G)}(A_{ii})$ where P is a finitely generated projective $\mathcal{N}(G)$ module and $A \in M_n(\mathcal{N}(G))$ is an idempotent such that $\text{im}(A) \cong P$.

Remark 1.4.5 *The invariance under the choice of projection is showed by considering a direct sum with a trivial matrix to unify the ranks and mapping through isomorphism both summand in the direct sum decomposition of P . The trace property ensures invariance under such consideration.*

Theorem 1.4.1 – Algebraic view on the von Neumann dimension

There is a \mathbf{C} -categories equivalence between $\text{Mod}_{f,g,p}^{\mathcal{N}(G)}$ and $\text{HMod}_{f,g}^{\mathcal{N}(G)}$ such that the dimension \dim corresponds with the von Neuman dimension $\dim_{\mathcal{N}(G)}$.

Proof. This fact is not trivial. The proof relies on an equivalence of \mathbf{C} -categories with involution between the category of finitely generated projective $\mathcal{N}(G)$ -module with a notion of inner product and $\text{HMod}_{f,g}^{\mathcal{N}(G)}$ through the Karoubi envelope (a definition can be found in [Bal01]) of their full subcategories respectively given by $\mathcal{N}(G)^n$ and $(l^2(G))^m$ for $n, m \in \mathbf{N}$. To have a glimpse on the argument see [Kam19] section 4.2 pages 71-72, the entire argument is presented in [Lüco4] section 6.2 pages 246-252. \square

Definition 1.4.5 – Extended dimension function

The *dimension function* \dim_N over the Hilbert $\mathcal{N}(G)$ -modules is defined as the extended dimension function relatively to \dim (see 1.4.4).

Chapter 2

l^2 -Betti numbers

As previously mentioned l^2 -Betti numbers are attempt to define Betti numbers for spaces where the usual Betti numbers are typically too often infinite. Even if l^2 -Betti numbers are not necessarily natural, or even rational, numbers they share similar properties with classical Betti numbers and thus can be computed with similar methods. One can notice that l^2 -Betti numbers behave well relatively to finite sheeted coverings and S^1 -action. Therefore, they can be obstruction to self-coverings, mapping torus constructions or circle actions.

2.1 Cellular l^2 -Betti numbers

Using G -orbits, one can reduce an infinite number of cells of a CW -complex with a suitable G -action to finitely many classes of equivariant cells. The cellular homology up to scalar extension yields finitely generated Hilbert $\mathcal{N}(G)$ -modules. Hence, the von Neumann dimension can be used to define an algebraic invariant: the cellular l^2 -Betti numbers.

The following section is a survey on G - CW -complexes needed in the definition of cellular l^2 -Betti numbers. They, somehow, generalise the notion of Galois covering. More details and demonstrations can be found in [Lüc89] sections II.1 and II.2, [Die87] sections 1 and 2. Let X be a CW -complex filtered by skeleta X_n and G a discrete countable group acting with a left action on X .

Definition 2.1.1 – Cellular action and G - CW -complex

The action of G on X is called *cellular* and X is called a *G - CW -complex*, if for any open cell E and $g \in G$:

- the translated set gE is an open cell.
- if $gE \cap E \neq \emptyset$ then g let E point-wise fixed.

Remark 2.1.1 *The condition on the trivial action ensures that the combinatorial data of the skeleta is sufficient to characterise the action of G .*

Proposition 2.1.1 – Pushout characterisation of cellular actions

The action of G is cellular if and only if the skeleta X_n are G -invariant and there exist a pushout

in the G -space category :

$$\begin{array}{ccc}
 \bigsqcup_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{q_n} & X_{n-1} \\
 \downarrow & & \downarrow \\
 \bigsqcup_{i \in I_n} G/H_i \times D^n & \xrightarrow{Q_n} & X_n
 \end{array}$$

