

RAPPORT DE STAGE DE M1

Surfaces minimales non locales dans variété riemannienne

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Abstract

J'ai effectué mon stage à l'ETH Zürich, au département de mathématiques, sous la direction de Joaquim Serra. Celui-ci s'est étendu du 22 février au 15 juillet. L'essentiel du stage s'est effectué depuis mon appartement à Zürich, l'ETH ne mettant pas – à mon grand regret – de bureau à disposition des élèves en stage.

Mon travail s'est fondé sur l'article « Nonlocal minimal surfaces » de Caffarelli, Roquejoffre et Savin. Les auteurs introduisent une notion de surface minimale non locale dans \mathbf{R}^n et démontrent un théorème de régularité à la De Giorgi. Le début de mon stage a été consacré à la compréhension de leur article, puis Joaquim m'a proposé de d'étendre ce résultat aux surfaces minimales non locales d'une variété riemannienne compacte sans bord, ce qui a constitué mon travail jusqu'à la fin du stage.

Au début de mon séjour, les rendez-vous avec Joaquim étaient espacés d'au moins deux semaines, ce qui a un peu compliqué mon travail, étant bloqué sur certains points, mais aussi en terme de motivation. Leur fréquence s'est progressivement accélérée, jusqu'à ce qu'on se voit plusieurs fois par semaine le dernier mois pour clore les derniers points qui posaient problème.

J'ai eu l'opportunité d'assister à un séminaire intitulé « Geometric analysis and calibrated geometries ». Un exercice d'humilité, une bonne part des exposés me semblant incompréhensibles après cinq minutes de parole. A la fin du stage, j'ai participé à un workshop, destiné à des étudiants de Master et des doctorants, sur le thème « Free boundary problems and related topics ». Ce fut l'occasion pendant une semaine d'assister à trois mini-cours enrichissants et de rencontrer des étudiants travaillant sur des sujets similaires au mien.

J'étais logé au nord de Zürich, à deux pas d'un lac ainsi que d'Elsa Maneval, élève de ma promotion aussi en stage au département de mathématiques de l'ETH, avec qui j'ai pu partager moult repas, déboires et joies du stage. Enfin, mon stage aura permis de sympathiques rencontres au gré de différents événements. Citons les magnifiques paysages suisses qui auront offert de belles randonnées autour des lacs et montagnes.

Je tiens enfin à remercier Joaquim Serra pour avoir encadré ce stage, et mon tuteur Cyril Imbert pour nous avoir mis en relation.

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1 Introduction

1.1 Fractional perimeter and Nonlocal minimal surfaces in \mathbf{R}^n

In their seminal paper [2], Caffarelli, Roquejoffre and Savin introduced and studied extensively *non local minimal surfaces* on \mathbf{R}^n . Given a suitable function f , the $H^{s/2}$ norm of f is defined as

$$\|f\|_{H^{s/2}}^2 = \iint \frac{|f(x) - f(y)|^2}{|x - y|^{n+s}} \, dx dy.$$

Then we define the fractional perimeter of ∂E as

$$\text{Per}_s(E) := (1 - s) \|\chi_E\|_{H^{s/2}}^2.$$

The constant $1 - s$ is used to guarantee $\text{Per}_s(E) \rightarrow \text{Per}(E)$ where $\text{Per}(E)$ denotes the classical perimeter of E and is usually defined as

$$\text{Per}(E) = \sup_X \left(\int_E \text{div}(X) \, dx \right),$$

where the supremum is taken over the set of \mathcal{C}^1 vector fields such that $|X| \leq 1$. One can check using Green formula that it coincides with the usual definition of perimeter for sets with smooth boundary. If ∂E is not bounded, the fractional perimeter $\text{Per}_s(E)$ may of course be infinite. Hence, if $\Omega \subset \mathbf{R}^n$ is a bounded set, we define $\text{Per}_s(E; \Omega)$ as the contribution of E in Ω to the $H^{s/2}$ norm of χ_E , that is we only consider the contribution of pair of points (x, y) for which at least one of them belongs to Ω .

$$\text{Per}_s(E; \Omega) := \iint_{\mathbf{R}^n \times \mathbf{R}^n \setminus \Omega^c \times \Omega^c} \frac{|\chi_E(x) - \chi_E(y)|}{|x - y|^{n+s}} \, dx dy.$$

Then,

Definition 1.1. *We say that a set E is minimal in Ω if $\text{Per}_s(E; \Omega)$ is minimal among sets which coincide with E outside Ω . More precisely, if $F \subset \mathbf{R}^n$ is such that $F \cap \Omega^c = E \cap \Omega^c$ then we have*

$$\text{Per}_s(E; \Omega) \leq \text{Per}_s(F; \Omega).$$

The boundary ∂E is called a nonlocal minimal surface in Ω .

The authors of [2] proved a flatness theorem, analogous to the De Giorgi Theorem for classical minimal surfaces. They showed

Theorem 1.1. *Assume E is minimal in B_1 . Then there exists $\varepsilon_0 > 0$ that may depend on n and s such that if*

$$\partial E \cap B_1 \subset \{|x \cdot e_n| \leq \varepsilon_0\},$$

then $\partial E \cap B_{1/2}$ is a $\mathcal{C}^{1,\gamma}$ graph in the x_n direction.

In [3], the authors manage to release the dependence in s as $s \uparrow 1$ hence obtaining a uniform flatness theorem. However it comes at the price of more intricate proofs, which rely on quite different techniques.

1.2 Fractional laplacian on a manifold

There are various ways to define the fractional laplacian on a manifold. One way is to use the representation formula

$$\lambda^\sigma = \frac{1}{|\Gamma(-\sigma)|} \int_0^{+\infty} (1 - e^{-\lambda t}) \frac{dt}{t^{1+\sigma}}. \quad (1)$$

We define Δ^σ using functional calculus by

$$-(-\Delta)^\sigma := \frac{1}{|\Gamma(\sigma)|} \int_0^{+\infty} (e^{t\Delta} - I) \frac{dt}{t^{1+\sigma}}.$$

If the manifold is *stochastically complete*, that is the heat kernel $p(t, x, y)$ on M verifies

$$\int_M p(t, x, y) \, dV(y) = 1$$

for any time t and $x \in M$, then we can rewrite, for suitable functions f ,

$$-(-\Delta)^\sigma f(x) = \text{P.V.} \int_M K(x, y) (f(y) - f(x)) \, dV(y),$$

where the principal value has to be understood as $\lim_{\varepsilon \rightarrow 0} \int_{M \setminus B_\varepsilon(x)}$, and we have defined

Definition 1.2. *Let M be a closed Riemannian manifold. The kernel of the $s/2$ -laplacian is defined by the formula*

$$K(x, y) := \int_0^{+\infty} \frac{dt}{t^{1+s/2}} H(t, x, y),$$

Notice that it is singular on the diagonal.

In a similar fashion, we can define the $H^{s/2}$ norm of a function by f

$$\|f\|_{H^{s/2}}^2 := \iint_{M \times M} (f(x) - f(y))^2 K(x, y) \, dV(x) dV(y),$$

where dV denotes the volume form on M . Since we are working on a closed manifold, we can also follow a spectral approach. Let (φ_k) be an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $(-\Delta)$, with eigenvalues (λ_k) . Then if

$$f = \sum_{k \geq 1} \langle f, \varphi_k \rangle \varphi_k$$

we define

$$(-\Delta)^{s/2} f = \sum_{k \geq 1} \lambda_k^{s/2} \langle f, \varphi_k \rangle \varphi_k$$

and

$$\|f\|_{H^{s/2}(M)}^2 = \sum_{k \geq 1} \lambda_k^{s/2} \langle f, \varphi_k \rangle_{L^2(M)}^2$$

This definition is equivalent to the previous one, up to dimensional constants.

Proof. We work with an eigenfunction φ_k . Then $e^{t\Delta} \varphi_k = e^{-t\lambda_k} \varphi_k$. Let us write using the first definition.

$$\|\varphi_k\|_{H^{s/2}}^2 := \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{+\infty} \frac{dt}{t^{1+s/2}} \iint_{M \times M} (\varphi_k(x) - \varphi_k(y))^2 H(t, x, y) dV(x) dV(y).$$

Using stochastic completeness and the fact that

$$e^{t\Delta} f(x) = \int_M H(t, x, y) f(y) \, dV(y)$$

we infer

$$\|\varphi_k\|_{H^{s/2}(M)}^2 := \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{+\infty} \frac{dt}{t^{1+s/2}} \int_M 2\varphi_k(x)^2 (1 - e^{-t\lambda_k}) \, dV(x).$$

Now $\|\varphi_k\|_{L^2(M)} = 1$, hence by (1),

$$\|\varphi_k\|_{H^{s/2}(M)}^2 = 2 \int_0^{+\infty} \frac{1 - e^{-\lambda_k t}}{t^{1+s/2}} dt = 2|\Gamma(-s/2)| \lambda_k^{s/2}.$$

□

1.3 Nonlocal minimal surfaces in a closed manifold

Definition 1.3. We say that a set $E \subset M$ is minimal if it is a critical point of the fractional perimeter functionnal, that is, for any \mathcal{C}^1 vector field X on M we have

$$\left. \frac{d}{dt} \right|_{t=0} \text{Per}_s(\varphi_t^X(E)) = 0$$

where φ_t^X denotes the flow of X .

