Swiss-European Mobility Programme / Erasmus +
Stage en géométrie énumérative

Méthodes arithmétiques en théorie de Donaldson-Thomas

Elsa Maneval
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Maître de stage : Rahul Pandharipande, ETHZ
Superviseur : Woonam Lim, ETHZ

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J'ai effectué mon stage à l'ETH Zurich en Suisse sous la supervision de Rahul Pandharipande, de mars à juillet 2022. Il m'a proposé un sujet puis a demandé à un post-doc de son équipe, Woonam Lim, de répondre à mes questions. J'ai vu Woonam une fois par semaine environ. Je n'avais pas de bureau à l'ETH, mais j'y allais plusieurs fois par semaine pour suivre les cours Intersection Theory (Pierrick Bousseau), Moduli of Stable Bundles on Curves (Woonam Lim) et au séminaire de l'équipe de Rahul Pandharipande. J'ai pu participer à des activités du groupe de recherche (repas, café, balades) et assister à 4 conférences de recherche[1], ce qui n'est pas habituel pour un stage de master. Ces expériences ont été très enrichissantes.

Résumé mathématique Le sujet de mon stage était assez ouvert et concernait au départ des méthodes de corps finis pour calculer des invariants en géométrie énumérative. Woonam m'a d'abord proposé de me familiariser avec la théorie des moduli (i.e. espaces de modules) sur $\mathbb{C}$, ce que j'ai fait les deux premiers mois. Ainsi, j'ai étudié les schémas de Hilbert qui sont les moduli les plus simples. Ce sont des objets géométriques attachés à une variété algébrique X où chaque point représente un sous-schéma. Dans certains cas, leur cohomologie est générée par des classes dites "tautologiques", qui viennent de X et d'un sous-schéma "universel". La formule de localisation d'Atiyah-Bott est alors un outil puissant pour réaliser des calculs cohomologiques sur les variétés équipées d'une action de $\mathbb{C}^*$. Ce type de calcul apparait dans l'étude des invariants énumératifs (c'est à dire invariants sur les moduli) comme les invariants de Donaldson-Thomas, définis comme des intégrales au-dessus d'une classe fondamentale virtuelle. Au lieu d'utiliser ce type de technique, Rahul Pandharipande a suggéré de regarder plutôt des méthodes de corps finis inspirées des conjectures de Weil. En effet, les invariants de Donaldson-Thomas peuvent dans certains cas être définis comme une caractéristique d'Euler pondérée par une certaine fonction, ce qui en fait un invariant "motivique". Cependant, ces méthodes de corps finis ne sont pas faciles à utiliser en dehors des cas lisses, où elles n'apportent rien en général. Lors d'une conférence, Francesca Carocci, post-doc à l'EPFL, nous a suggéré de regarder l'intégration p-adique, qui est une méthode nouvelle et prometteuse en géométrie énumérative. Elle permet d'éviter des difficultés techniques liées aux comptage de $\mathbb{F}_q$-points, en gardant une sensibilité à la structure du moduli en caractéristique p. Francesca et d'autres auteurs ont défini des intégrales p-adique dont les propriétés sont similaires à certains invariants énumératifs. Michael Groechenig, un autre spécialiste, a donné un mini-cours à une autre conférence. Grâce à ces rencontres, mon stage s'est progressivement tourné vers le calcul de ces intégrales p-adiques dans un cas (non encore traité) de moduli de fibrés vectoriels au dessus d'une courbe.

Bilan académique Un avantage de ce stage était de me permettre de suivre un cheminement (assez long) entre différentes situations et différentes méthodes jusqu'à arriver à un problème intéressant et accessible. Les conférences m'ont permis de rencontrer et parler avec beaucoup de gens, notamment des doctorants et postdoc[2], ce qui m'a non seulement aidé pour le stage mais aussi donné une idée plus précise du panorama des recherches en géométrie algébrique et énumérative, en plus des exposés de recherche auxquels j'ai assisté.

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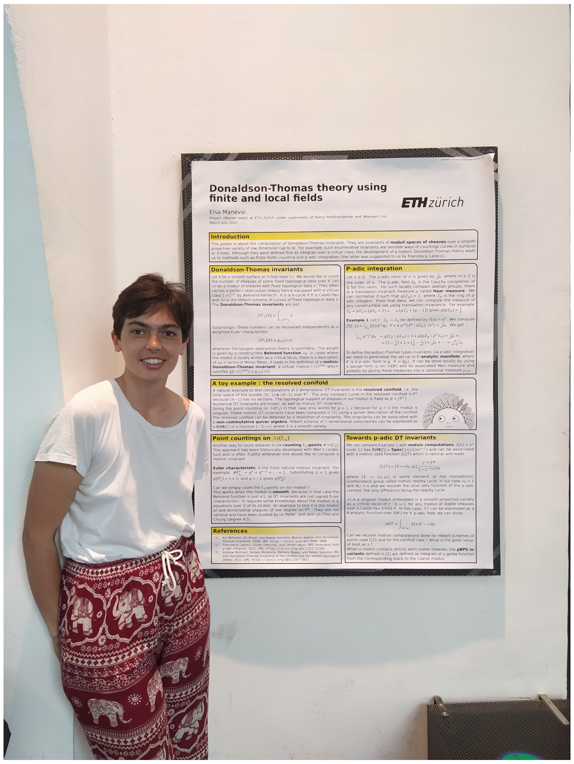
[1] Helvetic Algebraic Geometry Seminar, Les Diablerets, Suisse ; Birthday Conference of Alessandro Verra, Rome; Summer school on mirror symmetry and moduli spaces, Lisbonne ; ICM sectionals Number Theory and Algebraic Geometry, Zurich

[2] mais pas seulement, j'ai pu par exemple discuter avec Lothar Göttsche qui m'a raconté l'histoire de sa formule (voir 3.3)

Vie quotidienne J'ai beaucoup apprécié la vie à Zurich, notamment pour son calme et sa proximité avec les Alpes. Julien Moy, de ma promotion au DMA, habitait dans le même immeuble que moi et on travaillait tous les deux chez nous. Nous habitions à la limite des champs et de la forêt, tout près d'un agréable lac. J'ai pu faire de belles rencontres via le groupe de recherche, l'association locale Erasmus, les conférences ou bien le sport. Le prix de la vie était par contre démesuré, le cinéma coûtait 19 francs par exemple. Cela ne m'a pas empêchée d'apprécier grandement la Suisse, ses lacs et ses trains de montagne.Attention, la langue à Zurich n'est pas l'allemand, et il est vain d'aller dans cette ville pour progresser en allemand. Le suisse allemand est une langue vraiment distincte de l'allemand et incompréhensible pour quelqu'un parlant le "Hochdeutsch". Cependant, cela ne pose pas de problème dans la vie quotidienne où "Bonjour", "Merci" et "Excusez-moi" se disent "Grüzi", "Merci" et "Excusez".

Le rapport qui suit est rédigé en anglais et est en grande partie le rapport rendu à Rahul Pandharipande pour valider le projet au sein de l'ETHZ dans le cadre du programme de mobilité SEMP. J'ai aussi pu profiter d'une bourse Erasmus + de la DRI.



Présentation de poster à l'université Roma Tre

Abstract

Donaldson-Thomas theory gives enumerative invariants on Calabi-Yau 3-folds (meaning that there are invariants of moduli spaces over a fixed 3-fold). Since DT invariants are motivic in this case, arithmetic techniques such as finite fields countings and p-adic integration are promising methods for computations. Sometimes it is directly possible to use motivic decomposition of spaces, like the one of $\mathbf{GL}_n$ spaces is a fundamental brick in to compute virtual motives on the conifold [16]. Nonetheless in some cases, as for Hilbert schemes of points in [3], computations involves motives which could be constructed using finite fields methods. Recently, finite fields methods in algebraic geometry have been enhanced by the tool of p-adic integration. Some p-adic integrals behave like BPS invariants in [5]. For hyperkähler moduli spaces such as moduli of Higgs bundles, there is a work in progress by Wyss and Groechenig to define Donaldson-Thomas invariants in terms of p-adic integrals of gerbes. We investigate in the last section the gerbe involved in the case of moduli of vector bundles of rank 2 and degree 0 over a genus 2 curve.

I start with a very short introduction to concepts and notations of algebraic geometry. Then I report some useful background in moduli theory (Hilbert schemes of points and localisation formula). Finally I present arithmetic methods for Donaldson-Thomas theory and explain our example.

Acknowledgement I am thankful to Rahul Pandharipande for accepting my visit at ETHZ, suggesting this worthwhile topic and allowing me to participate in many conferences. I am also thankful to Woonam Lim for his valuable guidance throughout this project, showing me how to learn about these pieces of mathematics. I am grateful to Francesca Carocci and Michael Groechenig for their useful answers and suggestions.

Contents

1 Informal introduction to algebraic geometry

Algebraic geometry studies "shapes" X cut out by polynomial equations over a coefficient ring R. To understand these "shapes", a basic need is to compare them to other spaces using maps. Describing functions $X \to R$ is a key feature. It is crucial to choose the good set of functions to consider. The resulting trade-off between choosing a narrow set of functions and recording enough of the local information on X is as follows :

- Algebraic geometry is limited to algebraic functions, which are **polynomial or rational functions**.
- Algebraic functions well-defined on X as a whole are not enough. Recall the settings of complex analysis, which consider holomorphic functions $X \to \mathbb{C}$. For $X = \mathbf{S}^2$ the Riemann sphere, there are **only constant** functions[3]. In complex analysis one adds meromorphic functions to the story. Similarly, we add rational functions away from their poles. Each open set is a domain for certain rational functions. This information is stored in the **sheaf of regular functions** of X, denoted $\mathcal{O}_X$ (and also called structure sheaf).

1.1 Polynomials and Sheaves

Zariski Topology As we are limited to polynomial maps, we don't need the full euclidean topology on X, if X is defined on $\mathbb{R}$ or $\mathbb{C}$. The topology induced by polynomial functions (i.e. the weakest topology which makes polynomial functions continuous) is called **Zariski topology**. Closed sets are given by vanishing sets of families of polynomials (called algebraic sets).

Example 1.1 (Zariski topology on $\mathbb{R}$)**.** *Let $X = \mathbb{R}$. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a family of polynomials. The algebraic set defined by F is $V(F) = \{x \in X : \forall f \in F, f(x) = 0\}$. For the trivial case $F = \{0\}$ we get $V(F) = X$. By Hilbert's basis theorem, the ideal generated by F is generated by a finite family $f_1, \ldots, f_n$. Now each of the f_i's vanishes on a finite set of point so $V(F)$ is either empty, either a finite set of points.*

Zariski closed subsets of $\mathbb{R}$	$X, \emptyset,$ finite sets
Zariski open subsets of $\mathbb{R}$	$X, \emptyset,$ complements to finite sets

In particular, apart from $U = \emptyset$, all Zariski open sets are dense.

We defined the open sets in accordance to polynomial functions. We can now attach to each open U the algebra of rational functions without poles on U. We first need to admit that open sets of the form

$$U_f := \{x \in X : f(x) \neq 0\}$$

are a basis for Zariski topology. They are called **principal open** sets.

Definition 1.1 (Sheaf of regular functions on $\mathbf{k}^n$)**.** *Let $X = \mathbf{k}^n$, $\mathbf{k} = \overline{\mathbf{k}}$. The structure sheaf $\mathcal{O}_X$ is the data of a $\mathbf{k}$-algebra of functions for each open set $U \subset X$, such that there are restriction maps whenever $U \subset V$. This data is : $\mathcal{O}_X(X) := \mathbf{k}[X_1, \ldots, X_n]$, $\mathcal{O}_X(U_f) := \mathbf{k}[X_1, \ldots, X_n][f^{-1}]$ where the notation refers to localisation.*

The philosophy here is not so different from atlas and charts of differential geometry, but working with sheaves allows greater freedom since we can always choose a convenient open set to do the analysis, without fixing a specific atlas.

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[3] this happens as soon as objects are defined by charts glued in a twisted way, like projective spaces.

In general, sheaves $\mathcal{F}$ are functors from $\mathbf{Open_X}$ to $\mathbf{Set}$ with gluing conditions. We write $\mathcal{F}(U) = H^0(U;\mathcal{F})$ and call these subrings sections on U. Global sections are $H^0(X;\mathcal{F})$. There is a cohomology theory associated to sheaves.

We denote $\mathbb{A}^n := (\mathbf{k}^n, \mathcal{O}_{\mathbf{k}^n})$ and call it **affine space** in the category of algebraic varieties. A point in $X = \mathbf{k}^n$ is identified with a maximal ideal of $\mathbf{k}[X_1, \ldots, X_n]$. The corresponding ideal to $x = (x_1, \ldots, x_n)$ is given by $(X_1 - x_1, X_2 - x_2, \ldots, X_n - x_n)$. Let k be an algebraically closed field. Every maximal ideal $\mathfrak{m}$ corresponds to a point : there is a canonical map

$$\phi : \mathbf{k}[X_1, \ldots, X_n]/\mathfrak{m} \to \mathbf{k}$$

which gives a point $x = (\phi(X_1), \ldots, \phi(X_n))$.
Projective spaces $\mathbb{P}^n$ are sets given by lines of $\mathbb{A}^{n+1}$. Projective spaces are also defined by the gluing of $n+1$ affine open sets which provide them with Zariski topology and structure sheaves $\mathcal{O}_{\mathbb{P}^n}$ via gluing $\mathcal{O}_{\mathbf{k}^n}$.

To come back to our first point, **why is it relevant to restrict ourselves to polynomials ?** There is another fact hidden behind the example of the Riemann Sphere $\mathbf{S}^2$. We know that **all meromorphic functions $\mathbf{S}^2 \mapsto \mathbf{S}^2$ are rational functions** (i.e. fractions of polynomials). The algebraic structure of $\mathbb{P}^1_{\mathbb{C}}$ is in some sense sufficient to recover the analytic structure of $\mathbf{S}^2$. There is a more general statement by Serre in his famous GAGA[4] article, stating an equivalence of categories of coherent[5] sheaves and analytic coherent sheaves over a projective variety X. X defined over $\mathbb{C}$ carries an algebraic structure and an analytic structure. The fact highlighted by Serre's article is that this algebraic structure carries more information on the analytic structure than one would expect at first.

1.2 Spectra as geometric objects

After working on functions from X, we need to clarify the good set of points of X as a second step. We will be using the intuition of duality between point x and evaluation map ev_x. Scheme theory blur the role of points and functions. On algebraically closed fields, Hilbert's Nullstellensatz tells us that we can recover a point x in $\mathbb{C}^n$ from the ideal of polynomial functions vanishing at x. A point (or a set of points) is an object which can evaluate functions and which defines a **vanishing ideal**. The notion of algebraic variety is built on this correspondance.