Proof. Here is a construction of the pushout from the space X with a cellular action : since X is a CW-complex for $n \in \mathbf{N}$, one can choose a pushout such that $X_n \setminus X_{n-1} \cong \bigsqcup_{j \in J_n} \text{int}(D^n)$. Hence, G acts on J_n , define $I_n = G \backslash J_n$ and for any $i \in I$ define H_i its isotropy group. Then, $X_n \setminus X_{n-1} \cong \bigsqcup_{i \in I_n} G \backslash H_i \times \text{int}(D^n)$ this isomorphism is also a G -homeomorphism with the diagonal G -action given by left translation on $G \backslash H_i$ and the trivial action on $\text{int}(D)$. One can prove that this construction yields a pushout in the category of G -space, see [Kam19] theorem 3.2 pages 36-37. \square

Remark 2.1.2 – Cellular basis

A pushout as in 2.1.1 is determined by Q_n and therefore by G -equivariant by its action on the $H_i \times D^n$. We shall call a collection of one n -cell in each orbit together with their characteristic map a cellular basis of X .

Definition 2.1.2 – Adjectives on G -CW-complexes

A G -CW complex X is called :

- *finite type* if it has finitely many G - n -cells for every $n \in \mathbf{N}$
- *proper* if all isotropy groups are finite
- *finite* if it has finitely many G -cells
- *free* if all isotropy groups are trivial

Remark 2.1.3 The pushout characterisation of cellular action allows us to define it in the case of a locally compact Hausdorff group requiring the isotropy group to be closed and the action to be continuous see [Lüc89] sections I.1 and I.2.

Example 2.1.1 – Galois coverings

This example integrates the G -CW-complexes in our overarching direction. Let X be a connected, finite type CW-complex. Any Galois covering p of X is a connected, finite type, free $\text{Aut}(p)$ -CW-complex conversely if X is such a G -CW-complex then $X \rightarrow G \backslash X$ is a Galois covering.

Proposition 2.1.2 – Cellular chain complex of G -CW-complexes

The cellular chain complex defines a functor from the category of G -CW-complexes to left ZG -modules chain complexes.

Proof. Since the action is cellular any element of G yield a collection of self-homeomorphisms of the pair (X_n, X_{n-1}) and therefore automorphism of $H_n(X_n, X_{n-1})$ turning it into a left ZG -module.

For any CW-complex Y and any cellular map $f : X \rightarrow Y$ we consider the diagram :

$$\begin{array}{ccccc}
H_n(X_n, X_{n-1}) & \xrightarrow{f_*} & & \xrightarrow{f_*} & H_n(Y_n, Y_{n-1}) \\
\downarrow d_n & \searrow \delta_n & & \swarrow \delta_n & \downarrow d_n \\
& & H_{n-1}(X_{n-1}) & \xrightarrow{f_*} & H_{n-1}(Y_{n-1}) \\
& & \swarrow \partial_{n-1} & & \searrow \partial_{n-1} \\
H_{n-1}(X_{n-1}, X_{n-2}) & \xrightarrow{f_*} & & \xrightarrow{f_*} & H_{n-1}(Y_{n-1}, Y_{n-2})
\end{array}$$

Where $d_n, \delta_n, \partial_{n-1}$ arise respectively from long exact sequences of $(X_n, X_{n-1}, X_{n-2}), (X_n, X_{n-1})$ and (X_{n-1}, X_{n-2}) . Both triangles commute by definition of d_n see [Mun84] chapter 3, §26, page 148 and the square by naturality of the long exact sequence of pairs. Therefore this diagram commutes and the boundary map is natural. In particular specifying f to be a left translation we obtain that the cellular chain complex is a chain complex of $\mathbf{Z}G$ -modules and any cellular G -map yield a homomorphism of $\mathbf{Z}G$ -chain complex. \square

From now on we denote $C_*(X)$ the cellular chain complex associated to X .

