EDGE BEHAVIOR OF FREE CONVOLUTIONS
RAPPORT DE STAGE DE M1

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Déroulement et apports du stage

Après avoir effectué mon mémoire de L3 sur les matrices aléatoires et avoir participé à un Groupe de Travail sur les probabilités libres, j'ai effectué mon Stage à l'Institut Royal de Technologie (KTH) à Stockholm où travaille mon superviseur, Kevin Schnelli. J'ai, lors des premières semaines, concentré mon travail sur la lecture et l'analyse approfondie de plusieurs articles de recherche sur la convolution libre et le comportement sur les bords de son support ([13], [2] et [14]). Même si j'avais déjà quelques notions concernant les probabilités libres, j'ai dû, dans un premier temps, me familiariser avec les nouvelles notions évoquées. J'ai ensuite commencé à travailler sur ce même problème, le comportement sur les bords de mesures convoluées pour des mesures initiales de Jacobi avec des comportements en puissance sur les bords de leur domaine, avec des exposants plus grands que un, sujet non traité dans [2] ou [14]. C'est réellement à partir de ce moment que j'eus l'impression de découvrir le métier de chercheur. Je me suis rapidement rendu compte de l'importance du travail en groupe, du partage des idées ainsi que des échelles de temps mises en jeu. Par la suite, j'ai également compris l'importance d'aborder plusieurs fois le même problème sous des angles différents ou bien d'appliquer une méthode concluante à d'autres problèmes similaires. Plus mathématiquement, j'ai beaucoup appris sur les nombreuses méthodes analytiques permettant la résolution de problèmes initialement probabilistes comme la transformée de Stieltjes et plusieurs de ses variantes, le lemme de Plemelj ou encore les fonctions de Nevanlinna et leur représentation sous forme intégrale.

Durant ce stage, j'ai particulièrement aimé réfléchir à des problèmes complexes sur des échelles de temps assez longues. Même s'il a été parfois laborieux d'arriver à comprendre en profondeur le contenu et les objectifs des premiers articles que j'ai lu, la maîtrise progressive des outils utilisés dans ces domaines est une récompense largement suffisante. Chercher, et rarement trouver, des réponses à des questions, parfois très techniques, parfois plus abstraites mais bloquant toujours la résolution générale du problème est ce qui m'a le plus plu lors de ce stage. Je suis donc assez confiant sur ma volonté à continuer mes études dans cette direction. Cependant, je pense avoir plus d'affinités avec des approches et des outils plus probabilistes et moins analytiques que ce que j'ai pu rencontrer durant ces quatre mois.

Je suis très heureux d'avoir pu partir à l'étranger, particulièrement en Suède, et d'avoir eu une réelle première expérience professionnelle. Je pense être désormais convaincu que le métier de chercheur en mathématiques est celui que je veux exercer.

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Abstract. We consider the free additive and multiplicative convolution semi-groups as well as the free additive convolution of two distinct measures. For those three cases, provided the initial measures have power law behaviors near their edges of their support with exponents greater than -1, we prove that the freely convoluted measures also have a power law behavior. The freely convoluted measures are proven to have either a square root behavior or a power law behavior with the same exponent as one of the initial measures near their end points of their support. We give, for the additive semi group and the free additive convolution of two distinct measures, explicitly computable criteria that allow to distinguish the different behaviors.

1. Introduction

In probability, the notion of convolution (additive or multiplicative) is of great interest because it gives the law of the sum or the product of two independent random variables. In analogy, Voiculsecu introduced in [19] and [20] the notion of free probability, a new probability theory for non commutative random variables in which he defined the free additive and multiplicative convolution. Although similar in their purpose, they remain very different from one another. Indeed, it is easy to define the convolution of two random variables once one knows the laws of the two initial random variables, however, the free convolution of two freely independent, self-adjoint non-commutative random variables can only be found implicitly using certain systems of equations.

One can highlight the strong links between random matrices and free convolution in order to understand the purpose of such a notion. Indeed, if one takes an $N \times N$ symmetric matrix with independent Gaussian random entries, say $M^{(N)}$, one could ask how the eigenvalues of such a matrix are distributed. If we denote $(\lambda_1^{(N)}, ..., \lambda_N^{(N)})$ the eigenvalues, the object that contains a lot of information concerning these eigenvalues is the so called "empirical spectral measure" : $\frac{1}{N}\sum_1^N \delta_{\lambda_i^{(N)}}$. Once we understand the utility of this empirical probability measure, a question one may ask is whether this distribution converges to a certain measure as N goes to infinity. In our example, it converges to Wigner's semi-circle law. However, taking matrices with non-Gaussian random entries but instead A_n, B_n independent, $n \times n$ Hermitian, random matrices with at least one of them being invariant in law under conjugation by any unitary matrix and such that the empirical spectral measure of A_n (resp. B_n) converges toward μ_α (resp. μ_β), then the empirical spectral measure of A_n+B_n converges toward $\mu_\alpha \boxplus \mu_\beta$, the free additive convolution of μ_α and μ_β. Regarding the free multiplicative convolution, there is also a random matrix interpretation of $\mu_\alpha \boxtimes \mu_\beta$, the free multiplicative convolution of μ_α and μ_β. If C_n, D_n are two independent, $n \times n$ non-negative Hermitian, random matrices with at least one of them being invariant in law under conjugation by any unitary matrix and such that the empirical spectral measure of C_n (resp. D_n) converges toward μ_α (resp. μ_β), then the empirical spectral measure of C_nD_n converges toward $\mu_\alpha \boxtimes \mu_\beta$.

Another way to observe this is from the free probability point of view. Indeed, let us take two freely independent, self-adjoint, non commutative random variables X and Y with distributions μ_α and μ_β respectively, in a non-commutative unital tracial $*$-algebra, for instance a $\mathbb{C}^*$ or a Von Neumann algebra. These two variables are non commutative and therefore the distribution of $X+Y$ is not the classical convolution but the free additive convolution $\mu_\alpha \boxplus \mu_\beta$. Morevover, $\mu_\alpha \boxtimes \mu_\beta$ denoted the law of $X^{\frac{1}{2}}YX^{\frac{1}{2}}$ which is the same as $Y^{\frac{1}{2}}XY^{\frac{1}{2}}$ for positive X and Y. Now that we saw the link between free probability, random matrix and free convolution, one can both understand the profound similarities between the classical and the free convolution and why having simple results regarding the free convolution is much more difficult.

We can next extend these operations by defining the free additive convolution semigroup and the free multiplicative convolution semigroup. Indeed, if we denote for all $n \in \mathbb{N}$, the free convolution n-th power of μ, $\mu^{\boxplus n} := \underbrace{\mu \boxplus ... \boxplus \mu}_{n \text{times}}$, (resp. $\mu^{\boxtimes n} := \underbrace{\mu \boxtimes ... \boxtimes \mu}_{n \text{times}}$) we notice that for any $n, m \in \mathbb{N}$, we have $\mu^{\boxplus n} \boxplus \mu^{\boxplus m} = \mu^{\boxplus(n+m)}$ (resp. $\mu^{\boxtimes n} \boxtimes \mu^{\boxtimes m} = \mu^{\boxtimes(n+m)}$). Bercovici and Voiculescu first showed in [8] that the free additive convolution powers can be embedded in a continuous family $\{\mu^{\boxplus T} \,|\, T \text{ large enough}\}$ when μ is compactly supported. Then Nica and Speicher extended this result for μ compactly supported and for $T > 1$ in [16]. For the multiplicative case, Belinschi and Bercovici showed in [4] that the free multiplicative convolution powers can be embedded in a continuous family for $T \geq 1$ for measures supported on $\mathbb{R}_+$. For further details on that matter and on free probability in general, one may consult Mingo and Speicher's work : [13].

In this report, we are interested in the edge behavior of the free additive convolution of a large range of measures. More precisely, we are looking for a real exponent t, depending on certain parameters of the two initial measures, such that the free additive convolution of those two measures is equivalent (up to some multiplicative constant) to $(x - E_-)^t$ when x approaches E_-, the lower edge point of the support, by the right (and similarly near E_+, the upper edge point of the support). These measures will be assumed to be of "Jacobi type". We will give a precise definition later but the most important property is that they have a power behavior at the edges of their bounded support. In [2], it was shown that for compactly supported measures, their free convolution also had a compact support. The case where the exponents near the edges are less than one has already been studied in [2] and the free convolution of two such measures has a square root behavior on its edges. We will give a similar result giving the power behavior near the edges of the support, first for the free convolution semigroup and then for the free additive convolution of two distinct measures.

The keystone of our results will always be a "distance lemma" that gives an easy-to-compute criterion for the so-called subordination functions to stay at a strictly positive distance from the support of the measure. The distance between the subordination functions and the support of the measures is crucial to determine if the free convolution has either a square root behavior or a power law behavior with the same exponent as one of the original measures. Indeed, in [2], even if the measures are supposed to have exponents smaller than one, the proof of the square root behavior only requires a distance Lemma. The assumption on the exponents was sufficient to ensure a strictly positive distance between the subordination function and the support but it was not necessary.

We will, in this report, in the section about the semi group, show that, for T small enough, $\mu^{\boxplus T}$ has the same power behavior on its edges than μ and for T large enough, $\mu^{\boxplus T}$ will have a square root behavior, see Theorem 3.4. We will give a criterion on T for the change of behavior on the edges. For the free multiplicative semi group, Ji gave in [12] a result stating that for two Jacobi measures on $\mathbb{R}_+$, with exponents smaller than one near their edges, their free multiplicative convolution is also Jacobi type and has a square root decay near its edges. We will use its notations that allow to see the strong link (particularly in the subordination equations) between the additive case and the multiplicative one. We will give the power behavior of $\mu^{\boxtimes T}$ for $T > 1$ and for μ with exponents greater than one depending on the behavior on the subordination function in Theorem 3.8. For the free convolution of two distinct measures, depending on the behavior of the subordination functions (whether or not they touch the support of the other measure), we will show in Theorem 4.2 that the power behavior of the convolution entirely depends on which subordination function touches the support. However, we are not yet able to give a computable criterion in order to know the behavior of the subordination function near the lower-edge point of the support of the free convolution (in other words, whether or not they touch the support and if they do which one touches) although we are able to narrow down the possibilities with several criteria.

Throughout the paper, for the free additive convolution, we will always assume our measures to be Jacobi type. More precisely,

Assumption 1.1. *We examine the case where μ is* ***absolutely continuous, compactly supported and centered probability measures*** *with respect to Lebesgues measure. Its density is denoted by ρ. We also assume that*

- *The density is supported on a single non empty interval, say $[E_-; E_+]$.*
- *The density function has a power behavior on the edges of its support. More precisely, there exists exponents $t_-, t_+ > -1$ such that for a certain $C > 1$,*

$$\frac{1}{C} \leq \frac{\rho(x)}{(x-E_-)^{t_-}(E_+-x)^{t_+}} \leq C \,, \qquad \text{for almost every } x \in [E_-; E_+] \,.$$

The exponents being greater than -1 ensures the integrability near the edges.

In the additive case, we will always assume our measures to verify these assumptions and we will precise when the exponents t_-, t_+ are assumed to be greater than one.

For the free multiplicative convolution, we will always assume our measures to verify the following assumption.

Assumption 1.2. *We examine the case where μ is a probability measure on $\mathbb{R}_+$,* ***compactly supported, absolutely continuous,*** *with respect to Lebesgues measure. Its density is denoted ρ. We also assume that :*

- *The density is supported on a single non empty interval, say $[E_-; E_+]$ and $E_- > 0$.*
- *The measure has a finite second moment and a first moment equal to one,*

$$\int_{\mathbb{R}_+} x \mathrm{d}\mu(x) = 1 \qquad \int_{\mathbb{R}_+} x^2 \mathrm{d}\mu(x) < \infty$$

- *The density function has a power behavior on the edges of its support. More precisely, there exists exponents $t_-, t_+ > -1$ such that for a certain $C > 1$,*

$$\frac{1}{C} \leq \frac{\rho(x)}{(x-E_-)^{t_-}(E_+-x)^{t_+}} \leq C \,, \qquad \text{for almost every } x \in [E_-; E_+]$$

The exponents being greater than -1 ensures the integrability near the edges.

Let us note that the assumption "being centered" (resp. having a first moment equal to one) in 1.1 (resp. 1.2) does not add any restrictions as trivially shifting and rescaling the measures does not change the other assumptions.

Notational Remark. *Throughout this paper, we will denote $\mathbb{C}_+ := \{e + \mathrm{i}\eta \,|\, e \in \mathbb{R},\, \eta > 0\}$ the complex upper-half plane and $\mathbb{R}_+ := [0; \infty)$ the positive real line. We will also use the notation $B(z, R)$ for denoting $\{z' \in \mathbb{C} \,|\, |z - z'| <$*

$R\}$. We will very often use C to denote a constant even though they might change from one line to another or even in an equation with several constants.

