

# Rapport de stage

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## 1 Cadre du stage

J'ai effectué mon stage à l'Université de Stanford, en Californie, aux États-Unis, sous la direction de Ciprian Manolescu, durant les mois de mars à juin 2022 (inclus).

Après quelques premières semaines compliquées en termes de logement, j'ai pu obtenir une chambre en collocation sur le campus, dans une grande résidence réservée aux *graduate students* à l'atmosphère très conviviale. J'ai beaucoup aimé la vie sur le campus qui était rendue agréable par la météo californienne, la gentillesse de Américains et la richesse culturelle de la communauté internationale parmi les étudiants. Le campus de Stanford étant immense, très vert et bien équipé en termes d'infrastructures sportives. J'ai pu avoir un bureau partagé (dans lequel mes co-bureau ne venaient jamais) et j'ai beaucoup apprécié l'ambiance au Département de Mathématiques. Il dispose d'une grande terrasse sur laquelle les doctorants se retrouvent pour travailler, discuter, déjeuner, partager le "thé" quotidien (un analogue de notre thé du DMA, mais Stanford a un peu plus de budget...) ou les "*maths beers*" du vendredi soir (une tradition à instaurer au DMA?). J'ai donc passé beaucoup de temps sur cette terrasse car je préférais travailler au soleil et avec d'autres étudiants.

Ce cadre très convivial m'a permis de m'intégrer dès le début dans l'équipe des doctorants, en particulier au sein du groupe de ceux de première année (une quinzaine environ). J'ai pu partir avec eux pour visiter pendant quelques jours les deux parcs nationaux magnifiques que sont Yosemite et le Lassen volcanic park. J'ai également visité San Francisco à plusieurs reprises, et j'ai trouvé cette ville sympathique et animée, surtout le soir ou lors de célébrations comme la Pride ou la Bay to Breakers race (une course bon enfant traversant la ville). Je me suis fais de très bons amis et amies parmi les doctorants, ainsi que parmi les étudiants de ma résidence, notamment ceux de la communauté française très active. J'ai pu participer à des événements traditionnels de l'université comme le bal autrichien ou la cérémonie de *graduation*. J'ai aussi découvert les *social dances* (swing, valse, etc.) dans les clubs de Stanford qui ont été un moyen amusant de rencontrer encore d'autres étudiants. En somme, le fait d'être bien

entourée m'a permis de ne pas trop me sentir seule ou loin de chez moi (contrairement à ce que certains autres normaliens racontaient dans leur rapport), et de passer un très bon séjour.

Du point de vue académique, mon maître de stage Ciprian m'a assez vite considérée comme faisant partie de ses doctorants de première année, ce qui a eu des avantages et des inconvénients. J'ai pu participer à un groupe de lecture hebdomadaire avec trois autres de ses doctorantes, avec lesquelles j'ai pu échanger sur nos lectures et travailler sur des exercices. En revanche, je n'ai pas vraiment pu faire de recherche à proprement parler, puisque les doctorants de Stanford passent principalement leur première année à lire des livres de maths et à suivre des cours. J'ai d'ailleurs suivi un cours assez général de géométrie différentielle et riemannienne ainsi qu'un cours avancé de géométrie symplectique (qui avait pour but d'introduire les invariants de Gromov-Witten) que j'ai beaucoup apprécié. J'ai également assisté et participé au séminaire des doctorants en topologie en présentant une séance sur l'homologie de Seiberg-Witten (voir plus loin dans le rapport). Enfin, j'ai assisté à presque toutes les sessions du séminaire de topologie, qui était très orienté vers la topologie de basse dimension, domaine de recherche de Ciprian ainsi que de ses trois post-doctorants. J'ai beaucoup apprécié la tradition de ce séminaire qui consistait à aller dîner avec tous les membres qui le souhaitaient dans un des restaurants de Palo Alto : cela m'a permis de discuter avec de nombreux chercheurs et chercheuses venus présenter leurs travaux.

J'ai aussi pu participer au programme de *mentoring* du département en étant la "tutrice" d'un étudiant de première année (*undergraduate*) : je lui ai fait découvrir la topologie algébrique et la théorie des groupes, et c'était une expérience d'enseignement très enrichissante.

Enfin, je dois aussi mentionner que j'ai réalisé une enquête ethnographique du département de Stanford dans le cadre de mon mémoire de sociologie des sciences. Je dois dire que cela m'a particulièrement aidée à mieux comprendre le monde de la recherche en mathématiques et la façon dont les chercheurs et chercheuses travaillent, notamment en collaboration.

En conclusion, j'ai beaucoup aimé le cadre de vie de Stanford, j'ai rencontré de nombreuses personnes très sympathiques et intéressantes, et qui m'ont permis d'apprécier mon expérience au sein du département et de l'université. Même si cette expérience ne m'a pas permis de faire moi-même de la recherche en mathématiques, je trouve qu'elle m'a beaucoup éclairée sur son fonctionnement, ses enjeux, ainsi que sur le quotidien des chercheurs et chercheuses.

## 2 Introduction

In the following sections, I will try to give an overview of what I studied during my stay in the department. I will begin by presenting the unknotting problem on which I concentrated during the first month of my stay. I will present the main historical invariants that have been developed in that purpose before moving to a more recent one, knot Floer homology. This will then be the occasion to give more details about Floer theory which inspired it, along with other related invariants, Heegaard Floer homology and Seiberg-Witten (or monopole) homology.

## 3 The unknotting problem

### 3.1 Introduction to knots and the unknotting problem

**Definition 1** (Knot and link). A knot  $K$  is a smoothly embedded closed curve in a closed orientable 3-manifold  $M$ . A link  $L$  is a finite and disjoint collection of such curves.

*Remark.* In general, people consider links in  $S^3$ , and we will do so in the rest of this document.

*Remark.* The smoothness condition can equivalently be replaced by piecewise linearity. It is needed to avoid knots with an infinite number of kinks that get smaller and converge to a point.