Proposition 2.1.3 – Explicit description of the cellular chain complex

The choice of an explicit pushout as in 2.1.1 yields a description of the cellular chain complex with the following $\mathbf{Z}G$ -isomorphisms :

$$C_n(X) \cong \bigoplus_{i \in I_n} \mathbf{Z}(G/H_i).$$

Moreover we obtain an $\mathbf{Z}G$ -isomorphism with the description arising from any other cellular basis given by:

$$\bigoplus_{i \in I_n} r_{\pm i} : \bigoplus_{i \in I_n} \mathbf{Z}(G/\text{ad}_{g_i} H_i) \rightarrow \bigoplus_{i \in I_n} \mathbf{Z}(G/H_i)$$

where $(g_i H_i)_{i \in I_n}$ is the collection of coset sending the first cellular basis to the second and $r_{\pm i}$ the right translation by $\pm g_i H_i$.

Proof. The Mayer-Vietoris sequence yields :

$$C_n(X) \cong H_n\left(\bigsqcup_{i \in I_n} G/H_i \times S^{n-1}, \bigsqcup_{i \in I_n} G/H_i \times D^n\right) \cong \bigoplus_{i \in I_n} \bigoplus_{G/H_i} H_n(D^n, S^{n-1}) \cong \bigoplus_{i \in I_n} \mathbf{Z}(G/H_i).$$

Let $\phi_1, \phi_2 : D^n \rightarrow X_n$ be characteristic from the two cellular basis maps picking out the n -cells in the same i -orbit with $i \in I_n$. There is g_i sending $\text{im}\phi_1$ to $\text{im}\phi_2$, it is unique up to left translation with an element in the isotropy group H_i . Let γ the fundamental class of $H_n(D^n, S^{n-1})$ given by \mathbf{R}^n orientation. Since $g_i \text{im}\phi_1 = \text{im}\phi_2$ both $H_n(g_i \phi_1)(\gamma)$ and $H_n(\phi_2)(\gamma)$ generate the same direct summand i.e $H_n(g_i \phi_1)(\gamma) = \pm H_n(\phi_2)(\gamma)$. \square

Example 2.1.2 – Cellular chain complex of free G -CW-complex of finite type

A free G -CW-complex of finite type admit a cellular chain complex of the form :

$$\cdots \rightarrow (\mathbf{Z}G)^{m_n} \rightarrow \cdots \rightarrow (\mathbf{Z}G)^{m_1} \rightarrow (\mathbf{Z}G)^{m_0} \rightarrow 0.$$

Definition 2.1.3 – l^2 -chain complex of G -CW-complexes
The l^2 -chain complex of a G -CW-complex X is defined by

$$C_*^{(2)}(X) := l^2G \otimes_{\mathbb{Z}G} C_*(X).$$

In particular, $C_*^{(2)}$ define a functor from the category of G -CW-complexes to $\mathbb{C}G$ -chain complexes. Moreover, when restricted to the full subcategory of free G -CW-complexes of finite type, it takes image in $\text{HMod}_{f.g}^{\mathcal{N}(G)}$.

Remark 2.1.4 We can extend the hypothesis from free to proper G -CW complex of finite type keeping a preferred $\mathcal{N}(G)$ structure on the l^2 -chain complex see [Kam19] section 3.2 pages 41-45.

Definition 2.1.4 – l^2 -homology

Let X be a free G -CW-complex of finite type, its p^{th} reduced l^2 -homology for $p \in \mathbb{N}$ is defined as

$$H_p^{(2)}(X; \mathcal{N}(G)) := \ker d_p^{(2)} \Big/ \text{cl}(\text{im} d_{p+1}^{(2)}).$$

Remark 2.1.5 The closure on the quotient allows the reduced l^2 -homology to admit a Hilbert $\mathcal{N}(G)$ -module structure.

A more detailed description of the homological algebra related to Hilbert module is treated in [Lüco4] section 1.1.4 pages 24-30 where in particular a weak analog of the Hodge-de Rham theorem is proved, a long (weak) exact sequence is derived from short exact sequences of chain complexes and the tensor product of chain complex are tackled. The reader interested into the analytical/cellular l^2 -Betti numbers interaction is vividly advised to took a look at it.

Here is now a cornerstone is this presentation, one can with the preceding background give a first definition of l^2 -Betti numbers.