Algebraic Varieties On algebraically closed fields k, points are given by maximal ideals and **algebraic varieties** X are defined to be locally isomorphic to $\mathrm{Spm}(A)$ (the set of maximal ideals in A), where A is a ring of polynomial functions over k. $\mathrm{Spm}(A)$ comes with its Zariski topology. Closed sets have the form $V(I) = \{\mathfrak{m} \in \mathrm{Spm}(A) \ : \ I \subset \mathfrak{m}\}$ for $I \subset A$ ideals. To bring the local information with, we study algebraic varieties as k-ringed spaces, adding a **structure sheaf** $\mathcal{O}_X$ to the data of the underlying topological space.

Now, for non algebraically closed fields like $\mathbb{R}$, we see that certain maximal ideals (e.g. (X^2+1)) do not corresponds to any real points. What is the best geometric object ? Classical set of points or **ideals as a set of points** ? The latter choice is that of modern algebraic geometry.

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[4] Géometrie Algébrique et Géométrie Analytique, published in 1956

[5] Coherent sheaves are sheaves of $\mathcal{O}_X$-module with finite presentation. There are a category of sheaves closely related to the geometric structure of X. Moduli of sheaves are very often moduli of coherent sheaves

classical view(of)point	$X = \{\text{points } P : P \in X\}$	P defines ideal $V(P) = \{f \ : \ f(P) = 0\}$
modern points	$\{\text{maximal/prime ideal } I \subset k[X]\}$	is the good set of "points" to put a topology and a structure sheaf on

Here $k[X]$ is the ring of polynomial defined over X. If X is defined by equations $f_1 = 0, \ldots, f_n = 0$ in k^n, $k[X] = k[X_1, \ldots, X_n]/I$ with $I = (f_1, \ldots, f_n)$.

To understand the richness of this construction, we can look at the spectrum corresponding to the real line.

Example 1.2. *The real affine line scheme is defined by* $\mathbb{A}^1_{\mathbb{R}} := \mathbf{Spec}\ \mathbb{R}[X]$. *Points are given by prime ideals of* $\mathbb{R}[X]$. *They are of 3 different types :*

	corresponding prime ideals
classical point	$\mathfrak{p} = (X - a), a \in \mathbb{R}$
new point	$\mathfrak{p} = (X^2 + bX + c), b^2 - 4c < 0$
generic point η	(0)

Remark 1.1. *The generic point* η *is not closed because* (0) *is not a maximal ideal. It is even dense. Even when* $k = \bar{k}$, *scheme theory attaches a generic point to every irreducible algebraic set. Its properties are in general those true for "almost all" closed points is that set. In our example of affine line over* k *we can think of the generic point as having a coordinate which is transcendental over* k *(so it can "move" along the line).*

Remark 1.2. *We see that the point* $(X^2 + bX + c)$ *corresponds to a polynomial with* 2 *complex roots. In fact, this point comes with an action of* $Gal(\mathbb{C}/\mathbb{R})$ *and splits in* $X(\mathbb{C})$, *where* X *is identified with its functor of points.*

Schemes To extend the theory to arbitrary rings R (e.g. non algebraically closed fields, e.g. $\mathbb{R}$, $\mathbb{F}_q$), we consider spectra of **prime ideals** Spec R. Now, functions can vanish on points located in **extensions of** R. Moreover, R can have a non empty nilradical (set of nilpotent elements). Nilpotent functions "vanishes everywhere but are not 0", because the nilradical of R is contained in every prime ideal. We have to enrich our notion of (closed) points by distinguishing between classical geometric points (called **reduced points**) which corresponds to ideals without nilpotent functions and fuzzed/fat/**thick points** which can be seen as points with extra data of infinitesimal movements. For example the unique element of Spec $\mathbf{k}$ is a reduced point. The only element of Spec $\mathbf{k}[X]/(X^2)$ is a thick point. These prime spectra can be given a **Zariski topology** and associated with structure sheaves, making them local model for a geometric object named **Scheme**.

(Co)tangent Space Each point p in a scheme is associated with a local ring $\mathcal{O}_{X,p}$, the stalk of the structure sheaf. The cotangent space at p is the vector space $\mathbf{m}_p/\mathbf{m}_p^2$, where $\mathbf{m}_p$ is the maximal ideal in $\mathcal{O}_{X,p}$.

Remark 1.3. *Let* X *be a variety,* $p \in X$ *a point. The* ***punctual Zariski tangent space*** $T_p(X)$ *is defined locally : they depend only on* $\mathcal{O}_{X,p}$. *It is defined by an inductive limit over open sets* $p \in U$. *Thus we can study tangent spaces of* p *in any open chart instead of* X.

It gives a notion of **smoothness** and singularity. A point is smooth if the dimension of the Zariski tangent space equals the dimension of X.

2 Hilbert Schemes of points

The base field is $k = \mathbb{C}$ and algebraic varieties are reduced k-schemes of finite type.

Hilbert schemes are the most basic spaces in moduli theory. To get familiar with it, we follow [13] presentation. I also have to credit Felix Thimm who sent me his master thesis on Donaldson-Thomas invariants of Hilbert schemes of points for toric 3-folds [24] in which most of this section is detailed.

2.1 The Hilb functor

Let X be a smooth projective algebraic variety. We introduce the Hilbert scheme of n points, $\mathrm{Hilb}^n(X)$. It is a scheme whose underlying set is the set of all subschemes of X of dimension 0 and length n.

Definition 2.1 (Length of a subscheme)**.** *Let $Z \subset X$ be a zero-dimensional subscheme. The* ***length*** *of Z is the complex vector space dimension* $\mathbf{dim}_{\mathbb{C}}\ H^0(Z, \mathcal{O}_Z)$.

Example 2.1. *Length 2 subschemes of X are either two distinct reduced points, either a "thick" point of length 2. On a surface, it is a point with a "tangent direction". Informally, when one point merge with another, we remember the tangent direction and the location of the collision (see p. 3 of [13]).*

In $\mathbb{C}^2$, a subscheme Z corresponds to an ideal $I_Z \subset \mathbb{C}[x,y]$. Length 2 means that the quotient $\mathbb{C}[x,y]/I_Z$ is 2-dimensional. The maximal ideal at (a,b) is $\mathfrak{m}_{(a,b)} = (x-a, y-b)$. For $(a,b) \neq (c,d)$, $I_{Z_1} = \mathfrak{m}_{(a,b)} \cdot \mathfrak{m}_{(c,d)}$ defines a length 2 subscheme. However, when $(a,b) = (c,d) = (0,0)$, $\mathfrak{m}^2 = (x^2, y^2, xy)$ is such that $\mathbb{C}[x,y]/\mathfrak{m}^2 \simeq \mathbb{C} \oplus \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y$ is 3-dimensional. Thus, a length 2 subscheme supported at $(0,0)$ is for example given by $I_{Z_2} = (x^2, y^2, xy, y-ax)$, where the point has "moving" coordinates along a tangent line $y - ax$.

The fact that $\mathrm{Hilb}^n(X)$ has a natural scheme structure is highly non-trivial. It was proven by Grothendieck, as a consequence of the representability of $\underline{\mathrm{Quot}}$ and $\underline{\mathrm{Hilb}}$ functors.

Definition 2.2 (Flat families)**.** *Let S be a scheme. A flat family Z of proper subschemes in X parametrised by S is a closed subscheme $Z \subset S \times X$ such that $Z \xrightarrow{pr_1} S$ is flat and proper.*

Remark 2.1. *In the category of algebraic varieties, we have :*
- A morphism $f : X \to Y$ is flat at $x \in X$ if $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,f(x)}$-module[6]*. A flat morphism is in particular open.*
- f is proper if it is separated[7] *and for all variety Z the morphism $X \times Z \xrightarrow[f\times id_Z]{} Y \times Z$ is a closed map.*

Definition 2.3 ($\underline{\mathrm{Hilb}}$ functor)**.** *$\underline{\mathrm{Hilb}}(X)$ is a functor* $\mathrm{Schemes}^{op} \to \mathrm{Set}$ *such that :*

$$\underline{\mathrm{Hilb}}(X)(S) = \{Y \textit{ flat family in } X \textit{ parametrised by } S\ \}$$
$$\underline{\mathrm{Hilb}}(X)(f : S \to S') = (id_X \times f)^{-1}(-)$$

Using Hilbert polynomials, the representability of $\underline{\mathrm{Hilb}}(X)$ functor breaks into small pieces. Hilbert polynomials can be defined using ample Cartier divisors.

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[6] flat module : A module R for which tensor product functor $- \otimes R$ is exact

[7] separated morphism : the diagonal morphism $Y \to Y \times_f Y$ is a closed immersion

When D is a **Weil divisor** (a formal combination of irreducible subvarieties of codimension 1) there is an associated coherent sheaf $\mathcal{O}(D)$ given by $\mathcal{O}(D)(U) = \{f \in R(X)^* : \mathbf{div}(f)|_U + D|_U \geq 0\}$. A Cartier divisor is a divisor whose associated sheaf is a line bundle $\mathcal{O}_X(D)$ which is a sheaf of sections given by a trivialisation $\{U_i\}_i$ and transition maps $f_i f_j^{-1}$.

Definition 2.4 (Cartier divisor). *A Cartier divisor of X is an open covering $\{U_i\}_i$ of X together with local equations (rational functions f_i on U_i such that $f_i f_j^{-1}$ is a unit in $\mathcal{O}_X(U_{ij})$). We denote $D = \{(U_i, f_i)\}$*

The support $|D|$ of D is the union of subvarieties Z such that one of f_i is not a unit in $\mathcal{O}_{Z,X}$.

Definition 2.5 (Ample line bundle). *We say that a line bundle L (over X) is **very ample** if there is an embedding of X in a projective space for which L is a line bundle associated to the intersection with the hyperplane class H. L **ample** if one positive tensor power of L is very ample.*

On a projective curve, $\mathcal{O}_X(D)$ is ample if and only if $\deg([D]) > 0$.

Definition 2.6 (Hilbert polynomial). *Let H be an ample Cartier divisor, Z a proper subscheme of X.*

$$P_Z(m) = \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(mH)) \in \mathbb{Q}[m]$$

is the Hilbert polynomial of Z

Proposition 2.1. *Let Z be a flat family parametrised by S. Let Z_s denotes the fiber over $s \in S$ of the projection map. The map $s \in S \mapsto P_{Z_s}$ is locally constant.*

Thus, we can define a functor $\underline{\mathrm{Hilb}}^P$ which is the restriction of $\underline{\mathrm{Hilb}}$ to flat families of subschemes of Hilbert polynomial P. It depends on the choice of an ample Cartier divisor H.

Theorem 2.1 (Grothendieck). *The functor $\underline{\mathrm{Hilb}}^P$ is represented by a projective scheme $\mathrm{Hilb}^P(X)$.*

For $P = n$ constant, the underlying set of this scheme are the set of subschemes of length n and dimension 0. There is $\mathcal{Z}_n \subset \mathrm{Hilb}^n(X) \times X$ a "universal subscheme", that is :

$$\forall Z \subset S \times X \text{ (flat family of length } n \text{ and dimension 0 subschemes)}$$
$$\exists! f : S \mapsto \mathrm{Hilb}^n(\mathrm{X}) \text{ such that } Z = (f \times \mathrm{id}_X)^*(\mathcal{Z}_n)$$

2.2 Classical properties of Hilbert schemes of points

The following result gives a description of the tangent space of the Hilbert scheme in terms of data on X, which is a key tool in moduli theory. We give here a detailed proof, following [13] and [24] presentation.

Definition 2.7. *Each subscheme Z is associated with an ideal sheaf $\mathcal{I}_{\mathrm{Z}}$ for which its structure sheaf $\mathcal{O}_Z$ fits in the short exact sequence :*

$$0 \to \mathcal{I}_{\mathrm{Z}} \longrightarrow \mathcal{O}_{\mathrm{X}} \longrightarrow \mathcal{O}_Z \to 0$$

Theorem 2.2 (Grothendieck). *Let $[Z] \in \mathrm{Hilb}^n(X)$ be a closed point corresponding to a subscheme $Z \subset X$. There is an isomorphism*

$$T_{[Z]}\mathrm{Hilb}^n(\mathrm{X}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}_{\mathrm{Z}}, \mathcal{O}_{\mathrm{Z}})$$

Proof. A closed point in $\mathrm{Hilb}^n(\mathrm{X})$ can be seen as a morphism $\mathrm{Spec}(\mathbb{C}) \longrightarrow \mathrm{Hilb}^n(\mathrm{X})$, and a closed point together with one of its tangent vectors is a morphism $\mathrm{Spec}(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{Hilb}^n(\mathrm{X})$ sending the closed point to $[Z]$, where $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/(t^2)$ is the ring of dual numbers.
By modularity a tangent vector at $[Z]$ in $\mathrm{Hilb}^n(\mathrm{X})$ is equivalent to the data of a zero-dimensional length n subscheme in $\mathrm{Spec}(\mathbb{C}[\varepsilon]) \times \mathrm{X}$, flat over $\mathrm{Spec}(\mathbb{C}[\varepsilon])$, which restricts to Z at the closed point. This is an ideal sheaf $\tilde{\mathcal{I}} \subset \mathcal{O}_{\mathrm{X}} \otimes \mathbb{C}[\varepsilon] = \mathcal{O}_{\mathrm{X}}[\varepsilon]$ such that the quotient $\mathcal{O}_{\mathrm{X}}[\varepsilon]/\tilde{\mathcal{I}} = \tilde{\mathcal{O}}$ is flat over $\mathbb{C}[\varepsilon]$ and restricts to $\mathcal{O}_{\mathrm{Z}}$ over the point (ε). This defines indeed a length n subscheme since Z is length n and $\mathrm{Spec}(\mathbb{C}[\varepsilon])$ is a point. In 4 steps corresponding to each colors, we get the following diagram :

$$\begin{tikzcd}
\mathcal{I}_{\mathrm{Z}} \arrow[r, hook] & \mathcal{O}_{\mathrm{X}} \arrow[r, two heads] & \mathcal{O}_Z \\
\tilde{\mathcal{I}} \arrow[r, hook] \arrow[u, two heads] & \mathcal{O}_{\mathrm{X}}[\varepsilon] \arrow[r, two heads] \arrow[u, two heads] & \tilde{\mathcal{O}} \arrow[u, two heads] \\
\mathcal{I}_{\mathrm{Z}} \arrow[r, hook] \arrow[u, hook, "\cdot\varepsilon"] & \mathcal{O}_{\mathrm{X}} \arrow[r, two heads] \arrow[u, hook, "\cdot\varepsilon"] & \mathcal{O}_Z \arrow[u, hook, "\cdot\varepsilon"]
\end{tikzcd}$$

- By definition
- The restriction of $\tilde{\mathcal{O}}$ over (ε) is $\mathcal{O}_{\mathrm{Z}}$ and it is $\mathbb{C}[\varepsilon]$-flat.
- By definition of $\mathcal{O}_{\mathrm{Z}}$
- By commutativity of the right side.

