# Rapport de stage : géométrie symplectique à Uppsala

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# 1 Introduction

## 1.1 Déroulement du stage

J'ai effectué mon stage de deuxième année dans la ville d'Uppsala, une ville située à 70 km au nord de Stockholm, au sud-est de la Suède. Pendant près de 5 mois, j'ai travaillé au département de mathématiques de l'université d'Uppsala, situé au laboratoire Ångström, qui regroupe la plupart des facultés de sciences (mathématiques, physique, chimie, ingénierie, informatique...).

Le chercheur qui encadrait mon stage est Georgios Dimitroglou Rizell, un mathématicien suédois travaillant sur la géométrie symplectique et la géométrie de contact. J'ai pu joindre Georgios grâce à Emmanuel Giroux, mon tuteur à l'ENS, qui connaissait bien Georgios. Il m'a proposé de travailler sur l'homologie de Floer, ce qui m'a donné envie car j'ai bien aimé découvrir l'homologie l'an dernier dans les cours de topologie algébrique ainsi que de géométrie différentielle, puis l'utiliser cette année au sein du groupe de travail sur les sphères exotiques.

Georgios m'a donné une clef de la *Gästrum*, un bureau destiné aux chercheurs invités : je l'ai partagé du début à la fin avec Raziyeh, une doctorante iranienne travaillant sur les représentations d'algèbres non commutatives et les carquois, et avec Hassan, un étudiant pakistanais en master à Lund effectuant à Uppsala son mémoire sur les groupes exceptionnels et les algèbres d'octonions. Quelques personnes sont venues travailler pour quelques jours dans ce bureau, mais nous n'avons pas fait connaissance. La cohabitation était parfois un peu difficile, avec Hassan notamment, ce qui a créé quelques tensions, mais heureusement nous ne nous tenions pas rancune après nos heurts.

J'ai adoré la flexibilité de la vie de chercheur : je venais au bureau et j'en repartais à l'heure que je voulais, et Georgios ne passait jamais à mon bureau à l'improviste mais m'envoyait des mails pour fixer des rendez-vous hebdomadaires avec lui. Je pouvais aussi rester chez moi pour travailler, mais je préférais me mettre dans une ambiance de travail en allant au bureau. Pour l'anecdote, il y a eu au début de mon voyage une grosse tempête de neige qui a causé l'annulation de tous les bus d'Uppsala. J'ai donc dû faire appel à mon propriétaire (qui est aussi mon voisin) pour qu'il nous ramène (mon colocataire et moi) chez nous. Le lendemain, j'écris à Georgios pour lui signaler que je reste travailler à la maison pour éviter d'être bloqué à nouveau, et il me répond que ce n'est pas nécessaire de le prévenir : il dit que je devrais plutôt profiter de la neige pour aller skier, ce qu'il va d'ailleurs faire lui-même !

## 1.2 Vie en Suède

J'habitais dans un grand appartement situé en périphérie d'Uppsala, en colocation avec mon ami Thomas, aussi normalien et en stage de deuxième année. Le propriétaire de l'appartement, Jawad, parle couramment français, ce qui nous a été bien pratique pour entrer en contact avec lui. L'appartement fait partie de la maison dans laquelle il vit, donc il a pu nous aider à nous installer, faire des courses et trouver comment se déplacer jusqu'au centre-ville. En effet, il fallait prendre un premier bus pour se rendre au centre-ville ou au centre commercial, puis un deuxième pour aller jusqu'au laboratoire. J'avais aussi la possibilité de m'y rendre à vélo, car Jawad nous a prêté deux vélos, mais le trajet était long (45 min, donc 15 min de plus qu'en bus).

À mon arrivée en Suède, début février, il a neigé quelques rares fois, mais la neige qui restait tenait au sol pendant plusieurs semaines, c'était magnifique ! J'aimais beaucoup me promener dans la neige épaisse et repérer les traces des animaux dans le bois voisin. La rivière Fyris, qui coule à Uppsala, était gelée, tout comme les lacs, donc nous en avons profité pour faire du patin à glace.

Le temps est resté froid assez longtemps, mais je sortais souvent pour me promener, courir, faire du vélo ou jouer au frisbee. J'avais repéré des terrains de disc-golf, un sport qui ressemble beaucoup au golf mais avec des disques de différents poids et tailles à la place des balles. Nous avions aussi un groupe avec lequel nous jouions régulièrement à l'ultimate, de février jusqu'à juin, regroupant en tout une vingtaine de personnes. Ce groupe contenait les principales personnes avec qui j'avais des interactions régulières, hormis Thomas, Simon, Georgios et mes collègues de bureau. Simon est un autre normalien du département de physique qui faisait aussi son stage de deuxième année à Uppsala.