In  $S^3$  (which is the setting we will consider most of the time), knots are considered the same when they are *equivalent*, i.e. when there is a smooth orientation preserving automorphism  $f : M \rightarrow M$  taking the first knot (or link) to the other one. Another notion of equivalence between knots is being *isotopic*, i.e. knots are considered the same when there is an isotopy between them (in  $S^3$  if knots are isotopic, they are in particular equivalent).

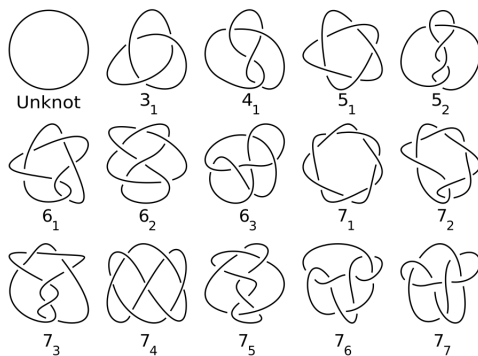


Figure 1: All prime knots with seven crossings or fewer

All links in  $\mathbb{R}^3$  admit a *regular projection* on a plane  $P \subset \mathbb{R}^3$  that is an orthogonal projection in which each point in the image of the link has at most two pre-images and the Jacobian of the projection map has rank 1 at all intersection points. From this projection we can draw diagrams of the link, and some examples for small prime knots are given in the figure 1.

A knot in  $S^3$  that is equivalent to the circle  $(\cos t; \sin t; 0), 0 \leq t \leq 2\pi$  is called the *unknot*. One of the main questions in knot theory is then to know, for a given knot (under the form of a diagram for example), if it is equivalent to the unknot. The principal technique that has been used in attempts to solve this problem is the construction of invariants, i.e. maps from the space of knots diagrams to another space such that diagrams of equivalent knots have same image. The goal is then to find invariants that are computable but also precise. In general, knots that have the same invariant are not necessarily equivalent. A good invariant would distinguish the unknot from other knots but it should also be computable. In the next part, we will present some famous knots invariants before moving to some more advanced and recent ones.

### 3.2 Presentation of some historical invariants

An invariant can be as simple as a yes-or-no answer to a question. A first example of an invariant is then **tricolorability**.

**Definition 2** (Tricolorability). We say that a knot is tricolorable if it has a knot diagram in which each strand can be colored one of three colors, according to the rules :

1. At least two colors must be used
2. At each crossing, the three incident strands are either all the same color or all different colors.

Tricolorability is invariant under isotopy, and is then an isotopy invariant. We can easily see that the unknot is not tricolorable, but the trefoil is : these two knots are then different.



Figure 2: The trefoil on the left and an example of a coloring for it. On the right, the figure eight knot which is not tricolorable : there is a strand that cannot be colored without breaking the rules.

However, this knot invariant is binary and thus not very precise. Indeed, we

can find knots that are not tricolorable but are not the unknot (the figure eight knot above for example).

More complicated but classic invariants would be given by polynomials. The two most useful and famous polynomial invariants are the Alexander(-Conway) polynomial and the Jones polynomial.

### 3.2.1 The Alexander polynomial

The Alexander polynomial can be defined in different but equivalent ways, and for the one we choose, we will use the notion of Seifert surfaces.

**Definition 3** (Seifert surface). A Seifert surface for an oriented knot in  $S^3$  is a connected compact oriented surface smoothly embedded in  $S^3$  with oriented boundary equal to the link. (Here, we consider the knot as being in  $\mathbb{R}^3$  as it is compact and in  $S^3 = \mathbb{R}^3 \cup \{\infty\}$ )

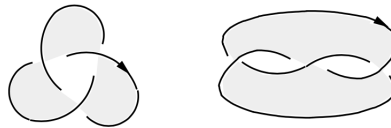


Figure 3: Two surfaces with boundary the trefoil, only the one on the right is a Seifert surface as the left one is the Möbius strip (thus not orientable)

*Remark.* This surface is not unique.

**Theorem 4.** *Every oriented knot in  $S^3$  bounds a Seifert surface.*

*Remark.* There is even an explicit algorithm to construct a Seifert surface of a knot.

To define the Alexander polynomial, we first need to define what a Seifert matrix is, and to do so we need a notion of linking number.

**Definition 5** (Linking number). Given two oriented links  $L_1, L_2$  and their planar projection, we define the linking number by the sum over all crossings (in which  $L_1$  goes under  $L_2$ ) of the signs given by the convention in figure 4. The notation for the linking number between  $L_1$  and  $L_2$  is  $\text{lk}(L_1, L_2)$ .



Figure 4: Conventions for the linking number

**Definition 6** (Seifert matrix). Given a Seifert surface  $F$  for a knot  $K$ , as  $F$  is oriented, it is possible to distinguish one side as a “top” side (i.e. we pick a non-zero normal vector field on  $F$  which orients  $F$ ). Given any simple closed oriented curve  $\gamma$  on  $F$ , one can form its *positive push off*,  $\gamma^+$ , which runs parallel to  $\gamma$ , and lies just above  $F$  in the direction of the normal field. We choose  $n$  simple closed curves  $\gamma_1, \dots, \gamma_n$  that generate a basis in  $H_1(F; \mathbb{Z})$  and we define the associated Seifert matrix as being the  $n \times n$  matrix  $S$  with  $S_{i,j} = \text{lk}(\gamma_i, \gamma_j^+)$ .

**Definition 7.** (Alexander polynomial) Then, for a knot  $K$ , we can define its Alexander polynomial by  $\Delta_K(t) = \det(t^{1/2}S - t^{-1/2}S^T)$  for  $S$  a Seifert matrix of a Seifert surface for  $K$ . ( $S^T$  is the transposed of  $S$ )

This polynomial does not seem to be well defined as we just noted that Seifert surfaces are not unique for a given knot. However, all the corresponding matrices are related by a certain notion of equivalence (‘S-equivalence’, that we will not describe here), and the formula in the definition of the Alexander polynomial is invariant under that equivalence. From this, we get the following theorem.