We define a morphism $t \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}_{\mathrm{Z}}, \mathcal{O}_{\mathrm{Z}})$ by going from the top left corner to bottom right of the diagram. Given $f \in \mathcal{I}_{\mathrm{Z}}$, we take a lift $f + \varepsilon g \in \tilde{\mathcal{I}}$, $g \in \mathcal{O}_{\mathrm{X}}$ and define $t(f) = \bar{g}$ where $\bar{g}$ denotes the image of g in the quotient $\mathcal{O}_Z$.
To see that it is well defined, take a second lift $f + \varepsilon g'$. Then $\varepsilon(g - g')$ maps to 0 in $\mathcal{I}_{\mathrm{Z}}$, thus $(g - g') \in \mathcal{I}_{\mathrm{Z}}$ and by considering the exact sequence attached to the subscheme Z :

$$0 \to \mathcal{I}_{\mathrm{Z}} \longrightarrow \mathcal{O}_{\mathrm{X}} \longrightarrow \mathcal{O}_Z \to 0$$

it is clear that $\overline{(g - g')} = 0$. Thus t is a well-defined morphism of $\mathcal{O}_{\mathrm{X}}$-modules.

Conversely, given $\psi \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}_{\mathrm{Z}}, \mathcal{O}_{\mathrm{Z}})$, we define $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{O}}$ by the following exact sequence :

$$\tilde{\mathcal{I}} = \{f + \varepsilon g : f \in \mathcal{I}_Z, \psi(f) = \bar{g}\} \hookrightarrow \mathcal{O}_{\mathrm{X}}[\varepsilon] \twoheadrightarrow \tilde{\mathcal{O}}$$

- $\tilde{\mathcal{I}}$ is an ideal : stability for addition follows from linearity of ψ and if $a + \varepsilon b \in \mathcal{O}_{\mathrm{X}}[\varepsilon]$ and $f + \varepsilon g \in \tilde{\mathcal{I}}$, then $(a + \varepsilon b)(f + \varepsilon g) = af + \varepsilon(ag + bf)$. Then $f \in \mathcal{I}_{\mathrm{Z}} \Rightarrow af \in \mathcal{I}_{\mathrm{Z}}$ and $bf \in \mathcal{I}_{\mathrm{Z}} \Rightarrow \overline{ag + bf} = \overline{ag}$. Finally $\psi(af) = \bar{a}\psi(f) = \overline{ag}$ because ψ is a morphism of $\mathcal{O}_{\mathrm{X}}$-modules.
- flatness of $\tilde{\mathcal{O}}$: we recall the following flatness criterium, derived from the flatness criterium for noetherian local rings of section 10.99 in [22] :

  Lemma 2.1 (Flatness criterium for sheaves)**.** *Let* F *be a sheaf on a scheme* S*.* F *is* S*-flat if and only if* $\mathrm{F}\big|_{S_{\mathrm{red}}}$ *is* S_{red}*-flat and the multiplication map* $I^{\mathrm{red}} \otimes \mathrm{F} \longrightarrow I^{\mathrm{red}} \cdot \mathrm{F}$ *is an isomorphism, where* I^{red} *is the defining ideal sheaf of* S_{red} *in* S*.*

In our situation, $\tilde{\mathcal{O}}$ is a sheaf on $\mathrm{Spec}(\mathbb{C}[\varepsilon]) \times \mathrm{X}$. Combining with the first projection map, it is also a sheaf on $S = \mathrm{Spec}(\mathbb{C}[\varepsilon])$. Moreover $\tilde{\mathcal{O}}$ is flat over S if it is S-flat with respect to that structure. Here S_{red} is the point so the first condition is automatic (because every k-module is a k-vector space) and the second condition is equivalent to $\tilde{\mathcal{O}}/\varepsilon\tilde{\mathcal{O}} \overset{.\varepsilon}{\hookrightarrow} \tilde{\mathcal{O}}$ being injective. Let $f + \varepsilon g \in \ker(.\varepsilon)$. It means $\varepsilon f \in \tilde{\mathcal{I}}$ so $\bar{f} = 0$ so $f \in \mathcal{I}_Z$. Let g' be a lift of $\psi(f)$ in $\mathcal{O}_{\mathrm{X}}$. If $g = g'$, we have shown that $f + \varepsilon g \in \tilde{\mathcal{I}}$ thus is $0 \in \tilde{\mathcal{O}}$. Else, the difference of $f + \varepsilon g$ and $f + \varepsilon g'$ lies in $\varepsilon\mathcal{O}_{\mathrm{X}}$, which surjects to $\varepsilon\tilde{\mathcal{O}}$, thus $[f + \varepsilon g] = [f + \varepsilon g'] = 0$ in $\tilde{\mathcal{O}}/\varepsilon\tilde{\mathcal{O}}$ because $f + \varepsilon g' \in \tilde{\mathcal{I}}$.

- $\tilde{\mathcal{O}}$ restricts to $\mathcal{O}_Z$ over (ε) : it comes directly from the definition : if $\varepsilon f \in \tilde{\mathcal{I}}$, then $\bar{f} = 0$ so $f \in \mathcal{O}_Z$.

We still have to show that this canonical bijection is an isomorphism. Let first understand how the linear structure of $T_{[Z]}\mathrm{Hilb}^n(\mathrm{X})$ is interpreted in terms of morphisms $\mathrm{Spec}(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{Hilb}^n(\mathrm{X})$. These morphisms all send the closed point of $\mathrm{Spec}(\mathbb{C}[\varepsilon])$ onto $[Z] \in \mathrm{Hilb}^n(\mathrm{X})$. Then each morphism is described by a induced map $\mathcal{O}_{\mathrm{Hilb}^n(\mathrm{X}),[Z]} \longrightarrow \mathbb{C}[\varepsilon]$. The linear structure on ring homomorphisms here is given by the following map : let $f, g : \mathcal{O}_{\mathrm{Hilb}^n(\mathrm{X}),[Z]} \longrightarrow \mathbb{C}[\varepsilon]$

$$f + g : \mathcal{O}_{\mathrm{Hilb}^n(\mathrm{X}),[Z]} \xrightarrow{f \otimes g} \mathbb{C}[\varepsilon_1] \otimes \mathbb{C}[\varepsilon_2] \longrightarrow \mathbb{C}[\varepsilon]$$

Specifically, this means that if we write $f = f_1 + \varepsilon f_2$, $g = g_1 + \varepsilon g_2$ (here f_i is a notation, there are not itself functions), $f + g = f_1 g_1 + \varepsilon(f_1 g_2 + f_2 g_1)$. It is possible to compute that $(f + g)(ab) = (f + g)(a)(f + g)(b)$ for a,$b \in \mathcal{O}_{\mathrm{Hilb}^n(\mathrm{X}),[Z]}$. For f, g as before, $\tilde{f}, \tilde{g} : \mathrm{Spec}(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{Hilb}^n(\mathrm{X})$ denotes the corresponding morphisms. $\tilde{\mathcal{I}}_f$ denotes the ideal sheaf of the subscheme $Y_f = (\tilde{f} \times \mathrm{id}_X)^*(\mathcal{Z}_n)$. Now $Y_{f\times g} = (\tilde{f} \times \tilde{g} \times \mathrm{id}_X)^*(\mathcal{Z}_n) \subset \mathrm{Spec}(\mathbb{C}[\varepsilon_1]) \times \mathrm{Spec}(\mathbb{C}[\varepsilon_2]) \times X$ is the result of gluing Y_f and Y_g along Y and this means that $Y_{f\times g}$ is a fiber product.

$$\begin{tikzcd}
Y_{f\times g} = Y_g \times_Y Y_f \arrow[r, "p"] \arrow[d, "q"] & Y_g \subset \mathrm{Spec}(\mathbb{C}[\varepsilon_2]) \times X \arrow[d, "pr_2"] \\
Y_f \subset \mathrm{Spec}(\mathbb{C}[\varepsilon_1]) \times X \arrow[r, "pr_2"] & Y \subset X
\end{tikzcd}$$

A function on $Y_{f\times g}$ is an element of $\mathcal{O}_{\mathrm{X}} \times \mathbb{C}[\varepsilon_1] \otimes \mathbb{C}[\varepsilon_2]$ such that the restriction to $\mathcal{O}_{\mathrm{X}} \times \mathbb{C}[\varepsilon_i]$ which send ε_j to 0 for $j \neq i$, $i = 0, 1$ gives a map on Y_f or Y_g respectively. Thus, $Y_{f\times g}$ has a defining ideal

$$\tilde{\mathcal{I}}_{f\times g} = \{a + b\varepsilon_1 + c\varepsilon_2 + d\varepsilon_1\varepsilon_2, a \in \mathcal{I}_{\mathrm{Z}}, a + b_1\varepsilon_1 \in \tilde{\mathcal{I}}_f, a + b_2\varepsilon_2 \in \tilde{\mathcal{I}}_g\}$$

Then, using the map $\mathbb{C}[\varepsilon_1] \otimes \mathbb{C}[\varepsilon_2] \longrightarrow \mathbb{C}[\varepsilon]$ which send each ε_i to ε, $\tilde{\mathcal{I}}_{f+g} = \{a + b\varepsilon, a \in \mathcal{I}_{\mathrm{Z}}, b = b_1 + b_2$ such that $a + b_1\varepsilon \in \tilde{\mathcal{I}}_f, a + b_2\varepsilon \in \tilde{\mathcal{I}}_g\}$. We have defined $t_{f+g}(a) = \bar{b} = \bar{b}_1 + \bar{b}_2 = t_f(a) + t_g(a)$ so the correspondence is linear thus an isomorphism. □

Hilbert schemes as a moduli of sheaves Hilbert schemes of points on a smooth projective variety of dimension larger than 2 can be identified with a moduli of sheaves of rank 1 with trivial determinant over X, mapping a subscheme Z to the ideal sheaf I_Z. Similarly, Hilbert schemes of curves is identified with a moduli of sheaves over X.

Hibert schemes and Symmetric product Let $S^n X = X^{\times n}/\mathfrak{S}_n$ the symmetric product of X. An element of $S^n X$ can be identified with a formal sum $\sum_{x\in X} \mathrm{mult}_x \cdot x$ such that the sum of multiplicities is n.

Theorem 2.3 (Hilbert-Chow map)**.**

$$\mathrm{Hilb}^n(X) \longrightarrow S^n X$$

$$Z \mapsto \sum_{x \in X} \mathrm{mult}_x(Z) \cdot x$$

is a scheme morphism, called ***Hilbert-Chow*** *morphism.*

The proof is in [24] for example. It requires first to give a scheme structure to the symmetric product, which can be done using Geometric Invariant Theory.

- if X a curve, it is an isomorphism.
- if X is a surface, it is a resolution of singularities and Hilbert schemes of points on a surface are smooth. We will study this case in the next section with Göttsche's formula.
- For X of dimension 3 and more, Hilbert schemes are even more singular than the symmetric product and can be reducible. However for Calabi-Yau 3-folds their sheaf-theoretic versions carries enumerative structure (such as virtual classes and virtual motives) we are interested in for Donaldson-Thomas theory.

3 Computations on Hilbert schemes of points over surfaces

In this section we investigate a powerful tool to compute integrals : the Atiyah-Bott localisation formula.

This is helpful for Donaldson-Thomas theory on toric 3-folds, because DT invariants are integrals over virtual classes, and toric action on X gives rise to toric action on moduli over X. An adapted version called virtual localisation formula has been used for many computations of Donaldson-Thomas invariants, as in MNOP paper [15].

3.1 Atiyah-Bott localisation formula

Definition 3.1 (Chow ring)**.** *Let X be a n-dimensional scheme.*

- *A formal combination of k-dimensional subvarieties in X is called a* ***k-cycle****. The free abelian group generated by k-cycles is denoted $Z_k(X)$*
- *A k-cycle is* ***rationnally equivalent to 0*** *if it is the divisor of a rational function over a $(k+1)$-dimensional subvariety.*
- *The* ***Chow group*** *$CH_k(X)$ is the quotient of $Z_k(X)$ by rational equivalence.*
- *If X is smooth, the* ***Chow ring*** *is the graded ring $CH^*(X)$ where $CH^k(X) = CH_{n-k}(X)$, the graded product is given by an intersection product.*

Proposition 3.1. *For X smooth, irreducible variety, there are* ***cycle maps*** *: for $0 \leq p \leq n$, there exists a ring morphism $CH^p(X) \to H^{2p}(X;\mathbb{Z})$.*

This proposition explains why we can abuse notation and write some divisors as cohomology classes, meaning their image by the cycle map. This allows us to use push-forward and pull back construction from intersection theory.

We denote by $\int$ the composition of cap-product and degree map, that is : if $i, j \in \mathbb{Z}$, $Y \in H_j(X)$ and $\alpha \in H^i(X)$, then $Y \cap \alpha \in H_{j-i}(X)$ is a class that we can send to $H_*(*, \mathbb{Z})$ via the push-forward along $\pi : X \to *$ (this is the degree map for $Y \cap \alpha$ in degree 0, and 0 otherwise).

$$\int_Y \alpha := \pi_*(Y \cap \alpha) \in H_*(*; \mathbb{Z})$$

If a group G acts on X, then we can define the **G-equivariant cohomology** of X. See [12] for an introduction. G-equivariant cohomology of the point is cohomology of the classifying space BG : $H^*_G(\{pt\}; \mathbb{Z}) = H^*(\mathrm{BG}; \mathbb{Z})$.

$\mathbb{C}^*$ viewed as a commutative algebraic group is a torus. A torus over a field k in general is locally a product of multiplicative groups k^* (also written $\mathbb{G}_m$).

Example 3.1. *For the torus* $G = T = \mathbb{C}^*$*, the classifying space is* $BG = \mathbb{CP}^\infty$ *and it is known that* $H^*_G(*; \mathbb{Z}) = \mathbb{Z}[\lambda]$.

If $Y \subset X$, let $\nu_{Y/X}$ denotes the **normal bundle** of Y along X.