1.4 The Euler-Lagrange equation in the viscosity sense

We refer to [4] for an exposure on regularity theory for viscosity solutions of integrodifferential equations. Recall we can write the fractional laplacian of a function f as

$$\Delta^{s/2}f(y) = \int_M K(x,y)(f(x) - f(y)) \, dV(x)$$

Denote P the subgraph of f . We say that $\Delta^{s/2}f = 0$ *in the viscosity sense* if the following holds

- If f is touched by below at y by a smooth function then

$$\int_M (f(x) - f(y))K(x,y) \, dV(x) \leq 0$$

- If f is touched by above at y by a smooth function then

$$\int_M (f(x) - f(y))K(x,y) \, dV(x) \geq 0$$

Morally speaking, we can say that $\chi_E = \chi_{CE}$ on the boundary of E . In [2] the authors showed that minimizers of the fractional perimeter are solutions of the following Euler-Lagrange equation

$$\Delta^{s/2}(\chi_E - \chi_{CE}) = 0 \quad \text{along } \partial E \tag{2}$$

in the viscosity sense. The proof of Theorem 1.1 in [2] actually shows that the result holds for sets E who are viscosity solutions of $\Delta^{s/2}(\chi_E - \chi_{CE}) = 0$ along ∂E , without requiring them to be minimizers.

Remark. The authors only proved that the integrals above for $f = \chi_E - \chi_{CE}$ are nonnegative (resp. nonpositive) only as supremum (resp. infimum) limits. Later, Cabré improved the result and showed in [1] that the integrals are actually defined as principal values (see also [7]).

In the following we will work with sets which are *nearly* viscosity solutions of the Euler-Lagrange equation. We assume, following the previously known results, that one can show that critical points of the fractional perimeter on a closed manifold are indeed viscosity solutions.

1.5 Main results

Theorem 1.2. *Let $s \in (0, 1)$. There exists $\varepsilon_0 > 0$ and $\alpha > 0$ such that the following hold. Let $g : \mathbf{R}^n \rightarrow \mathbf{R}$ be a Riemmanian metric satisfying*

$$\begin{cases} |g - I| \leq \varepsilon \\ |Dg| \leq C \end{cases}$$

Assume $E \subset \mathbf{R}^n$ satisfies:

We have $0 \in \partial E$ and for any $x \in \partial E \cap B_1$ such that E has an interior or exterior tangent ball at x we have

$$\left| \int_{\mathbf{R}^n} (\chi_E(y) - \chi_{CE}(y))K(x,y) \, dV(y) \right| \leq C,$$

in the viscosity sense, where C does not depend on the contact point $x \in B_1$. Moreover assume that ∂E is contained in a ε_0 -flat cylinder in B_1 , say

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon_0\}.$$

and E contains $B_1 \cap \{x_n \leq -\varepsilon_0\}$. Then ∂E is a $C^{1,\alpha}$ graph in the x_n direction in $B_{1/2}$.

More precisely, we consider sets E such that, if E as an interior tangent ball at $x \in \partial E$ then

$$\limsup_{\delta \rightarrow 0} \int_{\mathbf{R}^n \setminus B_\delta(x)} (\chi_E(y) - \chi_{cE}(y)) K(x, y) \, dV(y) \leq C.$$

and if E has an exterior tangent ball at $x \in \partial E$ then

$$\liminf_{\delta \rightarrow 0} \int_{\mathbf{R}^n \setminus B_\delta(x)} (\chi_E(y) - \chi_{cE}(y)) K(x, y) \, dV(y) \geq -C.$$

We will deduce

Corollary 1.1. *Let M be a n -dimensional closed manifold. Let E be a subset of M such that $\Delta^{s/2}(\chi_E - \chi_{cE}) = 0$ along ∂E . Let $x_0 \in \partial E$. Take $r < \text{inj}(x_0)$ and consider $\varphi : B_r(x_0) \rightarrow B_r(0) \subset \mathbf{R}^n$ be riemannian normal coordinates around x_0 . Then there exists $\varepsilon_0 > 0$ such that if $\varphi(\partial E)$ is included in a cylinder of flatness $\leq \varepsilon_0$ in $B_r(0)$, then ∂E is a $C^{1,\gamma}$ graph in $B_{r/2}$.*

2 Basic tools and definitions

2.1 The kernel $K(x, y)$

In this section, we provide some useful estimates on the kernel of the fractional laplacian. They essentially come down to integral or pointwise estimates of heat kernels. We refer to [5] for an exposure on heat kernel bounds.

Now we state a result, that comes down to the fact that the mass of $p(t, x, \cdot)$ is concentrated around x at small times.

Proposition 2.1. *Let M be a complete Riemannian manifold. Let $x \in M$ and fix r_0 a positive radius. Then given $r \leq r_0$, one has*

$$\int_{M \setminus B_r(x)} K(x, y) \, dV(y) \leq Cr^{-s}.$$

Proof. Fix $r > 0$ small and consider $\eta : M \rightarrow \mathbf{R}$ a cutoff function with $\eta \equiv 1$ in B_r and $\eta \equiv 0$ outside B_{2r} . We also require $\|\Delta\eta\| \leq \frac{1}{r^2}$. Using Green formula we have

$$\begin{aligned} \frac{d}{dt} \int_{B_{3r}} \eta(y) p_{B_{3r}}(t, x, y) \, dV(y) &= \int_{B_{3r}} \eta(y) \Delta_y p_{B_{3r}}(t, x, y) \, dV(y) \\ &= \int_{B_{3r}} p_{B_{3r}}(t, x, y) \Delta\eta(y) \, dV(y). \end{aligned}$$

By integrating over $[0, t_0]$ we infer

$$\int_{B_{3r}} \eta(y) p_{B_{3r}}(t_0, x, y) dV(y) \geq 1 - \|\Delta\eta\|_{L^\infty(B_{2r}\setminus B_r)} \int_0^{t_0} \int_{B_{2r}\setminus B_r} p_{B_r}(t, x, y) dV(y) dt.$$

Here we have used that $p_{B_{3r}}(t, x, y) \rightarrow \delta_x(y)$ as $t \rightarrow 0$. Using $\eta \leq 1$ and $p_{B_{3r}} \leq p$ and stochastic completeness (to get $\int_M p(t, x, y) dV(y) = 1$) we obtain

$$\int_{M\setminus B_{3r}} p(t_0, x, y) dV(y) \leq \|\Delta\eta\|_{L^\infty(B_{2r}\setminus B_r)} \int_0^{t_0} \int_{B_{2r}\setminus B_r} p_{B_{3r}}(t, x, y) dV(y) dt.$$

Using standard bounds for $p_{B_{3r}}$ we get

$$\int_{M\setminus B_{3r}} p(t_0, x, y) dV(y) \leq r^{-2} \int_0^{t_0} \int_{B_{2r}\setminus B_r} t^{-n/2} \exp\left(-\frac{r^2}{4t}\right) dV(y) dt.$$

Observe that for $r \leq r_0$, we have $\text{Vol}(B_{2r}) \lesssim r^n$. Then

$$\int_{M\setminus B_{3r}} p(t_0, x, y) dV(y) \leq r^{n-2} \int_0^{t_0} t^{-n/2} \exp\left(-\frac{r^2}{4t}\right) dt = \Gamma\left(\frac{n}{2} - 1, \frac{r^2}{t_0}\right),$$

where $\Gamma(s, x)$ denotes the incomplete gamma function

$$\Gamma(s, x) = \int_x^{+\infty} t^{s-1} e^{-t} dt,$$

and we have the asymptotics

$$\Gamma(s, x) \sim_{x \rightarrow +\infty} x^s e^{-x}.$$

Let $A \gg 1$ such that for $x \geq A$ we have $\Gamma(s, x) \lesssim x^s e^{-x}$. We want to evaluate

$$\int_{M\setminus B_{3r}} K(x, y) dV(y) = \int_{M\setminus B_{3r}} \int_0^{+\infty} \frac{dt}{t^{1+\sigma}} p(t, x, y) dV(y) dt.$$

We split \mathbf{R}_+ in two domains $\frac{r^2}{t} \geq A$ and $\frac{r^2}{t} \leq A$. Then,

$$\int_{M\setminus B_{3r}} K(x, y) dV(y) \leq \int_0^{r^2/A} \frac{dt}{t^{1+\sigma}} \left(\frac{r^2}{t}\right)^{n/2-2} e^{-r^2/t} + \int_{r^2/A}^{+\infty} \frac{dt}{t^{1+\sigma}} \leq C(A) r^{-s},$$

and the bound is uniform for $0 \leq r \leq r_0$. Notice that we recover the usual estimate on \mathbf{R}^n for $K(x, y) = \frac{C_n}{|x-y|^{n+s}}$. \square