**Theorem 8.** *The Alexander is a well-defined knot invariant for oriented knots.*

*Remark.* It is actually a link invariant, but we don’t need that property as we are interested in the unknotting problem.

The Alexander polynomial distinguishes most ‘small’ knots from the unknot but it does not work well for ‘bigger’ knots (i.e. a lot of them have the same Alexander polynomial as the unknot).

### 3.2.2 The Jones polynomial

Another famous polynomial knot invariant is the Jones polynomial. The Alexander polynomial was a polynomial in  $t^{1/2}$  and  $t^{-1/2}$ , but the Jones polynomial is a Laurent polynomial. It will also need several intermediary definitions in order to be introduced.

**Definition 9** (Kauffman bracket). It is a function from unoriented links diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate  $A$ . It maps a diagram  $D$  to  $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$  and is characterised by :

(i)  $\langle \bigcirc \rangle = 1$

(ii)  $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2) \langle D \rangle$

(iii) Where  $\bigcirc$  is the diagram of the unknot with no crossing, and  $D \sqcup \bigcirc$  is a diagram consisting of the diagram  $D$  together with an extra unknot on the side (unlinked and unknotted).

**Definition 10** (Writhe). The writhe  $w(D)$  of a diagram  $D$  of an oriented link is the sum of the signs of the crossings of  $D$ , where each crossing has sign  $+1$  or  $-1$  as defined (by convention) in figure 4.

Neither the Kauffman bracket nor the writhe are knot invariants, but they behave in a similar way when we do transformation of a diagram without changing the associated knot. Thus they can be combined to give a first invariant.

**Theorem 11.** *If  $D$  is a diagram for a link  $L$ , then  $(-A)^{-3w(D)} \langle D \rangle$  is an invariant of the oriented link .*

**Definition 12** (The Jones polynomial). The Jones polynomial  $V(L)$  of a knot is the Laurent polynomial in defined by

$$V(K) = ((-A)^{-3w(D)} \langle D \rangle)_{A^{-2}=t^{1/2}}$$

**Theorem 13.** *The Jones polynomial does not depend on the orientation of the knot. It has only integer powers of  $t$  and it is a knot invariant.*

The Jones polynomial distinguishes more knots than the Alexander polynomial but it is unknown if it always detects the unknot (this is the Jones unknot conjecture). It can be computed by algorithms but it takes a long time (it is  $\#P$ -hard for an alternating link).

### 3.2.3 The fundamental group

The complement of a knot in  $S^3$  is a 3-manifold with boundary a torus, and knots are determined by their complement (Gordon-Luecke). Studying it brings knot theory closer to the more general theory of 3-manifolds. Thus, the fundamental group of the complement of a knot is a good invariant, as the following theorem shows :

**Theorem 14.** *A knot  $K$  is the unknot if and only if  $\pi_1(S^3 \setminus K)$  is infinite cyclic.*

## 3.3 A more recent invariant : Knot Floer homology

Recently, Floer homology has been developed as a tool for symplectic geometry and low dimensional topology. A version of Floer homology for closed 3-manifolds, namely Heegaard Floer homology, was introduced by Ozsváth and Szabó in the beginning of the 2000's. Knot Floer homology was then defined as a relative version of Heegaard Floer homology to study oriented knots.

### 3.3.1 Definitions

The construction of Knot Floer homology is more complicated than the previous invariants, and I will begin by giving a few necessary definitions. In the following paragraphs,  $K$  will be an *oriented* knot in  $S^3$ . The first one is the definition of Heegaard diagrams, which are a way to describe 3-manifolds. We will only define the multi-pointed version that is used to describe oriented knots in  $S^3$ . Roughly, these diagrams tell how to glue two handlebodies along their identical boundary (which is then a surface) to obtain a 3-manifold.

**Definition 15** (Handlebody). A handlebody is an orientable 3-manifold with boundary, containing pairwise disjoint, properly embedded 2-discs such that the manifold resulting from cutting along the discs is a 3-ball (in other words, it looks like a 3-ball with handles attached to it). The *genus* of a handlebody is the genus of its boundary surface. Up to homeomorphism, there is exactly one handlebody of any non-negative integer genus.

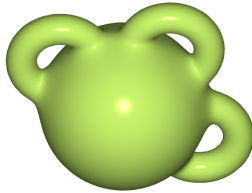


Figure 5: A genus 3 handlebody

From this definition we can explain what is a (multi-pointed) Heegaard diagram.

**Definition 16** (Heegaard diagram). A Heegaard diagram  $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$  for an oriented knot  $K$  consists in :

- a surface  $\Sigma \subset S^3$  of genus  $g \geq 0$ , splitting  $S^3$  into two handlebodies  $H_0$  and  $H_1$ , with  $\Sigma$  oriented as the boundary of  $H_0$ ,
- a collection  $\alpha = \{\alpha_1, \dots, \alpha_{g+k-1}\}$  consisting of  $g + k - 1$  pairwise disjoint, simple closed curves on  $\Sigma$ , such that each  $\alpha_i$  bounds a properly embedded disk  $D_i^\alpha$  in  $H_0$ , and the complement of these disks in  $H_0$  is a union of  $k$  balls  $B_1^\alpha, \dots, B_k^\alpha$ ,
- a curve collection  $\beta = \{\beta_1, \dots, \beta_{g+k-1}\}$  with similar properties, bounding disks  $D_i^\beta$  in  $H_1$ , such that their complement is a union of  $k$  balls  $B_1^\beta, \dots, B_k^\beta$ ,
- two collections of points on  $\Sigma$ , denoted  $\mathbf{w} = \{w_1, \dots, w_k\}$  and  $\mathbf{z} = \{z_1, \dots, z_k\}$ , all disjoint from each other and from the  $\alpha$  and  $\beta$  curves

such that the knot  $K$  intersects  $\Sigma$  exactly at the  $2k$  points  $w_i$  and  $z_i$ , with the intersections being positively oriented at  $w_i$  and negatively oriented at  $z_i$ . Additionally, the intersection of  $K$  with the handlebody  $H_0$  consists of  $k$  properly embedded intervals, one in each ball  $B_i^\alpha$ ; and similarly its intersection with  $H_1$  consists of  $k$  properly embedded intervals, one in each  $B_i^\beta$ .