Theorem 3.1 (Atiyah-Bott localisation theorem, [1])**.** *Let* X *be a smooth manifold over* $\mathbb{C}$ *with an action of a torus* T*. Let* X_a *be the irreducible components of the torus fixed locus in* X*. If one inverts some polynomials in* H^*_T *such that the Euler class become invertible, we have :*

$$\int_X \alpha = \sum_{X_a \subset X} \int_{X_a} \frac{\alpha|_{X_a}}{\mathbf{e}(\nu_{X_a/X})}$$

Remark 3.1. *To be more consistent, we could assume that the integral lives in the augmented* T*-equivariant cohomology of* X*,* $H^*_T(X; \mathbb{Z}) \otimes \mathrm{Frac}(H^*_T)$*. But we don't need that much. For example, for* $\mathbb{C}^*$*, the formula holds in* $\mathbb{Z}[\lambda, \lambda^{-1}]$.

For any monoid M, there is an associated **Grothendieck K-group**. The category of coherent sheaves on X gives rise to a K-group $K(X)$. If G is a group acting on X, $K_G(X)$ is the K-group associated to G-equivariant coherent sheaves. Equivariant characteristic classes are well-defined on this K-group and it is often convenient to work in this framework. K-theoretic Donaldson-Thomas theory is introduced in [18].

Computations of equivariant Euler classes : To use localisation formula, we need to compute Euler classes of the normal bundle restricted to T-fixed locus. In cases where this locus is 0-dimensional, normal bundle is tangent bundle, and tangent bundle restricted to a point is just a vector space. With the T-action, it is a finite dimensional T-representation and as such, it splits into 1-dimensional representations $\mathbf{t}^n$, $n \in \mathbb{Z}$. Whitney sum formula for Chern classes implies that for E, F vector bundles, $\mathbf{c}(E \oplus F) = \mathbf{c}(E) \cup \mathbf{c}(F) = \mathbf{c}(E) \cdot \mathbf{c}(F)$. So if $E = L_1 \oplus \cdots \oplus L_n$, L_i line bundles, we have $c(E) = \prod(1 + c_1(L_i))$. Euler classes are always top Chern classes[8]. The total Chern class of a T-equivariant bundle V with weights $\alpha_1, \ldots, \alpha_n$ is $c^G(V) = \prod_i(1 + \alpha_i\lambda)$ such that $\mathbf{e}(V) = (\prod_i \alpha_i)\lambda^n$

---

[8] It comes from the map $H^*(\mathbf{BSO}(2n)) \to H^*(\mathbf{BU}(n))$

Example 3.2 (application of localisation)**.** *Let h be a hyperplane class in $\mathbb{P}^2$. By Poincaré duality, it defines a generator of $H^2(\mathbb{P}^2) \cong \mathbb{Z}$. Two distinct lines in the projective plane intersects exactly once. Another way to write it :*

$$\int_{\mathbb{P}^2} h^2 = 1$$

Proof. We consider an action of the torus $\mathbb{C}^*$ on $\mathbb{P}^2$ given by $t \cdot [x_0 : x_1 : x_2] = [x_0 : tx_1 : t^2x_2]$ using homogenous coordinates. All Euler classes are T-equivariant Euler classes in the T-equivariant cohomology. We write sometimes $\mathbf{e}(.)$ instead of $\mathbf{e}^T(.)$. There are only 3 torus fixed points : $y_0 = [1 : 0 : 0], y_1 = [0 : 1 : 0], y_2 = [0 : 0 : 1]$. The normal bundle at a point is just the tangent bundle at the point. Then Atiyah-Bott reads

$$\int_{\mathbb{P}^2} h^2 = \frac{h^2|_{[1:0:0]}}{\mathbf{e}(\tau_{[1:0:0]})} + \frac{h^2|_{[0:1:0]}}{\mathbf{e}(\tau_{[0:1:0]})} + \frac{h^2|_{[0:0:1]}}{\mathbf{e}(\tau_{[0:0:1]})}$$

where τ is the tangent bundle. Global sections of $\mathcal{O}(1)$ are associated with the hyperplane class. Thus, $h = c_1(\mathcal{O}(1)) = -c_1(\mathcal{O}(-1))$. To be consistant with our chosen action of T, we use the following linearization of the tautological bundle : $\mathcal{O}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^2} \otimes (\mathbb{C} \oplus \mathbb{C}\mathbf{t} \oplus \mathbb{C}\mathbf{t}^2)$, where $\mathbf{t^n}$ is the one-dimensional T-representation of weight n. Denote $\lambda = c_1^T(\mathbb{C}\mathbf{t})$. Then we have the following T-equivariant Chern classes $h|_{[1:0:0]} = -c_1^T(\mathbb{C}) = 0$, $h|_{[0:1:0]} = -c_1^T(\mathbb{C}\mathbf{t}) = -\lambda$, $h|_{[0:0:1]} = -c_1^T(\mathbb{C}\mathbf{t}^2) = -2\lambda$. On the other hand, the tangent space at each points is respectively given by $\mathbb{C}\mathbf{t} \oplus \mathbb{C}\mathbf{t}^2$, $\mathbb{C}\mathbf{t}^{-1} \oplus \mathbb{C}\mathbf{t}$ and $\mathbb{C}\mathbf{t}^{-2} \oplus \mathbb{C}\mathbf{t}^{-1}$.

We get, $\mathbf{e}(\tau_{[1:0:0]}) = 2\lambda^2, \mathbf{e}(\tau_{[0:1:0]}) = -\lambda^2$ and $\mathbf{e}(\tau_{[0:0:1]}) = 2\lambda^2$.

$$\int_{\mathbb{P}^2} h^2 = 0 + \frac{\lambda^2}{-\lambda^2} + \frac{4\lambda^2}{2\lambda^2} = 1$$

□

3.2 A tautological class on $\mathrm{Hilb}^2(\mathbb{P}^2)$

Let us turn to a more sophisticated computation. Recall that we are given two projections maps

$$\begin{tikzcd}
\mathrm{Hilb}^n(X) \times X \arrow[r, "q"] \arrow[d, "p"] & X \\
\mathrm{Hilb}^n(X) &
\end{tikzcd}$$

We use Atiyah-Bott to compute the degree of a chosen **tautological class** in $\mathrm{Hilb}^2(\mathbb{P}^2)$. A tautological class is defined here as a polynomial in classes of the form $p_*(ch_k(\mathcal{O}_{\mathcal{Z}_n}) \cup q^*\gamma)$ for $\gamma \in H^*(X; \mathbb{Q})$. Pull-backs and push-forwards are well defined since p is proper and q is flat.

Remark 3.2. *These tautological classes plays a special role in moduli theory. There are many known cases of moduli space for which they generate the cohomology. Examples are Hilbert schemes of points on surfaces [21] ; Grassmannians ; moduli of stable bundles on a smooth projective curves of fixed coprime ranks and degrees [14].*

We use here both notations $X^{[n]} = \mathrm{Hilb}^n(X)$ for Hilbert Schemes of points.

Example 3.3 (Tautological insertion)**.** *Let γ be $c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^2)$*

$$\int_{\mathbb{P}^{2[2]}} p_*(ch_5(\mathcal{O}_{\mathcal{Z}_2}) \cup q^*\gamma) = \frac{11}{24}$$

Proof. The torus $T = \mathbb{C}^*$ acts on $\mathbb{P}^2$ as $t \cdot [x_0 : x_1 : x_2] = [x_0 : tx_1 : t^2x_2]$. Set $\lambda := c_1(\mathbf{t})$ as before. It induces an action of the torus $T = \mathbb{C}^*$ on the Hilbert scheme of points. The fixed locus is given by ideal sheaves supported at the T-fixed points y_j in $\mathbb{P}^2$. These ideal sheaves are given by monomial ideals in affine variables X, Y over each fixed point, which can be identified with sets of 3 partitions. Thus Atiyah-Bott formula relates this integral to a sum over sets of 3 partitions $\{\pi_0, \pi_1, \pi_2\}$ of total length 2. There are 9 such sets. A corresponding T-fixed ideal $I_{\{\pi_0,\pi_1,\pi_2\}}$ is thus either supported at two distinct fixed points (3 cases, denoted $I_{1,1,0}, I_{1,0,1}, I_{0,1,1}$), or at one point (6 cases, 2 at each fixed point, denoted $I_{0,1}, I_{0,2}, I_{1,1}, I_{1,2}, I_{2,1}, I_{2,2}$).

$$\int_{\mathbb{P}^{2[2]}} p_*(ch_5(\mathcal{O}_{\mathcal{Z}_2}) \cup q^*\gamma) = \sum_{\substack{\pi = \{\pi_0, \pi_1, \pi_2\},\\ |\pi| = 2}} \frac{p_*(ch_5(\mathcal{O}_{\mathcal{Z}_2}) \cup q^*\gamma)|_{I_\pi}}{\mathbf{e}(\nu_{[I_\pi]/\mathbb{P}^{2[2]}})}$$

Euler classes : $\mathrm{Hilb}^2(\mathbb{C}^2)$ is smooth, so the normal bundle restricted to a point is the tangent space at this point[9].

Using Theorem 2.2, we see that the tangent space at a 0-subscheme $Z \subset \mathbb{P}^2$ in $\mathrm{Hilb}^2(\mathbb{P}^2)$ is isomorphic to the tangent space at the image in an affine chart $\tilde{Z} \subset \mathbb{C}^2$ in $\mathrm{Hilb}^2(\mathbb{C}^2)$. Let I_{π_i} corresponds to a subscheme only supported at x_i. We have

$$T_{[I_{\pi_i}]}\mathbb{P}^{2[2]} \simeq T_{[\tilde{I}_{\pi_i}]}\mathbb{C}^{2[2]}.$$

The usual chart maps are denoted $(U_0, \phi_0)(U_1, \phi_1), (U_2, \phi_2)$ where for example

$$\phi_0 : U_0 \mapsto \mathbb{C}^2, [x_0 : x_1 : x_2] \mapsto (\frac{x_1}{x_0}, \frac{x_2}{x_0}) = (X, Y).$$

$(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ as $(t_1, t_2) \cdot (x, y) = (t_1x, t_2y)$. Denote $\mathbf{t}_1$ and $\mathbf{t}_2$ the associated 1-dimensional representations of $\mathbb{C}^*$. Now we can describe the torus action induced on each chart in terms of pairs of weighted representations $(\mathbf{t}^m, \mathbf{t}^n)$ coming from the action of $t \in T$ on $\mathrm{Hilb}^2(\mathbb{P}^2)$.

$\mathbb{C}^2$ chart	$(t_1, t_2) \cdot (x, y) = (t_1x, t_2y)$ torus action	$\mathbf{t}_1$	$\lambda_1 = c_1(\mathbf{t}_1)$	$\mathbf{t}_2$	$\lambda_2 = c_1(\mathbf{t}_2)$
U_0	$t \cdot (X, Y) = (tX, t^2Y)$	$\mathbf{t}$	λ	$\mathbf{t}^2$	2λ
U_1	$t \cdot (X, Y) = (t^{-1}X, tY)$	$\mathbf{t}^{-1}$	$-\lambda$	$\mathbf{t}$	λ
U_2	$t \cdot (X, Y) = (t^{-2}X, t^{-1}Y)$	$\mathbf{t}^{-2}$	-2λ	$\mathbf{t}^{-1}$	$-\lambda$

Table 1: Weights on each affine chart

T-equivariant Euler classes will be given by the T-equivariant structure of tangent spaces at fixed points in $\mathrm{Hilb}^2(\mathbb{C}^2)$, which are given by sets of partitions. [18] provides a formula for the T-equivariant K-theory class of the tangent space at a T-fixed point in $\mathrm{Hilb}^n(\mathbb{C}^2)$ corresponding to a partition λ. For a partition $\lambda = \{(i, j) \in \mathbb{Z}^2_{\leq 0} \mid x^iy^j \notin I_\lambda\}$, we define a generating function $V = \sum_{(i,j)\in\lambda} \mathbf{t}_1^{-i}\mathbf{t}_2^{-j}$.

Proposition 3.2 ([18], Proposition 3.4.17)**.** *In T-equivariant K-Theoretical ring,*

$$T_{I_\lambda} = V + \bar{V}\mathbf{t}_1\mathbf{t}_2 - V\bar{V}(1 - \mathbf{t}_1)(1 - \mathbf{t}_2),$$

where V is the generating function for λ and $\bar{V}$ its dual.

---

[9] Indeed, a vector bundle over a point is just a vector space

This formula follows from the description of the tangent space of the Hilbert scheme we proved in 2.2.
For the fixed subscheme supported at 1 fixed point,

- Subscheme supported at x_0 :
  - $I_{0,1} := (X^2, Y)$, $T_{I_{0,1}} = \mathbf{t} + \mathbf{t} + \mathbf{t}^2 + \mathbf{t}^2$ so $\mathbf{e}(T_{I_{0,1}}) = 4\lambda^4$
  - $I_{0,2} := (X, Y^2)$, $T_{I_{0,1}} = \mathbf{t} + \mathbf{t}^2 + \mathbf{t}^4 + \mathbf{t}^{-1}$ so $\mathbf{e}(T_{I_{0,2}}) = -8\lambda^4$
- Subscheme supported at x_1 :
  - $I_{1,1} := (X^2, Y)$, $T_{I_{1,1}} = \mathbf{t} + \mathbf{t}^{-1} + \mathbf{t}^2 + \mathbf{t}^{-2}$ so $\mathbf{e}(T_{I_{1,1}}) = 4\lambda^4$
  - $I_{1,2} := (X, Y^2)$, $T_{I_{1,1}} = \mathbf{t} + \mathbf{t}^{-1} + \mathbf{t}^2 + \mathbf{t}^{-2}$ so $\mathbf{e}(T_{I_{1,2}}) = 4\lambda^4$
- Subscheme supported at x_0 :
  - $I_{2,1} := (X^2, Y)$, $T_{I_{2,1}} = \mathbf{t}^{-4} + \mathbf{t} + \mathbf{t}^{-2} + \mathbf{t}^{-1}$ so $\mathbf{e}(T_{I_{2,1}}) = -8\lambda^4$
  - $I_{2,2} := (X, Y^2)$, $T_{I_{2,2}} = \mathbf{t}^{-1} + \mathbf{t}^{-1} + \mathbf{t}^{-2} + \mathbf{t}^{-2}$ so $\mathbf{e}(T_{I_{2,2}}) = 4\lambda^4$

For fixed subschemes supported at two closed points, we use the formula at each point and use the decomposition in direct sum of the tangent space.