Remark. We can give an other proof of Proposition 2.1 on manifolds with nonnegative curvature (See [6]). Indeed, on such manifolds we have upper bounds on heat kernel

$$p(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} e^{-c \frac{d(x,y)^2}{t}} \quad (3)$$

where $V(x, r)$ denotes the volume of the metric ball $B(x, r)$. Moreover the following volume doubling property holds: for any radius R ,

$$V(x, 2R) \leq CV(x, R).$$

From (3) we deduce

$$K(x, y) \leq \frac{\Lambda}{V(x, d(x, y))d(x, y)^s}.$$

We infer

$$\begin{aligned} \int_{M \setminus B_r(x)} K(x, y) \, dV(y) &\leq \sum_{k=0}^{\infty} \int_{B_{r2^{k+1}} \setminus B_{r2^k}} \frac{\Lambda}{V(x, d(x, y))d(x, y)^s} \, dV(y) \\ &\lesssim \sum_{k=0}^{\infty} \frac{V(x, 2^{k+1}r)}{V(x, 2^k r)(r2^k)^s}, \end{aligned}$$

and using Volume Doubling property we get

$$\int_{M \setminus B_r(x)} K(x, y) \, dV(y) \lesssim \sum_{k=0}^{\infty} (r2^k)^{-s} \lesssim r^{-s}.$$

Next we show that the kernel of the fractional Laplacian is comparable to $|x - y|^{-(n+s)}$ (that is the kernel of the fractional laplacian in the euclidean case)

Proposition 2.2. *Under the hypotheses of Theorem 1.2 There exists a constant $\Lambda > 0$ such that we have the upper bound*

$$K(x, y) \leq \frac{\Lambda}{|x - y|^{n+s}}$$

Proof. It follows from the fact that g is comparable to the euclidean metric, hence we have upper bounds on heat kernel

$$p(t, x, y) \leq \frac{C}{t^{n/2}} e^{-c \frac{|x-y|^2}{t}},$$

and the conclusion follows. □

3 Non local mean curvature

Because the kernel K is singular, we can not define $\int_{B_r(x)} (\chi_E - \chi_{CE})(y) K(x, y) \, dV(y)$ in the principal value sense for any set $E \subset M$. Indeed if the set is too irregular around x then we do not get sufficient cancellations between E and CE .

But under suitable regularity we have

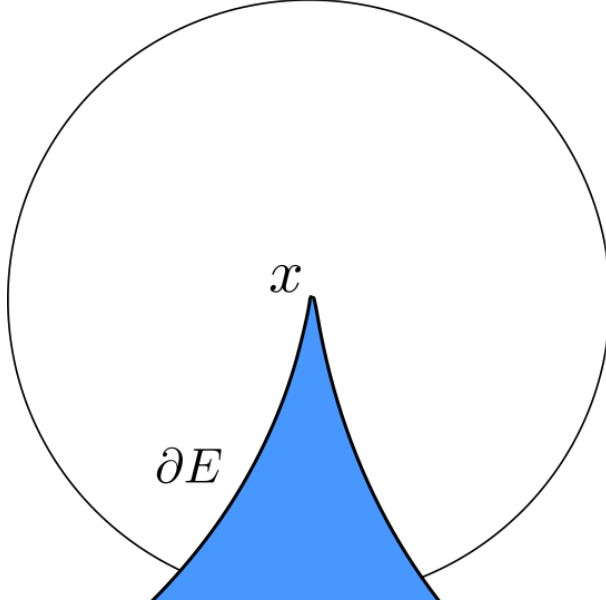


Figure 1: typical situation where the integral does not converge

Proposition 3.1. *Let M be a closed manifold and consider $E \subset M$. If ∂E is touched by above and below at x by graphs of \mathcal{C}^2 functions, say to fix ideas parabolooids, then*

$$H_K[\partial E](x) := \int_M (\chi_E(y) - \chi_{cE}(y))K(x, y) \, dV(y)$$

is well defined in the principal value sense and is called the fractional mean curvature at point x

Proof. Let $r < \text{inj}(x)$ be small. Outside of $B_r(x)$ we simply bound

$$\left| \int_{\mathcal{C}B_r(x)} (\chi_E(y) - \chi_{cE}(y))K(x, y) \, dV(y) \right| \leq C(r).$$

Let $\varphi : B_r(x) \rightarrow \mathbf{R}^n$ be normal coordinates around x . By a change of variables we can write the local contribution as

$$\int_{B_r(0)} (\chi_{E_\varphi} - \chi_{cE_\varphi})K(x, \varphi^{-1}(z)) (1 + O(|z|^2)) \, dz =: I_1 + I_2,$$

where $E_\varphi = \varphi(E)$. The error term I_2 is bounded by

$$|I_2| \leq \int_{B_r(0)} \Lambda |z|^{2-(n+s)} dz \leq C\Lambda r^{2-s}$$

where C depends on the curvature. To control I_1 , we use that

$$\partial E \cap B_r(x) \subset \{u : |u_n| \leq M|u'|^2\}.$$

As we will see, $K(x, \varphi^{-1}(z)) \approx K(x, \varphi^{-1}(-z))$ with an error term that is integrable. Hence we have

$$I_1 \leq C + \left| \int_{B_r(x) \cap \{z: |z_n| \leq M|z'|^2\}} \Lambda |z|^{-(n+s)} dz \right|.$$

Denote by Σ_ρ the surface

$$\Sigma_\rho = \{y \in \partial B_\rho: |y_n| \leq M|u'|^2\}.$$

Then $\mathcal{H}^{n-1}(\Sigma_\rho) \leq C\rho^n$ (See [7]) and we get

$$|I_1| \leq C + \int_0^r C\mathcal{H}^{n-1}(\Sigma_\rho)\Lambda\rho^{-(n+s)} d\rho \leq C(1 + \Lambda r^{1-s}).$$

Together it shows that the fractional curvature of ∂E is well defined. Observe the necessity of having $s \in (0, 1)$. \square

4 Improvement of flatness

In this section we wish to prove an improvement of flatness result. We follow the proof of [CRS]

Theorem 4.1 (Improvement of flatness). *Assume ∂E is a nearly viscosity solution in the sense of Theorem 1.2, and fix $0 < \alpha < s$. There exists k_0 depending on s, α such that if $0 \in \partial E$ and*

$$\partial E \cap B_{2^{-l}} \subset \{|x \cdot \nu_l| \leq 2^{-l(\alpha+1)}\}$$

with unit vectors ν_l , for $l = 0, \dots, k_0$, then there exists unit vectors ν_l for any $l \in \mathbf{N}$ such that the above inclusion remains valid.

Let us give a sketch the proof of the theorem before diving into the details.

Sketch of the proof. The proof works by contradiction. We assume that we have a sequence of nearly viscosity solutions in $\{\partial E_k\}$ in B_1 such that the inclusions in the statement of Theorem 4.1 hold for E_k , for $j = 0, \dots, k$; say

$$\partial E_k \cap B_{2^{-j}} \subset \{|x \cdot \nu_j^k| \leq 2^{-j(\alpha+1)}\}$$

but the conclusion fails. Up to rotations we can always assume that $\nu_k^k = e_n$ for every surface ∂E_k . Then, we dilate the sets ∂E_k by a factor 2^k to get a sequence of surfaces $\{\partial \tilde{E}_k\}$, and we stretch $\partial \tilde{E}_k$ by a factor $\frac{1}{a_k}$ in the x_n direction to get a surface $\partial \tilde{E}_k^*$. Now the point is that $\{\partial \tilde{E}_k^*\}$ converge on compact sets to the graph of a function that is a viscosity solution of a certain non local equation, and has some kind of growth condition. This implies that this limit function is linear, and that yields a contradiction when $k \rightarrow +\infty$. \square

The main difference between the proof in the euclidean case is that the kernel $K(x, y)$ is not symmetric in x with respect to y . Hence we cannot get some cancellations by using $K(y + z, y) = K(y - z, y)$. However the following lemma shows that we can get rid of the antisymmetric part of the kernel.

Lemma 4.1 (Removing singularities). *Let $y \in \mathbf{R}^n$ and denote H_y the heat kernel associated to the constant metric $g(y)$. Also denote $K_y(x, z) := \int_0^{+\infty} \frac{dt}{t^{1+\sigma}} H_y(t, x, z)$. Then we have*

$$\int_{\mathbf{R}^n} |K(x, y) - K_y(x, y)| dx < +\infty$$

and the bound is uniform for y in a compact set.