2. Definitions, notations and useful results

For the additive case as well as the multiplicative one, we will need several functions defined after the measure itself.

Definition 2.1. *Let ν be a Borel probability measure on $\mathbb{R}$ (resp. on $\mathbb{R}_+$), its **Stieltjes Transform** (resp. M-transform) is defined as follows. For any $z \in \mathbb{C}_+$,*

$$m_\nu(z) := \int_{\mathbb{R}} \frac{1}{x-z} \mathrm{d}\nu(x) \qquad \textit{resp.} \qquad M_\nu(z) := \frac{z m_\nu(z)}{1 + z m_\nu(z)},$$

where we notice that $m_\nu : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ and $M_\nu : \mathbb{C} \setminus \mathbb{R}_+ :\longrightarrow \mathbb{C} \setminus \mathbb{R}_+$ are analytic.
We will also use, for z in $\mathbb{C}_+$,

$$F_\nu(z) : = -\frac{1}{m_\nu(z)}, \qquad I_\nu(z) : = \int_{\mathbb{R}} \frac{\mathrm{d}\nu(x)}{|x-z|^2},$$
$$H_\nu(z) : = \frac{M_\nu(z)}{z}, \qquad J_\nu(z) : = \int_{\mathbb{R}_+} \frac{x}{|x-z|^2} \mathrm{d}\nu(x),$$

where we notice that $F_\nu, H_\nu : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ are analytic.

for absolutely continuous measures, we have a way to extract the density from the Stieltjes transform using the Siteltjes inversion formula :

$$\rho_\nu(e) = \lim_{\eta \to 0^+} \frac{1}{\pi} \operatorname{Im} m_\nu(e + \mathrm{i}\eta). \tag{2.1}$$

Thanks to this equation, in order to study the density of the free additive convolution, we can now study the imaginary part of the Stieltjes transform. Furthermore, for the measures considered in Assumption 1.1 ans 1.2, we can take any non tangential limit in 2.1

Furthermore, we will also need the integral representation for Nevanlinna functions that gives us, for any analytic self-mappings f of $\mathbb{C}_+$, the existence of a Borel measure σ on $\mathbb{R}$ and real numbers $a \in \mathbb{R}$ and $b > 0$ such that f can be written as

$$f(w) = a + bw + \int_{\mathbb{R}} \left(\frac{1}{x-w} - \frac{x}{1+x^2} \right) \mathrm{d}\sigma(x), \qquad w \in \mathbb{C}_+.$$

This theorem is originally attributed to Nevanlinna [15] (see [9] for the classical proof using Ries-Herglotz theorem). The numbers a and b and the measure σ are uniquely determined as follows,

$$\sigma(]\lambda_1, \lambda_2]) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \operatorname{Im} f(\lambda + \mathrm{i}\epsilon) \mathrm{d}\lambda,$$
$$a = \operatorname{Re} f(\mathrm{i}),$$
$$b = \lim_{y \to \infty} \frac{f(\mathrm{i}y)}{\mathrm{i}y}.$$

We will use this representation for the F and H functions defined earlier in Definition 2.1.

Lemma 2.1. *Let μ be a Borel measure satisfying Assumptions 1.1 (resp. Assumptions 1.2), there exists a Borel measure $\widehat{\mu}$ (resp. $\tilde{\mu}$) on $\mathbb{R}$ (resp. $\mathbb{R}_+$), such that,*

$$F_\mu(\omega) = \omega + \int_{\mathbb{R}} \frac{1}{x-\omega} \mathrm{d}\widehat{\mu}(x) \quad \textit{resp.} \quad H_\mu(\omega) = 1 + \int_{\mathbb{R}_+} \frac{\mathrm{d}\tilde{\mu}(x)}{x-\omega} \qquad \forall \omega \in \mathbb{C}_+, \tag{2.2}$$

with

$$\widehat{\mu}(\mathbb{R}) = \operatorname{Var} \mu = \int_{\mathbb{R}} x^2 \mathrm{d}\mu < \infty \quad \textit{resp.} \quad \tilde{\mu}(\mathbb{R}_+) = \operatorname{Var} \mu = \int_{\mathbb{R}_+} x^2 \mathrm{d}\mu(x) - 1 < \infty. \tag{2.3}$$

Moreover, we also have $\operatorname{supp} \widehat{\mu} = \operatorname{supp} \mu$ *(resp.* $\operatorname{supp} \tilde{\mu} = \operatorname{supp} \mu$*).*

Proof. As seen in Definition 2.1, F_μ (resp. H_μ) is an analytic function mapping the upper plan to itself and therefore has a Nevanlinna representation. We then compute the values of a and b by evaluating $F(\omega)$ (resp. $H(\omega)$) in i and in iη in the limit $\eta \to \infty$. A more detailed proof is given in Lemma 3.2 and in Lemma 4.4 of [2] (resp. Lemma 2.2 in [12]). ☐

Moreover, we can also deduce the edge behavior of the "hat measures" $\widehat{\mu}$ and the "tilde measures" $\tilde{\mu}$ we just defined from the original measure μ.

Lemma 2.2. *Let μ be a Borel measure satisfying Assumption 1.1 (resp. 1.2) with exponents $t_-, t_+ > 0$ and with support $[E_-; E_+]$. The measure $\widehat{\mu}$ (resp. $\tilde{\mu}$) defined in Lemma 2.1 by equation* (2.2) *also verifies Assumptions 1.1 (resp. Assumptions 1.2) with the same exponents.*

Proof. The main idea is to use Definition 2.1 for F_μ (resp. H_μ) and equation (2.2) to express the density, ρ_μ, of μ in terms of the density, $\rho_{\widehat{\mu}}$, (resp. $\rho_{\tilde{\mu}}$) of $\widehat{\mu}$ (resp. $\tilde{\mu}$). Let $z := e + \mathrm{i}\eta \in \mathbb{C}_+$ with $e \in \mathbb{R}, \eta > 0$,

$$z + m_{\widehat{\mu}}(z) = F_\mu(z) = \frac{-1}{m_\mu(z)}, \quad \text{resp.} \quad 1 + m_{\tilde{\mu}}(z) = H_\mu(z) = \frac{m_\mu(z)}{1 + z m_\mu(z)}. \tag{2.4}$$

Taking the imaginary part on both sides yields

$$\eta + \operatorname{Im} m_{\widehat{\mu}}(e + \mathrm{i}\eta) = \frac{\operatorname{Im} m_\mu(e + \mathrm{i}\eta)}{|m_\mu(e + \mathrm{i}\eta)|^2}, \quad \text{resp.} \quad \operatorname{Im} m_{\tilde{\mu}}(e + \mathrm{i}\eta) = \frac{\operatorname{Im} m_\mu(z) - \eta |m_\mu(z)|^2}{|1 + z m_\mu(z)|^2}. \tag{2.5}$$

We know from Lemma 2.1 that μ and $\widehat{\mu}$ have the same support. We now take z in a neighborhood of E_-, the lower endpoint of the support, and take the limit $z \to E_-$. We just have to not take a tangential limit in order to assure that the Cauchy-Stieltjes transform has a finite limit. The terms with imaginary parts will converge to the density of μ, $\widehat{\mu}$ or $\tilde{\mu}$. We will distinguish the additive and the multiplicative case from now on.

- ⊞ Here, $|m_\mu(z)|$ stays finite and strictly positive when z approaches E_- non tangentially (and similarly for E_+). Indeed, for $m_\mu(z)$ to be equal to 0, z must be real (otherwise, the imaginary part would always be strictly positive if $z \in \mathbb{C}_+$) and because μ is supported on a single interval, m_μ is strictly is either strictly positive, either strictly negative depending on which side of the support we are. Therefore,

$$\frac{\rho_{\widehat{\mu}}(e)}{(e - E_-)^{t_-}} \longrightarrow C \text{ as } e \to E_-. \tag{2.6}$$

  Thus $t_- = t_-$ (a similar result is obtained when e approaches E_+).
- ⊠ Here, since we assumed the exponents to be greater than one, we know that $m_\mu(z)$ approaches a constant as z approaches E_- and since we assumed $E_- > 0$, then $|1 + z m_\mu(z)|$ also approaches a strictly positive constant as z tends to E_-. Moreover, using the Stieltjes inversion formula, we know that $\operatorname{Im} m_{\tilde{\mu}}(e + \mathrm{i}\eta)$ converges toward $\pi \rho_{\tilde{\mu}}(e)$ as η approaches 0. Therefore, we have,

$$\frac{\rho_{\tilde{\mu}}(e)}{(e - E_-)^{t_-}} \sim C \frac{\rho_\mu(e)}{(e - E_-)^{t_-}} \longrightarrow cste \quad \text{as} \quad e \to E_-. \tag{2.7}$$

This concludes the proof of Lemma 2.2. □

Remark 2.1. *We restricted our proof to measure with strictly positive exponents because it is sufficient for our further results. However, for measures with exponents t_-, t_+ in $(-1; 0)$, following the same guidelines, we find that the new measures have exponents $|t_-|, |t_+|$.*

The last Lemma we will need gives us an expansion of the Cauchy-Stieltjes transform of a measure near the edge points of its support.

Lemma 2.3. *(See Appendix 6 in* [18] *for further reading.) For a measure μ satisfying Assumptions 1.1 with positive exponents and support $[E_-; E_+]$, we have,*

$$m_\mu(z) = m_\mu(E_-) + C(E_- - z)^{t_-} + o((z - E_-)^{t_-}). \tag{2.8}$$

A similar property holds near E_+ : $E_- - z$ becomes $z - E_+$

Proof. We will use Plemelj's lemma to show this result. Let us recall a simplified version of Plemelj's lemma (one can look at Lemma 1 in [17] for a more precise definition).

Lemma 2.4 (Plemelj's Lemma)**.** *Let f be an integrable function having some regularities (Holder continuous for instance), then its Cauchy transform $\mathcal{C}f(z) := \frac{1}{2\mathrm{i}\pi} \int_{\mathbb{R}} \frac{f(x)}{x - z} \mathrm{d}x$ is the unique analytic function $\Phi(z)$ decaying at infinity such that*

(1) $\Phi_\pm(x) := \lim_{\epsilon \searrow 0} \Phi(x \pm \mathrm{i}\epsilon)$ is continuous in x real,
(2) Φ satisfies the jump condition : $\Phi_+(x) - \Phi_-(x) = f(x)$.

Since μ satisfies Assumptions 1.1 (or Assumptions 1.2), it is absolutely continuous, $m_\mu(z)$ can be viewed as

$$2\mathrm{i}\pi \mathcal{C}\rho_\mu(z) = \int_{E_-}^{E_+} \frac{\Psi(x)(E_+ - x)^{t_+}(x - E_-)^{t_-}}{x - z} \mathrm{d}x.$$

We are only interested in the integral in a small neighborhood of E_- so we write

$$\int_{E_-}^{E_-+\delta}\frac{\mathrm{d}\mu}{x-z}=\int_{E_-}^{E_-+\delta}\frac{\Psi(x)(E_+-x)^{t_+}-\Psi(E_-)(E_+-E_-)^{t_+}}{x-z}(x-E_-)^{t_-}\mathrm{d}x + \Psi(E_-)(E_+-E_-)^{t_+}\int_{E_-}^{E_-+\delta}\frac{(x-E_-)^{t_-}\mathrm{d}x}{x-z}.$$

The function $\phi(z):=C(E_--z)^{t_-}$ satisfies all the above conditions. The jump condition is verified for a C to be chosen later. Indeed, we have for all x in $\mathbb{R}$

$$\begin{aligned}\phi_+(x)-\phi_-(x)&=C\lim_{\epsilon\to0^+}\left((E_--x-\mathrm{i}\epsilon)^{t_-}-(E_--x+\mathrm{i}\epsilon)^{t_-}\right)\\&=C\left(\exp(t_-\log(x-E_-)+\mathrm{i}\pi t_-)-\exp(t_-\log(x-E_-)-\mathrm{i}\pi t_-)\right)\\&=(x-E_-)^{t_-}2\mathrm{i}C\sin(\pi t_-),\end{aligned}$$

with $C:=\frac{1}{2i\sin(\pi t_-)}$, we obtain the wanted result. The second equality follows from

$$\begin{aligned}\lim_{\epsilon\searrow0}(E_--x\pm\mathrm{i}\epsilon)^{t_-}&=\lim_{\epsilon\searrow0}\exp(t_-\log_\pm(E_--x\mp\mathrm{i}\epsilon)\\&=(E_--x)^{t_-}\exp(\pm\mathrm{i}\pi t_-).\end{aligned}$$

This concludes the proof of Lemma 2.3. ☐

3. Free convolution semi-group

In this section, we will first be interested in the free additive semi-group in Subsection 3.1, then we will study the free multiplicative semi-group in Subsection 3.2.