It is important to remark that a Heegaard diagram determines the knot, but is not unique. Furthermore, every oriented knot in  $S^3$  admits a multi-pointed Heegaard diagram and several algorithms can be used to find such a diagram. In particular, there exists an algorithm yielding a *doubly pointed* diagram, i.e. a diagram with only two marked points  $w$  and  $z$  ( $k = 1$ ), and another algorithm yielding a genus 0 diagram ( $g = 0$ ) for every oriented knot.

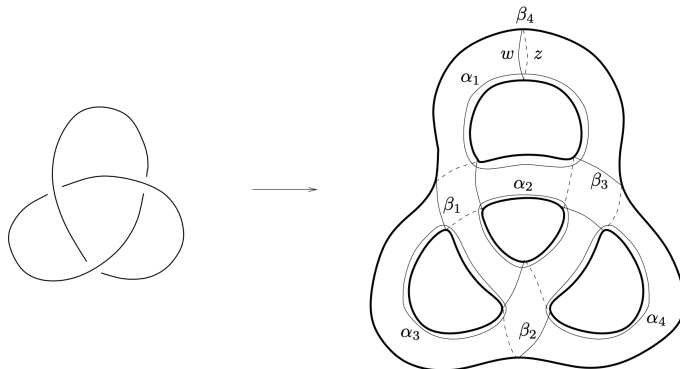


Figure 6: A doubly pointed diagram for the trefoil, obtained from its planar projection

Once we have a multi-pointed diagram for an oriented knot, we have to define the symmetric product of the corresponding surface.

**Definition 17** (Symmetric product). The  $d^{\text{th}}$  symmetric product  $\text{Sym}^d(\Sigma)$  of the manifold  $\Sigma$  is the quotient  $\Sigma^{\times d}/\mathfrak{S}_d$  where  $\Sigma^{\times d}$  is the Cartesian product of  $d$  copies of  $\Sigma$  on which the symmetric group  $\mathfrak{S}_d$  acts by permuting the factors.

One can prove that it is a  $2d$ -manifold by looking at the local model and distinguishing between points that have redundant coordinates and others. Here we will take  $d = g + k - 1$ . For a given Heegaard diagram, we will consider the symmetric product of its genus  $g$  surface  $\text{Sym}^d(\Sigma)$ . We can then define two submanifolds of the symmetric product :  $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_d$  and  $\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_d$  which are unordered tuples of points on each of the curves.

Two important notions we have to define now are *symplectic structures* and *almost complex structures*, along with a few associated notions.

**Definition 18** (Symplectic structure). A symplectic structure on a manifold  $M$  is a closed nondegenerate 2-form  $\omega$ .

A manifold admitting a symplectic structure is always even dimensional. Additionally, the symplectic form  $\omega$  can restrict in different ways to a submanifold of  $M$ . One will be particularly useful :

**Definition 19** (Lagrangian manifold). A submanifold of a symplectic manifold  $N \subset (M, \omega)$  is said to be Lagrangian if  $\omega|_N = 0$  and  $N$  is half-dimensional.

*Example.* It can be shown that the manifold  $\text{Sym}^d(\Sigma)$  and its submanifolds  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$  are respectively (and canonically) symplectic and Lagrangian. This can be done by finding a symplectic structure on  $\text{Sym}^d(\Sigma)$  which coincides with the form induced by a symplectic structure on  $\Sigma$  outside of the diagonal  $\Delta$  (it always exists).

**Definition 20** (Almost complex structure). An almost complex structure on a manifold  $M$  is a bundle map  $J : TM \rightarrow TM$  covering the identity map on  $M$  such that  $J^2 = -\text{id}$ .

We say that an almost complex structure  $J$  and a symplectic structure  $\omega$  are *compatible* if  $\omega = J^*\omega$  and  $\omega(v, Jv) > 0$  for all non-zero  $v$ .

From this definition, we can define a generalization of holomorphic maps, that will prove useful in the definition of holomorphic disks and strips.

**Definition 21** (Pseudo-holomorphic map). A map  $f : (N, j) \rightarrow (M, J)$  between two almost complex manifolds is said to be  $(j, J)$ -holomorphic if it satisfies the *Cauchy-Riemann equation* :  $df \circ j = J \circ df$ .

When the manifold  $(N, j)$  is actually a Riemann surface, we just say that  $f$  is  $J$ -holomorphic. For example,  $S^2 = \mathbb{C}\mathbb{P}^1$  and  $\mathbb{R} \times [0, 1]$  admit canonical almost complex structures (and are diffeomorphic to each other), so we can use them alternatively to define the following objects :

**Definition 22** ( $J$ -holomorphic disk and strip). Let  $L_1, L_0 \subset (M, \omega)$  be two Lagrangians, and  $J$  be a compatible almost complex structure. A  $J$ -holomorphic disk is a map  $u : D^2 \rightarrow M$  such that the two points  $-1, 1 \in D^2$  are mapped to  $x, y \in L_0 \cap L_1$  and the top half of the boundary of  $D^2$  is mapped into  $L_0$  and the bottom half into  $L_1$ , and  $u$  is  $J$ -holomorphic.

Alternatively a  $J$ -holomorphic strip is a map  $u : \mathbb{R} \times [0, 1] \rightarrow M$  such that  $u(-, 0) \in L_0, u(-, 1) \in L_1, \lim_{s \rightarrow -\infty} u(s, t) = x$  and  $\lim_{s \rightarrow \infty} u(s, t) = y$ , and  $u$  is  $J$ -holomorphic.