Proof. Our goal is to bound the difference $\tilde{H}(t, x, y) := H(t, x, y) - H_y(t, x, y)$ for $x \in \mathbf{R}^n$. For simplicity of notations we assume $y = 0$ and $g(0) = \text{Id}$ so H_0 is the usual heat kernel on \mathbf{R}^n . First observe that

$$\partial_t \tilde{H}(t, x, 0) = \Delta_g \tilde{H}(t, x, 0) + (\Delta_g - \Delta) H_0(t, x, 0)$$

Denote $F(t, x) := (\Delta_g - \Delta) H_0(t, x, 0)$. By Duhamel's principle we have

$$\tilde{H}(t, x, 0) = \int_0^t e^{(t-s)\Delta_g} F(s, x) ds = \int_0^t \int_{\mathbf{R}^n} H(t-s, x, z) F(s, z) dV_g(z) ds$$

Write

$$F(s, z) dV_g(z) = \text{div}((\sqrt{g}g^{ij} - I)\nabla H_0) + (1 - \sqrt{g})\Delta H_0 dz$$

Now we claim that $g(z) = I + O(|z|)$. This comes from the smoothness of g and the fact that g is bounded. We also have that ∇g is bounded. Hence we obtain after calculations

$$|F(s, z)| dV_g(z) \lesssim |\nabla_z H| + |z| |D^2 H_0| \lesssim \frac{\max(|z|, |z|^2)}{s} H_0(s, z, 0)$$

Integrating over \mathbf{R}^n we get, using $\int_{\mathbf{R}^n} H(t, x, 0) dV(x) = 1$, and recalling the measures dx and $dV(x)$ are comparable,

$$\int_{\mathbf{R}^n} |\tilde{H}(t, x, 0)| dx \lesssim \int_0^t \int_{\mathbf{R}^n} \frac{\max(|z|, |z|^2)}{s} H_0(s, z, 0) \lesssim t + \sqrt{t}$$

But we can also brutally bound $|\tilde{H}(t, x, 0)| \leq H(t, x, 0) + H_0(t, x, 0)$ and using that heat kernels have mass 1 (against the appropriate volume forms, but recall $dV_g(x)$ is comparable to dx), we can just bound $\int |\tilde{H}(t, x, 0)|$ by a constant C . It yields

$$\int_{\mathbf{R}^n} |\tilde{H}(t, x, 0)| dx \leq C \max(1, \sqrt{t}).$$

Finally we deduce

$$\int_{\mathbf{R}^n} |\tilde{H}(t, x, 0)| dx \lesssim \int_{\mathbf{R}^+} \frac{dt}{t^{1+s/2}} \max(1, \sqrt{t}) \lesssim \frac{1}{s} + \frac{1}{1-s}.$$

□

Remark. Observe that $\frac{1}{s} + \frac{1}{1-s}$ goes up to infinity as $s \uparrow 1$. Hence we can probably not expect to get a uniform flatness theorem in our setup

Definition 4.1. For any $y \in \mathbf{R}^n$ we define the symmetric and antisymmetric parts of the kernel $K(x, y)$ by

$$K_s(x, y) := \frac{K(x, y) + K(\mathcal{T}_y(x), y)}{2} \quad \text{and} \quad K_a(x, y) := \frac{K(x, y) - K(\mathcal{T}_y(x), y)}{2},$$

where $\mathcal{T}_y(x)$ denotes the reflection of the point x with respect to y .

Then we have the following

Corollary 4.1. For any $y \in \mathbf{R}^n$, we have

$$\int_{\mathbf{R}^n} |K_a(x, y)| \, dx < +\infty$$

and the bound is uniform for y in a compact set.

Proof. Using the symmetry property $K_y(x, y) = K_y(\mathcal{T}_x(y), y)$, we write

$$K_a(x, y) = \frac{K(x, y) - K_y(x, y)}{2} + \frac{K_y(\mathcal{T}_x(y), y) - K(\mathcal{T}_x(y), y)}{2}$$

and conclude using Lemma 4.1. □

Remark. One could actually introduce splittings $K(x, y) = K_s(x, y) + K_a(x, y)$ and $dV(x) = dV_s(x) + dV_a(x)$ on a general manifold, for $x \in B_{\text{inj}(y)}(y)$ (where $\mathcal{T}_x(y)$ is defined as the geodesic reflection of x with respect to y , i.e. if $x = \exp_y(u)$ then $\mathcal{T}_x(y) := \exp_y(-u)$). In [6], the authors investigate regularity of solutions to certain integrodifferential equations on manifolds with nonnegative curvature, in the spirit of [4]. They manage to get precise quantitative estimates on $|dV_a(x)|$ depending on local bounds on the curvature. Now we are ready to prove a Harnack-type inequality.

Proposition 4.1 (Harnack inequality). *There exists δ, k_1 such that the following hold. Let $k > k_1$ and $r \leq 1$. Moreover assume*

$$\partial E \cap B_{r2^{-l}}(0) \subset \{|x \cdot \nu_l| \leq r2^{-l(1+\alpha)}\} \quad \text{for } l = 0, \dots, k$$

with $\nu_k = e_n$. Then either

$$\partial E \cap B_{r2^{-k\delta}}(0) \subset \{x_n \leq r2^{-k(1+\alpha)}(1 - \delta^2)\}$$

or

$$\partial E \cap B_{r2^{-k\delta}}(0) \subset \{x_n \geq r2^{-k(1+\alpha)}(-1 + \delta^2)\}$$

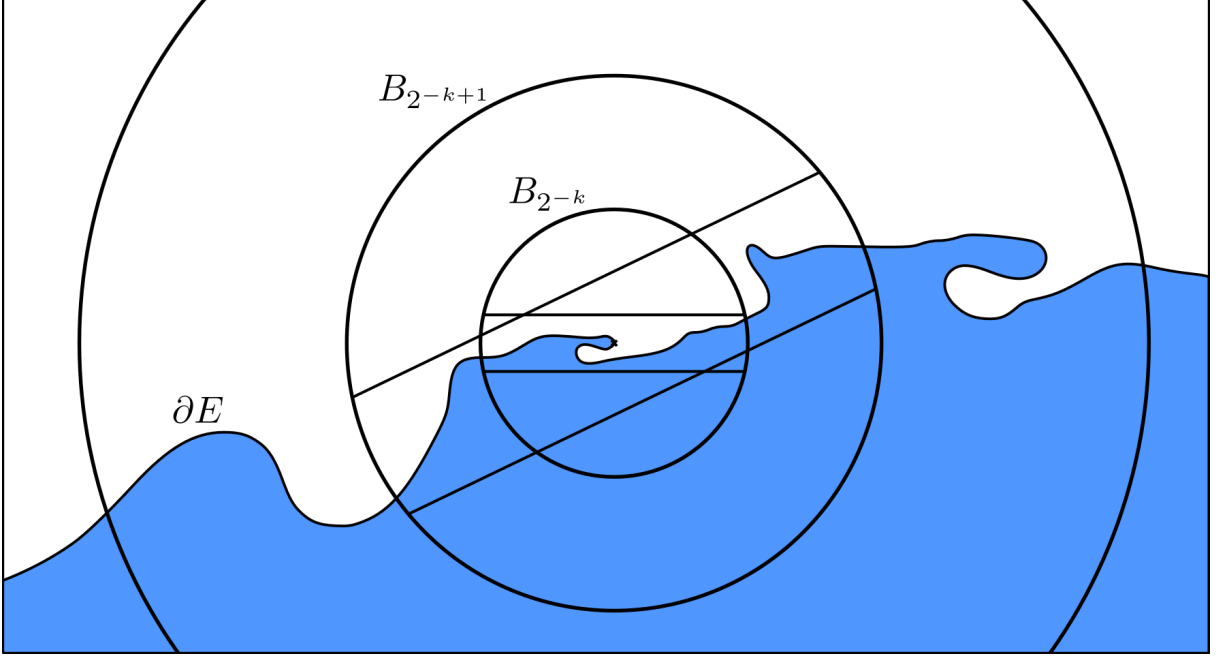


Figure 2: The boundary ∂E is included in flatter and flatter cylinders as we zoom in. Observe that the normal vectors cannot be too far apart.

Proof. We recall the Euler-Lagrange equation at a point $y \in \partial E \cap B_{r_{2^{-k-1}}}(0)$:

$$\left| \int_{\mathbf{R}^n} (\chi_E(x) - \chi_{CE}(x)) K(x, y) \, dV(x) \right| \leq C.$$

We wish to estimate the non local contribution

$$\left| \int_{\mathbf{R}^n \setminus B_{r_{2^{-k-1}}}(y)} (\chi_E(x) - \chi_{CE}(x)) K(x, y) \, dV(x) \right|.$$

First, we start by controlling the tail

$$\left| \int_{CB_r(y)} (\chi_E(x) - \chi_{CE}(x)) K(x, y) \, dV(x) \right| \leq 2 \int_{CB_r(y)} K(x, y) \, dV(x) \leq Cr^{-s}. \quad (4)$$

Now by Corollary 4.1, we have

$$\int_{B_r(y)} |K_a(x, y)| \, dx \leq C.$$

Also split $dV(x) =: dV_s(x) + dV_a(x)$, where we have defined

$$dV_s(x) := \frac{dV(x) + dV(\mathcal{T}_y(x))}{2} \quad \text{and} \quad dV_a(x) := \frac{dV(x) - dV(\mathcal{T}_y(x))}{2}$$

the symmetric and antisymmetric parts of the measure with respect to y . Notice that we have $|\mathrm{d}V_a(x)| = |\sqrt{g(x)} - \sqrt{g(\mathcal{T}_y(x))}| \lesssim |x - y|$. Using the bound $K(x, y) \leq \frac{\Lambda}{d(x, y)^{n+s}}$ we infer

$$\int_{B_r(y)} K(x, y) |\mathrm{d}V_a(x)| \leq \frac{r^{1-s}}{1-s} \leq Cr^{-s}.$$

Finally we obtain

$$\left| \int_{B_r(y)} (\chi_E(y) - \chi_{CE}(y)) K_s(x, y) \mathrm{d}V_s(x) \right| \leq Cr^{-s}.$$

It is way more convenient to work with this formula as we can use $K_s(x, y) \mathrm{d}V_s(x) = K_s(\mathcal{T}_y(x), y) \mathrm{d}V_s(\mathcal{T}_y(x))$ to get some cancellations for sets whose boundaries are trapped in flat cylinders. Indeed we kind of recover the usual setting on \mathbf{R}^n .