3.1. Free additive convolution semi group.

Let μ be a Borel probability measure that satisfies Assumptions 1.1. We are interested in the measures $\mu^{\boxplus T}$ for $T>1$. The existence of a subordination function ω_T and the properties it verifies was studied by Belinschi and Bercovici in [3] and [4]. More precisely, for $T\geq1$, $\mu^{\boxplus T}$ is the unique measure verifying $F_{\mu^{\boxplus T}}=F_\mu\circ\omega_T$. From [3], [4] and [11] we have the following theorem giving us the existence of the subordination function and of the equations it verifies.

Theorem 3.1. *For a single compactly supported, Borel probability measure μ, for every $T>1$, there exists a unique analytic function ω_T, mapping the upper complex plane to itself, such that*

$$\lim_{\eta\to\infty}\frac{\omega_T(\mathrm{i}\eta)}{\mathrm{i}\eta}=1,\qquad \mathrm{Im}(\omega_T(z))\geq\mathrm{Im}(z)\quad\forall z\in\mathbb{C}_+,\tag{3.1}$$

$$\frac{T\omega_T(z)-z}{T-1}=F_\mu(\omega_T(z)),\qquad F_{\mu^{\boxplus T}}(z)=F_\mu(\omega_T(z)),\quad\forall z\in\mathbb{C}_+.\tag{3.2}$$

Moreover, the subordination function ω_T extends analytically to the real line with values in $\mathbb{C}_+\cup\mathbb{R}\cup\{\infty\}$.

We now use the Nevanlinna representation for F_μ as in Lemma 2.1 but combining it with equation (3.2) we now obtain

$$\frac{\omega_T(z)-z}{T-1}=\int_{\mathbb{R}}\frac{\mathrm{d}\widehat{\mu}(x)}{x-\omega_T(z)}.\tag{3.3}$$

This equation allows us to establish a lemma that bounds the subordination function on any compact subset of $\mathbb{C}_+$.

Lemma 3.2. *Let μ be a Borel measure satisfying Assumptions 1.1 and let $T>1$ be a real number. For any compact subset $\mathcal{K}$ of $\mathbb{C}_+$, there exists a constant C_0 depending on the subset, T and the measure μ such that,*

$$|\omega_T(z)|\leq C_0,\qquad \text{for all } z\in\mathcal{K}.$$

Proof. Since $\mathcal{K}$ is compact, there is R such that $\mathcal{K}\subset B(0,R)$. Let C be a some constant and let us assume that there is z_0 such that $|\omega_T(z_0)|>C$.
First, we have

$$\left|\frac{\omega_T(z_0)-z_0}{T-1}\right|>\frac{C-R}{T-1}.\tag{3.4}$$

Second, for all x in $\operatorname{supp}\mu=\operatorname{supp}\widehat{\mu}$, we have the following inequalities,

$$|x-\omega_T(z_0)|\geq C-\max(E_+,-E_-).\tag{3.5}$$

Let $C_0 \in \mathbb{R}$ be such that for all $C \geq C_0$,

$$C - \max(E_+, -E_-) \geq \frac{T-1}{C-R}.$$

Then, for all $C \geq C_0$, using equation (3.3) in $z = z_0$, equations (3.4),(3.5) and the fact that $\widehat{\mu}$ is a probability measure, we obtain the wanted contradiction,

$$\frac{C-R}{T-1} < \left| \frac{\omega_T(z_0) - z_0}{T-1} \right| \leq \int_{\operatorname{supp} \mu} \frac{\mathrm{d}\widehat{\mu}(x)}{|x - \omega_T(z_0)|} \leq \frac{C-R}{T-1}.$$

Therefore, for all $z \in \mathcal{K}$, $|\omega_T(z)| \leq C_0$. □

Now, let us show that the hypothesis "ω_T stays at a strictly positive distance from the support of μ" can be expressed in a simpler way. The following lemma is a criterion for ω_T to either touch or either stay away from the lower end-point of the support, but a corresponding criterion can be established for the upper end-point of the support.

Lemma 3.3. *Let $T > 1$, for a measure μ verifying Assumptions 1.1 with exponents greater than one, we have the following equivalences,*

$$I_{\widehat{\mu}}(E_-) \leq \frac{1}{T-1} \iff \exists e_T \in \mathbb{R},\, \omega_T(e_T) = E_- \iff \inf\{\omega_T(E), E \in \mathbb{R} \,|\, \operatorname{Im} \omega_T(E) > 0\} = E_-.$$

Moreover, if it exists, $e_T = E_-^{\boxplus T} = T\left(E_- - F_\mu(E_-)\right) + F_\mu(E_-)$.

Proof. Let us give the proof of the first equivalence. ($\Longrightarrow$) Let us assume $I_{\widehat{\mu}}(E_-) \leq \frac{1}{T-1}$. We want to show that there exists a real e_T such that $\omega_T(e_T) = E_-$. By contradiction, if it is not the case, since ω_T is continuous, there exists a real E such that $\operatorname{Re} \omega_T(E) = E_-$ and $\operatorname{Im} \omega_T(E) > 0$. Taking the imaginary part in equation (3.3) in $z = E$ and dividing by $\operatorname{Im} \omega_T(E)$ leads to

$$\begin{aligned} \frac{1}{T-1} &= I_{\widehat{\mu}}(E) \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}\widehat{\mu}(x)}{(x-E_-)^2 + \operatorname{Im}(\omega_T(E))^2} \\ &< \int_{\mathbb{R}} \frac{\mathrm{d}\widehat{\mu}(x)}{(x-E_-)^2} = I_{\widehat{\mu}}(E_-) \leq \frac{1}{T-1}, \end{aligned}$$

and gives us the wanted contradiction.
($\Longleftarrow$) Let us assume $I_{\widehat{\mu}}(E_-) > \frac{1}{T-1}$. For all z in $\mathbb{C}_+$ such that $\operatorname{Im} \omega_T(z) > 0$, taking the imaginary part of (3.3) yields

$$\frac{1}{T-1}\left(1 - \frac{\operatorname{Im}(z)}{\operatorname{Im} \omega_T(z)}\right) = I_{\widehat{\mu}}(\omega_T(z)). \tag{3.6}$$

Recalling from Theorem 3.1 that $\operatorname{Im} \omega_T(z) \geq \operatorname{Im}(z)$, we can now say that $\forall z \in \mathbb{C}_+$, $I_{\widehat{\mu}}(\omega_T(z)) \leq \frac{1}{T-1}$. Therefore, there is no real e_T such that $\omega_T(e) = E_-$. Indeed, if there was, taking a sequence of complex $(z_n)_{n\in\mathbb{N}}$ in the upper-half plane converging non tangentially to e leads to a contradiction because $\omega_T(z_n)$ would also converge non tangentially to $\omega_T(e_T)$ and therefore $\frac{1}{T-1} \geq I_{\widehat{\mu}}(\omega_T(z_n) \to I_{\widehat{\mu}}(E_-) > \frac{1}{T-1}$.

The second equivalence follows directly from the fact that inside the support of $\mu^{\boxplus T}$, ω_T has a strictly positive imaginary part (because $m_{\mu^{\boxplus T}}(z) = m_\mu(\omega_T(z))$) and from the continuity of $\omega_T : \mathbb{R} \longrightarrow \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$. Last, we will show that if a real e_T is such that $\omega_T(e_T) = E_-$ then, $e_T = E_-^{\boxplus T}$. In order to do so, we recall from the definition of $F_{\mu^{\boxplus T}}$ that $m_{\mu^{\boxplus T}}(z) = m_\mu(\omega_T(z))$. Therefore, taking the imaginary part on both sides, we obtain

$$\operatorname{Im} m_{\mu^{\boxplus T}}(z) = \operatorname{Im} \omega_T(z)\, I_\mu(\omega_T(z)). \tag{3.7}$$

Since, $I_\mu > 0$ and $E_-^{\boxplus T} := \inf\{E \in \mathbb{R} \,|\, \operatorname{Im} m_{\mu^{\boxplus T}}(E) > 0\}$ by the Stieltjes inversion formula, we have $E_-^{\boxplus T} = \inf\{E \in \mathbb{R} \,|\, \operatorname{Im} \omega_T(E) > 0\} = e_T$ the real such that $\omega(e_T) = E_-$. Next, using $\omega_T(E_-^{\boxplus T}) = E_-$ and equation (3.2) we can now express $E_-^{\boxplus T}$ in terms of variables depending only on the initial measure μ as follows,

$$E_-^{\boxplus T} = TE_- - (T-1)F_\mu(E_-) = T\left(E_- - F_\mu(E_-)\right) + F_\mu(E_-). \tag{3.8}$$

This concludes the proof of Lemma 3.3. □

Now that we know how to characterize the fact that the subordination function touches or avoid the support of μ, we can establish a theorem giving the behavior of the density of the free additive convolution semi group depending of the parameter T.

Theorem 3.4. *Let μ be a Borel probability measures satisfying Assumptions 1.1 with exponents greater than one (in fact, we only make assumptions on the exponent of the lower edge side to obtain results on the behavior of the convolution measure at the lower edge side, but similar assumptions on the upper edge lead to corresponding results at the upper edge). Keeping the previous notations,*

(1) *If $T \leq \frac{1}{I_{\widehat{\mu}}(E_-)} + 1$, then $\exists C > 1, \delta > 0$ such that $\forall x \in [E_-^{\boxplus T}; E_-^{\boxplus T} + \delta]$,*

$$\frac{1}{C} \leq \frac{\rho_{\mu^{\boxplus T}}(x)}{(x - E_-^{\boxplus T})^{t_-}} \leq C, \qquad E_-^{\boxplus T} = TE_- - (T-1)F_\mu(E_-).$$

(2) *If $T > \frac{1}{I_{\widehat{\mu}}(E_-)} + 1$, then $\exists C > 1, \delta > 0$ such that $\forall x \in [E_-^{\boxplus T}; E_-^{\boxplus T} + \delta]$,*

$$\frac{1}{C} \leq \frac{\rho_{\mu^{\boxplus T}}(x)}{\sqrt{x - E_-^{\boxplus T}}} \leq C.$$

In that case, $E_-^{\boxplus T}$ is the solution of $|F'_\mu(\omega_T(z)) - 1| = \frac{1}{T-1}$.

Proof. Rewriting (3.3) gives us our key equation for the proof of this Theorem,

$$\omega_T(z) - z = (T-1)m_{\widehat{\mu}}(\omega_T(z)). \tag{3.9}$$

If we are in the first case of Theorem 3.4, using Lemma 3.3, we now know that $\omega_T(E_-^{\boxplus T}) = E_-$ Let $\widehat{\mu}$ be the Borel measure defined in Lemma 2.1. Because $\widehat{\mu}$ also verifies Assumption 1.1 with the sames exponents as μ, according to Lemma 2.2, we can use the result in Lemma 2.3 for the measure $\widehat{\mu}$ in $\omega_T(z)$ (which converges to E_- as z approaches $E_-^{\boxplus T}$) in order to obtain

$$m_{\widehat{\mu}}(\omega_T(z)) = m_{\widehat{\mu}}(E_-) + C(E_- - \omega_T(z))^{t_-}(1 + o(1)). \tag{3.10}$$

Using (3.9) in $z := E_-^{\boxplus T}$ gives us

$$E_- = \omega_T(E_-^{\boxplus T}) = E_-^{\boxplus T} + (T-1)m_{\widehat{\mu}}(E_-).$$

When combined with (3.10), we obtain

$$(\omega_T(z) - E_-) = (z - E_-^{\boxplus T}) + C(T-1)(E_- - \omega_T(z))^{t_-}(1 + o(1)). \tag{3.11}$$

In order to find ω_T in terms of z we will define some new real variables u, v, k, η such that

$$\begin{cases} \omega_T(z) - E_- = u + \mathrm{i}v \\ z - E_-^{\boxplus T} = k + \mathrm{i}\eta. \end{cases} \tag{3.12}$$

We first have $v > 0$ and $\eta > 0$ because we are interested in $z \in \mathbb{C}_+$ and $\operatorname{Im}\omega_T(z) \geq \operatorname{Im} z$. We then need to express v in terms of k as k approaches 0. We still write C instead of $C(T-1)$ because T is fixed. Equation (3.11) can now be written as a set of two equations when taking the real and imaginary parts,

$$\begin{cases} v - \eta = C\operatorname{Im}(-u - \mathrm{i}v)^{t_-}(1 + o(1)) \\ -k + u = C\operatorname{Re}(-u - \mathrm{i}v)^{t_-}(1 + o(1)). \end{cases} \tag{3.13}$$