$J$ -holomorphic strips and disks are just two ways of describing the same object. Instead of  $J$ -holomorphic, they are sometimes said to be *pseudo-holomorphic*. Finally, we can also define a 'time-dependent' version :

**Definition 23** (Time-dependent pseudo-holomorphic strip). If  $J_t$  is a path of almost complex structures, a map  $u : \mathbb{R} \times [0, 1] \rightarrow M$  is a time-dependent pseudo-holomorphic strip if it satisfies the same conditions as above but instead of being  $J$ -holomorphic, we require that  $u$  satisfy the equation :

$$\partial u / \partial s + J_t(u(s, t)) \partial u / \partial t = 0$$

Alternatively we can also call it a disk if the domain is a disk and the conditions are similar to the previous definition.

In our case (up to homotopy of the curves),  $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$  is a finite set of points. We define, for  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ ,  $\pi_2(\mathbf{x}, \mathbf{y})$  as the relative homotopy classes of time-dependant pseudo-holomorphic disks (or strips) from  $\mathbf{x}$  to  $\mathbf{y}$ , relatively to a suitable family  $J = (J_t)_{t \in [0, 1]}$  of compatible almost complex structures on  $\text{Sym}^d(\Sigma)$ .

For a  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ , one can define the *Maslov index*  $\mu(\phi) \in \mathbf{Z}$  but I will not give details about it here. Using Gromov compactness, one can also show that when  $\mu(\phi) = 1$  and  $J$  is a generic family, the set  $\widehat{\mathcal{M}}(\phi)$  of unparametrized

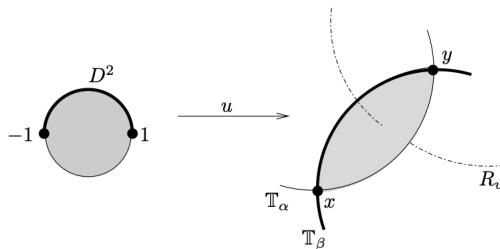


Figure 7: An illustration of a pseudo-holomorphic disk between  $x$  and  $y$

pseudo-holomorphic representatives for  $\phi$  is compact and of dimension 0. This means that  $\#\widehat{\mathcal{M}}(\phi)$  is a well defined integer.

We have a few other notation and definitions to review before giving the definition of Knot Floer homology. Recall that a multi-pointed Heegaard diagram has two families of marked points  $\mathbf{w}$  and  $\mathbf{z}$ . For each point  $v$  in these families we can define  $R_v = \{v\} \times \text{Sym}^{d-1}(\Sigma)$  which is a submanifold inside  $\text{Sym}^d(\Sigma)$ , of codimension 2.

By construction,  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$  are disjoint of  $R_v$ . Given  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  and  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  we can define  $n_v(\phi)$  to be the intersection number between  $\phi$  and  $R_v$ . The illustration of figure 7 can be found in [Man14] and shows a potential intersection between a pseudo-holomorphic disk and  $R_v$ . If this number is zero, then a generic pseudo-holomorphic representative of  $\phi$  does not intersect  $R_v$ .

Moreover, we can associate a relative bigrading  $(M, A) \in \mathbf{Z}^2$  to the points  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ . The *Maslov grading*  $M$  is defined by

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2 \sum_{i=1}^k n_{w_i}(\phi)$$

independently of the choice of  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ . The *Alexander grading*  $A$  is defined by

$$A(\mathbf{x}) - A(\mathbf{y}) = \sum_{i=1}^k n_{z_i}(\phi) - \sum_{i=1}^k n_{w_i}(\phi)$$

and

$$\sum_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (-1)^{M(\mathbf{x})} q^{A(\mathbf{x})} = (1 - q^{-1})^{k-1} \Delta_K(q)$$

Recall here that  $\Delta_K$  is the Alexander polynomial of the knot  $K$ . There is a way to fix the grading instead of having a relative grading, but I will not detail it.

With these definitions, we are finally ready to introduce the simplest version of the knot Floer complex, in which we consider disks that do not pass over any basepoint (i.e.  $n_v(\phi) = 0$ ).

**Definition 24** (Knot Floer complex). Let  $\mathcal{H}$  be a multipointed Heegaard diagram for the oriented knot  $K$ . The knot Floer complex  $\widetilde{gCFK}(\mathcal{H})$  is a bigraded

abelian group freely generated by the intersection points  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ . The differential is given by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1; n_{z_i}(\phi)=n_{w_i}(\phi)=0, \forall i}} (\#\widehat{\mathcal{M}}(\phi)) \mathbf{y}$$

There is a lot of verification I will not detail, in particular the fact that it is a well-defined complex with  $\partial^2 = 0$ . This fact comes from Gromov compactness again. Moreover, one can show that the differential  $\partial$  decreases the Maslov grading by one, and keeps the Alexander grading constant. The homology of this complex is denoted  $\widehat{HFK}(\mathcal{H})$ . The historical case was defined for a doubly pointed Heegaard diagram ( $k = 1$ ) and denoted  $\widehat{gCFK}(\mathcal{H})$  and  $\widehat{HFK}(\mathcal{H})$ . In the general case, we have the following relation :

$$\widehat{HFK}(\mathcal{H}) = \widehat{HFK}(\mathcal{H}) \otimes V^{\times(k-1)}$$

where  $V$  is an abelian group freely generated by an element in bi-degree  $(-1, -1)$  and one in bi-degree  $(0, 0)$ .

A very important result is that the hat version  $\widehat{HFK}$  does not depend on the choice of a multipointed Heegaard diagram for the knot :

**Theorem 25** (Ozsváth-Szabó, in [OS04b]). *The isomorphism class of  $\widehat{HFK}(\mathcal{H})$ , as a bigraded abelian group, is an invariant of the knot  $K \subset S^3$ .*

We can then denote this invariant  $\widehat{HFK}(K)$ .

As we have been focusing on the unknotting problem, it can be interesting to compute this invariant for it.

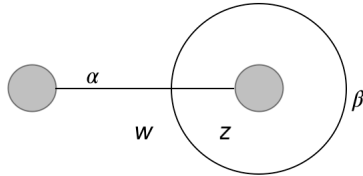


Figure 8: The genus 1 Heegaard diagram for the unknot

A convenient Heegaard diagram will be the genus 1 Heegaard splitting for the sphere, with two marked points. As  $d=1$ , the symmetric product of the surface  $\Sigma$  is actually just itself. There is only one intersection point and no disk between the point and itself so  $\widehat{HFK}(\text{unknot})$  is  $\mathbb{Z}$  in bi-degree  $(0, 0)$  and 0 otherwise.