Now we aim to bound the contribution in dyadic annuli $B_{2^{-l}}(y) \setminus B_{2^{-l-1}}(y)$, that we denote I_l . Recall

$$\partial E \cap B_{r2^{-l}}(0) \subset \{|x \cdot \nu_l| \leq r2^{-l(1+\alpha)}\} \quad \text{for } l = 0, \dots, k$$

By using symmetry $K_s(x, y) \mathrm{d}V_s(x) = K_s(\mathcal{T}_y(x), y) \mathrm{d}V_s(\mathcal{T}_y(x))$ we get

$$|I_l| \leq \int_{B_{2^{-l}r}(y) \setminus B_{2^{-l-1}r}(y)} \mathbf{1}_{\{|x \cdot \nu_l| \leq r2^{-l(1+\alpha)}\}} \frac{\Lambda}{|x - y|^{(n+s)}} \mathrm{d}V_s(x) \leq \Lambda r^{-s} 2^{l(s-\alpha)} \quad (5)$$

Summing for $l = 0, \dots, k$ we obtain

$$\left| \int_{B_r \setminus B_{r2^{-k-1}}} (\chi_E - \chi_{CE}) K_s(x, y) \mathrm{d}V_s(x) \right| \leq r^{-s} 2^{k(s-\alpha)}.$$

Finally the non local contribution is bounded by

$$\left| \int_{\mathbf{R}^n \setminus B_{r2^{-k-1}}} (\chi_E - \chi_{CE}) K_s(x, y) \mathrm{d}V_s(x) \right| \leq Cr^{-s} 2^{k(s-\alpha)}.$$

Let us work with the rescaled version of $\frac{2^k}{r} E$ that we denote F .

Up to reversing F , assume $\{x_n < -a\} \cap B_1 \subset F$. We also assume F contains more than half of the measure of the cylinder

$$D = \{|x'| \leq \delta\} \times \{|x_n| \leq a\}.$$

Let us show that it implies $\partial F \cap B_\delta \subset \{x_n \geq (-1 + \delta^2)a\}$. Indeed, if it is not verified, we slide by below the paraboloid $x_n = -\frac{a}{2}|x'|^2$. We touch ∂F at a point \tilde{y} such that

$$|\tilde{y}'| \leq 2\delta, \quad |\tilde{y}_n| \leq 2a\delta^2 \quad (6)$$

Indeed, denote $(x', -a/2|x'|^2 + t)$ the sliding graphs. Then we have contact for $t \leq a(-1 + 3/2\delta^2)$. Indeed, looking at $|x'| = \delta$ if $-\frac{a}{2}\delta^2 + t \geq (-1 + \delta^2)a$ then contact has *already* happened. But at the contact point we have $\tilde{y}_n > -a$ so $|\tilde{y}'|^2 \leq 3\delta^2$. Also,

$|\tilde{y}_n + a| \leq |h + a| \leq 3a/2\delta^2$. Denote \tilde{P} the subgraph of the paraboloid, and P the rescaled version $2^{-k}\tilde{P}$ scaled back by a factor 2^{-k} . Then one has

$$\begin{aligned} \int_{B_{2^{-k-1}(y)}} (\chi_E - \chi_{CE})K(x, y) dV_s(x) &= \int_{B_{2^{-k-1}(y)}} (\chi_P - \chi_{CP})K_s(x, y)dV_s(x) \\ &+ 2 \int_{B_{2^{-k-1}(y)}} \chi_{E \setminus P}(x)K_s(x, y) dV_s(x) \\ &=: I_3 + I_4 \end{aligned}$$

We can assume that y lies at the top of the paraboloid (this is where $|I_3|$ is maximal), Observe

$$\partial\tilde{P} \cap B_r(\tilde{y}) \subset \{|x_n| \leq ar^2\}.$$

Then as in Lemma 3.1 we get a lower bound

$$I_3 \geq -C2^{ks} \int_0^{1/2} \frac{a\rho^n}{\rho^{n+s}} d\rho \geq -C(n, s)2^{k(s-\alpha)}. \quad (7)$$

Moreover, $|\tilde{P} \cap D| \leq \delta^2|D|$ so $|(F \setminus \tilde{P}) \cap D| \geq (1/2 - \delta^2)|D|$. Also if y verifies (6) and assuming $a \leq \delta$ we have

$$|\tilde{y} - x| \leq \sqrt{(3\delta)^2 + (2a)^2} \leq 4\delta,$$

it follows that

$$I_4 \geq C2^{ks}(1/2 - \delta^2) \frac{a\delta^{n-1}}{(4\delta)^{n+s}} \geq C(n)\delta^{-1-s}2^{k(s-\alpha)}.$$

If $\delta > 0$ is chosen small enough in function of n, s, α (and $k_1(\delta)$ is large enough to guarantee $a \leq \delta$). Combining these two estimates we get

$$\int_{B_{2^{-k-1}(y)}} (\chi_E - \chi_{CE})K_s(x, y) dV_s(x) \geq r^{-s}2^{k(s-\alpha)}(-C(n, s) + C(n)\delta^{-1-s}).$$

given $a \leq \delta$. Finally, we obtain

$$\begin{aligned} \int_{\mathbf{R}^n} (\chi_E(x) - \chi_{CE}(x))K(x, y) dV(x) &\geq r^{-s}2^{k(s-\alpha)}(-C(n, s) + C(n)\delta^{-1-s}) \\ &\geq 2^{k(s-\alpha)}(-C(n, s) + C(n)\delta^{-1-s}). \end{aligned}$$

And we reach a contradiction to Euler-Lagrange equation as $\delta \rightarrow 0$. \square

Remark. The crucial point of the theorem is that δ does not depend on the radius r and can be chosen uniformly for any contact point y . This will allow to iterate Harnack inequality and get the same improvement δ at each step. This is where the hypothesis of Theorem 1.2 intervenes.

Applying Harnack inequality, we get that ∂E is trapped in a cylinder of flatness $\frac{\theta}{8}a$, where $\theta := 1 - \frac{\delta^2}{2}$. Applying Harnack inequality again, we get that ∂E is trapped in cylinder of flatness $(\theta/\delta)^j a$ in a ball of radius δ^j . Unfortunately, we can not keep going forever

as we can apply Harnack inequality only as long as $(\theta/\delta)^j a \leq a_0$ where $a = 2^{-k\alpha}$ and $a_0 = 2^{-k_0\alpha}$. Denote $k(a)$ the maximum value of j which satisfies the previous inequality. The height of the cylinder at the last step is given by

$$\theta^{k(a)} a \approx a^{\gamma+1}$$

where γ is defined by the relation $\theta = \delta^\gamma$, so when we dilate by a factor $\frac{1}{a_k}$ in the x_n direction we have a cylinder of height $\approx a_k$

Then we observe that we can actually shift the result to other base points $y \in B_{r2^{-k-1}}$. Indeed for such points we have

$$|y \cdot \nu_l| \leq r2^{l(1+\alpha)}.$$

Hence for any $x \in \partial E \cap B_{2^{l-1}}(y)$, we have $x \in \partial E \cap B_{2^l}(0)$ whence

$$|(y-x) \cdot \nu_l| \leq 2r2^{l(1+\alpha)}$$

We deduce that

Proposition 4.2. *We have for $x, y \in \partial \tilde{E}_k^*$*

$$|y_n - x_n| \leq C \max(b_k^\gamma, |y' - x'|^\gamma)$$

in $B'_{1/2}$ with $b_k \rightarrow 0$

Proof. It follows from the previous remark. If we zoom in by a factor $1/\delta$ then flatness is improved by a factor δ^γ/δ . This gives the Hölder estimates when x' and y' are far apart. The problem is that we can not keep zooming in indefinitely because we need flatness $\leq a_0$ to iterate Harnack inequality. That is why we only get $|y_n - x_n| \leq Cb_k^\gamma$ when x' and y' are too close. \square

4.1 Convergence to a limit function

From the previous results we obtain

Proposition 4.3. *In $B_{1/2}$, the sequence $\{\partial \tilde{E}_k^*\}$ converges to the graph of a Hölder continuous function.*

This proposition follows from the next lemma, that is a variant of Arzela-Ascoli Theorem.