Let us denote $\theta = \arg(-u - \mathrm{i}v)$ and let us also assume $u > 0$ (we are interested in what happens when we approach the lower-edge of the support by the right). The first equation in (3.13) allows us to bound v as follows,

$$\begin{aligned} |v| &\leq \eta + C_1|u + \mathrm{i}v|^{t_-}|\sin(\theta t_-)| \\ |v| &\leq \eta + C_1|u + \mathrm{i}v|^{t_-} \\ |v| &\leq \eta + C_1|u + v|^{t_-}. \end{aligned}$$

The last inequality holds because $t_- > 1$. Here, we see that if $0 < u \leq v$, having $\eta \to 0$ we would have a contradiction when v goes to 0 since t_- is greater than one (we would have $|v| \leq C|v|^{t_-}$ which is impossible). Therefore, since $u > v$,

$$|v| \leq \eta + C_2|u|^{t_-}. \tag{3.14}$$

From the second equation in (3.13) and using the estimate (3.14), we have $|u-k| = o(|u|)$ and therefore $|u| \sim |k|$. In order to have a lower bound on y, we go back to (3.9) using the notations defined above to obtain

$$\begin{aligned} v &= \eta + (T-1)\int_{\mathbb{R}} \frac{\mathrm{d}\widehat{\mu}(x)v}{|x-(E_- + u + \mathrm{i}v)|^2} \\ &= \eta + (T-1)\int_{E_-}^{E_+} \frac{\mathrm{d}\widehat{\mu}(x)v}{(x-E_- - u)^2 + v^2} \\ &= \eta + (T-1)\int_{0}^{E_+ - E_-} \frac{\mathrm{d}\widehat{\mu}(x)v}{(x-u)^2 + v^2}. \end{aligned}$$

The latter equality is just a change of variables. From now on, we take u, v small enough such that $u+v < E_+ - E_-$.

$$\begin{aligned} v &\geq \eta + (T-1)\int_{u}^{u+v} \frac{\mathrm{d}\widehat{\mu}(x)v}{(x-u)^2 + v^2} \\ &\geq \eta + (T-1)\int_{u}^{u+v} \frac{\mathrm{d}\widehat{\mu}(x)v}{2v^2} \\ &= \eta + \frac{c}{v}\left(\left(v+u\right)^{t_-+1} - u^{t_-+1}\right) \\ &= \eta + \frac{c}{v}u^{t_-+1}\left((1+\frac{v}{u})^{t_-+1} - 1\right) \\ &\geq \eta + \frac{c}{v}u^{t_-+1}c'\frac{v}{u} = \eta + cu^{t_-}. \end{aligned}$$

The second inequality is clear since for all $x \in [u, u+v], (x-u)^2 \leq v^2$. The second equality comes from (3.14) stating $v = o(u)$. We saw when studying the imaginary part that $0 < u < v$ is impossible and the second assumption does not add any problems since we are in the limit $u, v \to 0$.

Combining our previous observations, we obtain the complementing result to the square root behavior. Using the notations from (3.12), we have for k small enough (such that $u + v < E_+ - E_-$) the following results.

$$ck^{t_-} \leq \lim_{\eta\to 0} \operatorname{Im} \omega_T(E_-^{\boxplus T} + k + \mathrm{i}\eta) \leq Ck^{t_-} \tag{3.15}$$

$$ck^{t_-} \leq \lim_{\eta\to 0} \operatorname{Im} m_{\mu^{\boxplus T}}(E_-^{\boxplus T} + k + \mathrm{i}\eta) \leq Ck^{t_-} \tag{3.16}$$

$$ck^{t_-} \leq \rho_{\mu^{\boxplus T}}(E_-^{\boxplus T} + k) \leq Ck^{t_-} \tag{3.17}$$

Here, the estimate (3.17) follows from (3.16) and the Stieltjes inversion formula. The inequality (3.16) derives from (3.15) and from

$$\operatorname{Im} m_{\mu^{\boxplus T}}(z) = \operatorname{Im} \omega_T(z)\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{|x-\omega_T(z)|^2}. \tag{3.18}$$

This equation is the imaginary part of $m_{\mu^{\boxplus T}}(z) = m_\mu(\omega_T(z))$. The latter integral is bounded in a neighborhood of $E_-^{\boxplus T}$ since $t_- > 1$, the exponent on the denominator is smaller than one.

For the second case of Theorem 3.4, using Lemma 3.3, we see that ω_T stays at a strictly positive distance from the lower end-point of the support of μ. We can use [2] and [14] to conclude that we have a square root behavior near the lower end-point of the support. For result on $E_-^{\boxplus T}$, one can find a detailed proof in Proposition 3.15 of [14] A similar proof can be done when z approaches $E_+^{\boxplus T}$. □

Example 3.5. *Through this example, we want to highlight the fact that, even if the exponent of a measure are greater than one we can still have a square root behavior near the edge. First, in order to compute $I_{\widehat{\mu}}(E_-)$ we need to express it with integrals in terms of μ. Let us recall*

$$I_{\widehat{\mu}}(z) = \frac{\operatorname{Im} m_{\widehat{\mu}}(z)}{\operatorname{Im}(z)} = \frac{\operatorname{Im} F_\mu(z) - \operatorname{Im}(z)}{\operatorname{Im}(z)} = \frac{\operatorname{Im} m_\mu(z)}{|m_\mu(z)|^2 \operatorname{Im}(z)} - 1 = \frac{\int \frac{\mathrm{d}\mu}{|x-z|^2}}{|\int \frac{\mathrm{d}\mu}{x-z}|^2} - 1.$$

The first equality comes from the definition of I_μ, the second and third ones come from (2.4) and the last one is just the definition of m_μ. These equalities are valid for all $z \in \mathbb{C}_+$ but if we just keep

$$I_{\widehat{\mu}}(z) = \frac{\int \frac{\mathrm{d}\mu}{|x-z|^2}}{|\int \frac{\mathrm{d}\mu}{x-z}|^2} - 1, \tag{3.19}$$

this equation remains valid for $z \in \mathbb{C}_+ \cup \mathbb{R} \setminus \operatorname{int}\operatorname{supp}\mu$ provided we assume that μ has exponents greater than one.

Let us study a general family of centered probability measures $\mathrm{d}\mu := \frac{1}{C}(1-x)^a(x-(-1))^a\mathrm{d}x\,, a>1$ *supported on* $[-1,1]$ *where* $C := \int_{-1}^1 (1-x^2)^a \mathrm{d}x$ *is chosen such that* $\int_{-1}^1 \mathrm{d}\mu = 1$*. For every* $a>1$*, these measures verify Assumptions 1.1. If we compute* $I_{\widehat{\mu}}(E_-)$ *using* (3.19)*, we find*

$$\begin{aligned} I_{\widehat{\mu}}(-1) &= \frac{C\int_{-1}^1 (1+x)^{a-2}(1-x)^a \mathrm{d}x}{|\int_{-1}^1 (1+x)^{a-1}(1-x)^a\mathrm{d}x|^2} - 1 \\ &= \frac{C\left(\frac{1}{a-1}|(1+x)^{a-1}(1-x)^a|_{-1}^1 + \frac{a}{a-1}\int_{-1}^1(1-x^2)^{a-1}\mathrm{d}x\right)}{|\int_{-1}^1(1-x^2)^{a-1}\mathrm{d}x - \int_{-1}^1 x(1-x^2)^{a-1}\mathrm{d}x|^2} - 1 \\ &= \frac{\frac{aC}{a-1}\int_{-1}^1(1-x^2)^{a-1}\mathrm{d}x}{|\int_{-1}^1(1-x^2)^{a-1}\mathrm{d}x|^2} - 1 \\ &= \frac{aC}{(a-1)\int_{-1}^1 (1-x^2)^{a-1}\mathrm{d}x} - 1 = \frac{a+1}{2a^2-a-1}. \end{aligned}$$

Where the last equality comes from the computation of C *and of the integral at the denominator. Therefore, if* $1 < a < \frac{T+\sqrt{T^2+8T}}{4}$ *(or equivalently* $T > \frac{2a^2}{a+1}$*), then* $I_{\widehat{\mu}}(E_-) > \frac{1}{T-1}$ *and* $\mu^{\boxplus T}$ *has a square root behavior on its lower edge even though* μ *verifies Assumptions 1.1 with exponent greater than one. However, if* $a \geq \frac{T+\sqrt{T^2+8T}}{4}$ *(or equivalently* $1 < T \leq \frac{2a^2}{a+1}$*), then* $I_{\widehat{\mu}}(E_-) \leq \frac{1}{T-1}$ *and* $\mu^{\boxplus T}$ *has a a-power behavior on its lower edge.*

3.2. **Free multiplicative convolution semi-group.** After Subsection 3.1, we know how to determine the edge behavior of the additive semi-group knowing the edge behavior of the initial measure. Let us now look at the multiplicative version.

Let μ be a measure satisfying Assumptions 1.2 and let $T>1$ be a certain real. Let us also define the Ψ-transform and the η-transform as follows,

Definition 3.1 (See [4] or [12] for further reading on these transforms)**.** *Let* ν *be a Borel measure on* $\mathbb{R}_+$*. The* Ψ*-transform is the analytic function on* $\mathbb{C}\setminus\mathbb{R}_+$ *defined as follows. For any* z *in* $\mathbb{C}\setminus\mathbb{R}_+$

$$\Psi_\nu(z) := \int_{\mathbb{R}_+} \frac{zx}{1-zx}\mathrm{d}\nu(x),$$

and we define $\eta_\nu(z) := \Psi_\nu(z)/(1+\Psi_\nu(z))$.

In [4], the following subordination theorem is given.

Theorem 3.6. *For every* $T>1$*, there exists a subordination function* $\omega_T : \mathbb{C}\setminus\mathbb{R}_+ \longrightarrow \mathbb{C}\setminus\mathbb{R}_+$ *such that*

(1) $\lim_{z\nearrow 0}\omega_T(z) = 0$,
(2) ω_T *maps* $\mathbb{C}_+$ *into* $\mathbb{C}_+$ *and for every* $z\in\mathbb{C}_+$, $\omega_T(\overline{z}) = \overline{\omega_T(z)}$ *and* $\arg(\omega_T(z)) \geq \arg(z)$.
(3) *For every* $z\in\mathbb{C}\setminus\mathbb{R}_+$,

$$\eta_\mu(\omega_T(z)) = \eta_{\mu^{\boxtimes T}}(z) \qquad \text{and} \qquad z = \left(\frac{\omega_T(z)}{\eta_\mu(\omega_T(z))}\right)^{T-1}\omega_T(z). \tag{3.20}$$

In order to make this look more alike the additive case (especially for the first two claims), we will reformulate this result. For every measure ν verifying Assumptions 1.2 and every $z\in\mathbb{C}\setminus\mathbb{R}_+$, from Definition 2.1, we notice that

$$M_\nu(z) := \frac{1}{\eta_\nu(\frac{1}{z})} \quad \text{and we therefore define} \quad \Omega_T(z) := \frac{1}{\omega_T(\frac{1}{z})}. \tag{3.21}$$

Note that it is only a matter of definition : we can start by defining M, prove the existence of subordination function Ω and then define η,ω using M,Ω instead. In [4], we have Theorem 2.6 that gives us the existence of subordination functions for all $T>1$.