Definition 2.1.5 – Cellular l^2 -Betti numbers

Let X be a free G -CW-complex of finite type, its p^{th} cellular l^2 -Betti number for $p \in \mathbb{N}$ as

$$b_p^{(2)}(X; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)} H_p^{(2)}(X; \mathcal{N}(G)).$$

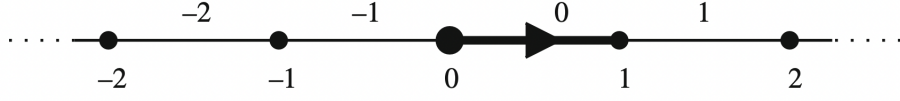
Remark 2.1.6 – Notation for Galois coverings The mention to $\mathcal{N}(G)$ shall be removed when the context allows it. For instance, for \tilde{X} a Galois covering the group is understood to be the deck transformations in particular $\pi_1(X)$ for the universal covering.

Example 2.1.3 – Extension of classical Betti numbers

Let G be finite then X is proper and if X is of finite type :

$$C_*^{(2)}(X) \cong l^2G \otimes_{\mathbb{Z}G} C_*(X) \cong \mathbb{C}G \otimes_{\mathbb{Z}G} C_*(X) \cong \mathbb{C} \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(X) \cong \mathbb{C} \otimes_{\mathbb{Z}} C_*(X) \cong C_*(X; \mathbb{C}).$$

The characterisation of $\dim_{\mathcal{N}(G)}$ in 1.3.3 yields we obtain $|G|b_p^{(2)}(X) = b_p(X)$. Unfortunately, in the case of a finite group the l^2 considerations do not provide any more information. However the whole interest is exactly lying in the other framework i.e G infinite.



Example 2.1.4 Let X be the \mathbf{Z} -CW complex arising from the circle S^1 with its classical CW-complex structure (one 0-cell and one 1-cell). The \mathbf{Z} -CW complex X is free, finitely generated (one equivariant 0-cell and one equivariant 1-cell, one can consider a cellular basis given by the thickened part in the following image shamelessly robbed from Kammeyer's book [Kam19] p.46. The only non-trivial differential map is characterised by its action on the "0" equivariant 1-cell i.e $0 \mapsto 1 - 0$. Using both the characterisation of $l^2\mathbf{Z}$ and $\mathcal{N}(\mathbf{Z})$ provided in 1.1.2, the preceding differential map seen as a $L^\infty(S^1)$ operator becomes the left multiplication by $(z - 1)$. This map is a weak isomorphism i.e it is injective with dense image, thus X is l^2 -acyclic i.e admit only trivial l^2 -homology groups.

Clearly, the philosophy of classical Betti numbers still applies. One must try to avoid references to actual complexes to compute l^2 -Betti numbers. Ergo, the following properties constitute a precious computational toolbox.

Theorem 2.1.1 – Properties of cellular l^2 -Betti numbers

The cellular l^2 -Betti numbers admit the following properties :

- (Homotopy invariance) If X and Y are two free G -CW complex of finite type that are G -homotopic equivalent then they admit the same cellular l^2 -Betti numbers.
- (0th-Betti number) If X a connected non empty free G -CW-complex of finite type then $b_0^{(2)}(X) = |G|^{-1}$.
- (Künneth Formula) If X_i is a free G_i -CW complex of finite type $i \in \{1, 2\}$ then $X_1 \times X_2$ is a free $G_1 \times G_2$ -CW complex of finite type and for any $n \in \mathbf{N}$:

$$b_n^{(2)}(X_1 \times X_2) = \sum_{p+q=n} b_p^{(2)}(X_1)b_q^{(2)}(X_2).$$

- (Restriction) If X is a free G -CW complex of finite type and G_0 a finite index subgroup of G , then $\text{res}_{G_0}^G X$ is a free G_0 -CW complex of finite type and for any $p \in \mathbf{N}$:

$$b_p^{(2)}\left(\text{res}_{G_0}^G X\right) = |G : G_0| b_p^{(2)}.$$

- (Wedge) If X_i are pointed connected G -CW complexes of finite type $i \in [r]$ and $X = \bigvee_{i \in [r]} X_i$, then for any $2 \leq p$:

$$b_1^{(2)}(\tilde{X}) - b_0^{(2)}(\tilde{X}) = r + \sum_{j=0}^r \left(b_1^{(2)}(\tilde{X}_j) - b_0^{(2)}(\tilde{X}_j) \right); \quad b_p^{(2)}(\tilde{X}) = \sum_{j=0}^r b_p^{(2)}(\tilde{X}_j).$$