Lemma 4.2 (Arzelà-Ascoli). *Consider a sequence of functions $f_k : B_1 \rightarrow [-1, 1]$ such that $|f_k(x) - f_k(y)| \leq \max(b_k^\gamma, |x - y|^\gamma)$ with $b_k \rightarrow 0$. Then there exists a subsequence $(f_{\varphi(n)})$ that converges in L^∞ to a γ -Hölder continuous function.*

Proof. Denote I the (countable) set of rational points in B_1 . Since the functions f_k are uniformly bounded, by a process of diagonal extraction, we get a subsequence $(f_{\varphi(n)})$ such that $(f_{\varphi(n)}(x))$ converges to a point $f(x) \in [-1, 1]$ for every $x \in I$. Fix $\varepsilon > 0$. For $x_k \in I$, take an integer $N(\varepsilon, x_k)$ such that for every $n, m \geq N(\varepsilon, x_k)$ we have

$$|f_{\varphi(n)}(x_k) - f_{\varphi(m)}(x_k)| \leq \varepsilon^\gamma.$$

Let N_0 be large enough such that for all $n \geq N_0$, $|b_n| \leq \varepsilon$. Then for $t, s \in B_1$ one has

$$|f_n(t) - f_n(s)| \leq \max(\varepsilon^\gamma, |t - s|^\gamma).$$

Take x_1, \dots, x_K in B_1 such that $\{B(x_k, \varepsilon)\}_{k=1, \dots, K}$ covers B_1 :

$$B_1 \subset \bigcup_{1 \leq k \leq K} B(x_k, \varepsilon).$$

Let t in B_1 . Then we can take $k \in \{1, \dots, K\}$ such that $|t - x_k| \leq \varepsilon$. Then

$$|f_{\varphi(n)}(t) - f_{\varphi(m)}(t)| \leq |f_{\varphi(n)}(t) - f_{\varphi(n)}(x_k)| + |f_{\varphi(n)}(x_k) - f_{\varphi(m)}(x_k)| + |f_{\varphi(m)}(x_k) - f_{\varphi(m)}(t)|,$$

Hence, for every $n, m \geq N_0 + \max\{N(\varepsilon, x_1), \dots, N(\varepsilon, x_K)\}$ we have

$$|f_{\varphi(n)}(t) - f_{\varphi(m)}(t)| \leq 2 \max(\varepsilon^\gamma, |t - x_k|^\gamma) + \varepsilon^\gamma \leq 3\varepsilon^\gamma.$$

Thus $(f_{\varphi(n)})$ is Cauchy, uniformly in t , so for any $t \in B_1$, the sequence $(f_{\varphi(n)}(t))$ converges to a point $f(t) \in B_1$. But $|f_{\varphi(n)}(t) - f(t)| \leq 3\varepsilon^\gamma$ for n large enough, so we have convergence in L^∞ . Finally note that

$$|f(t) - f(s)| = \lim |f_{\varphi(n)}(t) - f_{\varphi(n)}(s)| \leq \lim \max(b_{\varphi(n)}^\gamma, |t - s|^\gamma) \leq |t - s|^\gamma,$$

so f is γ -h\"older continuous. □

Proof of Proposition 4.3. Denote

$$f_k^+(x') := \sup\{x_n \mid (x', x_n) \in \partial \tilde{E}_k^*\}$$

and

$$f_k^-(x') := \inf\{x_n \mid (x', x_n) \in \partial \tilde{E}_k^*\}.$$

By Proposition 4.2 we have

$$|f_k^+ - f_k^-| \leq b_k^\gamma. \tag{8}$$

But by Proposition 4.2, f_k^+ and f_k^- satisfy the hypotheses of Lemma 4.2, so up to taking subsequences, (f_k^+) and (f_k^-) converge in L^∞ to γ -h\"older continuous functions f^+ and f^- . But from (8) it follows that $f^+ = f^-$ and we conclude by using that $\partial \tilde{E}_k^*$ is trapped between the graphs of f_k^- and f_k^+ in $B_{1/2}$. □

Actually, such estimates can be conducted in larger and larger balls. Indeed, there exists a constant C independent of j and k such that

$$|\nu_j^k - \nu_{j+1}^k| \leq C2^{-j\alpha}.$$

Hence we find

$$\partial \tilde{E}_k \cap B_{2^j(0)} \subset \{|x_n| \leq C2^{\alpha(j-k)}\}$$

where C is independent of E .

Proposition 4.4. *Up to a subsequence, $\{\partial\tilde{E}_k^*\}$ converges uniformly on compact sets to the graph of a Hölder continuous function $f : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$. Moreover, $f(0) = 0$ and we have a growth condition*

$$|f(x')| \leq C(1 + |x'|^{1+\alpha}).$$

Proof. It only remains to prove the growth estimate. We work with the rescaled set $\partial\tilde{E}$. Recall $|e_l - e_{l+1}| \leq a2^{(l+1)\alpha}$, hence $|e_l - e_0| \leq Ca2^{l\alpha}$. Then

$$\partial\tilde{E} \cap B_{2^k} \subset \{|x \cdot e_k| \leq a2^{k(1+\alpha)}\}.$$

Let $p_k := \frac{1}{a}(e_k - \langle e_k, e_0 \rangle e_0)$. Then we have

$$\{|x \cdot e_k| \leq a2^{k(1+\alpha)}\} \subset \{|x \cdot (ap_k + \langle e_k, e_0 \rangle e_0)| \leq a2^{k(1+\alpha)}\}.$$

Observe

$$|p_k|^2 = \frac{1 - \langle e_k, e_0 \rangle^2}{a^2}.$$

and $\langle e_k, e_0 \rangle \geq 1 - ca^22^{2k\alpha}$, thus,

$$|p_k|^2 \leq \frac{1 - (1 - ca^22^{2k\alpha})^2}{a^2} \leq \frac{2ca^22^{2k\alpha}}{a^2} = c2^{2k\alpha}.$$

Consequently, $|p_k| \leq c2^{k\alpha}$. We infer

$$\{|x_n + x' \cdot ap_k| \leq a2^{k(1+\alpha)} + |x_n| |\langle e_k, e_0 \rangle - 1|\}.$$

But in B_{2^k} , as $2^{k\alpha}a \leq 1$ we get

$$|x_n| |\langle e_k, e_0 \rangle - 1| \leq ca2^{k(1+\alpha)}a2^{k\alpha} \leq ca2^{k(1+\alpha)}.$$

We finally obtain

$$\partial\tilde{E} \cap B_{2^k} \subset \left\{ \left| \frac{x_n}{a} + p_k \cdot x' \right| \leq C2^{k(1+\alpha)} \right\}.$$

As $a_m \rightarrow 0$, up to a subsequence, $p_k^{(m)}$ converges to $p_k \in \mathbf{R}^{n-1}$ and we find

$$|f(x') + p_k \cdot x'| \leq C2^{k(1+\alpha)}.$$

We conclude that in B'_{2^k}

$$|f(x')| \leq C2^{k\alpha}(|x'| + 2^k), \quad k \geq 0,$$

hence the growth condition on f . □

5 The limit function is linear

We recall the process that yields the function f . Consider a minimal surface ∂E_k such that we have $\partial E_k \cap B_{2^{-l}}(0) \subset \{|x \cdot e_l^k| \leq 2^{-l(1+\alpha)}\}$ for $l = 0, \dots, k$. We rescale by a factor 2^k to get

$$\partial \tilde{E}_k \cap \tilde{B}_{2^j} \subset \{|x \cdot \nu_j^k| \leq a_k 2^{j(1+\alpha)}\}$$

where $a_k = 2^{-k\alpha}$. Then we dilate in the x_n direction by a factor $\frac{1}{a_k}$ to get a surface $\partial \tilde{E}_k^*$. Then the claim is that $\{\partial \tilde{E}_k^*\}$ converges to the graph of a function f satisfying a growth condition $|f(x')| \leq 1 + |x'|^{1+\alpha}$.