Theorem 3.7. *For every* $T>1$*, there exists a subordination function* $\Omega_T : \mathbb{C}\setminus\mathbb{R}_+ \longrightarrow \mathbb{C}\setminus\mathbb{R}_+$ *such that*

(1) $\lim_{z\to\infty, z<0}\Omega_T(z) = \infty$,
(2) Ω_T *maps* $\mathbb{C}_+$ *into* $\mathbb{C}_+$ *and for every* $z\in\mathbb{C}_+$, $\Omega_T(\overline{z}) = \overline{\Omega_T(z)}$ *and* $\arg(\Omega_T(z)) \geq \arg(z)$.
(3) *For every* $z\in\mathbb{C}\setminus\mathbb{R}_+$,

$$M_\mu(\Omega_T(z)) = M_{\mu^{\boxtimes T}}(z) \qquad \text{and} \qquad \Omega_T(z)^T = zM_{\mu^{\boxtimes T}}(z)^{T-1}. \tag{3.22}$$

Proof. Assuming Theorem 3.6 to be true, the first two claims are clear given the definition of Ω_T in (3.21). The third one is also easily obtained using equation (3.20) at $\frac{1}{z}$. ☐

Let us first establish a link between the imaginary part of Ω_T and the density of $\mu^{\boxtimes T}$ using the Stieltjes transform. Let us notice

$$1 - \frac{1}{1 + z m_{\mu^{\boxtimes T}}(z)} = M_{\mu^{\boxtimes T}}(z) = M_\mu(\Omega_T(z)) = 1 - \frac{1}{1 + \Omega_T(z) m_\mu(\Omega_T(z))}.$$

We now have $z m_{\mu^{\boxtimes T}}(z) = \Omega_T(z) m_\mu(\Omega_T(z))$. Let now $z := e + \mathrm{i}\eta$ for some strictly positive numbers e, η (the support is still strictly included in $\mathbb{R}_+$ and we will take the limit as η goes to 0 by the right).

$$\begin{aligned}
\operatorname{Im} m_{\mu^{\boxtimes T}}(z) &= \operatorname{Im}\left(\frac{\Omega_T(z)}{z}\right) \int_{\mathbb{R}_+} \frac{x - \operatorname{Re}\Omega_T(z)}{|x - \Omega_T(z)|^2} \mathrm{d}\mu(x) + \operatorname{Re}\left(\frac{\Omega_T(z)}{z}\right) \operatorname{Im}\Omega_T(z) \int_{\mathbb{R}_+} \frac{\mathrm{d}\mu(x)}{|x - \Omega_T(z)|^2} \\
&= \operatorname{Im}\left(\frac{\Omega_T(z)}{z}\right) J_\mu(\Omega_T(z)) + \int_{\mathbb{R}_+} \frac{\mathrm{d}\mu(x)}{|x - \Omega_T(z)|^2} \left[\operatorname{Re}\left(\frac{\Omega_T(z)}{z}\right) \operatorname{Im}\Omega_T(z) - \operatorname{Im}\left(\frac{\Omega_T(z)}{z}\right) \operatorname{Re}\Omega_T(z)\right] \\
&= \operatorname{Im}\left(\frac{\Omega_T(z)}{z}\right) J_\mu(\Omega_T(z)) + \operatorname{Im} z \left[\frac{|\Omega_T(z)|^2}{|z|^2} \int_{\mathbb{R}_+} \frac{\mathrm{d}\mu(x)}{|x - \Omega_T(z)|^2}\right],
\end{aligned}$$

where J_μ is defined in Definition 2.1.
Now, we can get rid of the $\operatorname{Im} z$ term because we are interested in $\lim_{\eta \searrow 0} \operatorname{Im} m_{\mu^{\boxtimes T}}(e + \mathrm{i}\eta)$ and the terms inside the brackets are bounded by above because the exponents are greater than one and $E_- > 0$. Indeed, $|\Omega_T(z)|$ and $|z|$ stay bounded uniformly on any compact subset of $\mathbb{C} \cup \mathbb{R}_+^*$ and if Ω_T stays away from $\operatorname{supp}\mu$, then the integral term is bounded by above by $1/\operatorname{dist}(\Omega_T, \operatorname{supp}\mu)^2$ and otherwise, it is bounded by $\int_{\mathbb{R}_+} \frac{\mathrm{d}\mu(x)}{|x - E_-|^2} < \infty$. We now have

$$\lim_{\eta \searrow 0} \operatorname{Im} m_{\mu^{\boxtimes T}}(e + \mathrm{i}\eta) = \lim_{\eta \searrow 0} \operatorname{Im}\left(\frac{\Omega_T(z)}{z}\right) J_\mu(\Omega_T(z)) = \lim_{\eta \searrow 0} \frac{\operatorname{Im}\Omega_T(e + \mathrm{i}\eta)}{e} J_\mu(\Omega_T(e + \mathrm{i}\eta)). \tag{3.23}$$

This equation gives us a key link between the density of the free multiplicative convolution and the imaginary part of the subordination function.

If we still denote $\tilde{\mu}$ the unique measure defined in Lemma 2.1, we have the following key equation for all z in $\mathbb{C}_+$,

$$\frac{\Omega_T(z)}{z} = H_\mu(z)^{T-1} = (1 + m_{\tilde{\mu}}(\Omega_T(z)))^{T-1}. \tag{3.24}$$

Now that we have this key equation, we can state the main result of this subsection.

Theorem 3.8. *For a measure μ be a Borel probability measure satisfying Assumptions 1.2 with exponents t_-, t_+ greater than one, for every $T > 1$, $\mu^{\boxtimes T}$ is supported on a single interval denoted $[E_-^{\boxtimes T}; E_+^{\boxtimes T}] \subset \mathbb{R}_+^*$. Moreover, we have*

(1) If the subordination function touches the support, we have a t_--power behavior near the lower edge. More precisely, there exists $C > 1$, $\delta > 0$ such that for every x in $[E_-^{\boxtimes T}; E_-^{\boxtimes T} + \delta]$,

$$\frac{1}{C} \leq \frac{\rho_{\mu^{\boxtimes T}}(x)}{(x - E_-^{\boxtimes T})^{t_-}} \leq C, \qquad E_-^{\boxtimes T} = \frac{E_-^T}{M_\mu(E_-)^{T-1}} = \left(\frac{E_-}{M_\mu(E_-)}\right)^T M_\mu(E_-).$$

(2) If the subordination function stays away from the support, we have a square root behavior near the lower edge. More precisely, there exists $C > 1$, $\delta > 0$ such that for every x in $[E_-^{\boxtimes T}; E_-^{\boxtimes T} + \delta]$,

$$\frac{1}{C} \leq \frac{\rho_{\mu^{\boxtimes T}}(x)}{\sqrt{x - E_-^{\boxtimes T}}} \leq C.$$

Proof. See [12], Theorem 3.3 for the proof of the second case. Let us focus on the first case : $\Omega_T(E_-^{\boxtimes T}) = E_-$. We can still use Lemma 2.2 and Plemelj's Lemma (Lemma 2.3) in order to approximate $m_{\tilde{\mu}}$ near E_- as follows

$$m_{\tilde{\mu}}(\Omega_T(z)) = m_{\tilde{\mu}}(E_-) + C(E_- - \Omega_T(z))^{t_-}(1 + o(1)). \tag{3.25}$$

Using equation (3.24) in $z = E_-^{\boxplus T}$ and combining it with the upper equation gives us

$$\frac{\Omega_T(z)}{z} = \left(\left(\frac{E_-}{E_-^{\boxtimes T}}\right)^{\frac{1}{T-1}} + C(E_- - \Omega_T(z))^{t_-}(1 + o(1))\right)^{T-1} = \frac{E_-}{E_-^{\boxtimes T}} \left(1 + C(E_- - \Omega_T(z))^{t_-}(1 + o(1))\right)^{T-1}. \tag{3.26}$$

Because we are in case 1 of Theorem 3.8, $\Omega_T(z) - E_-$ goes to 0 as z converges toward $E_-^{\boxtimes T}$ and we can therefore develop the right-hand side as follows,

$$\begin{aligned}
\frac{\Omega_T(z)}{z} &= \frac{E_-}{E_-^{\boxtimes T}}\left(1+(T-1)C(E_- - \Omega_T(z))^{t_-}(1+o(1))\right)\\
\frac{\Omega_T(z)}{E_-^{\boxtimes T}}\frac{1}{1+\frac{z-E_-^{\boxtimes T}}{E_-^{\boxtimes T}}} - \frac{E_-}{E_-^{\boxtimes T}} &= C(E_- - \Omega_T(z))^{t_-}(1+o(1))\\
\frac{\Omega_T(z)-E_-}{E_-^{\boxtimes T}} &= \Omega_T(z)\frac{z-E_-^{\boxtimes T}}{(E_-^{\boxtimes T})^2}(1+o(1)) + C(E_- - \Omega_T(z))^{t_-}(1+o(1)).
\end{aligned}$$

Multiplying both sides by $E_-^{\boxtimes T}$ and using

$$\Omega_T(z)(z-E_-^{\boxtimes T}) = (\Omega_T(z)-E_-)(z-E_-^{\boxtimes T}) + E_-(z-E_-^{\boxtimes T}) = o(z-E_-^{\boxtimes T}) + E_-(z-E_-^{\boxtimes T}),$$

gives us our key equation for the multiplicative semi-group,

$$\Omega_T(z) - E_- = C(z-E_-^{\boxtimes T})(1+o(1)) + C(E_- - \Omega_T(z))^{t_-}(1+o(1)). \tag{3.27}$$

As in the additive case, let us introduce the notations

$$\begin{cases} \Omega_T(z) - E_- =: u + \mathrm{i}v \\ \quad z - E_-^{\boxtimes} =: k + \mathrm{i}\eta. \end{cases} \tag{3.28}$$

in order to rewrite equation (3.27) as follows

$$(u+\mathrm{i}v)(1+o(1)) = C(k+\mathrm{i}\eta)(1+o(1)) + C(-u-\mathrm{i}v)^{t_-}(1+o(1)). \tag{3.29}$$

Here, we denoted two constants with the symbol C even if they might not be the same. Separating the imaginary and the real part gives us

$$\begin{cases} o(u) + u = Ck + o(k) + C|u+\mathrm{i}v|^{t_-}\cos(t_-\theta)(1+o(1)) \\ o(v) + v = C\eta + o(\eta) + C|u+\mathrm{i}v|^{t_-}\sin(t_-\theta)(1+o(1)), \end{cases} \tag{3.30}$$

where we denoted $\theta := \arg(-u-\mathrm{i}v)$.

We want to bound v on both sides. First, the second line of (3.30) yields

$$\begin{aligned}
|v| &\le C\eta + C|u+\mathrm{i}v|^{t_-}\\
&\le C\eta + C|u+v|^{t_-}\\
&\le C\eta + C|u|^{t_-}.
\end{aligned}$$

The constants may vary from one line to another. These inequality are very similar to those of the additive case. The last one is obtained after noticing that we cannot have $0 < u < v$ because we would obtain a contradiction when $\eta \to 0$. This inequality also allows us to use the first line of (3.30) in order to show $u \sim Ck$. Therefore, we have obtained the following upper bound,

$$|v| \le C\eta + Ck^{t_-}. \tag{3.31}$$

In order to find the lower bound, we go back to (2.2) using the subordination equation given in Theorem 3.7

$$1 + m_{\tilde\mu}(\Omega_T(z)) = \frac{m_{\mu^{\boxtimes T}}(z)}{\frac{\Omega_T(z)}{z} + \Omega_T(z)m_{\mu^{\boxtimes T}}(z)}.$$

Taking the imaginary part and using $zm_{\mu^{\boxtimes T}}(z) = \Omega_T(z)m_\mu(\Omega_T(z))$, after several lines of calculations, we find

$$\operatorname{Im}\Omega_T(z)\; I_{\tilde\mu}(\Omega_T(z)) = \frac{\operatorname{Im}\Omega_T(z)\left[I_\mu(\Omega_T(z)) - |m_\mu(\Omega_T(z))|^2\right]}{|1+zm_{\mu^{\boxtimes T}}(z)|^2}.$$

We can now write, using the notations introduced in (3.28),

$$\frac{v\left(I_\mu(\Omega_T(z)) - |m_\mu(\Omega_T(z))|^2\right)}{|1+zm_{\mu^{\boxtimes T}}(z)|^2} = v\int_{E_-}^{E_+}\frac{\mathrm{d}\widehat{\mu}(x)}{(x-E_- - u)^2 + v^2}.$$

Because $I_\mu(\Omega_T(z))$, $m_\mu(\Omega_T(z)$ and $|1+zm_{\mu^{\boxtimes T}}(z)|^2$ are all bounded above and below by strictly positive constants, we can rewrite the above equation as follows,

$$\begin{aligned} v &\geq Cv\int_{E_-}^{E_+}\frac{\mathrm{d}\widehat{\mu}(x)}{(x-E_--u)^2+v^2} \\ &\geq Cu^{t_-}. \end{aligned}$$

The second inequality is valid as long as u, v are small enough. The details of this inequality are done in the additive case (see Proof of Theorem 3.4). We now have an upper bound for v,

$$v \geq Ck^{t_-}. \tag{3.32}$$

Using equation (3.23), because $J_\mu(\Omega_T(z))$ is bounded above and below by two strictly positive constants (the exponents are greater than one), we now have the following result,

$$Ck^{t_-} \leq \lim_{\eta\searrow 0}\operatorname{Im} m_{\mu^{\boxtimes T}}(E_-^{\boxtimes}+k+\mathrm{i}\eta) \leq Ck^{t_-}. \tag{3.33}$$

Thanks to the Stieltjes inversion formula given in 2.1, this ends the proof of the edge behavior. We now just have to prove the relation for $E_-^{\boxtimes T}$. Using the hypothesis of case 1 of the theorem gives us $\Omega_T(E_-^{\boxtimes T}) = E_-$ and using this equality in (3.22) gives us $E_-^{\boxtimes T} = E_-^T/M_\mu(E_-)^{T-1}$ and concludes the proof. □

For several reasons, we were not able to find an equivalent criterion for the subordination function to stay away from the support in a general case. The case $T=2$ can be done using $|z|I_{\widehat{\mu}}(\Omega_2(z)) \leq 1$ for every z in $\mathbb{C}_+$ (see [12] for the details) and with a similar proof as Lemma 3.3. However, even in the case T=2, we find a criteria that involve $E_-^{\boxtimes T}$ which is not a known quantity.