- For $I_0 = (X, Y)$ in U_0, we get $T_{I_0} = \mathbf{t} + \mathbf{t}^2$,
- for $I_1 = (X, Y)$ in U_1, we get $T_{I_1} = \mathbf{t}^{-1} + \mathbf{t}$,
- for $I_2 = (X, Y)$ in U_2, we get $T_{I_2} = \mathbf{t}^{-2} + \mathbf{t}^{-1}$,

Using the fact that the tangent space at I_π is $T_{\pi_i} \oplus T_{\pi_j}$,

- for $\pi_0 = 1, \pi_1 = 1$, $\mathbf{e}(T_{I_{1,1,0}}) = -2\lambda^4$,
- for $\pi_2 = 1, \pi_1 = 1$, $\mathbf{e}(T_{I_{0,1,1}}) = -2\lambda^4$,
- for $\pi_0 = 1, \pi_2 = 1$, $\mathbf{e}((T_{I_{1,0,1}}) = 4\lambda^4$.

Gamma classes : as for the previous example, the chosen action of T provides a T-equivariant linearization of $\mathcal{O}(-1)$. Thus

$$\begin{aligned} q^*\gamma|_{[1:0:0]} &= q^*c_1(\mathcal{O}(1))|_{[1:0:0]} = 0 \\ q^*\gamma|_{[0:1:0]} &= q^*c_1(\mathcal{O}(1))|_{[0:1:0]} = -\lambda \\ q^*\gamma|_{[0:0:1]} &= q^*c_1(\mathcal{O}(1))|_{[0:0:1]} = -2\lambda \end{aligned}$$

As the gamma class is 0 for subschemes supported at x_0, we do not need to compute Chern characters of $I_{0,1}$ and $I_{0,2}$.

Chern characters : We compute equivariant $ch_5(\mathcal{O}_{\mathcal{Z}_2})$ at a fixed point corresponding to a subscheme Z either supported at one point, either at two points. Thanks to the smoothness of $\mathrm{Hilb}^2(\mathbb{P}^2)$, every coherent sheaf admits a finite locally free resolution. Moreover Chern characters are well-defined on Grothendieck groups and additive on exact sequences so we can use these resolutions to compute them as a sum.

- Let $\mathrm{Z_{i,j}}$ be supported at one point. By universal property,

  $$ch_5(\mathcal{O}_{\mathcal{Z}_2})|_{I_{i,j}} = ch_5(\mathcal{O}_{Z_{i,j}}).$$

  We assume that the associated ideal in the chart is (x^2, y). We can use the following T-equivariant resolution

  $$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3h) \otimes \mathbf{t}_1^2\mathbf{t}_2 \underset{P \mapsto (-yP, x^2P)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2}(-2h) \otimes \mathbf{t}_1^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-h) \otimes \mathbf{t}_2 \underset{(a,b) \mapsto ax^2+by}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

  Then, Chern character being well-defined on T-equivariant K-theory classes, we get :

  $$\begin{aligned} ch(\mathcal{O}_Z) &= ch(\mathcal{O}_{\mathbb{P}^2}) - ch(\mathcal{O}_{\mathbb{P}^2}(-2h) \otimes \mathbf{t}_1^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-h) \otimes \mathbf{t}_2) + ch(\mathcal{O}_{\mathbb{P}^2}(-3h) \otimes \mathbf{t}_1^2\mathbf{t}_2) \\ &= 1 - e^{2(\lambda_1 - h)} - e^{\lambda_2 - h} + e^{2\lambda_1 + \lambda_2 - 3h} \end{aligned}$$

  Thus,

  $$ch_5(\mathcal{O}_Z) = \frac{1}{5!}\left((-2\lambda_1 + 2h)^5 + (-\lambda_2 + h)^5 + (2\lambda_1 + \lambda_2 - 3h)^5\right)$$

  For $Z_{i,j}$ defined by $I_{i,j}$ we have to specify what is λ_1, λ_2, x and y in each of the 4 cases, using Table 1. It gives :

  $$\begin{aligned} ch_5(\mathcal{O}_{Z_{1,1}}) &= \frac{1}{5!}\left((2\lambda + 2h)^5 + (-\lambda + h)^5 + (-\lambda - 3h)^5\right) \\ ch_5(\mathcal{O}_{Z_{1,2}}) &= \frac{1}{5!}\left((-2\lambda + 2h)^5 + (\lambda + h)^5 + (\lambda - 3h)^5\right) \\ ch_5(\mathcal{O}_{Z_{2,1}}) &= \frac{1}{5!}\left((4\lambda + 2h)^5 + (\lambda + h)^5 + (-5\lambda - 3h)^5\right) \\ ch_5(\mathcal{O}_{Z_{2,2}}) &= \frac{1}{5!}\left((2\lambda + 2h)^5 + (2\lambda + h)^5 + (-4\lambda - 3h)^5\right) \end{aligned}$$

- When Z is supported at two points, $Z = Z_1 \cup Z_2$, corresponding to I_1, I_2,

  $$\begin{aligned} ch_5(\mathcal{O}_{\mathcal{Z}_2} \cup q^*\gamma)|_I &= ch_5(\mathcal{O}_{\mathcal{Z}_2} \cup q^*\gamma)|_{I_1} + ch_5(\mathcal{O}_{\mathcal{Z}_2} \cup q^*\gamma)|_{I_2} \\ &= ch_5(\mathcal{O}_{Z_1} \cup q^*\gamma_{y_i}) + ch_5(\mathcal{O}_{Z_2} \cup q^*\gamma_{y_j}) \end{aligned}$$

  and we use the following resolution :

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2h) \otimes \mathbf{t}_1\mathbf{t}_2 \underset{P\mapsto(-yP,xP)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2}(-h) \otimes \mathbf{t}_1 \oplus \mathcal{O}_{\mathbb{P}^2}(-h) \otimes \mathbf{t}_2 \underset{(a,b)\mapsto ax+by}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

It gives :

$$ch(\mathcal{O}_{Z_i}) = 1 - e^{\lambda_1-h} - e^{\lambda_2-h} + e^{\lambda_1+\lambda_2-2H}$$

where we specialize again λ_i in each chart :

$$\begin{aligned}
ch_5(\mathcal{O}_{Z_0}) &= \frac{1}{5!}\big((-\lambda+h)^5 + (-2\lambda+h)^5 + (3\lambda-2h)^5\big) \\
ch_5(\mathcal{O}_{Z_1}) &= \frac{1}{5!}\big((\lambda+h)^5 + (-\lambda+h)^5 + (-2h)^5\big) \\
ch_5(\mathcal{O}_{Z_2}) &= \frac{1}{5!}\big((2\lambda+h)^5 + (\lambda+h)^5 + (-3\lambda-2h)^5\big)
\end{aligned}$$

Conclusion The pushforward along p is the same as integrating along $\mathbb{P}^2$, so we get exactly coefficients of h^2 of $(ch_5(\mathcal{O}_{\mathcal{Z}_2}) \cup q^*\gamma)_{|I_\pi}$, which are polynomials of degree 4 in λ.

- For subscheme supported at 1 point, these coefficient are :

$$\begin{aligned}
p_*(ch_5(\mathcal{O}_{Z_{1,1}}) \cup q^*\gamma)_{|I_{1,1}} &= \frac{1}{12}\big(2^3 \cdot 2^2 \cdot \lambda^3 - \lambda^3 - 3^2 \cdot \lambda^3\big) \cdot (-\lambda) \\
&= \frac{11}{6}\lambda^4 \\
p_*(ch_5(\mathcal{O}_{Z_{1,2}}) \cup q^*\gamma)_{|I_{1,2}} &= -\frac{11}{6}\lambda^4 \\
p_*(ch_5(\mathcal{O}_{Z_{2,1}}) \cup q^*\gamma)_{|I_{2,1}} &= \frac{434}{3}\lambda^4 \\
p_*(ch_5(\mathcal{O}_{Z_{2,2}}) \cup q^*\gamma)_{|I_{2,2}} &= \frac{272}{3}\lambda^4
\end{aligned}$$

- For subschemes supported at 2 points, the coefficients are sums of coefficients corresponding to each point :

$$\begin{aligned}
p_*(ch_5(\mathcal{O}_{Z_{[0:1:0]}}) \cup q^*\gamma)_{|I_{[0:1:0]}} &= \frac{1}{5!}\big(-10 \cdot \lambda^3 + 10 \cdot \lambda^3\big)(-\lambda) \\
&= 0 \\
p_*(ch_5(\mathcal{O}_{Z_{[0:0:1]}}) \cup q^*\gamma)_{|I_{[0:0:1]}} &= \frac{1}{5!}\big(10 \cdot 2^3 \cdot \lambda^3 + 10 \cdot \lambda^3 - 10 \cdot 3^3 \cdot 2^2 \cdot \lambda^3\big) \cdot (-2\lambda) \\
&= \frac{33}{2}\lambda^4
\end{aligned}$$

Finally, we divide by Euler classes. Contributions of $I_{1,1}$ and $I_{1,2}$ cancels because $\mathbf{e}(T_{I_{1,1}}) = \mathbf{e}(T_{I_{1,2}})$.

We get only these 4 terms :

$$\begin{aligned}\int_{\mathbb{P}^{2[2]}} p_*(ch_5(\mathcal{O}_{\mathcal{Z}_2}) \cup q^*\gamma) &= \frac{p_*(ch_5(\mathcal{O}_{Z_{2,1}}) \cup q^*\gamma)_{|I_{2,1}}}{\mathbf{e}(T_{I_{2,1}})} + \frac{p_*(ch_5(\mathcal{O}_{Z_{2,2}}) \cup q^*\gamma)_{|I_{2,2}}}{\mathbf{e}(T_{I_{2,2}})} \\ &+ \frac{p_*(ch_5(\mathcal{O}_{Z_{[0:0:1]}}) \cup q^*\gamma)_{|I_{[0:0:1]}}}{\mathbf{e}(T_{I_{0,1,1}})} + \frac{p_*(ch_5(\mathcal{O}_{Z_{[0:0:1]}}) \cup q^*\gamma)_{|I_{[0:0:1]}}}{\mathbf{e}(T_{I_{1,0,1}})} \\ &= \frac{\frac{434}{3}}{-8} + \frac{\frac{272}{3}}{4} + \frac{\frac{33}{2}}{-2} + \frac{\frac{33}{2}}{4} \\ &= \frac{11}{24}\end{aligned}$$

□

Remark 3.3. *All the Cherns classes have palindroms in the numerator. However, I did not find any pattern (for example the ch_4 are not palindromic in base $2 \times 4 = 8$). Thus, I believe it is random rather than a new form of mirror symmetry.*

3.3 Göttsche's formula

Betti numbers of Hilbert schemes of points on smooth surfaces (which are smooths) are described by Göttsche's formula. To study the homolgy of Hilbert schemes of n points, it is much easier to consider all values of n at once with generating series. This is what Göttsche's formula do. There are two versions, one which compute Betti numbers of Hilb^n in term of (shifted around 0) Betti numbers of X, and one in terms of Euler characteristics $\chi(X) = \sum_{i\geq 0}(-1)^i b_i(X)$. They are proven in Göttsche's original paper [9].

Theorem 3.2 (Göttsche, 1990)**.** *Let X be a projective surface. For p and q formal parameters, the Betti numbers satisfies :*

$$\sum_{n\geq 0}\sum_{i\geq 0}(-1)^i b_i(X^{[n]})y^i t^n = \prod_{k\geq 1}\prod_{i\geq 0}(1 - y^i t^k)^{(-1)^{i+1}b_i(X)}$$

Euler characteristics satisfies :

$$\sum_{n\geq 0}\chi(X^{[n]})t^n = \prod_{m\geq 1}(1 - t^m)^{-\chi(X)}$$

These formulae was first proved using Weil's conjecture, without localisation formula.

proof for toric surface, second formula. X a toric surface (smooth, projective). Then localisation formula implies that $\chi(X) = \#X^T$. Torus action on X induces a torus action on $X^{[n]}$. Each torus fixed subscheme has to be supported at fixed points and thanks to Atiyah-Bott :

$$\chi(X^{[n]}) = \int_{X^{[n]}} \mathbf{e}(T_{X^{[n]}}) = \sum_{\text{I fixed ideal}} \frac{\mathbf{e}(T_{X^{[n]}}|_I)}{\mathbf{e}(N_{I/X^{[n]}})} = \sum_{\text{I fixed ideal}} \frac{\mathbf{e}(T_{X^{[n]}}|_I)}{\mathbf{e}(T_{I/X^{[n]}})} = \#\{\text{fixed ideal}\} = \chi((X^{[n]})^T)$$

using that the normal bundle over a smooth point is just the tangent bundle (it is crucial here that X is a surface so its Hilbert scheme of point is smooth), and $(X^{[n]})^T$ is zero-dimensional. Thus $\chi(X^{[n]}) = \chi((X^{[n]})^T)$ and we just have to count the torus fixed points in the Hilbert schemes. A T-fixed ideals of length n in X corresponds to a partition of $n = \sum_{i\in X^T} l_i$ in terms of length l_i at a fixed point and monomial ideals in two variables over each fixed points corresponding to partitions of l_i. Denote $p(n)$ the partition number of n. For $\chi = \# X^T$, the contribution of $(X^{[n]})^T$ is

$$\sum_{\sum l_i = n} p(l_1) + \cdots + p(l_\chi).$$

In total,

$$\sum_{n \geq 0} \chi(X^{[n]}) q^n = \sum_n \sum_{\sum l_i = n} p(l_1) + \cdots + p(l_\chi) q^n = (\sum_m p(m) q^m)^\chi = \prod_{m \geq 1} (1 - q^m)^{-\chi}$$

. □

Remark 3.4. *Here X is projective and so are $X^{[n]}$, but the formula works as well for the affine plane $X = \mathbb{A}^2$, taking Borel-Moore homology (this was proved independently from the general case by Ellingsrud and Strømme [8]). Using localisation, the proof works as for the projective case, the contribution of $\chi((X^{[n]})^T)$ is $p(n)$ and $\chi(\mathbb{C}^2) = 1$.*

Remark 3.5. *Further work on this formula has shown that it is in fact a character formula for an irreducible representation of an infinite dimensional Lie algebra (Heisenberg-Clifford algebra). The homology rings of Hilbert schemes $\oplus_n H_*(X^{[n]})$ together form a graded highest weight module for this algebra. Authors in [6] used this as a starting point to prove the formula. This new proof is more direct as it doesn't use the decomposition theorem.*

Remark 3.6. *There is the modular form η in the right hand side of the formula.*

In the case of smooth Hilbert scheme, (signed) Euler characteristic is the Donaldson-Thomas invariant. Thus, Göttsche's formula can be seen as a formula for certain generating series of DT invariants. The original ambitious goal for my project was to investigate if similar formula in terms of DT series could be obtained for 3-folds via finite fields counting.

4 Donaldson-Thomas Theory

Donaldson-Thomas theory gives **enumerative invariants** on 3-folds. This means that they are invariants of moduli spaces of sheaves over the 3-fold. These moduli are often singular and difficult to describe geometrically.