- (Euler Poincaré) If X is a free finite G -CW complex then

$$\chi(G \setminus X) = \sum_{p \in \mathbb{N}} (-1)^p b_p^{(2)}(X).$$

Proof. For sake of conciseness the proof of this statement will be omitted. Albeit, the reader is strongly advise to look for it in [Lüco4] theorem 1.35 pages 37-40 and [Kam19] theorem 3.18 pages 48-50. Indeed, a considerable classical argument in the study of l^2 -invariant are there used. \square

Remark 2.1.7 – l^2 -Euler characteristic

The preceding Euler-Poincaré formula yield the definition of an other l^2 -invariant called l^2 -Euler characteristic. Here the distinction with $\chi(G \setminus X)$ is not clear. Although, the extension of l^2 -Betti numbers to any G -space allows a broader interpretation of the l^2 -Euler characteristic. More information on this invariant can be found in [Lüco4] section 6.6 pages 276-285.

2.2 Application of cellular l^2 -Betti numbers

Now that a first version of l^2 -Betti numbers has been defined, one can appreciate some applications in diverse areas. One must notice that those application are well known and arise from [Lüco4], [Kam19] and some oral explanations and sketches provided by Lück.

First, l^2 -Betti numbers admit a nice behavior relatively to finitely sheeted cover spaces. Moreover, they realise an obstruction to finite-sheeted self coverings.

Proposition 2.2.1 – Cellular l^2 -Betti numbers and self-covers

Let $p : X \rightarrow Y$ a d -sheeted covering of connected CW-complexes ($1 \leq d$) then $b_n^{(2)}(X) = d \cdot b_n^{(2)}(Y)$ for all $n \in \mathbb{N}$

Proof. Considering the tower $\tilde{X} \rightarrow X \rightarrow Y$ and recalling that $\text{Aut}(p) \cong p_*(\pi_1 X)$. Passing to the universal cover $\tilde{X} = \text{res}_{p_*(\pi_1 X)}^{\pi_1 Y}(\tilde{Y})$ therefore $b_p^{(2)}(\tilde{X}) = [\pi_1 X : p_*(\pi_1 Y)] \cdot b_p^{(2)}(\tilde{Y})$. The desired formula is obtained recalling that the number of sheets in a covering is given by the previously mentioned index. \square

Corollary 1 – Obstruction on trivial self coverings

If X admit a non-trivial connected finite-sheeted self-coverings then for any $p \in \mathbb{N}$, $b_p^{(2)}(\tilde{X})$ vanishes.

A vastly studied object in the context of 3-manifold is the *mapping torus*. If f is a continuous self-map on a topological space X the mapping torus $T(f)$ is the quotient of $X \times [0, 1]$ by the relation $(x, 0) \sim (f(x), 1)$. The mapping torus comes naturally with a map $p : T(f) \rightarrow S^1$ induced by the projection $X \times [0, 1] \rightarrow [0, 1]$.

Inspired by a conjectures of Gromov on the nature l^2 -Betti numbers of manifolds fibered over the circle [Gro92] pages 152. Lück showed in [Lüc94] theorem 2.1 page 7 that l^2 -Betti numbers obstruct some mapping torus structures.

Proposition 2.2.2 – l^2 -acyclicity of $\overline{T(f)}$

Let X a connected CW-complex of finite type and X a cellular self-map of X . All the l^2 -Betti numbers arising from the universal covering of $T(f)$ vanishes.