4. Free additive convolution of two distinct measures

As in the beginning of the previous sections, we give a theorem corresponding to Theorem 3.1 (a detailed proof is given in [6]).

Theorem 4.1 (Theorem 4.1 in [6])**.** *Let us keep the notations and assumptions on μ_α and μ_β from the previous section. Then there exist two unique analytic functions mapping the upper plane to itself called subordination functions, denoted ω_α and ω_β, such that*

$$\lim_{\eta\to\infty}\frac{\omega_\alpha(\mathrm{i}\eta)}{\mathrm{i}\eta} = 1, \qquad \operatorname{Im}(\omega_\alpha(z)) \geq \operatorname{Im}(z) \quad \forall z\in\mathbb{C}_+, \tag{4.1}$$

$$\lim_{\eta\to\infty}\frac{\omega_\beta(\mathrm{i}\eta)}{\mathrm{i}\eta} = 1, \qquad \operatorname{Im}(\omega_\beta(z)) \geq \operatorname{Im}(z) \quad \forall z\in\mathbb{C}_+, \tag{4.2}$$

$$\omega_\alpha(z)+\omega_\beta(z)-z = F_{\mu_\alpha}(\omega_\beta(z)) = F_{\mu_\beta}(\omega_\alpha(z)) = F_{\mu_\alpha\boxplus\mu_\beta}(z), \quad \forall z\in\mathbb{C}_+. \tag{4.3}$$

Remark 4.1. *Both ω_α and ω_β extend continuously to the real line with values in $\mathbb{C}_+\cup\mathbb{R}\cup\{\infty\}$ (see Theorem 3.3 in* [7]*).*

Remark 4.2. *In the setup of Assumptions 1.1, for any compact subsets K of $\mathbb{C}_+$, $\sup_{z\in K}|\omega_\alpha(z)|, \sup_{z\in K}|\omega_\beta(z)| < C$. This claim follows directly from Lemma 3.2 of* [1].

From now on, we will be interested in the free additive convolution of two distinct measures verifying Assumptions 1.1. We will keep intuitive notations : our two measures will be denoted μ_α and μ_β, their support are respectively $[E_-^\alpha; E_+^\alpha]$ and $[E_-^\beta; E_+^\beta]$. They have respectively t_-^α and t_-^β power behaviors on their lower edge side and t_+^α and t_+^β on their upper edge side which we no longer assume to be greater than one. We still have the Nevanlinna representation for μ_α and μ_β with measures we will denote $\widehat{\mu}_\alpha$ and $\widehat{\mu}_\beta$ that have the same support as their original measure. More precisely, for every ω in $\mathbb{C}_+$, we have the following equations,

$$\begin{aligned} F_{\mu_\alpha}(\omega)-\omega &= \int_{\operatorname{supp}\mu_\alpha}\frac{\mathrm{d}\widehat{\mu}_\alpha(x)}{x-\omega}, \\ F_{\mu_\beta}(\omega)-\omega &= \int_{\operatorname{supp}\mu_\beta}\frac{\mathrm{d}\widehat{\mu}_\beta(x)}{x-\omega}. \end{aligned} \tag{4.4}$$

Combined with (4.3) and using the notation $m_\lambda(z) := \int_{\operatorname{supp}\lambda}\frac{\mathrm{d}\lambda(x)}{x-z}$ from Definition 2.1, we obtain our crucial result similar to (3.3) :

$$\begin{aligned} \omega_\alpha(z)-z &= m_{\widehat{\mu}_\alpha}(\omega_\beta(z)), \\ \omega_\beta(z)-z &= m_{\widehat{\mu}_\beta}(\omega_\alpha(z)). \end{aligned} \tag{4.5}$$

We will start by proving a theorem that gives the power behavior of the free additive convolution once we know which subordination function touches the support.

Theorem 4.2. *For two Borel probablity measures μ_α and μ_β verifying Assumptions 1.1, their free additive convolution measure $\mu_\alpha \boxplus \mu_\beta$ is also supported on a single interval denoted $[E_-^\boxplus; E_+^\boxplus]$. Moreover, we have the following results.*

(1) *If both subordination functions stay away from the lower edge : $\omega_\alpha(E_-^\boxplus) < E_-^\beta$ and $\omega_\beta(E_-^\boxplus) < E_-^\alpha$, then we have a square root behavior near the lower edge point of the support. More precisely, there exists $C > 1$ and $\delta > 0$ such that for every $x \in [E_-^\boxplus; E_-^\boxplus + \delta]$,*

$$\frac{1}{C} \leq \frac{\rho_{\mu_\alpha \boxplus \mu_\beta}(x)}{\sqrt{x - E_-^\boxplus}} \leq C.$$

(2) *If only one subordination function touches the lower endpoint of the support : $\omega_i(E_-^\boxplus) = E_-^j$ and $\omega_j(E_-^\boxplus) < E_-^i$, (with $(\alpha, \beta) = (i, j)$ or $(\alpha, \beta) = (j, i)$), then we have a t_-^j power behavior near the lower edge point of the support. More precisely, there exists $C > 1, \delta > 0$ such that for all $x \in \left[E_-^\boxplus; E_-^\boxplus + \delta\right]$,*

$$\frac{1}{C} \leq \frac{\rho_{\mu_\alpha \boxplus \mu_\beta}(x)}{(x - E_-^\boxplus)^{t_-^j}} \leq C.$$

(3) *If both subordination functions touch the support : $\omega_\alpha(E_-^\boxplus) = E_-^\beta$ and $\omega_\beta(E_-^\boxplus) = E_-^\alpha$, then we must have $t_-^\alpha = t_-^\beta =: t_-$ and we have a t_- power behavior near the lower edge point of the support. More precisely, there exists $C > 1, \delta > 0$ such that for all $x \in \left[E_-^\boxplus; E_-^\boxplus + \delta\right]$,*

$$\frac{1}{C} \leq \frac{\rho_{\mu_\alpha \boxplus \mu_\beta}(x)}{(x - E_-^\boxplus)^{t_-}} \leq C, \qquad \text{with} \quad E_-^\boxplus = F_{\mu_\alpha}(E_-^\alpha) - E_-^\beta - E_-^\alpha = F_{\mu_\beta}(E_-^\beta) - E_-^\beta - E_-^\alpha.$$

Proof. We will need Holder's inequality throughout the proof. Let us give a simple version of Holder's inequality. For any $t \geq 1$,

$$|\sum_{k=1}^n a_k|^t \leq (\sum_{k=1}^n |a_k|)^t \leq n^{t-1}(\sum_{k=1}^n |a_k|^t), \quad \forall a_k > 0. \tag{4.6}$$

Case 1 has already been proven in [2] even though in there it was that the exponents are smaller than one, the necessary assumption was for the subordination function to stay away of the support (distance lemma).

In order to simplify the proof of Case 2, let us assume that $\omega_\alpha(E_-^\boxplus) = E_-^\beta$ while $\omega_\beta(E_-^\boxplus) < E_-^\alpha$ (the other case is exactly the same proof provided we exchange all α and β occurrences). This involves that t_-^β is bigger than one, otherwise, as shown in [2], ω_α would stay away from $\operatorname{supp} \mu_\beta$. Lemma 2.2 tells us that the measure $\widehat{\mu}_\beta$ also has a power behavior near its lower end point (and verifies Assumptions 1.1) with exponents t_-^β. Using Lemma 2.3 with the measure $\widehat{\mu}_\beta$ in $\omega_\alpha(z)$ gives us the following approximation as z approaches $E_-^\boxplus$,

$$m_{\widehat{\mu}_\beta}(\omega_\alpha(z)) = m_{\widehat{\mu}_\beta}(E_-^\beta) + C(E_-^\beta - \omega_\alpha(z))^{t_-^\beta}(1 + o(1)).$$

Using equation (4.5) to rewrite the terms in $m_{\widehat{\mu}}$ gives us our first relation between the two subordination functions

$$\omega_\beta(z) - \omega_\beta(E_-^\boxplus) = z - E_-^\boxplus + C(E_-^\beta - \omega_\alpha(z))^{t_-^\beta}(1 + o(1)). \tag{4.7}$$

Equation (4.3) then yields

$$\begin{aligned} \omega_\alpha(z) - z &= F_{\mu_\alpha}(\omega_\beta(z)) - \omega_\beta(z) \\ &= F_{\mu_\alpha}(\omega_\beta(E_-^\boxplus)) + \left(F'_{\mu_\alpha}(\omega_\beta(E_-^\boxplus)) - 1\right)\left(\omega_\beta(z) - \omega_\beta(E_-^\boxplus)\right) + o\left(\omega_\beta(z) - \omega_\beta(E_-^\boxplus)\right) - \omega_\beta(E_-^\boxplus). \end{aligned} \tag{4.8}$$

Here, we used the fact that, because $\omega_\beta(z)$ stays away from the support of μ_α when z is in a neighborhood of $E_-^\boxplus$, we can use the Taylor expansion of F_{μ_α} in $\omega_\beta(E_-^\boxplus)$.

Using the Nevannlina representation now gives us $F_{\mu_\alpha}(\omega) - \omega = m_{\widehat{\mu}_\alpha}(\omega)$ which is differentiable in ω and leads to

$$F'_{\mu_\alpha}(\omega_\beta(E_-^\boxplus)) - 1 = \int_{\operatorname{supp} \mu_\alpha} \frac{\mathrm{d}\widehat{\mu}_\alpha(x)}{(x - \omega_\beta(E_-^\boxplus))^2} = I_{\widehat{\mu}_\alpha}(\omega_\beta(E_-^\boxplus)) =: C' > 0, \tag{4.9}$$

where the last equality comes from the fact that $\omega_\beta(E_-^\boxplus)$ is real.

Using $F_{\mu_\alpha}(\omega_\beta(E_-^\boxplus)) - \omega_\beta(E_-^\boxplus) = \omega_\alpha(E_-^\boxplus) - E_-^\boxplus$, in equation (4.8) and using the notation introduced in (4.9) gives us our wanted result,

$$\omega_\alpha(z) - \omega_\alpha(E_-^\boxplus) = z - E_-^\boxplus + C'\left(\omega_\beta(z) - \omega_\beta(E_-^\boxplus)\right)(1 + o(1)). \tag{4.10}$$

Let us introduce the notations

$$\text{(4.11)}\qquad \begin{cases} \omega_\alpha(z) - \omega_\alpha(E_-^{\boxplus}) = u_\alpha + \mathrm{i}v_\alpha \\ \omega_\beta(z) - \omega_\beta(E_-^{\boxplus}) = u_\beta + \mathrm{i}v_\beta \\ \qquad\qquad z - E_-^{\boxplus} = \quad k + \mathrm{i}\eta, \end{cases}$$

where the new variables on the right-hand side are all real. Here, we are only interested in the case where k, η are positive. Indeed η being strictly positive is to ensure that the subordination functions are evaluated in $\mathbb{C}_+$ and k is positive because we are interested in the case where z approaches $E_-^{\boxplus}$ by the right hand side. Moreover, since the subordination functions take values in $\mathbb{C}_+$, v_α and v_β are also strictly positive. Finally, we also have u_α and u_β positive because the subordination functions approaches the lower edge point of the support of the other measure by the right hand side as z approaches $E_-^{\boxplus}$ by the right hand side.