DT invariants are in particular defined on moduli spaces of stable coherent sheaves on smooth projective Calabi-Yau 3-folds. In the following part, we focus only on moduli spaces $\mathcal{M}$ being a Hilbert scheme of points or curves, following [19].

Definition 4.1. *Let X be a smooth quasi-projective 3-fold.* ***Hilbert schemes of curves*** $\mathbf{Hilb}_{\beta,n}(X)$ *parameterizes subcurves (subscheme of dimension 1) $Z \subset X$ of fixed curve class support $\beta \in H_2(X;\mathbb{Z})$ with $\chi(\mathcal{O}_Z) = n$ for $n \in \mathbb{N}$*

What are these invariants ? The most basic invariant we can think of is cardinality. As a moduli space $\mathcal{M}$ has no reason to be 0-dimensional, there is a procedure to "reduce to dimension 0". Behrend and Fantechi [4] introduced a **virtual class** $[\mathcal{M}]^{vir}$ using a sheaf-theoretic tangent-obstruction theory. It gives a 0-cycle if the canonical bundle of X is trivial (X is **Calabi-Yau**) and

$\mathcal{M}$ is for example $\mathbf{Hilb}_{\beta,n}(X)$. In this case, we simply define a **Donaldson-Thomas invariant** by integrating 1 against a virtual class.

$$I_{n,\beta} = \int_{[\mathbf{Hilb}_{\beta,n}(X)]^{vir}} 1$$

Behrend showed that in this case these invariants can be computed independently from the chosen tangent-obstruction theory, as a weighted Euler characteristic χ_{vir}. The weight is given by an algebraic constructible "Behrend function" ν.

4.1 Donaldson-Thomas partition function

Donaldson-Thomas theory gives invariants of moduli of coherent sheaves on X. Using the short exact sequence of sheaves

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

a rank-1 sheaf $\mathcal{I}_Z$ can be associated to each closed subscheme Z. This is in fact an equivalence of functor between Hilbert schemes of points or curves and moduli of rank-1 coherent sheaves with trivial determinant denoted $I_n(X,\beta)$. In the case of β having **genus 0** with no deformation to higher genus curves[10], these subcurves have $n-1$ roaming points (either **free** or **embedded**). By representability of $\mathbf{Hilb}_{\beta,n}(X)$, the corresponding moduli of sheaves $I_n(X,\beta)$ is also represented so it exists a universal rank-1 sheaf denoted $\mathcal{F}$.

To define the **virtual class** of $I_n(X,\beta)$, there are 3 steps :
1 - Define $I_n(X,\beta)$ as being a representation of a moduli functor of coherent sheaves of rank 1 and trivial determinant with Chern character $(1,0,-\beta,-n)$ (in degree $(0,2,4,6)$).
2 - Construct a tangent-obstruction theory on $I_n(X,\beta)$ and show that it is perfect.
3 - Define a virtual cone in a total space of a vector bundle over $I_n(X,\beta)$, use Gysin morphisms to pull back the cone's fundamental class over the zero section. This defines a cycle called **fundamental virtual class**.

Fundamental class is in the degree of the "expected dimension of $I_n(X,\beta)$", which is
$\mathrm{Ext}^1(\mathcal{F},\mathcal{F})_0 - \mathrm{Ext}^2(\mathcal{F},\mathcal{F})_0$. Serre duality asserts $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})_0 \simeq \mathrm{Ext}^2(\mathcal{F},\mathcal{F}\otimes K_X)_0^\vee$. For Calabi-Yau 3-folds, $K_X \simeq \mathcal{O}_X$ so the expected dimension is 0.

This requires working with sheaves because some vanishings happening in traceless Ext groups of sheaves makes the tangent-obstruction theory simpler. In general tangent spaces are given by Ext^1 groups while obstruction spaces are Ext^2 groups related to the universal object $\mathcal{F}$ in the moduli, if exists. We do not give details nor complete definitions since it requires derived categories formalism, and we prefer going forward applications. The construction is done in [4] original article. We admit that this construction provides us with a cycle which has an enumerative meaning for curve countings and we want to compute Donaldson-Thomas numerical invariants given by

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1 \ \in \mathbb{Z}$$

Among other properties, Donaldson-Thomas invariants are invariant on **deformation** of X.

---

[10] In the general case, $\chi(\mathcal{O}_Z) = n - g + 1$, and there can be phenomena of trade-off between genus and free points [19].

In section 3.3, we saw that it is fruitful to study families of Hilbert schemes of points. This applies also here : it is a generating function which is involved in MNOP conjectures, relating Donaldson-Thomas and Gromov-Witten theory.

Definition 4.2. *Let X be a Calabi-Yau 3-fold. For a fixed curve class $\beta \in H_2(X, \mathbb{Z})$, a partition function for Donaldson-Thomas invariants is*

$$Z_\beta^{DT}(t) = \sum_{n \in \mathbb{Z}} I_{n,\beta} t^n$$

When the moduli space $\mathcal{M} = I_n(X, \beta)$ is **smooth**, the sheaf of Kähler differential $\Omega_{\mathcal{M}}$ (which is the obstruction part of the deformation theory) is a vector bundle and the virtual class is given by its Euler class.

$$\int_{[\mathcal{M}]^{vir}} 1 = \int_{[\mathcal{M}]} \mathbf{e}(\Omega_X) = (-1)^{dim \mathcal{M}} \chi(\mathcal{M})$$

So in this case DT invariants are just signed Euler characteristic. The idea of Behrend was to extend this formula by finding a formulation of DT invariants in terms of a weighted Euler characteristic.

4.2 Motivic Donaldson-Thomas invariants

Let Y be any scheme. In [2], the **Behrend function** $\nu : Y \to \mathbb{Z}$ is introduced.

Definition 4.3. *Let Y be a scheme.*
*- a **constructible set** is a finite union of intersections of open and closed sets.*
*- a **constructible partition** is a locally finite, disjoint union of locally constructible sets.*
*- a **constructible function** $f : Y \to \mathbb{Z}$, meaning that $\bigcup_{n \in \mathbb{Z}} \nu^{-1}(n)$ is a constructible partition.*

The Behrend function ν is a constructible function over any scheme. It was defined as the MacPherson local Euler obstruction of a distinguished cycle related to normal cones of embeddings of Y in smooth S. It is difficult to compute in general.
If Y is smooth, it is just the topological Euler characteristic with a sign : $\nu(Y) = (-1)^{\mathbf{dim}(X)}$.

Definition 4.4. *The **virtual Euler characteristic** (or ν-weighted Euler characteristic) is*

$$\chi_{vir}(Y) := \sum_{n \in \mathbb{Z}} n \cdot \chi(\nu^{-1}(n))$$

In the case of X being Calabi-Yau 3-fold, tangent-obstruction theories associated to a moduli of sheaves $\mathcal{M}$ are **symmetric**. This means that it is coming with a non-degenerate bilinear form given by Serre duality for sheaf cohomology on trace-less Ext groups. For $[F] \in \mathcal{M}$,
$\mathrm{Ext}^1(F, F)_0 \simeq \mathrm{Ext}^2(F, F)_0^\vee$, because $K_X \simeq \mathcal{O}_X$.

Theorem 4.1 (Behrend, [2])**.** *For $\mathcal{M}$ a moduli equipped with a symmetric tangent-obstruction theory,*

$$\int_{[\mathcal{M}]^{vir}} 1 = \chi_{vir}(\mathcal{M})$$

In other words, DT invariants of Calabi-Yau 3-folds can be computed independently from tangent-obstruction theories and virtual classes.

The formulation of Donaldson-Thomas invariant as a (weighted) Euler characteristic brings us toward **motives**. Euler characteristic satisfies inclusion-exclusion principle. For $A, B \subset X$ subspaces, $\chi(A \cup$

$B) = \chi(A) + \chi(B) - \chi(A \cap B)$. Euler characteristic has many ties with cardinality. For example, we can compute it by counting cells of CW-complexes. This was pushed further away by Grothendieck. As for K-theory classes, Euler characteristic in an invariant for Motives. Motives are complicated to define formally. For us, a **motive** (or a pure Chow motive) lies in a K-group $\mathcal{M}_{\mathbb{C}}$ whose generators are isomorphism classes of algebraic varieties. There is an equivalence : $X = U \amalg V \Rightarrow [X] \sim [U] + [V]$ for U, V topological subspaces. The $+$ is given by disjoint union and the multiplication $\times$ is cartesian product. Extra operations $\oplus$ and $\otimes$ are well-defined. The ring is extended with $\mathbb{L}^{\frac{1}{2}}$, where the Lefschetz motive $\mathbb{L}$ is the image of $\mathbb{A}^1$.

Example 4.1. *We can write in terms of motives that* $[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + [1]$, *where* $[1]$ *corresponds to the point.*

Remark 4.1. ***Euler characteristic*** *is a natural motivic invariant.* $\#\mathbb{P}^n_{\mathbb{F}_q} = q^n + q^{n-1} + \cdots + 1$. *Substituting* $q = 1$ *gives the well-known result that* $\chi(\mathbb{P}^n_{\mathbb{C}}) = n + 1$. *Moreover* $q = -1$ *gives* $\chi(\mathbb{P}^n_{\mathbb{R}})$, *which follows from* $\chi_{BM}(\mathbb{R}) = -1$, *where* χ_{BM} *is the Euler characteristic for Borel-Moore homology (i.e. compactly supported homology)*

There are various rings associated to motives. There are a completion $\tilde{\mathcal{M}_{\mathbb{C}}} = \mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-1}]$ and a ring of equivariant motives $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$, with $\hat{\mu}$ group of roots of unity.

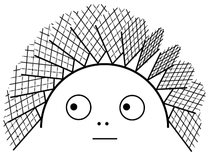
Definition 4.5 (Virtual Motive of Y, in the sense of [3]). *Let* Y *be a finite type* $\mathbb{C}$*-scheme. A* ***virtual motive*** *of* Y *is* $[Y]^{mot} \in \mathcal{M}_{\mathbb{C}}$ *such that* $\chi([Y]^{mot}) = \chi_{vir}(Y)$.

Following Weil's conjecture philosophy and the spirit of Example 4.1, there is an expectation of recovering motivic data such as χ_{vir} using finite fields $\mathbb{F}_q$. Can we compute Donaldson-Thomas invariants by counting $\mathbb{F}_q$-points of moduli spaces ?

5 Arithmetic techniques for Donaldson-Thomas theory

5.1 Finite fields countings

The first thing we could do to answer the question was to try finite fields countings with a concrete example whose DT invariants are already known.



We study the case of a smooth 3-fold called the **resolved conifold**[11], given by the total space X of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$.
Over $\mathbb{C}$, this variety is smooth, quasi-projective and Calabi-Yau. This variety has only one closed subcurve, given by its 0-section $\mathbb{P}^1$ because $\mathcal{O}(-1)$ has no sections. We can compare results with already computed numerical DT-invariants.

Proposition 5.1 (Adapted from [19]). *On the resolved conifold,*

$$I_{n,\mathbb{P}^1} = \frac{qM(-q)^2}{(1+q)^2}$$

where $M(-q)$ *is the Mac-Mahon partition function. The first terms are given by :*

$$I_{n,\mathbb{P}^1} = q - 4q^2 + 14q^3 - 35q^4 \ldots$$

---

[11] the illustration was made by Oscar Bouverot-Dupuis as a free interpretation of my command

X is well-defined over $\mathbb{F}_q$ and the Hilbert schemes of curves can be stratified such that we can count the closed $\mathbb{F}_q$ points. This naive counting is $\#\mathcal{M}_{\mathbb{P}^1,n+1}(\mathbb{F}_q)$.

$$\#\mathcal{M}_{\mathbb{P}^1,1}(\mathbb{F}_q) = 1$$
$$\#\mathcal{M}_{\mathbb{P}^1,2}(\mathbb{F}_q) = q^3 + 2q^2 + q$$
$$\#\mathcal{M}_{\mathbb{P}^1,3}(\mathbb{F}_q) = (q+1)^2(\frac{1}{2}q^4 + 2q^3 + \frac{3}{2}q^2)$$

Taking the limit $q \to 1$ gives $1, 4, 16$. We see that it does not corresponds to the answer series from the third term. This is not surprising, since Hilbert schemes are expected to be singular, and doing this computation gives the Euler characteristic, which matches Donaldson-Thomas invariants only in smooth case up to a sign.

Now, the idea is to embed our singular moduli in a smooth space such that we can do finite fields countings on the smooth space.

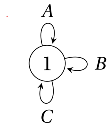
5.2 Non-commutative embedding

Definition 5.1 (Quivers)**.**
*- A **quiver** Q is the data of a set Q_0 of vertices and a set Q_1 of arrows. An arrow is a pair of vertices (h, t) where h is the head and t is the tail of the arrow. A quiver is an oriented multigraph with possible loops.*
*- a **potential** W on Q is a formal sum of cycles in Q_1.*
*- The **quiver algebra** kQ is the algebra of paths in Q, where the product is given by concatenation when it makes sense.*
- We denote A the quotient of kQ by the ideal I_W generated by formal derivatives of W.

Quiver algebras are non commutative in general. Let $\underline{v} = (v_0, \dots, v_n)$ be a dimension vector for Q. A representation of Q of dimension $\underline{v}$ is the data of vector spaces V_i of dimension v_i for each $i \in Q_0$ and linear maps $V_i \to V_j$ for all arrows $i \to j$. A representation is in turn a kQ-module. Representations satisfying potential relations I_W are A-modules. **Moduli spaces of representations of quivers** are constructed as GIT quotient. They behave similarly to geometric moduli spaces and are sometimes easier to understand.

Quiver of the Hilbert scheme of points, [3] and [23] A subscheme $[Z]$ of length n in $\mathbb{C}^3$ is the data of a $\mathbb{C}[x, y, z]$-module $H^0(\mathcal{O}_Z)$, which is a n dimensional vector space equipped with 3 automorphisms x, y, z and a cyclic vector 1. So we can think of it as a vector space V_n with $A, B, C \in M_n(\mathbb{C})$ and a cyclic vector v.

The associated quiver is the triple loop quiver (illustration from [7]), with potential $W = [A, B]C$. The quiver algebra is the non commutative free algebra $\mathbb{C}Q = \mathbb{C}\langle x, y, z\rangle$. A representation of dimension n is a triple of non-necessarily commuting matrices A, B, C of size n. The Hilbert scheme of points is described as $\mathbf{Crit}(f)$ for $f = \mathbf{tr}(W) : \mathbf{M}_{1,n} \to \mathbb{C}$, where $\mathbf{M}_{1,n}$ is the (**smooth**) moduli of representations of dimension $(1, n)$ associated to the framed quiver $\tilde{Q}$ (it justs collect the data of a cyclic vector for each representations, imitating the data of $1 \in H^0(\mathcal{O}_Z)$).