Proof. Exceptionally, the universal covering of $T(f)$ and all analog spaces is denoted $\overline{T(f)}$ by convenience. First, some notations altogether : the fundamental group of $T(f)$ is denoted π , since X is connected $\pi_1(p)$ is surjective and denote π_k the group $\pi_1(p)^{-1}(k\mathbf{Z})$ for $k \in \mathbf{N}^*$, observe that π_k has index k in π . Constructing the pullback $T(f)_k$ of the finite k -sheeted covering $S^1 \rightarrow S^1$ by writing S^1 as the k -fold mapping telescope of the interval, since pullbacks commute to colimits $T(f)_k$ is the k -fold mapping telescope of X along f . One can verify that is $T(f)_k$ is G -homotopy equivalent to $T(f^k)$. Thus, the following expression holds for any $n \in \mathbf{N}$:

$$b_n^{(2)}(\overline{T(f)}) = k^{-1} b_n^{(2)}(\text{res}_{\pi_k}^{\pi} \overline{T(f)}) = k^{-1} b_n^{(2)}(\overline{T(f)_k}) = k^{-1} b_n^{(2)}(\overline{T(f^k)})$$

For any $p \in \mathbf{N}$ let β_p the number of p -cell in the CW-complex decomposition of X . Then $T(q)$ for any self-map q admit a CW-complex decomposition with $\beta_p + \beta_{p-1}$ p -cells. By additivity of von Neumann dimension the following estimate holds :

$$b_n^{(2)}(\overline{T(f)}) = k^{-1} b_n^{(2)}(\overline{T(f^k)}) \leq \frac{\beta_p + \beta_{p-1}}{k}$$

Since β_p is k -independent taking the limit $k \rightarrow \infty$ concludes. □

There would be a lot more to say about application of l^2 -Betti numbers in different fields altogether : as obstruction to existence of specific S^1 -action see [Lüc04] theorem 1.40 pages 43-44, as obstruction to amenable infinite normal subgroup of the fundamental group for an aspherical complex see [Lüc04] theorem 1.44 page 48. Finally but not least they are related to *analytic l^2 -Betti numbers* in the case of cocompact free proper G -manifold with G -invariant metric i.e they can be defined using a heat kernel in a similar fashion as classical Betti numbers.

2.3 G-Space l^2 -Betti Numbers

A classical use of homotopy invariants in algebraic topology is to derive an isomorphism invariant by the mean of classifying spaces. However the classifying space of a group fits rarely in the framework of free G -CW complex of finite type i.e where l^2 -Betti numbers are yet defined. Motivated by this reason but not uniquely this one, one should try to extend the notion of l^2 -Betti numbers to general G -spaces by the use of the extended dimension function defined previously.

The G -action on X induced a left $\mathbf{Z}G$ -module structure on C_*^{sing} ; therefore, by scalar extension the functor $\mathcal{N}(G) \otimes_{\mathbf{Z}G} -$ yields a $\mathcal{N}(G)$ -module structure. From now on define X to be any G -space.

Definition 2.3.1 – l^2 -Betti number of a G -space

The l^2 -Betti numbers of X are defined for any $p \in \mathbf{N}$ by

$$b_p^{(2)}(X; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)} H_p(\mathcal{N}(G) \otimes_{\mathbf{Z}G} C_*^{\text{sing}}(X)).$$

Remark 2.3.1 The definition of the extended dimension function 1.4.5 allows the l^2 -Betti numbers of a general G -space to take ∞ as a value.

Proposition 2.3.1 – Extension property

If X is a free G -CW complex of finite type then both definition of Betti-numbers agree.