We start by proving a lemma that makes precise the value of $K(x, y)$ for any constant metric

Lemma 5.1. *Let A be a positive definite symmetric matrix and denote $\Phi(t, x)$ the heat kernel associated to the Riemannian manifold (\mathbf{R}^n, A) . Then*

$$\Phi(t, x) = \frac{(4\pi t)^{-n/2}}{\sqrt{\det(A)}} e^{-\frac{\langle Ax, x \rangle}{4t}}.$$

Hence for this kernel

$$K(x, y) = \frac{C_n}{\det(A)^{1/2}} \langle A(x - y), x - y \rangle^{-\frac{n+s}{2}}$$

Proof. Consider u smooth that solves

$$\begin{cases} \partial_t u = \operatorname{div}(A \nabla u) \\ u(0, x) = u_0(x) \end{cases}$$

Using Fourier transform we get

$$\partial_t \hat{u}(t, \xi) = -\langle A\xi, \xi \rangle \hat{u}(t, \xi).$$

This yields

$$\hat{u}(t, \xi) = e^{-t\langle A\xi, \xi \rangle} \hat{u}_0(\xi).$$

It follows that

$$\Phi(t, x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x, \xi \rangle - t\langle A\xi, \xi \rangle} d\xi,$$

where it remains to compute the inverse Fourier Transform of the gaussian. Write $B = tA$ and

$$i\langle x, \xi \rangle - \langle B\xi, \xi \rangle = -\frac{1}{4}\langle B^{-1}x, x \rangle - \left\langle B\left(\xi + iB^{-1}\frac{x}{2}\right), \xi + iB^{-1}\frac{x}{2}\right\rangle.$$

We infer

$$\Phi(t, x) = (2\pi)^{-n} e^{-\frac{1}{4}\langle B^{-1}x, x \rangle} \int_{\mathbf{R}^n} e^{-\langle B(\xi + iB^{-1}\frac{x}{2}), \xi + iB^{-1}\frac{x}{2} \rangle} d\xi.$$

By doing an orthonormal change of variables and denoting $B \sim \text{diag}(\lambda_1, \dots, \lambda_n)$, we obtain

$$\int_{\mathbf{R}^n} e^{-\langle B(\xi+iB^{-1}\frac{x}{2}), \xi+iB^{-1}\frac{x}{2} \rangle} d\xi = \int_{\mathbf{R}^n} e^{-\sum \lambda_i \xi_i^2} dx = \prod_{i=1}^n \left(\int_{-\infty}^{+\infty} e^{-\lambda_i \xi_i^2} dx_i \right) = \prod_{i=1}^n \sqrt{\frac{\pi}{\lambda_i}} = \frac{\pi^{n/2}}{\det(B)}.$$

Finally

$$\Phi(t, x) = \frac{(4\pi t)^{-n/2}}{\sqrt{\det(A)}} e^{-\frac{(Ax, x)}{4t}}.$$

□

Then we are ready to prove

Proposition 5.1. *The limit function f satisfies*

$$D^{\frac{1+s}{2}} f = 0$$

where $D = \Delta_{g(0)}$, hence is linear (See [4]).

Proof. We refer again to [2] for the proof in the euclidean setup. We assume for simplicity that $g(0) = \text{Id}$. Fix $\eta > 0$. Assume $\varphi + |x'|^2$ is a \mathcal{C}^2 function that touches f by below, say for simplicity at the origin. Then one can find ∂E minimal, and a very small, such that $\partial \tilde{E}$ is included in a $a\eta$ -neighbourhood of $\{(x', a f(x'))\}$ for $|x'| \leq R$, $\partial \tilde{E}$ is touched by below at $\tilde{y} := 2^k y$ with $|\tilde{y}'| \leq \eta$ by a vertical translation of $a\varphi$. Then we can write

$$\partial \tilde{E} \subset \{y + z : |z_n - a(f(y' + z') - f(y'))| \leq 2a\eta\}.$$

Define the cylinder

$$D_r := \{|x' - y'| \leq r\} \times \{|x_n - y_n| \leq r\}$$

Fix $R = 2^l > 0$ large, $\delta > 0$ small. Recall the Euler-Lagrange equation at point y :

$$\int (\chi_E - \chi_{CE}) K_s(x, y) dV_s(x) \leq C,$$

where C is uniform for y in a compact set. First we estimate

$$\int_{D_{\delta 2^{-k}}} (\chi_E - \chi_{CE}) K_s(x_0, x) dV_s(x).$$

Observe that in $D_{\delta 2^{-k}}$, E contains the subgraph P of a translation of $a2^{-k}\varphi(2^k \cdot)$. It follows using (7) that

$$\int_{D_{\delta 2^{-k}}} (\chi_E - \chi_{CE}) K_s(y, x) dV_s(x) \geq \int_{D_{\delta 2^{-k}}} (\chi_P - \chi_{CP}) K_s(y, x) dV_s(x) \geq 2^{ks} C(\varphi) a \delta^{1-s}.$$

Now combining (4) and (5) we have

$$\int_{D_{R 2^{-k}}} (\chi_E - \chi_{CE}) K_s(y, x) dV_s(x) \leq C + C 2^{(k-l)(s-\alpha)} = C + C 2^{k(\alpha-s)} R^{\alpha-s}.$$

It remains to estimate

$$\int_{D_{R2^{-k}} \setminus D_{\delta 2^{-k}}} (\chi_E - \chi_{CE}) K_s(y, x) dV_s(x).$$

It is however complicated to work with this formula as $K_s(y, x)$ can have a wild behaviour. However by Proposition 4.1 we know that we can replace $K_s(x, y)$ by $K_y(x, y)$ in the formula. Also we can replace $dV_s(x)$ by $\sqrt{g(y)} dx$. It only changes the value of the integral by a constant C that does not depend on R, δ , and is uniform if y chosen in a compact set. Now, by Lemma 5.1 we know that

$$K_y(y, x) \simeq \frac{1}{\det(Q)^{1/2}} \langle Q(y-x), y-x \rangle^{-\frac{n+s}{2}}$$

where $Q = g(y)$. Also, $\|Q - I\| = \|g(y) - g(0)\| \leq C|y| \leq C2^{-k}\eta$. It follows that

$$|K_y(y, x) - |y-x|^{-(n+s)}| \leq C2^{-k}\eta|y-x|^{-(n+s)},$$

hence

$$\int_{D_{R2^{-k}} \setminus D_{\delta 2^{-k}}} |K_y(y, x) - |y-x|^{-(n+s)}| dx \leq 2^{k(s-1)}\eta\delta^{-s} \leq 2^{k(s-\alpha)}\eta\delta^{-s}.$$

Eventually, we have

$$\begin{aligned} & \int_{D_{R2^{-k}} \setminus D_{\delta 2^{-k}}} (\chi_E - \chi_{CE}) K_s(x, y) dV_s(x) \\ &= 2^{k(s-\alpha)} \left(\int_{B'_R \setminus B'_\delta} \frac{2(f(z' + \bar{y}') - f(\bar{y}))}{|z'|^{n+s}} dz' + O(\eta)\delta^{-s} + O(a) \right). \end{aligned}$$

Let $k \rightarrow +\infty, \eta \rightarrow 0$ to obtain, using Euler-Lagrange equation

$$\int_{B'_R \setminus B'_\delta} \frac{(f(z') - f(0))}{|z'|^{n+s}} dz' \leq C(R^{\alpha-s} + \delta^{1-s}),$$

and we get the desired result as $\delta \rightarrow 0, R \rightarrow +\infty$.

Remark. The constant C becomes irrelevant as the factor $2^{k(s-\alpha)}$ grows to infinity. \square

Proof of improvement of flatness. By contradiction, assume we have a sequence ∂E_m of minimal surfaces, with $a_m \rightarrow 0$, such that ∂E_m^* converges on compact sets to the graph of a linear function f and ∂E_m does not include in any cylinder of flatness $2^{-\alpha}a_m$ in $B_{1/2}$, i.e.

$$\partial E_m \cap B_{1/2} \not\subseteq \{|x \cdot e| \leq a_m 2^{-(\alpha+1)}\}, \quad \forall e \in \mathbb{S}^{n-1}$$

Let $u \in \mathbf{R}^{n-1}$ such that $f(x') = u \cdot x'$. Let v an orthogonal vector to the hyperplane $\{(x', u \cdot x'), x' \in \mathbf{R}^{n-1}\}$. Since $\partial E_m^* \rightarrow \text{Graph}(f)$, we have in $B_{1/2}$

$$\partial E_m^* \subset \{|x \cdot v| \leq \varepsilon_m\}$$

where $\varepsilon_m \rightarrow 0$. Let w orthogonal to $\{(x', a_m u \cdot x'), x' \in \mathbf{R}^{n-1}\}$, and $\Psi(x) = (x', a_m x_n)$. Then, in $B_{1/2}$,

$$|\Psi(x) \cdot w| \leq \frac{a_m \varepsilon_m |w_n|}{|v_n|} \leq \frac{a_m \varepsilon_m}{|v_n|}$$

Hence

$$\partial E_m \cap B_{1/2} \subset \{|x \cdot w| \leq \frac{\varepsilon_m}{|v_n|} a_m\}$$

and we reach a contradiction when $a_m \rightarrow 0$. Observe $v_n \neq 0$ because an orthogonal vector to the graph of u cannot be in $\mathbf{R}^{n-1} \times \{0\}$. \square

6 Back to main results

6.1 Proof of theorem 1.2

Now we are ready to prove our main theorem. The following proof is quite standard.