Using such a "non-commutative" embedding, authors in [3] computed the virtual motives $[\mathbf{Hilb}^n(X)]^{mot}$ for Hilbert schemes of points on Calabi-Yau 3-folds, first by reducing to $\mathbb{C}^3$ case. Once we know that $\mathbf{Hilb}^n(\mathbb{C}^3) = \mathbf{Crit}(f)$, there is a result which assert that a virtual motive is given by

$$[\mathbf{Crit}(f)]^{mot} = -\mathbb{L}^{-\frac{\dim \mathbf{M}_{1,n}}{2}}[\varphi_f],$$

where $[\varphi_f]$ is a Denef-Loeser vanishing cycle. This cycle itself decomposes in terms of general fiber and central fiber $[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)]$ because of the T-equivariant structure. The computation of the virtual motive reduces to a motivic serie $C(t)$ involving the motive $[C_n]$ where C_n is the variety of pairs of commuting matrices of size n. This serie

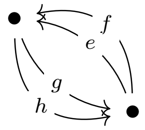
$$C(t) = \sum_{n \geq 0} \frac{[C_n]}{[GL_n]} t^n$$

lies in an extended ring of motives. Over finite fields $\mathbb{F}_q$, this is an explicit infinite product $C(t,q)$, computed before in a "motivic way" which allows authors to replace q by $\mathbb{L}$ and use the formula. Thus $C(t)$ is computed as an infinite product of in terms of t and $\mathbb{L}$.

Theorem 5.1 (Behrend, Bryan, Szendrői [3])**.** *The generating series of the virtual motives of Hilbert schemes of points on* $\mathbb{C}^3$ *is given by :*

$$Z^{mot\ DT}_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=1}^{m-1} (1 - \mathbb{L}^{k+2-\frac{m}{2}} t^m)^{-1}$$

Resolved Conifold Let X be the resolved conifold. This name comes from the fact that it is a blow-up above a singular variety, a conifold $Z = \{xy - wz = 0\} \subset \mathbb{C}^4$. For such singularities (called crepant), Van den Bergh developed a theory of non-commutative resolution which associates the singularities with a non-commutative algebra A [25]. This algebra is associated to the following "conifold quiver" :



Szendrői defined Donaldson-Thomas type invariants of moduli of quiver representations with potential, frames and cyclic vectors, called **non-commutative Donaldson-Thomas invariants** [23]. Using this non-commutative framework, authors in [16] obtained the motivic Donaldson-Thomas invariants (or virtual motives) for moduli of coherent sheaves over the resolved conifold X. Quiver moduli spaces carry a chamber structure with respect to stability conditions.

Definition 5.2 (stability)**.** *If E is a vector bundle of rank r and degree d, then the slope of E is* $\mu(E) = \frac{d}{r} \in (-\infty, \infty]$*. A vector bundle is semi-stable if for all subbundle* $F \neq (0) \subset E$, $\mu(F) \leq \mu(E)$ *and stable if the inequality is strict.*

There are lot of notions of stability in moduli theory and representation theory. When the stability condition is defined with respect to parameters, it produces chambers of stability condition. Wall-crossing formulae are links between moduli with stability parameter in different chambers.
For the conifold quiver, authors identified a chamber which corresponds to the geometric moduli over the conifold and computed the motivic DT series using wall-crossing formula. They had to describe a quiver moduli in terms of critical locus of (Chern-Simons) holomorphic functional and use a T-equivariant structure. The problem then reduced to motivic series given in terms of spaces of matrices, which they could directly decompose in terms of motives $[\mathbf{GL}_n]$ and then $\mathbb{L}$.

Ext-quiver For a Calabi-Yau 3-fold X, there is a construction of Ext-quiver associated to a set of sheaves $E_0, \dots, E_n$ such that $\mathrm{Hom}(E_i, E_j) = \delta_{ij}$. The set of vertices is $\{0, \dots, n\}$ and the numbers of arrow $i \to j$ is dim $\mathrm{Ext}(E_i, E_j)$. Then, there is a procedure to determine a potential W.
Joyce and Song showed that moduli spaces of coherent sheaves over Calabi-Yau 3-folds can be locally described as critical locus of holomorphic function on a smooth variety [11]. Locally at a polystable sheaf $\oplus_{i=0}^{n} V_i \otimes E_i$ (E_i's have same slope and are non isomorphic, V_i vector space) the moduli can be locally embedded in the smooth stack of representation of a Ext-quiver as a critical locus of the trace $\mathbf{Tr}(W)$. This result was extended by Toda such that the embedding is given over a dense open set. Is the last section, vanishings of all Ext^2 groups of vector bundles over curves implies $W = 0$, so that we can identify both moduli locally.

We can use local models given by quiver moduli to study our geometric moduli. These quiver moduli give rise to non-commutative invariants that are in some cases calculable by direct motivic decomposition or finite fields techniques and additional results (toric structure and dimensional reduction). Now p-adic integration can be used as a bridge between characteristic 0 and p thanks to Weil formula and thus may be helpful to compute motivic enumerative invariants.

5.3 P-adic integration

Direct motivic computations are often complicated and finite fields countings are not much easier in general. However, p-adic integration is a way to bypass this difficulty by computing a global p-adic volume which in turn gives the finite fields countings thanks to a Weil's theorem. This method could be helpful to compute Donaldson-Thomas invariants of moduli of sheaves containing strictly semi-stable sheaves via the construction of a gerbe function coming from the moduli stack.

For introduction to p-adic integration, the reference is Popa's lecture notes [20].

Let $x \in \mathbb{Q}$. The p-adic norm of x is given by $\frac{1}{p^m}$, where $m \in \mathbb{Z}$ is the order of p in x. The p-adic field $\mathbb{Q}_p$ is the Cauchy completion of $\mathbb{Q}$ for this norm. For such locally compact abelian groups, there is a translation-invariant measure μ called **Haar measure**. We can normalise it such that $\mu(\mathbb{Z}_p) = 1$, where $\mathbb{Z}_p$ is the ring of p-adic integers (it is compact). From that data, we can compute the measure of any constructible set using translation invariance. For example, $\mathbb{Z}_p = p\mathbb{Z}_p \sqcup (p\mathbb{Z}_p + 1) \sqcup \dots \sqcup (p\mathbb{Z}_p + (p-1))$ gives $\mu(p\mathbb{Z}_p) = \frac{1}{p}$.

Example 5.1. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be defined by $f(x) = x^n$. We compute*

$$Z(f, s) = \int_{\mathbb{Z}_p} |f(x)|^s d\mu$$

If $x \in p^m(\mathbb{Z}_p \setminus p\mathbb{Z}_p)$, $|x^n| = \frac{1}{p^{mn}}$. We get

$$\begin{aligned}\int_{\mathbb{Z}_p} |x^n|^s d\mu &= \mu(\mathbb{Z}_p \setminus p\mathbb{Z}_p) \cdot 1 + \mu(p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \cdot \frac{1}{p^{ns}} + \dots \\ &= (1 - \frac{1}{p}) \cdot 1 + (\frac{1}{p} - \frac{1}{p^2}) \cdot \frac{1}{p^{ns}} + \dots \\ &= \frac{p-1}{p - p^{-ns}}\end{aligned}$$

To define Donaldson-Thomas types invariants via p-adic integration, we need to generalise the set-up to any F**-analytic manifold**, where F is a p-adic field (e.g. $F = \mathbb{Q}_p$). It can be done locally by using gauge forms and their associated Weil measures and then globally by gluing these measures into a canonical measure μ_{can}. Now, the link with finite fields counting is given by :

Theorem 5.2 (Weil)**.** *Let Y be smooth scheme of relative dimension n over $\mathcal{O}_F$ (the ring of integers of a p-adic field F). Then*

$$\int_{Y(\mathcal{O}_F)} d\mu_{can} = \frac{Y(\mathbb{F}_q)}{q^n}$$

5.4 Enumerative invariants as p-adic integrals

When a moduli space $\mathcal{M}$ contains strictly semi-stable sheaves, there is an interesting object which contains the extra data about automorphisms of sheaves. It is called a moduli **stack**. The **pBPS invariants** defined in [5] are some integrals of a gerbe function $\mathcal{M} \to \mathbb{C}$ coming from the structure of the stack. BPS invariants are closely related to DT invariants, see [19].

Definition 5.3 (vague definition)**.** *A moduli stack of sheaves $\mathfrak{M}$ is a geometric object where each point is a sheaf together with its automorphisms. There is a map to the "coarse" moduli space $\mathfrak{M} \to \mathcal{M}$ ("coarse" space is scheme-theoretic).*

A **gerbe** is a generalisation of the notion of G-torsor (or G-principal bundle). A trivial gerbe is a projection $[BG] \times X \to X$, where $[BG]$ is the classifying stack of G : it is the point together with $Aut(*) = G$. For a moduli $\mathcal{M}$ of sheaves, we can define open sets $\mathcal{M}^{ss}$ of semi-stable sheaves and $\mathcal{M}^{st}$ of stable sheaves. As stable sheaves only have scalar automorphisms, $\mathfrak{M}^{st} \to \mathcal{M}^{st}$ is a $\mathbb{G}_m$-gerbe[12].

We study a smooth stack $\mathcal{M}^{ss}$ of semi-stable sheaves, with strictly semi-stable sheaves. [5] defines a **gerbe function** $h_\alpha : \mathcal{M}^{ss}(\mathcal{O}_F) \cap \mathcal{M}^{st}(F) \to \mathbb{C}$ for a p-adic field F.

$x \in \mathcal{M}^{ss}(\mathcal{O}_F) \cap \mathcal{M}^{st}(F)$ is a morphism $x : \text{Spec } \mathcal{O}_F \to \mathcal{M}^{ss}$ such that the generic point is sent to $\mathcal{M}^{st}$.

Remark 5.1. *$\mathcal{O}_F$ is a local ring so* Spec $\mathcal{O}_F$ *has two points : - a closed point : the unique maximal ideal* m_F *of residue field $\mathbb{F}_p$ (where $p = char(F)$)*
- a generic point : a prime ideal (0) *corresponding to the fraction field of $\mathcal{O}_F$ which is the p-adic local field F.*

Definition 5.4 (gerbe function h_α)**.** *Let $x \in \mathcal{M}^{ss}(\mathcal{O}_F) \cap \mathcal{M}^{st}(F)$. We define $h_\alpha(x)$ via the following steps :*

1 - associate a class in the Brauer group $[\alpha] \in H^2(F_{fppf}, \mathbb{G}_m)$[13] *to the pullback via x of the gerbe $\mathfrak{M}^{st} \to \mathcal{M}^{st}$. For algebraic varieties X, $H^2_{\acute{e}t}(X; \mathbb{G}_m)$ are isomorphic to the set of $\mathbb{G}_m$-gerbe over X.*
2 - send $[\alpha]$ to its Hasse invariant $h_\alpha \in \mathbb{Q}/\mathbb{Z}$. Hasse invariant $H^2(F_{fppf}, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ defines an isomorphism in this case.
3 - apply $exp(2i\pi -)$

---

[12] $\mathbb{G}_m$ denotes the multiplicative group k^*

[13] *fppf* is a Grothendieck topology on the stack. It is a topology with enriched notion of open sets, such that there are good descent properties.

Definition 5.5. *The p-adic BPS invariants of [5] are given by the following p-adic integral :*

$$pBPS = \int_{\mathcal{M}^{ss}(\mathcal{O}_F)\cap\mathcal{M}^{st}(F)} h_\alpha \; d\mu_{can}$$

In [5] authors shows that in the case of moduli of 1-dimensional sheaves over del Pezzo surfaces these p-adic integrals satisfy known properties of BPS invariants. Thus, there is hope to prove that they are in fact the same. The key property is χ-independance, that is, for moduli of sheaves $\mathcal{M}_\nu$ and $\mathcal{M}_{\nu'}$, $\nu \neq \nu'$ Chern characters, if points $[F]$ in both moduli corresponds to sheaves F with the same Euler characteristic $\chi(X, F)$, then their BPS invariants have to be related (for example they should satisfy an equation involving a correction term). This property is expected to come from a certain relationship between Borel-Moore and BPS cohomology, whenever it can be given a good definition.

5.5 The case of moduli of vector bundles on curves

We investigate an example of gerbe function. We fix a curve C of genus $g = 2$ defined over $\mathbb{Z}$. There are quasi-projective **moduli of** semi-stable **vector bundles** of rank r and degree d over C denoted $\mathcal{M}^{ss}_C(r, d)$ (see [14]), and also corresponding stacks.

The moduli spaces $\mathcal{M}^{ss}_C(r, d)$ are such that each of its points are S-equivalence classes of sheaves. Two semi-stable sheaves are S-equivalent if the graded quotients in Hölder-Jordan filtration are the same, see [14]. Each S-equivalence class has a polystable representative (i.e. a sheaf that is a direct sum of stable sheaves of same slope).

If V is a vector bundle of rank r and degree d over a curve, Riemann-Roch formula implies that $\chi(C, F) = \sum_{i=0}^{\infty} \mathrm{h}^i(C, F) = d + r(1 - g)$. We fix[14] $r = 2$ and want to test χ-independence by varying degree d.

As C is defined over $\mathbb{Z}$, so are the moduli $\mathcal{M}^{ss}_C(r, d)$.

When (r, d) are coprime, all semi-stable sheaves are stable (because the slope in non-integer) and the gerbe function of [5] is trivial.

We are thus interested in the gerbe functions involved in non-coprime case. We study $\mathcal{M}^{ss}_C(2, 0)$.
There is a smooth stack $\pi : \mathfrak{M}^{ss}_C(2, 0) \to \mathcal{M}^{ss}_C(2, 0)$ with the restriction over the stable locus
$\pi_{st} : \mathfrak{M}^{st}_C(2, 0) \to \mathcal{M}^{st}_C(2, 0)$ being a $\mathbb{G}_m$-gerbe. Using [5], we know that there is a well-defined canonical measure μ_{can} on $\mathcal{M}^{ss}_C(2, 0)(\mathcal{O}_F) \cap \mathcal{M}^{st}_C(2, 0)(F)$ for a local p-adic field F.
Assume $F = \mathbb{Q}_p$ for simplicity. An $\mathcal{O}_F$-point in the moduli is a map $x : \mathrm{Spec}\ \mathcal{O}_F \to \mathcal{M}^{ss}_C(2, 0)$. Taking the intersection with $\mathcal{M}^{st}_C(2, 0)(F)$ means that we restrict to maps x which send the generic point to the stable locus.