Proof. The proof relies on the fact that the fact that the extended dimension agrees with the dimension on finitely generated projective $\mathbf{N}(G)$ module and a $\mathbf{Z}G$ -chain homotopy equivalence $C_*(X) \rightarrow C_*^{\text{sing}}(X)$ unique up to $\mathbf{Z}G$ -homotopy and natural in X constructed in [Lüc98] lemma 4.2 page 152. A complete proof is available in [Lüc04] lemma 6.53 page 264. \square

Proposition 2.3.2 – Properties of general l^2 -Betti numbers

Most properties of cellular l^2 -Betti numbers extend to general l^2 -Betti number in such manner :

- (Homotopy invariance) If X and Y are two homotopic equivalent G -spaces then they admit the same l^2 -Betti numbers.
- (0^{th} -Betti number) If X is a path-connected G -space then $b_0^{(2)}(X) = |G|^{-1}$.
- (Künneth Formula) If X_i is a G_i -space for $i \in \{1, 2\}$ then $X_1 \times X_2$ is a $G_1 \times G_2$ -space and for any $n \in \mathbf{N}$ (the convention being $\infty \cdot 0 = 0$) :

$$b_n^{(2)}(X_1 \times X_2) = \sum_{p+q=n} b_p^{(2)}(X_1) b_q^{(2)}(X_2).$$

- (Restriction) If X is a G -space and G_0 a finite index subgroup of G , then $\text{res}_{G_0}^G X$ is G_0 -space and for any $p \in \mathbf{N}$:

$$b_p^{(2)}(\text{res}_{G_0}^G X) = |G : G_0| b_p^{(2)}.$$

Proof. The proof can be found in [Lüc04] theorem 6.54 pages 265-270. Most assertions are proved by reducing to the theorem 2.1.1 writing the G -space as "weak homotopic equivalent" to a direct colimit of a G -CW complexes of finite type. \square

The general l^2 -Betti numbers admit similar behavior as the cellular ones. In particular, they constitute an obstruction to mapping torus structure and S^1 -actions similarly to proposition 2.2.2 (see [Lüc04] theorems 6.63 and 6.65 pages 270-271).

The definition of l^2 -Betti numbers is now almost accessible, the last piece being an extremely brief survey on classifying spaces in the G -CW complex framework. A more detailed survey can be found in [Kam19] section 4.1 page 67 and a more detailed treatment of classifying spaces is presented in [Die87] section I.6 page 46.

Definition 2.3.2 – Family of subgroups

A family of subgroups of G is a set of subgroups of G closed under conjugation and finite intersection.

Example 2.3.1 – Trivial examples for families of subgroups

Three trivial examples of families of subgroups that always exist are : \mathcal{TRIV} the family consisting of the trivial subgroup, \mathcal{FIN} the family consisting of finite subgroups and \mathcal{ALL} the family consisting of all subgroups.

Definition 2.3.3 – Classifying space of a group for a family

Let \mathcal{F} be a family of subgroups of G a G -CW complex $E_{\mathcal{F}}G$ with isotropy groups in \mathcal{F} satisfying the following equivalent condition is called a *classifying space of G for \mathcal{F}* :

- For any G -CW complex X with isotropy groups in \mathcal{F} there exist a G -map $X \rightarrow E_{\mathcal{F}}G$ unique up to G -homotopy.
- For any H in \mathcal{F} the fixed pointset $(E_{\mathcal{F}}G)^H$ is n -connected for any $n \geq -1$.

Theorem 2.3.1 – Existence classifying space

For any group G and any family of subgroups \mathcal{F} there exists a $E_{\mathcal{F}}G$ classifying space of G for \mathcal{F} unique up to G -homotopy equivalence.

Example 2.3.2 – Usual classifying spaces

For sake of simplicity the classifying space for free G -action is denoted $EG := E_{\mathcal{TRIV}}G$ and the classifying space for proper G -action $\underline{EG} := E_{\mathcal{FIN}}G$. By definition G acts freely on EG , hence $EG \rightarrow G/EG$ is a covering. Usually G/EG is denoted BG , the plain mention of "classifying space" generally refers to BG .

Here is the cornerstone of this brief introduction to l^2 -Betti numbers. The definition of l^2 -Betti number for a general group G . It embeds one of the many bridges between the realm of l^2 -invariants and more standard fields, here group theory.