A very interesting result, in our context of the unknotting problem, is the following :

**Theorem 26** (Ozsváth-Szabó, Theorem 1.2 in [OS04a]). *If  $K \subset S^3$  has the same bi-graded knot Floer homology groups  $\widehat{HFK}(K)$  as the unknot, then  $K$  is the unknot.*

This theorem is proven using a stronger result which says that the *knot genus* (the smallest genus of a Seifert surface of the knot) can be computed from its knot Floer homology. As the unknot is the only knot of genus 0, we get the above result.

This invariant has been very useful in the past years and other related knot Floer complexes (and associated sub-complexes) have been developed ( $\widehat{CFK}$ ,  $CFK^\infty$ ,  $CFK^+$ ,  $CFK^-$ , etc.). They are all more or less related and can prove useful in different contexts, and they correspond to different flavor of the Heegaard Floer homology (see 4.2).

### 3.3.2 A combinatorial method

Thus, we were able to define a very powerful invariant, but computing it can be hard and tedious when we have a complicated planar diagram of the knot. Indeed, the algorithm yielding a Heegaard splitting from a planar regular projection has genus equal to the number of crossing  $-1$ , even if it is only doubly pointed, that gives a lot of curves and a lot of intersection points. A way to ensure that the knot Floer homology could always be computed and studied has been explained in [Man+07] : building Heegaard splittings from *grid diagrams* gives a combinatorial method to compute the knot Floer homology of a given knot.

**Definition 27** (Grid diagram). *A planar grid diagram lies on an  $n \times n$  grid of squares in the plane. Each square is decorated either with an  $X$ , an  $O$ , or nothing. Moreover, the decorations are arranged so that every row contains exactly one  $X$  and one  $O$ , and every column contains exactly one  $X$  and one  $O$ .*

From a planar grid diagram, one can construct the corresponding oriented, planar knot projection by drawing horizontal segments from the  $O$ 's to the  $X$ 's in each row, and vertical segments from the  $X$ 's to the  $O$ 's in each column. At every intersection point, we let the horizontal segment be the underpass and the vertical one the overpass.

In the opposite direction, it is easily shown that a planar projection of a knot can be transformed into a grid presentation, to give a grid diagram corresponding to the knot (see figure 9). We say that the grid diagram *represent* the corresponding knot (it can also be a link)

Such a diagram can then be transposed on the torus by gluing the topmost segment of the grid to the bottom-most one, and the leftmost segment to the rightmost one. This process gives a Heegaard diagram  $\mathcal{G}$  for the corresponding knot : the surface  $\Sigma$  is the torus, the  $O$ 's are the  $w$  marked points in the previous

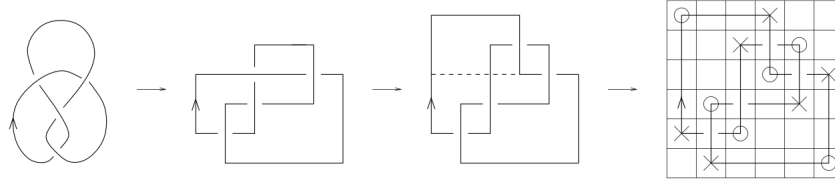


Figure 9: A planar grid diagram for the figure eight knot (on the right), corresponding to a grid presentation (on the left)

construction, the  $X$ 's are the  $z$  marked points, the horizontal and vertical lines of the grid become circles on the torus and they correspond to the  $\alpha$  and  $\beta$  curves.

In this perspective, intersection points  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  are unoriented  $n$ -tuples of points on the grid, with one point on each  $\alpha$  curve and one point on each  $\beta$  curve. The bigrading can be defined in this setting, using combinatorial methods that are a little bit tedious to write, for more details see [Man+07]. As in the previous section, these intersection points span the bigraded knot Floer complex associated  $\widetilde{gCFK}(\mathcal{G})$ .

To recover the full knot Floer complex, we have to re-define the differential of this complex in the new setting. Counting pseudo-holomorphic disks (that do not cross any marked point) between two points  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  can be done by counting *empty rectangles* that do not have any  $X$ 's or  $O$ 's in it.

**Definition 28.** An *empty rectangles* from  $\mathbf{x}$  to  $\mathbf{y}$  is an embedded rectangle in the grid diagram, such that its bottom edge is an arc on an  $\alpha$  curve from a point of  $\mathbf{x}$  (on the left) to a point of  $\mathbf{y}$  (on the right), and its top edge is an arc on an  $\alpha$  curve from a point of  $\mathbf{y}$  (on the left) to a point of  $\mathbf{x}$  (on the right). Furthermore, for it to be *empty* we require that the remaining  $n-2$  coordinates of  $\mathbf{x}$  coincides with the remaining  $n-2$  coordinates of  $\mathbf{y}$ .

We denote by  $\text{Rect}(\mathbf{x}, \mathbf{y})$  the set of empty rectangles from  $\mathbf{x}$  to  $\mathbf{y}$  that contain no  $X$  nor  $O$ . The differential of the complex is then given by :

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{r \in \text{Rect}(\mathbf{x}, \mathbf{y})} \epsilon(r) \mathbf{y}$$

where  $\epsilon$  associate a sign to every empty rectangle (I will not detail how, see [Man+07]). One can prove that  $\partial^2 = 0$  using this description and combinatorial arguments instead of Gromov compactness as above. Thus the knot Floer complex  $\widetilde{gCFK}$  is computable with combinatorial methods, which can be transformed into algorithms to automatize the computation of this invariant. Indeed, the number of rectangles is finite and it suffices to count which ones are empty and do not contain any  $X$ 's and  $O$ 's. Remember however that this is not an invariant of the knot, because only  $\widetilde{HFK}$  is one. To recover it from this combinatorial method we have to use the formula  $\widetilde{HFK}(\mathcal{H}) = \widetilde{HFK}(\mathcal{H}) \otimes V^{\times(k-1)}$ .