Let $x \in \mathcal{M}^{ss}_C(2, 0)(\mathcal{O}_F) \cap \mathcal{M}^{st}_C(2, 0)(F)$. We have the following diagram :

$$\begin{tikzcd}
x^*(\mathfrak{M}^{st}_C(2,0)) \arrow[r, "x"] \arrow[d, "x^*\alpha"] & \mathfrak{M}^{st}_C(2,0) \arrow[d, "\alpha"] \\
\mathrm{Spec}\ F \arrow[r, "x"] & \mathcal{M}^{st}_C(2,0)
\end{tikzcd}$$

---

[14] because the $r = 1$ case is trivial

which provides a $\mathbb{G}_m$-gerbe over Spec F, i.e. an element $x^*\alpha$ in the Brauer group $Br(F) \simeq \mathbb{Q}/\mathbb{Z}$.

There are 3 kinds of points in the locus $\mathcal{M}_C^{ss}(2,0)(\mathcal{O}_F) \cap \mathcal{M}_C^{st}(2,0)(F)$, depending on the image under x of the closed point (of Spec $\mathcal{O}_F$).

- Case 1 : $x(\mathbb{F}_p)$ is contained in the stable locus $\mathcal{M}_C^{st}(2,0)$
- Case 2 : $x(\mathbb{F}_p)$ is contained in the strictly semi-stable locus in the strata isomorphic to $\mathcal{M}_C^{ss}(1,0)$, i.e. the strata contains S-equivalence classes with a polystable representative of the form $L \oplus L$ for a line bundle L of degree 0.
- Case 3 : $x(\mathbb{F}_p)$ is contained in the strictly semi-stable locus, it is a S-equivalence class with a polystable representative of the form $L_1 \oplus L_2$ for non-isomorphic line bundles L_1, L_2 of degree 0.

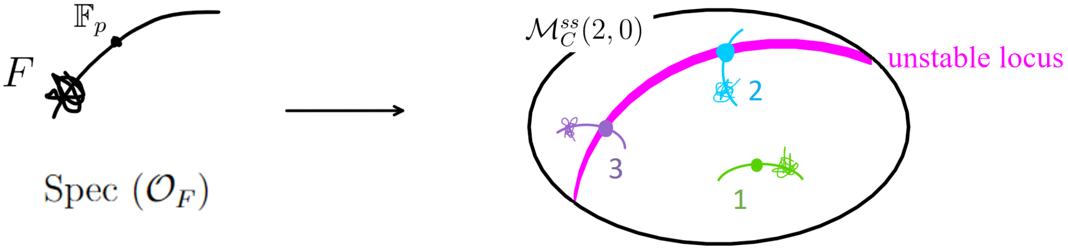
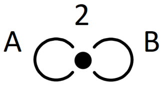


Illustration of the locus $\mathcal{M}_C^{ss}(2,0)(\mathcal{O}_F) \cap \mathcal{M}_C^{st}(2,0)(F)$

To understand the gerbe function over these 3 kinds of points, we have the following remark :

- Case 1 : locally around that closed point there are only stable closed points, so the gerbe is trivial[15] and $h_\alpha(x) = 1$
- Case 2 : analytic-locally around that point the moduli should coincide with the moduli of representations $\mathcal{M}_Q$ of the Ext-quiver corresponding to $L \oplus L$ [11]. This quiver is the double loop quiver with dimension vector $d = 2$.



Ext-quiver associated to case 2

- Case 3 : same for the Ext-quiver corresponding to $L_1 \oplus L_2$. The corresponding quiver moduli stack has a trivial $\mathbb{G}_m$-gerbe, so $h_\alpha(x) = 1$. The triviality of the gerbe follows from [10] and the fact that the map from stack quotient to GIT quotient $\pi : \mathbb{A}^2 \setminus (0,0) \to \mathbb{A}^1 : (x,y) \to xy$ is such that for all $x \in \mathbb{Q}_p$, $(x,1)$ is a rational pre-image.

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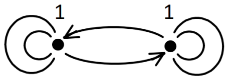
[15] According to F. Carocci but we could not find it in the literature.



Ext-quiver associated to case 3

Thus, the points which could define a non trivial $\mathbb{G}_m$-gerbe over F belong to Case 2. They can be locally modelled by the double loop quiver with dimension vector 2. Representations are given by pairs of matrices of size 2. The moduli stack $\mathfrak{M}^{ss}_Q(2)$ is the stack quotient of the space of representations by a GL_2-action given by simultaneous conjugation ($\mathbb{G}_m \hookrightarrow GL_2$ acts trivially). The coarse moduli space $\mathcal{M}^{ss}_Q(2)$ is given by the GIT quotient. Its stable points are closed orbits of simple representations of Q, so they are precisely pairs of matrices of size 2 up to simultaneous conjugation without common eigenvectors. Up to a literature check for the isomorphism $F^5 \simeq \mathcal{M}^{ss}_Q(2)$, we have the following map:

$$\begin{tikzcd} \mathfrak{M}^{ss}_Q(2) = \{(A,B) \in \mathrm{Mat}_2(F)\}/^{stack}GL_2(F) \arrow[d, "\text{(detA, trA, detB, trB, trAB)}"] \\ F^5 \simeq \mathcal{M}^{ss}_Q(2) \end{tikzcd}$$

The stable locus $\mathcal{M}^{st}_Q(2)$ is the image of $\mathfrak{M}^{st}_Q(2)$. We can describe the unstable locus $F^5 \setminus \mathcal{M}^{st}_Q(2)$: it is the space of (detA, trA, detB, trB, trAB) which come from A, B with a common eigenvector. They have a representative(under simultaneous GL_2 action) of the form :

$$\{A = \begin{pmatrix} a & 0 \\ * & d \end{pmatrix}, B = \begin{pmatrix} e & 0 \\ * & h \end{pmatrix}\}$$

Its image in the coarse moduli is given by $\{(ad, a+d, eh, e+h, ae+dh)\}$. Direct computations shows that it is the irreducible closed subset of dimension 4 defined by the equation

$$Z_{unst} := \mathcal{M}^{ss}_Q(2) \setminus \mathcal{M}^{st}_Q(2) = \{(y^2-4x)z = wty - w^2 - xt^2 \ : \ (x,y,z,t,w) \in F^5\}$$

For each $\mathcal{O}_F$-point x, the gerbe $x^*\alpha$ is the pull-back via x of the $\mathbb{G}_m$-gerbe

$$\pi : \mathfrak{M}^{st}_Q(2) \to \mathcal{M}^{st}_Q(2)$$

An obstruction for the triviality of π is the non-exisence of rational representation in the fibre over a rational point. In fact, the main result of [10] states that the Hasse invariant associated to a point in $\mathcal{M}^{st}_Q(2)(\mathbb{Q}_p)$ is 0 if there is a rational representation in the pre-image and $\frac{1}{2}$ otherwise (in our case). After applying exponential, we get $h_\alpha(x) = \pm 1$.

In general, the GL_2-bundle map is the following :

$$\begin{tikzcd} \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\} \arrow[d, "\text{(detA, trA, detB, trB, trAB)}"] \\ (ad-bc, a+d, eh-fg, e+h, ae+bg+cf+dh) \end{tikzcd}$$

The existence of a rational representation in the pre-image of (x,y,z,t,w) in the stable locus is equivalent to the existence of $(a,b,c,d,e,f,g,h) \in F^8$ solution of the following system :

$$\begin{cases} ad - bc = x \\ a + d = y \\ eh - fg = z \\ e + h = t \\ ae + bg + cf + dh = w \end{cases}$$

Substituting a in the first equation makes d solution of a degree 2 polynomial of discriminant $\Delta_1 = y^2 - 4(bc+x)$. Rational solution for the first matrix implies that Δ_1 is square. We let $\delta^2 = \Delta_1$, $\delta \leq 0$. c is free, we can take $c \neq 0$[16]. Then b is constraint as follows :

$$\begin{cases} a = y - d \\ d^2 - yd + (bc + x) = 0 \text{ of discriminant } \Delta_1 \end{cases} \qquad \begin{cases} b = \frac{-\delta^2+y^2-4x}{4c} \\ a = \frac{y\pm\delta}{2} \\ d = \frac{y\mp\delta}{2} \end{cases}$$

The system for the second matrix is now written in terms of x, y, z, t, w, c, δ :

$$\begin{cases} eh - fg = z \\ e + h = t \\ \frac{y\pm\delta}{2} e + g\frac{-\delta^2+y^2-4x}{4c} + cf + h\frac{y\mp\delta}{2} = w \end{cases}$$

Substituting e, f in the first equation using second and third equation leads to a degree 2 equation in h, whose determinant Δ_2 depends in g. Taking any g gives a a pre-image in a quadratic extension of F given by $F(\sqrt{\Delta_2})$.

$$\begin{cases} P_h = -h^2 + th - g\frac{w + \frac{t(-y\mp\delta)}{2} \pm \delta h + \frac{g(\delta^2-y^2+4x)}{4c}}{c} - z = 0 \text{ of discriminant } \Delta_2 = \gamma^2 \\ e = t - h \\ f = \frac{w + \frac{t(-y\mp\delta)}{2} \pm \delta h + \frac{g(\delta^2-y^2+4x)}{4c}}{c} \end{cases}$$

We can simplify P_h :

$$P_h = -h^2 + h(t \mp \frac{g\delta}{c}) - \frac{g}{c}\Big(w + \frac{t}{2}(-y \pm \delta) + g\frac{\delta^2 - y^2 + 4x}{4c}\Big) - z$$

so that

$$\Delta_2 = \Big(t \mp \frac{g\delta}{c}\Big)^2 - 4\Big[\frac{g}{c}\Big(w + \frac{t}{2}(-y \pm \delta) + g\frac{\delta^2 - y^2 + 4x}{4c}\Big) + z\Big]$$

Now if $\Delta_2 = \gamma^2$, $\gamma \in F$ (which implies existence of a solutions $h, f, e \in F$), then g is a root of a degree 2 polynomial P_g of discriminant Δ_3 which depends on γ, δ.

$$P_g = -\gamma^2 + \Big(t \mp \frac{g\delta}{c}\Big)^2 - 4\Big[\frac{g}{c}\Big(w + \frac{t}{2}(-y \pm \delta) + g\frac{\delta^2 - y^2 + 4x}{4c}\Big) + z\Big]$$

To sum up, existence of rational pre-image reduces to existence of $(\delta, \gamma, \phi) \in F^3$ such that

$$\Delta_3' = \frac{\Delta_3}{4c^2} = \phi^2 = (2(\mp t\delta - w) + ty)^2 + (4x - y^2)(t^2 - \gamma^2 - 4z).$$

For simplicity, we study the situation over $\mathbb{Q}$.

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[16] If $c = 0$, existence of rational solutions fails when $y^2 - 4x$ is not a square, which will be the case in our example of points without rational pre-image

- The following algebraic map

$$f : \mathbb{Q}^5 \to \mathbb{Q}^5$$
$$(x, y, t, w, \gamma) \mapsto (x, y, \frac{t^2 - \gamma^2}{4}, t, w)$$

is such that points in $Im(f)$ defines trivial gerbe. Indeed $(\mp\delta t + 2(t - w))^2$ is a square for any $\delta \in \mathbb{Q}$. Thus $h_\alpha(y) = 1$ on this 5-dimensional locus.

- A positivity argument shows that points in

$$\{(x, y, z, t, w) \in \mathbb{Q}^5 \mid t = 0,\ y^2 > 4x,\ z > \frac{w^2}{y^2 - 4x} - 1\}$$

do not have pre-image in $\mathbb{Q}$ and thus define non-trivial gerbe. In that case, the corresponding Hasse invariant is non trivial and 2-torsion (because of the existence of solution in quadratic extension) so $h_\alpha(x) = -1$.

- (7,0,1,0,0) has no pre-image in $\mathbb{Q}$. Indeed, $\Delta_3'' = \frac{\Delta_3'}{4} = 7(\gamma^2 + 4)$ being a square would imply $\gamma^2 \equiv 3 \mod 7$ which is a contradiction.

Remark 5.2 ($\mathbb{P}^3$-fibration structures)**.** *There is a determinant map*

$$det\ :\ \mathcal{M}_C^{ss}(2, 0) \to \mathrm{Pic}_C^0$$

where Pic_C^0 *is the group of degree* 0 *line bundles over* C*. The fibres are* $\mathcal{M}_C^{ss}(2, L)$*, the moduli of rank* 2 *bundles with fixed determinant* L *of degree* 0 *and there is an exceptional isomorphism for genus* 2 *curves* $\mathcal{M}_C^{ss}(2, L) \simeq \mathcal{M}_C^{ss}(2, \mathcal{O}_C) \simeq \mathbb{P}^3$ *proved in [17]. Thus* $\mathcal{M}_C^{ss}(2, 0)$ *has a structure of* $\mathbb{P}^3$*-fibration over* Pic_C^0*.*

On the other hand, we have in the coprime case a moduli $\mathcal{M}_C^{ss}(2, 1) = \mathcal{M}_C^{st}(2, 1)$ *which is also a* $\mathbb{P}^3$*-fibration over the abelian variety* Jac_C^1 *of line bundles of degree* 1*. In this case, the gerbe is trivial so the gerbe function is constantly* 1*.*

We expect that

$$\int_{\mathcal{M}_C^{ss}(2,0)(\mathcal{O}_F) \cap \mathcal{M}_C^{st}(2,0)(F)} 1\ \mu_{can} - \int_{\mathcal{M}_C^{ss}(2,1)(\mathcal{O}_F)} h_\alpha\ \mu_{can}$$

can be expressed in terms of integral over $\mathcal{M}_C^{ss}(1, 0)(\mathcal{O}_F)$. It would be a degree-independence statement for *pBPS* in this case (analogous to result in [5] for the case of moduli of curves embedded in a Del Pezzo surface). Indeed,

- the $\mathbb{P}^3$-fibration structure over a jacobian variety of both moduli $\mathcal{M}_C^{ss}(2, 1)$ and $\mathcal{M}_C^{ss}(2, 0)$ should imply that their p-adic volume (as F-analytic manifolds) are the same.

- The locus $h_\alpha(x) \neq 1$ is contained in a subspace identified with $\mathcal{M}_C^{ss}(1, 0)$ (points corresponding to Case 2).

To achieve this, we have to understand the p-adic measure of the set of points with $h_\alpha(x) = -1$. Some missing steps are :

- Starting with a $\mathcal{O}_F$-point in the quiver moduli, link the location of the closed point $\mathbb{F}_q$ (as in cases 1, 2, 3 but for the quiver moduli) and existence of pre-image in $\mathbb{Q}_p$. It would be a finite fields criterion to determine the gerbe.

- Study how the pull-back of gerbes from quiver moduli to geometric moduli behave.

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