Definition 2.3.4 – l^2 -Betti number of a group for a family

Let G be a group and \mathcal{F} be a family of subgroups. For any $p \in \mathbb{N}$ the p^{th} l^2 -Betti number of G with respect to \mathcal{F} is defined by : $b_p^{(2)}(G, \mathcal{F}) := b_p^{(2)}(E_{\mathcal{F}}G)$. Usually, $b_p^{(2)}(G)$ denotes $b_p^{(2)}(G, \mathcal{TRIV}) = b_p^{(2)}(EG)$.

Remark 2.3.2 – Role of families

It is known that for any family \mathcal{F} of l^2 -acyclic subgroups the l^2 -Betti numbers are invariant i.e $b_p^{(2)}(G) = b_p^{(2)}(G, \mathcal{F})$. Therefore, the family \mathcal{F} can be tuned to obtain nicer classifying space $E_{\mathcal{F}}G$. Concretely the pair used is often $(\mathcal{FIN}, \underline{EG})$, indeed constructing \underline{EG} is typically better behaved than the construction of EG .

The l^2 -Betti number of group G can be applied to obtain various information on it. Altogether: they are able to detect finitely co-Hopfian groups i.e not isomorphic to a subgroup of finite index, they yield a bound on the deficiency of a finitely presented group i.e maximal difference in between generators and relations and many more...

The

Notations

Here is all the notations that are possibly unknown for a l^2 -invariants theory layman :

$b_p^{(2)}(G, \mathcal{F})$ p^{th} l^2 -Betti number of G with respect to \mathcal{F} 2.3.4.

\underline{EG} Classifying space for proper G -actions 2.3.2.

BG Quotient G/EG 2.3.2.

EG Classifying space for free G -actions 2.3.2.

$E_{\mathcal{F}}G$ Classifying space of G for \mathcal{F} 2.3.3.

$[n]$ $\{0, \dots, n\}$.

$B(H)$ Set of bounded endomorphism of H . 1.1

$B(H, K)$ Set of bounded linear operator from H to K .

$b_p^{(2)}(X; \mathcal{N}(G))$ p^{th} l^2 -Betti number of X 2.1.5, 2.3.1.

CG Complex group algebra. 1.1.2

$C_*^{(2)}$ l^2 -complex chain 2.1.3.

$\chi(X)$ Euler-Poincaré characteristic of the CW-complex X .

D^n Unit closed disc of \mathbf{R}^n .

$\dim_{\mathcal{N}(G)}$ von Neumann dimension 1.3.3, 1.4.5

$H_*^{(2)}$ l^2 -homology 2.1.4.

$\text{HMod}_{f.g}^{\mathcal{N}(G)}$ Category of finitely generated Hilbert $\mathcal{N}(G)$ -modules. 1.2.1

ι_* Induction functor. 1.2.3

\mathcal{ALL} Family of all subgroups 2.3.1.

\mathcal{FIN} Family of finite subgroups 2.3.1.

\mathcal{TRIV} Family of the trivial subgroup 2.3.1.

Dim Extended dimension 1.4.5
 T(f) Mapping torus of f 2.2.
 $\text{Mod}_{f.g.p}^{\mathcal{N}(G)}$ Category of finitely generated projective $\mathcal{N}(G)$ -modules.
 $\mathcal{N}(G)$ group von Neumann algebra. 1.1.3
 \overline{M} Closure submodule 1.4.1.
 π_H orthogonal projection onto H .
PM 1.4.1.
 res Restriction functor. 1.2.2
 \tilde{X} Universal covering of X .
TM 1.4.1.
 $\text{tr}_{\mathcal{N}(G)}$ von Neumann trace 1.3.2, 1.3.1.
 l^2G Hilbert space induced from the group algebra. 1.1.2
 M'' , $M \subset B(H)$ Bicommutant. 1.1.1
 M' , $M \subset B(H)$ Commutant. 1.1.1
 M^* , dual of M 1.4.
 $R^{\oplus n}$ $\bigoplus_{i=1}^n R$
 $s \leq t$, $s, t \in B(H)$ The operator $t - s$ is positive see [Lüco4] p.16-17 definition 1.8 1.2.2

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