Using those notations in equations (4.7) and (4.10) gives us the following system of equations

$$\text{(4.12)}\qquad \begin{cases} \mathrm{i}v_\alpha + u_\alpha = \quad k + \mathrm{i}\eta + C'\left(u_\beta + \mathrm{i}v_\beta\right)(1 + o(1)) \\ \mathrm{i}v_\beta + u_\beta = k + \mathrm{i}\eta + C\left(u_\alpha + \mathrm{i}v_\alpha\right)^{t_-^\beta}(1 + o(1)), \end{cases}$$

for $u_\alpha, v_\alpha, u_\beta, v_\beta$ small enough. We will rewrite this system as follows :

$$\text{(4.13)}\qquad \begin{cases} \mathrm{i}v_\alpha + u_\alpha = \qquad\qquad k + \mathrm{i}\eta + C'\left(u_\beta + \mathrm{i}v_\beta\right)(1 + o(1)) \\ \mathrm{i}v_\beta + u_\beta = k + \mathrm{i}\eta + C\left(k + \mathrm{i}\eta + C'\left(u_\beta + \mathrm{i}v_\beta\right)\right)^{t_-^\beta}(1 + o(1)). \end{cases}$$

Using Holder's inequality after taking the imaginary part of the second equation yields

$$\begin{aligned} |v_\beta| &\leq \eta + C|k + \mathrm{i}\eta|^{t_-^\beta} + C|u_\beta + \mathrm{i}v_\beta|^{t_-^\beta} \\ &\leq \eta + C|k|^{t_-^\beta} + C|\eta|^{t_-^\beta} + C|u_\beta + v_\beta|^{t_-^\beta}. \end{aligned}$$

If we had $u_\beta < v_\beta$, taking the limit in the previous inequality would lead to a contradiction. Therefore, we can assume $v_\beta < u_\beta$ and this leads to

$$\text{(4.14)}\qquad \begin{aligned} |v_\beta| &\leq \eta + C|k|^{t_-^\beta} + C|\eta|^{t_-^\beta} + C u_\beta^{t_-^\beta} \\ &\leq c\eta + C|k|^{t_-^\beta}. \end{aligned}$$

Even if being denoted the same, the constants change from one line to another. The last inequality requires $u_\beta \sim k$. This is verified by taking the real part of both equations of system (4.12) (we also obtain $u_\alpha \sim c\,k$ for a certain c>1).

Using the imaginary part of equation (4.5) we obtain a lower-bound on v_β provided we take u_α, v_α small enough such that $u_\alpha + v_\alpha \leq E_+^\beta - E_-^\beta$,

$$\text{(4.15)}\qquad \begin{aligned} v_\beta &= \eta + v_\alpha \int_{E_-^\beta}^{E_+^\beta} \frac{\mathrm{d}\widehat{\mu}_\beta(x)}{(x - E_-^\beta - u_\alpha)^2 + v_\alpha^2} \\ &= \eta + v_\alpha \int_0^{E_+^\beta - E_-^\beta} \frac{\mathrm{d}\widehat{\mu}_\beta(x)}{(x - u_\alpha)^2 + v_\alpha^2} \\ &\geq \eta + v_\alpha \int_{u_\alpha}^{u_\alpha + v_\alpha} \frac{\mathrm{d}\widehat{\mu}_\beta(x)}{(x - u_\alpha)^2 + v_\alpha^2} \\ &\geq \eta + \frac{1}{2v_\alpha} \int_{u_\alpha}^{u_\alpha + v_\alpha} \mathrm{d}\widehat{\mu}_\beta(x) \\ &\geq \eta + \frac{1}{2v_\alpha}\left((u_\alpha + v_\alpha)^{t_-^\beta} - u_\alpha^{t_-^\beta}\right) \\ &\geq \eta + c\frac{u_\alpha^{t_-^\beta + 1}}{v_\alpha}\left((1 + \frac{v_\alpha}{u_\alpha})^{t_-^\beta + 1} - 1\right) \\ &\geq \eta + \frac{c}{v_\alpha} u_\alpha^{t_-^\beta + 1} c' \frac{v_\alpha}{u_\alpha} \\ &\geq \eta + C u_\alpha^{t_-^\beta} \\ &\geq \eta + C k^{t_-^\beta}. \end{aligned}$$

Combining this lower bound with the upper bound proven in (4.14) gives us

$$c\,k^{t_-^\beta} \le \lim_{\eta \searrow 0} \operatorname{Im} \omega_\beta(E_-^{\boxplus} + k + \mathrm{i}\eta) \le C\,k^{t_-^\beta}. \tag{4.16}$$

Recalling $\operatorname{Im} m_{\mu_\alpha \boxplus \mu_\beta}(z) = \operatorname{Im} \omega_\beta(z) I_{\widehat{\mu}_\alpha}(\omega_\beta(z))$ and using the fact that because ω_β does not approach the support of μ_α, $I_{\widehat{\mu}_\alpha}(\omega_\beta(z))$ stays bounded near $E_-^{\boxplus}$, we can now conclude that for x close enough to $E_-^{\boxplus}$,

$$c \le \frac{\rho_{\mu_\alpha \boxplus \mu_\beta}(x)}{(x - E_-^{\boxplus})^{t_-^\beta}} \le C. \tag{4.17}$$

Let us now prove Case 3 of the theorem. Let us assume $\omega_\alpha(E_-^{\boxplus}) = E_-^\beta$ and $\omega_\alpha(E_-^{\boxplus}) = E_-^\beta$. Here, we know that both exponents are greater than one, otherwise, we would be able to establish a distance lemma that bounded by below the distance between the subordination functions and the support. We will later obtain a contradiction if the two exponents are not the same.
Using Lemma 2.3 for $\widehat{\mu}_\alpha$ and $\widehat{\mu}_\beta$ and recalling equations (4.5) gives us

$$\omega_\alpha(z) - z = m_{\widehat{\mu}_\alpha}(E_-^\alpha) + C_\alpha (E_-^\alpha - \omega_\beta(z))^{t_-^\alpha}(1 + o(1)), \tag{4.18}$$

$$\omega_\beta(z) - z = m_{\widehat{\mu}_\beta}(E_-^\beta) + C_\beta (E_-^\beta - \omega_\alpha(z))^{t_-^\beta}(1 + o(1)). \tag{4.19}$$

Using equation (4.5) in $z = E_-^{\boxplus}$ allows us to rewrite the two previous equations as follows

$$\omega_\alpha(z) - E_-^\beta = z - E_-^{\boxplus} + C_\alpha (E_-^\alpha - \omega_\beta(z))^{t_-^\alpha}(1 + o(1)), \tag{4.20}$$

$$\omega_\beta(z) - E_-^\alpha = z - E_-^{\boxplus} + C_\beta (E_-^\beta - \omega_\alpha(z))^{t_-^\beta}(1 + o(1)). \tag{4.21}$$

As in the proof of the previous case, we will introduce several notations to simplify our previous system. Let us denote

$$\begin{cases} \omega_\alpha(z) - E_-^\beta = u_\alpha + \mathrm{i}v_\alpha \\ \omega_\beta(z) - E_-^\alpha = u_\beta + \mathrm{i}v_\beta \\ \quad\;\; z - E_-^{\boxplus} = \quad k + \mathrm{i}\eta. \end{cases} \tag{4.22}$$

Using (4.22) to rewrite (4.20) and (4.21) gives us, after rearrangement,

$$u_\alpha + \mathrm{i}v_\alpha = k + \mathrm{i}\eta + C_\alpha \left(-k - \mathrm{i}\eta - C_\beta(-u_\alpha - \mathrm{i}v_\alpha)^{t_-^\beta}(1 + o(1))\right)^{t_-^\alpha}(1 + o(1)), \tag{4.23}$$

$$u_\beta + \mathrm{i}v_\beta = k + \mathrm{i}\eta + C_\beta \left(-k - \mathrm{i}\eta - C_\alpha(-u_\beta - \mathrm{i}v_\beta)^{t_-^\alpha}(1 + o(1))\right)^{t_-^\beta}(1 + o(1)). \tag{4.24}$$

Let us focus on the first equation, because of the symmetry of these equations, the result we obtain on the first one can easily be obtained the same way on the second equation. We start by taking the imaginary part on both sides, and denoting θ the argument of $-k - \mathrm{i}\eta - C_\beta(-u_\alpha - \mathrm{i}v_\alpha)^{t_-^\beta}$ we have,

$$|v_\alpha| = \eta + C_\alpha \left|k + \mathrm{i}\eta + C_\beta(-u_\alpha - \mathrm{i}v_\alpha)^{t_-^\beta}(1 + o(1))\right|^{t_-^\alpha} |\sin(\theta t_-^\alpha)|(1 + o(1)) \tag{4.25}$$

$$|v_\alpha| \le \eta + C_\alpha^{'} \left(|k + \mathrm{i}\eta|^{t_-^\alpha} + C\,|u_\alpha + \mathrm{i}v_\alpha|^{t_-^\alpha t_-^\beta}(1 + o(1))\right)(1 + o(1)) \tag{4.26}$$

$$|v_\alpha| \le \eta + C_\alpha^{'}\,|k + \eta|^{t_-^\alpha} + C\,|u_\alpha + v_\alpha|^{t_-^\alpha t_-^\beta}. \tag{4.27}$$

Here, we used Holder for the first inequality and the arithmetic-geometric inequality for the second. The last inequalities are valid only for k, η small enough because we slightly increase the constants in order to remove the $(1 + o(1))$ term. As in the proof of Case 2, if we assume $u_\alpha \le v_\alpha$, we would have a contradiction when $|z - E_-^{\boxplus}|$ approaches 0. Therefore, we have for k, η small enough, $v_\alpha \le u_\alpha$. Now that we have inequality (4.27), we will examine the real part of (4.23) before coming back to this inequality as follows,

$$\begin{aligned} |u_\alpha - k| &= C_\alpha \left|k + \mathrm{i}\eta + C_\beta(-u_\alpha - \mathrm{i}v_\alpha)^{t_-^\beta}(1 + o(1))\right|^{t_-^\alpha}(1 + o(1))|\cos(\theta t_-^\alpha)| \\ |u_\alpha - k| &\le C_1|k + \eta|^{t_-^\alpha} + C_2|u_\alpha + v_\alpha|^{t_-^\alpha t_-^\beta}(1 + o(1)) \\ &\le C_1|k|^{t_-^\alpha} + C_2|u_\alpha|^{t_-^\alpha t_-^\beta}. \end{aligned}$$

Here, the constants (even if being denoted the same) may change from one line to another. The last inequality comes from $v_\alpha \le u_\alpha$ and leads to $u_\alpha \sim k$ because t_-^α and $t_-^\alpha t_-^\beta$ are greater than one, therefore, $k^{t_-^\alpha} = o(k)$ and $u_\alpha^{t_-^\alpha t_-^\beta} = o(u_\alpha)$.

Let us now go back to the study of the imaginary part of (4.23). Using $u_\alpha \sim k$ in (4.27) and the fact that the exponents are all greater than one, we find that

$$|v_\alpha| \leq \eta + C\,|k+\eta|^{t_-^\alpha} + C\,|u_\alpha + v_\alpha|^{t_-^\alpha t_-^\beta}$$
$$|v_\alpha| \leq \eta + C(|k|^{t_-^\alpha} + |\eta|^{t_-^\alpha}) + C|k|^{t_-^\alpha t_-^\beta}$$
$$|v_\alpha| \leq c\eta + C|k|^{t_-^\alpha}.$$

The last inequality comes from $k^{t_-^\alpha t_-^\beta} = o(k^{t_-^\alpha})$ and $\eta^{t_-^\alpha} = o(\eta)$. The symmetry of (4.23) and (4.24) allows us to say $u_\beta \sim k$ and $|v_\beta| \leq c\eta + C|k|^{t_-^\beta}$. Provided we find a link between the imaginary part of the subordination functions and the density of $\mu_\alpha \boxplus \mu_\beta$, we have an upper bound. Taking the imaginary part of (4.3), we have

$$\operatorname{Im} m_{\mu_\alpha \boxplus \mu_\beta}(z) = \operatorname{Im} \omega_\alpha(z)\; I_{\mu_\beta}(\omega_\alpha(z)) \tag{4.28}$$
$$= \operatorname{Im} \omega_\beta(z)\; I_{\mu_\alpha}(\omega_\beta(z)). \tag{4.29}$$

Because $I_\mu(\omega)$ stays bounded as the subordination function approaches the support (due to the exponents being greater than one), we therefore have two upper-bounds for k small enough,

$$\lim_{\eta \searrow 0} \operatorname{Im} m_{\mu_\alpha \boxplus \mu_\beta}(E_-^{\boxplus} + k + \mathrm{i}\eta) \leq C|k|^{t_-^\alpha} \tag{4.30}$$
$$\leq C|k|^{t_-^\beta}. \tag{4.31}$$

The two constants may be different. The last inequality is obtained exactly the same way because equation (4.24) is the same equation as (4.23) when replacing α by β.
In order to obtain a lower bound, we will use equations (4.5) with the notations defined in (4.22). Taking the imaginary part yields

$$v_\alpha = \eta + \int_{E_-^\alpha}^{E_+^\alpha} \frac{\mathrm{d}\widehat{\mu}_\alpha(x) v_\beta}{|x - (E_-^\alpha + u_\beta + \mathrm{i}v_\beta)|^2}$$
$$= \eta + \int_0^{E_+^\alpha - E_-^\alpha} \frac{\mathrm{d}\widehat{\mu}_\alpha(x) v_\beta}{(x-u_\beta)^2 + v_\beta^2}.$$