Proof of Theorem 1.2. Fix $\varepsilon > 0$, say $\varepsilon < \frac{1}{4}2^{-k_0(\alpha+1)}$. Then for any $k = 0, \dots, k_0$ and $x \in B_{1/2}(0)$ we have

$$\partial E \cap B_{1/2}(x) \subset \{|(z-x) \cdot e_n| \leq \frac{1}{2}2^{-k(\alpha+1)}\}.$$

Thus we can apply Theorem 4.1 to get a sequence of unit vectors $\nu_k(x)$ such that

$$\partial E \cap B_{2^{-k}}(x) \subset \{|(z-x) \cdot \nu_k^x| \leq \frac{1}{2}2^{-k(\alpha+1)}\}.$$

These inclusions imply that $|\nu_k(x) - \nu_{k+1}(x)| \leq C2^{-k\alpha}$ with C independent of k . Hence $(\nu_k(x))$ converges with a geometric rate to a unit vector $\nu(x)$. More precisely we have $|\nu_k(x) - \nu(x)| \leq C2^{-k\alpha}$. It follows that

$$\partial E \cap B_{2^{-k}}(x) \subset \{|(z-x) \cdot \nu(x)| \leq C2^{-k(\alpha+1)}\}.$$

This yields for any $\rho \in (0, 1/2)$

$$\partial E \cap B_\rho(x) \subset \{|(z-x) \cdot \nu(x)| \leq C\rho^{1+\alpha}\}.$$

We can take ε small enough to have $C \ll 1$. We infer that in $B_{1/2}$,

$$|\nu(x) - \nu(y)| \leq C|x-y|^\alpha.$$

Actually we get that $\nu(x) \cdot \nu(0)$ stays positive for $x \in B_{1/2}(0)$. It implies that ∂E is the graph of a (differentiable from results above) function f in the direction $\nu(0)$. We write $x = (x', f(x'))$ a point on ∂E . Now observe that

$$\nu(x', f(x')) = \frac{(-\nabla_{x'} f, 1)}{\sqrt{1 + |\nabla_{x'} f|^2}}.$$

By projecting the vectors onto the n -th coordinate we get

$$\left| \frac{1}{\sqrt{1 + |\nabla_{x'} f|^2}} - \frac{1}{\sqrt{1 + |\nabla_{y'} f|^2}} \right| \leq C|x - y|^\alpha.$$

Then write

$$\nu(x) - \nu(y) = \left(\frac{1}{\sqrt{1 + |\nabla_{x'} f|^2}} - \frac{1}{\sqrt{1 + |\nabla_{y'} f|^2}} \right) (-\nabla_{x'} f, 1) + \frac{1}{\sqrt{1 + |\nabla_{y'} f|^2}} (\nabla_{y'} f - \nabla_{x'} f).$$

We get

$$C(1 + \sqrt{1 + |\nabla_{x'} f|^2})|x - y|^\alpha \geq \frac{1}{\sqrt{1 + |\nabla_{y'} f|^2}} (\nabla_{y'} f - \nabla_{x'} f). \quad (9)$$

Now notice that in coordinates, we have $\nu(0) = e_n$ hence $\nabla_0 f = 0$. This yields

$$C2^{-\alpha}(1 + \sqrt{1 + |\nabla_{x'} f|^2}) \geq |\nabla_{x'} f|.$$

Taking C small enough so that $C < 2^\alpha$ it shows that $\nabla_{x'} f$ is bounded in $B'_{1/2}$, in particular f is Lipschitz and $|f(x') - f(y')| \leq L|x' - y'|^\alpha$, so that for $x, y \in \partial E \cap B_{1/2}$ we have $|x - y|^\alpha \leq C|x' - y'|^\alpha$. Using (9) with these results we get

$$|\nabla_{x'} f - \nabla_{y'} f| \leq C|x' - y'|^\alpha.$$

and f is $\mathcal{C}^{1,\alpha}$ in $B'_{1/2}$, which concludes the proof. \square

6.2 Proof of Corollary 1.1

To prove our corollary, we take a small metric ball $B_r \subset M$ that we map to a ball $B_r(0) \subset \mathbf{R}^n$. Then we define a new metric \bar{g} on \mathbf{R}^n that is equal to g around 0 and the identity away from the origin. Then we show that we can apply Theorem 1.2 to conclude. We use ideas similar to those of the proof of Lemma 4.1.

Proof of Corollary 1.1. Take $B_{3r}(p) \subset M$ with $3r < \text{inj}(p)$. Then take $x \in B_r(p)$. We have in the viscosity sense

$$\int_{B_{2r}(x)} (\chi_E(y) - \chi_{CE}(y)) K(x, y) \, dV(y) + \int_{M \setminus B_{2r}(x)} (\chi_E(y) - \chi_{CE}(y)) K(x, y) \, dV(y) = 0,$$

where the second integral actually converges and is bounded by Cr^{-s} . Hence

$$\left| \int_{B_{2r}(x)} (\chi_E(y) - \chi_{CE}(y)) K(x, y) \, dV(y) \right| \leq Cr^{-s}.$$

Now we take Riemannian normal coordinates $\varphi : B_{3r}(p) \rightarrow B_{3r}(0) \subset \mathbf{R}^n$. We denote $E_\varphi := \varphi(E)$ and $H(t, x, y)$ for $H(t, \varphi^{-1}(x), \varphi^{-1}(y))$ for the heat kernel in coordinates. By a change of coordinates in the integral we obtain

$$\left| \int_{\varphi(B_{2r}(x))} (\chi_{E_\varphi}(y) - \chi_{E_\varphi}(y)) K(x, \varphi^{-1}(y)) \sqrt{g(y)} \, dy \right| \leq Cr^{-s}.$$

where g denotes the metric in coordinates. Now take $0 < \lambda < 1$ depending on the curvature such that $B_{\lambda r}(\varphi(x)) \subset \varphi(B_{2r}(x))$. We obtain

$$\left| \int_{B_{\lambda r}(\varphi(x))} (\chi_{E_\varphi}(y) - \chi_{E_\varphi}(y)) K(x, \varphi^{-1}(y)) \sqrt{g(y)} \, dy \right| \leq C(1 + \lambda^{-s})r^{-s}.$$

We define a new set $F \subset \mathbf{R}^n$ by letting

$$F = E_\varphi \text{ in } B_{3r} \quad \text{and} \quad F \cap B_{3r}^c = \emptyset.$$

In normal coordinates we have $g(0) = I$ and $Dg(0) = 0$. Now let $\eta \equiv 1$ in $B_{2r}(0)$ and $\eta \equiv 0$ outside B_{3r}^c . Define another metric \bar{g} by setting

$$\bar{g} = \eta g + (1 - \eta)I = I + \eta(g - I).$$

Denote \bar{H} the associated heat kernel and $\bar{K}(x, y) = \int_0^{+\infty} \frac{dt}{t^{1+\sigma}} \bar{H}(t, x, y)$. Also let $\tilde{H} := H - \bar{H}$ and $\tilde{K} := \int \frac{dt}{t^{1+s/2}} |\tilde{H}|$. Fix $y \in B_{2r}(0)$. We have

$$\begin{cases} \partial_t \tilde{H}(t, x, y) = \Delta_g \tilde{H}(t, x, y) & \text{for } x \in B_{2r}(0) \\ \tilde{H}(t, x, y) \rightarrow 0 & \text{as } t \rightarrow 0 \text{ in } B_{2r} \end{cases}$$

Now by parabolic maximum principle

$$\sup_{B_{2r} \times [0, T]} |\tilde{H}(t, x, y)| \leq \sup_{\partial B_{2r} \times [0, T]} |\tilde{H}(t, x, y)|.$$

That yields by standard heat kernel estimates

$$\sup_{B_{2r}(0) \times [0, T]} |\tilde{H}(t, x, y)| \leq \sup_{[0, T]} \frac{1}{t^{n/2}} \exp\left(-C \frac{(2r - |y|)^2}{t}\right).$$

We deduce that for any $x, y \in B_r(0)$ and $t \leq T$ we have

$$\tilde{H}(t, x, y) \leq \sup_{[0, T]} \frac{1}{t^{n/2}} \exp\left(-C \frac{r^2}{t}\right).$$

Finally, for $t \lesssim \frac{r^2}{n}$ we obtain

$$\tilde{H}(t, x, y) \leq \frac{1}{t^{n/2}} \exp\left(-C \frac{r^2}{t}\right).$$

So, if $x \in B_{r/2}$ we have

$$\int_{B_{r/2}(x)} |\tilde{H}(t, x, y)| \, dy \leq \frac{r^n}{t^{n/2}} \exp\left(-C \frac{r^2}{t}\right)$$

for small times, otherwise we just bound the integral by 2. Integrating over \mathbf{R}_+ we get

$$\int_{B_{r/2}(x)} \tilde{K}(x, y) \, dy \leq Cr^{-s}$$

It means we can replace K by \bar{K} and only change the value of the integral by an uniform constant. Also, $g = \bar{g}$ around 0 so we obtain for $x \in B_{r/2}$

$$\left| \int_{B_{\lambda r}(\tilde{x})} (\chi_F - \chi_{CF})(y) \bar{K}(x, y) \sqrt{\bar{g}(y)} \, dy \right| \leq C(1 + \lambda^{-s})r^{-s}$$

Lastly we get

$$\left| \int_{\mathbf{R}^n} (\chi_F - \chi_{CF})(y) \tilde{K}(\tilde{x}, y) \sqrt{\bar{g}(y)} \, dy \right| \leq Cr^{-s}$$

in the viscosity sense, and we can apply the main theorem to the set F . □

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