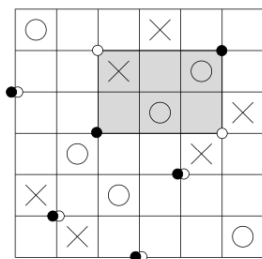


Figure 10: An example of an empty rectangle (shaded) between  $\mathbf{x}$  (represented by black dots) and  $\mathbf{y}$  (represented by white dots), on the diagram of the figure eight knot. However, this rectangle contains one  $X$  and two  $O$ 's.

### 3.4 Conclusion

Knot Floer homology is a powerful invariant that detects the unknot. It is complicated to define but pretty easy to compute using grid diagrams and the combinatorial methods we described above.

As mentioned in the introduction, it is actually a version of a more general invariant for 3-manifolds, Heegaard Floer homology. This other invariant is itself an element of a broader theory, Floer homology, developed to study symplectic geometry and low dimensional topology (we saw that both intervened in the previous definitions). In the next section, I will give an overview of other aspects of this theory that I could study during my internship and how they relate to each other, without getting too much into the exact details of the definition.

## 4 A more general context for invariants : 3-manifolds and 4-manifolds, Floer theory

### 4.1 General idea of Floer theory

Floer theory has been developed by Andreas Floer according to a general principle inspired by Morse theory : constructing complexes in which the generators are either solutions to some equations or intersection points of Lagrangians (like above for the knot Floer complex), and the differential counts the ways of 'going from one point to another'. The homology of such complexes is in general interesting and gave many invariants. In Morse theory, it meant having a flow between the two points of the manifold, in Floer theory it can mean having a pseudo-holomorphic disk (or curve) between the two points (like in knot Floer theory), or even more complicated things, as we will see. If the spaces we considered in the definition of knot Floer homology were finite, in some cases Floer theory gives an analogue of Morse theory in infinite dimension.

It has also proven very useful in different settings and one of the current

project is to develop Floer homotopy (instead of homology) by introducing one more step in the process : instead of associating a homology to the object one wants to study, the goal would be to associate a spectrum, whose homology would be the corresponding Floer homology. This spectrum could give more information and invariants.

## 4.2 Heegaard Floer homology

Heegaard-Floer homology is very similar to knot Floer homology as it also counts pseudo-holomorphic disks. It has been introduced by Ozsváth and Szabó in the 2000s (in particular in [OS04c]). To define it we need to use another version of Heegaard diagram, this time not multipointed. We mentioned Heegaard splitting in the example of the unknot and we give here the definition for it, along with the one of Heegaard diagram.

**Definition 29** (Heegaard splitting). A *Heegaard splitting* of a closed oriented 3-manifold  $Y$  is a decomposition  $Y = H_g \sqcup_f H'_g$  where  $H_g$  and  $H'_g$  are handlebodies of genus  $g$  glued along their boundary according to the diffeomorphism  $f : \partial H_g \rightarrow \partial H'_g$

Similarly to the result we had for knots in the 3-sphere, we have the following result :

**Theorem 30.** *Every closed oriented 3-manifold has a Heegaard splitting.*

This can be shown by using a triangulation of such a manifold and 'thickening' the vertices and edges (by considering a neighborhood of them) : this gives a handlebody, whose complement in the manifold is also a handlebody. Then we get a variation of Heegaard diagrams :

**Definition 31** (Heegaard diagram). A *Heegaard diagram*  $\mathcal{H} = (\Sigma, \alpha, \beta)$  of genus  $g$  for a closed oriented 3-manifold  $Y$  consists in :

- a surface  $\Sigma$  of genus  $g \geq 0$ ,
- a collection  $\alpha = \{\alpha_1, \dots, \alpha_g\}$  consisting of  $g$  linearly independent simple disjoint closed curves on  $\Sigma$  that determine a handlebody  $H_g$  by filling in the disks that the curves bound on one side of the surface (as it is oriented) and filling the remaining space with a 3-ball,
- a curve collection  $\beta = \{\beta_1, \dots, \beta_g\}$  with similar properties, determining this time the handlebody  $H'_g$  on the other side of the surface,

such that the two handlebodies  $H_g, H'_g$  form a Heegaard splitting of the manifold  $Y$ . If we want the diagram to be *based* we can add a base point  $z$  somewhere on  $\Sigma$ .

The construction of the complex is then very similar to the knot Floer case (there is also different flavors of it but I will only detail the hat version).

We begin by considering the symmetric product  $\text{Sym}^g(\Sigma)$ . As before, it is a canonically symplectic manifold and the submanifolds  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta \subset \text{Sym}^g(\Sigma)$  are Lagrangian.

**Definition 32.** The Heegaard Floer complex  $CF(\mathcal{H})$  associated to a Heegaard diagram  $\mathcal{H}$  for a closed oriented 3-manifold  $Y$  is freely generated (as an abelian group) by the intersection points  $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$  and the differential is given by :

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1; n_z(\phi)=0}} (\#\widehat{\mathcal{M}}(\phi)) \mathbf{y}$$

for a point  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ . The notations are identical to the knot case, except that this time we only have one basepoint.

**Definition 33** (Hatted version of Heegaard Floer homology). The homology of the complex  $CF(\mathcal{H})$  is noted  $\widehat{HF\mathcal{K}}(\mathcal{H})$ .

As in the previous section, there is a lot of verification to do to check that this homology is well defined, but again, we have the following result :

**Theorem 34** (Ozsváth-Szabó, in [OS04c]). *The hatted Heegaard Floer homology does not depend on the choice of an admissible Heegaard diagram and it is an invariant of closed oriented 3-manifolds.*

Here admissible is a small restriction on the diagrams, which have to verify a condition of nullity of the area for periodic domains (details are given in [Manb]).