Taking k, η small enough such that $u_\beta + v_\beta < E_+^\alpha - E_-^\alpha$ gives us,

$$v_\alpha \geq \eta + \int_{u_\beta}^{v_\beta + u_\beta} \frac{\mathrm{d}\widehat{\mu}_\alpha(x) v_\beta}{(x-u_\beta)^2 + v_\beta^2}$$
$$\geq \eta + \int_{u_\beta}^{v_\beta + u_\beta} \frac{\mathrm{d}\widehat{\mu}_\alpha(x) v_\beta}{2v_\beta^2}$$
$$= \eta + \frac{c}{v_\beta}\left((v_\beta + u_\beta)^{t_-^\alpha + 1} - u_\beta^{t_-^\alpha + 1}\right)$$
$$\geq \eta + c u_\beta^{t_-^\alpha}.$$

We also have the corresponding inequality for $\beta :\ v_\beta \geq \eta + c u_\alpha^{t_-^\beta}$. Using equation (4.28) and the fact that $u_\alpha \sim u_\beta \sim k$, we have our lower-bounds,

$$\lim_{\eta \searrow 0} \operatorname{Im} m_{\mu_\alpha \boxplus \mu_\beta}(E_-^{\boxplus} + k + \mathrm{i}\eta) \geq c k^{t_-^\alpha} \tag{4.32}$$
$$\geq c k^{t_-^\beta}. \tag{4.33}$$

Now, if we assume that $t_-^\alpha < t_-^\beta$, let us show a contradiction. Using (4.31) and (4.32) and dividing both sides by $k^{t_-^\beta}$ yields,

$$\frac{c}{k^{t_-^\beta - t_-^\alpha}} \leq \lim_{\eta \searrow 0} \operatorname{Im} m_{\mu_\alpha \boxplus \mu_\beta}(E_-^{\boxplus} + k + \mathrm{i}\eta) \leq C. \tag{4.34}$$

Here, the left-hand side goes to infinity as k approaches 0 while it should stay bounded by above according to the right-hand side. Our conclusion is that if we are in the third case of the theorem, then the exponents near the lower edge of the two measures must be the same. Then, once we know they are the same, we can directly conclude thanks to the Stieltjes inversion formula and (4.30) and (4.32). □

Now that we can deduce the power behavior of the free additive convolution from the behavior of the subordination function, we want to be able to find the behavior of the subordination function given only the two measures. Indeed, there is no way of determining explicitly those subordination functions and we will therefore need some easy to compute criterion that gives us the behavior of the subordination functions. However, so far, we can only know whether there is at least one of the subordination function that touches the support or wheter both of them stay away.

Lemma 4.3. *For two Borel measures μ_α and μ_β satisfying Assumptions 1.1 with exponents greater than one, we have the following implications.*

(1) If $I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(E_-^\alpha) \leq 1$, then at least one subordination function touches the support of the other measure.
(2) If $I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(E_-^\alpha) > 1$, then at most one subordination function touches the support of the other measure.

Remark 4.3. *Here, even though we cannot directly compute those integrals because of the "hat measures", we can use equation* (3.19) *in Example 3.5 to express $I_{\widehat{\mu}}$ in terms of integrals with respect to μ.*

Proof. We will need a key equation in order to prove our lemma. Taking the imaginary part in the system of equation (4.5) and combining both lines yields

$$I_{\widehat{\mu}_\alpha}(\omega_\beta(z))I_{\widehat{\mu}_\beta}(\omega_\alpha(z)) = \left(1 - \frac{\operatorname{Im} z}{\operatorname{Im}\omega_\alpha(z)}\right)\left(1 - \frac{\operatorname{Im} z}{\operatorname{Im}\omega_\beta(z)}\right) \leq 1, \quad \forall z, \operatorname{Im}\omega_\alpha(z) > 0. \tag{4.35}$$

The condition on ω_α can also be a condition on ω_β because these two conditions would be equivalent since their imaginary part are positive at the same time.

(1) Let us assume that the product of the two $I_{\widehat{\mu}}$ is smaller or equal to one as in the Lemma. Let us also assume that both subordination functions stay away from the support of the other measure and let us find a contradiction. Since both subordination functions are away from the support, we have a square root behavior near the lower edge point of the free convolution measure and therefore, we are able to take the following limit since it is not tangential,
$$I_{\widehat{\mu}_\alpha}(\omega_\beta(E_-^\boxplus))I_{\widehat{\mu}_\beta}(\omega_\alpha(E_-^\boxplus)) = \lim_{\mathbb{E}_-^\boxplus \leftarrow E} I_{\widehat{\mu}_\alpha}(\omega_\beta(E))I_{\widehat{\mu}_\beta}(\omega_\alpha(E)) = 1.$$
We have equality thanks to equation (4.35) in $E \in \operatorname{supp}\mu_\alpha \boxplus \mu_\beta$. We further notice that $I_{\widehat{\mu}_\alpha}(E)$ (resp. $I_{\widehat{\mu}_\beta}(E)$) is an increasing function for $E \in (-\infty; E_-^\alpha]$ (resp. $(-\infty; E_-^\beta]$) and because $\omega_\beta(E_-^\boxplus) < E_-^\alpha$ (and the corresponding statement replacing α and β), we obtain the wanted contradiction :
$$I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(E_-^\alpha) > I_{\widehat{\mu}_\alpha}(\omega_\beta(E_-^\boxplus))I_{\widehat{\mu}_\beta}(\omega_\alpha(E_-^\boxplus)) = 1.$$

(2) Let us now assume the product of the $I_{\widehat{\mu}}$ functions to be strictly greater than one as in the Lemma and also assume that both subordination functions touch the support in order to find a contradiction. Using (4.35), we have,
$$I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(E_-^\alpha) = I_{\widehat{\mu}_\alpha}(\omega_\beta(E_-^\boxplus))I_{\widehat{\mu}_\beta}(\omega_\alpha(E_-^\boxplus)) = \lim_{\eta\searrow 0} I_{\widehat{\mu}_\alpha}(\omega_\beta(E_-^\boxplus + \mathrm{i}\eta))I_{\widehat{\mu}_\beta}(\omega_\alpha(E_-^\boxplus + \mathrm{i}\eta)) \leq 1.$$
Here, the limit is valid since $\operatorname{Im}\omega(z) \geq \operatorname{Im} z$ for all $z \in \mathbb{C}_+$, we do not have a tangential limit inside the $I_{\widehat{\mu}}$ functions and therefore, the proof is complete.

□

Now, in order to know which of the subordination function is able to touch the support first (or in other words, which subordination function cannot touch the support), we will need the following Lemma.

Lemma 4.4. *Let μ_α, μ_β be two Borel probability measures satisfying Assumptions 1.1, then we have the implications,*

$$F_{\mu_\alpha}(E_-^\alpha) < F_{\mu_\beta}(E_-^\beta) \Longrightarrow \omega_\alpha(E_-^\boxplus) < E_-^\beta,$$
$$F_{\mu_\beta}(E_-^\beta) < F_{\mu_\alpha}(E_-^\alpha) \Longrightarrow \omega_\beta(E_-^\boxplus) < E_-^\alpha,$$
$$F_{\mu_\beta}(E_-^\beta) = F_{\mu_\alpha}(E_-^\alpha) \Longrightarrow \left(\omega_\alpha(E_-^\boxplus) < E_-^\beta \text{ and } \omega_\beta(E_-^\boxplus) < E_-^\alpha\right) \text{ or } \left(\omega_\alpha(E_-^\boxplus) = E_-^\beta \text{ and } \omega_\beta(E_-^\boxplus) = E_-^\alpha\right).$$

Proof. The proof will mainly rely on $F_{\mu_\alpha}(\omega_\beta(z)) = F_{\mu_\beta}(\omega_\alpha(z))$ for all $z \in \mathbb{C}_+ \cup (-\infty; E_-^\boxplus] \cup [E_+^\boxplus; +\infty)$ and on the fact that both F functions (as real-valued functions) are strictly increasing on $(-\infty; E_-^\boxplus]$.
For the first implication, let us assume that $\omega_\alpha(E_-^\boxplus) = E_-^\beta$ (we obviously have $E_-^\alpha \geq \omega_\beta(E_-^\boxplus)$), we would have
$$F_{\mu_\alpha}(E_-^\alpha) \geq F_{\mu_\alpha}(\omega_\beta(E_-^\boxplus)) = F_{\mu_\beta}(\omega_\alpha(E_-^\boxplus)) = F_{\mu_\beta}(E_-^\alpha).$$
This inequality is a contradiction with our hypothesis and therefore, we must have $\omega_\alpha(E_-^\boxplus) < E_-^\alpha$. The second implication can be proven exactly the same way interchanging α and β.
For the last implication, it is clear that if exactly one of the two subordination functions touches the support while the other stays away, using the monotonicity of the F functions would lead to a contradiction. □

Lemma 4.3 and Lemma 4.4 allows us to do some computable tests in order to know in which of the cases from Theorem 4.2 we are in. For anys two measures μ_α and μ_β verifying Assumptions 1.1 (with exponents only assumed to be greater than -1, the situation can be summarized as follows,

(a) If $I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(E_-^\alpha) \leq 1$,
  (i) If $F_{\mu_\alpha}(E_-^\alpha) < F_{\mu_\beta}(E_-^\beta)$, then $\omega_\alpha(E_-^\boxplus) < E_-^\beta$ and $\omega_\beta(E_-^\boxplus) = E_-^\alpha$ (case 2),
  (ii) If $F_{\mu_\alpha}(E_-^\alpha) > F_{\mu_\beta}(E_-^\beta)$, then $\omega_\alpha(E_-^\boxplus) = E_-^\beta$ and $\omega_\beta(E_-^\boxplus) < E_-^\alpha$ (case 2),
  (iii) If $F_{\mu_\alpha}(E_-^\alpha) = F_{\mu_\beta}(E_-^\beta)$, then $\omega_\alpha(E_-^\boxplus) = E_-^\beta$ and $\omega_\beta(E_-^\boxplus) = E_-^\alpha$ (case 3).

(b) If $I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(E_-^\alpha) > 1$,
  (i) If $F_{\mu_\alpha}(E_-^\alpha) < F_{\mu_\beta}(E_-^\beta)$, then $\omega_\alpha(E_-^\boxplus) < E_-^\beta$ and ω_β can either approach or stay away from $\operatorname{supp}\mu_\alpha$ (case 1 or 2),
  (ii) If $F_{\mu_\alpha}(E_-^\alpha) > F_{\mu_\beta}(E_-^\beta)$, then $\omega_\beta(E_-^\boxplus) < E_-^\alpha$ and ω_α can either approach or stay away from $\operatorname{supp}\mu_\beta$ (case 1 or 2),
  (iii) If $F_{\mu_\alpha}(E_-^\alpha) = F_{\mu_\beta}(E_-^\beta)$, then $\omega_\alpha(E_-^\boxplus) < E_-^\beta$ and $\omega_\beta(E_-^\boxplus) < E_-^\alpha$ (case 1).

Furthermore, if we are in the case b-(i), (resp. b-(ii)), we con show that $I_{\widehat{\mu}_\alpha}(E_-^\alpha)I_{\widehat{\mu}_\beta}(F_{\mu_\beta}^{-1} \circ F_{\mu_\alpha}(E_-^\alpha)) \leq 1 \iff \omega_\beta(E_-^\boxplus) = E_-^\alpha$ (resp. $I_{\widehat{\mu}_\beta}(E_-^\beta)I_{\widehat{\mu}_\alpha}(F_{\mu_\alpha}^{-1} \circ F_{\mu_\beta}(E_-^\beta)) \leq 1 \iff \omega_\alpha(E_-^\boxplus) = E_-^\beta$) but since F_μ^{-1} is not easy to compute in a general case, this criterion is not a computable one.

5. Open questions and future work

Throughout this paper, several questions were left unanswered. We give here some ideas to extend the results we found.

First, one could find some criteria for the multiplicative semi-group in order to distinguish case 1 from case 2 in Theorem 3.8.

Second, one could also do a similar work as the one done in Section 4 but for the multiplicative free convolution of two distinct measures.

Last, even though we found several criteria for the subordination functions to stay away from the support of the other measure, one might find a simpler criterion that would include all the other or just find a explicitly computable criterion to distinguish the t_-^α decay from the square root behavior once we are in case (b)(i) (and similarly for case (b)(ii)). Even though one could numerically compute the criterion to distinguish cases (b)(i) and (b)(ii), it requires to locally inverse some function and might therefore be improved.

All the work done in this paper refers to absolutely continuous and compactly supported measures but what happens if we now allow our initial measures to be supported on several disjoint real intervals ? What happens if we authorize the initial measures to have some pure points ?

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