*Remark.* Heegaard Floer homology is actually a special case of the Lagrangian Floer homology, in which the definitions are almost identical but  $\text{Sym}^g(\Sigma)$  is replaced by an arbitrary symplectic manifold (of any dimension), while  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$  are replaced by two compact transverse Lagrangian submanifolds  $L_0, L_1$ . We also have to add a few conditions, which were automatically verified in the case above, to make sure that the homology is well defined (in particular the existence of spin structures and conditions on the *energy* of pseudo-holomorphic strips).

Similarly to the knot Floer homology, a combinatorial algorithm has been given (in [OSS12]) to compute this homology from a Heegaard splitting of the manifold  $Y$ , which is exceptional among Floer theories (most of the time they are very hard to compute). Furthermore, many results can be recovered from the combinatorial description of this homology.

### 4.3 Seiberg-Witten Floer homology

The Seiberg-Witten Floer homology (or monopole Floer homology) is another tool coming from Floer theory, and is historically older than Heegaard Floer homology. It has been developed to study 3-manifolds and cobordisms between them given by 4-manifolds. Roughly, it is a generalization of Morse homology for the Chern–Simons–Dirac functional on the space of connections on a

3-manifold. I will give a few elements to understand better what this means.

But first, recall that the main idea of Morse theory is to build a complex spanned by critical points of a function from a closed Riemannian manifold to  $\mathbb{R}$ , with a differential that counts the trajectories of the flow going from a point of index  $k$  to points of index  $k - 1$ . The homology obtained from this construction is independent of the function chosen (if it is 'Morse-Smale') and is equal to the ordinary homology of the manifold. A version for manifolds with boundaries has also been developed, and generalized to a version for compact manifolds with an  $S^1$  action, using a process called *blow-up*. These different versions are well described in [KM07].

An aspect of this that is useful to understand the construction of the monopole Floer homology is that in the case of manifolds with an  $S^1$  action (or with a boundary), Morse theory gives three different complexes :  $\hat{C}, \bar{C}$  and  $\check{C}$ . Each one has its own differential and they are spanned by different choices of critical points either from the interior, stable on the boundary or unstable on the boundary. Their respective homologies correspond to the relative and ordinary homologies of the manifold and its boundary.

Similarly, there is three flavors of monopole floer homology :  $\widehat{HM}, \overline{HM}$  and  $\widetilde{HM}$ . To define them we need many notions that I will just mention without detailing them, as the goal of this section is not to define everything in detail ([Lin16] and [KM07] have done it very well and it takes lots of pages) but to give a big picture of the construction. In italics are the objects or notions I did not define.

- We start with a 3-manifold  $Y$  and a *spin<sup>c</sup> structure*  $\mathfrak{s}$ .
- From them, we get the *configuration space*  $C(Y, \mathfrak{s})$  which consists of pairs  $(B, \Psi)$  of a *spin<sup>c</sup> connection* and a spinor.
- We define the gauge group  $\mathcal{G}(Y, \mathfrak{s}) = \{u : Y \rightarrow S^1\}$  that is the group of automorphisms of the spin<sup>c</sup> structure. It acts on the configuration space but not freely.
- We blow-up the configuration space to obtain a space  $C^\sigma(Y, \mathfrak{s})$ , on which the action of the gauge group is free, similarly to what we can do for manifolds with an  $S^1$  action in Morse theory.
- We define the *Chern-Simons-Dirac* functional  $\mathcal{L}$  that goes from the configuration space  $C(Y, \mathfrak{s})$  to  $\mathbb{R}$  and we compute its formal gradient  $\text{grad}\mathcal{L}$ . The critical points of this functional are actually solutions to a system called *monopoles*, which is a version of the Seiberg-Witten equations on 3-manifolds. Roughly, this construction is a way to use these equations from physics to code the information of 3-manifolds and cobordisms between them.

- We extend this functional to  $C^\sigma(Y, \mathfrak{s})$  : we can then distinguish three types for its critical points, similarly to what is done in Morse theory with boundaries.
- These three types of points can then be arranged to span the three complexes  $\widehat{C}$ ,  $\widetilde{C}$  and  $\check{C}$  , whose differential count unparametrized trajectories between critical points in the blown-up configuration space  $C^\sigma(Y, \mathfrak{s})$ . Everything is very similar to Morse theory, except that this configuration space is infinite dimensional.
- Computing the homology of these complexes give the monopole or Seiberg-Witten Floer homologies  $\widetilde{HM}$ ,  $\widehat{HM}$  and  $\overline{HM}$ .

A priori these homologies seem to depend on a lot of choices, but we have the remarkable following theorem (again an analogue of Morse theory) :

**Theorem 35** (Mrowka-Kronheimer, in [KM07]). *The monopole Floer homologies (also known as Seiberg-Witten Floer homologies) are invariants of the pair  $(Y, \mathfrak{s})$ .*

Another very interesting result is the link that has been established between monopole Floer homology and Heegaard Floer homology that we described in a previous section (in which versions that we did not fully detail intervene) :

**Theorem 36** (Mrowka-Kronheimer, in [KM07]). *With the previous notations, we have, for a closed, oriented, Riemann 3-manifold :*

$$\begin{aligned}\widetilde{HF} &\cong HF^+ \\ \widehat{HF} &\cong HF^- \\ \overline{HF} &\cong HF^\infty\end{aligned}$$

## 5 Conclusion

We began with a focus on the unknotting problem and described several invariants that have been introduced in this setting. Considering some of the most recent invariants (namely knot Floer homology), we saw that developments in low dimensional topology and symplectic geometry yielded powerful tools to study knots. We took some distance with knots to study in more details some aspects of this theory. In our quick review of some invariants for 3-manifolds and symplectic manifolds, we saw that Floer theory laid the foundations for many invariants which mainly consisted in building various complexes and their homology. If they were in general hard to compute and to define, we saw that a combinatorial algorithm existed for knot Floer homology and Heegaard Floer homology. Finally, we saw that the latter one also related to an older invariant, Seiberg-Witten homology, that incorporated intuitions and elements from physics to produce an invariant in three flavors, that took into account cobordisms.

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*Remark.* Most illustrations in this report are not mine but come from one of the papers in the bibliography.