J'ai pu découvrir pendant mon voyage quelques traditions et plats typiques de la Suède. Pendant la veille de la Walpurgis, que les Suédois appellent Valborg, il y avait des concerts partout dans la ville, et les étudiants lançaient leurs chapeaux en l'air, mais c'est surtout la fête de la Saint-Jean, *Midsommar* en suédois, qui était pour moi la plus mémorable : des "mâts de mai" sont dressés, et les enfants dansent autour avec des couronnes de fleurs sur la tête pendant que les plus grands jouent au *kubb*, un jeu de quilles suédois. Les Suédois aiment beaucoup le poisson, et notamment le hareng, qu'ils mangent sur des toasts à la Saint-Jean, ou en sandwich à la fin de l'été quand il est fermenté. J'ai goûté au hareng fermenté, le *surströmming*, qu'il est d'ailleurs nécessaire d'ouvrir à l'extérieur de la maison pour éviter que les odeurs restent... ce n'est pas un délice mais c'est une expérience amusante. Les pâtisseries suédoises sont en revanche délicieuses : on peut noter les *kanelbullar*, des brioches à la cannelle, ou les *semlor*, des choux à la cardamome et à la pâte d'amande surmontés de crème fouettée, qu'on mange traditionnellement à Mardi gras.

Enfin, j'ai visité les alentours d'Uppsala ainsi que ses musées, la ville de Stockholm, et le village côtier d'Öregrund. J'ai adoré le musée Vasa à Stockholm, où se trouve un navire du XVIIe siècle parfaitement conservé, puisqu'il a coulé dans le port juste après sa mise à flot. J'aurais aimé voyager en Laponie, mais notre projet de voyage n'a pas pu se concrétiser.

## 1.3 Occupations mathématiques au cours du stage

Je ne connaissais rien de la géométrie symplectique avant de choisir d'aller à Uppsala pour ce stage, donc j'ai commencé par apprendre les bases avec le livre *Introduction to Symplectic Topology* de Salamon et McDuff. Pendant ce temps, Georgios me présentait la théorie de l'homologe de Floer et me proposait des pistes de réflexion, ou des problèmes. Le premier vrai problème que j'ai étudié est le suivant :

**Problème 1.1.** *Soit $L$ un lagrangien exact de la variété symplectique exacte $(T^*\mathbb{S}^1, \mathrm{d}\lambda)$. Montrer qu'il y a une isotopie hamiltonienne $\phi_H^t$ telle que $L = \phi_H^1(L_0)$, où $L_0$ est la section nulle du fibré $T^*\mathbb{S}^1$.*

Ce problème m'a conduit à m'intéresser au flot de courbure moyenne, et Georgios m'a proposé d'utiliser cet outil pour décrire les lagrangiens spéciaux de certaines variétés de Calabi-Yau, mais j'ai abandonné ce projet car je l'avais réduit à des EDP, et ce n'était pas ce que je voulais faire.

Georgios m'a présenté un deuxième projet, qui consiste en la démonstration d'une conjecture qu'il a faite :

**Problème 1.2.** *Soit $(M, \mathrm{d}\lambda)$ une variété symplectique exacte, où $M$ est la sphère percée de trois trous, et soit $L$ un lagrangien exact de $M$. Montrer que la norme spectrale du couple $(L, \phi_H^1(L))$ est bornée pour toute isotopie hamiltonienne $\phi_H^t$ de $(M, \mathrm{d}\lambda)$.*

J'ai pu résoudre ce problème avec l'aide de Georgios, et écrire un article à ce propos, dont je joins une version remaniée (en anglais).

Je vais à présent introduire les prérequis nécessaires à la compréhension de l'article.

## 1.4 Introduction au contenu de l'article

**Définition 1.3.** *Une* ***variété symplectique*** *est un couple $(M, \omega)$, où $M$ est une variété différentielle, et où $\omega$, appelée* ***forme symplectique****, est une 2-forme différentielle fermée, i.e $\mathrm{d}\omega = 0$, telle que pour tout $p \in M$, la forme bilinéaire alternée $\omega_p\colon T_pM \times T_pM \longrightarrow \mathbb{R}$ est non dégénérée.*

Dans le cadre de cet article, les variétés symplectiques sont bidimensionnelles ; on peut donc penser aux formes symplectiques comme à des formes d'aire, i.e une façon de mesurer les aires infinitésimales au voisinage de chaque point de la variété. Elles vérifieront aussi la définition suivante :

**Définition 1.4.** *Une variété symplectique* $(M, \omega)$ *est dite* ***exacte*** *lorsque sa forme symplectique* $\omega$ *est exacte, i.e s'il existe une 1-forme différentielle* $\lambda$ *telle que* $\omega = \mathrm{d}\lambda$*.*

La classe de sous-variétés qui suscite ici notre intérêt est celle des sous-variétés lagrangiennes.

**Définition 1.5.** *Soit* $(M, \omega)$ *une variété symplectique. Une sous-variété* $L \subseteq M$ *est dite* ***lagrangienne*** *si* $\omega_{|TL} = 0$*. Si de plus* $\omega = \mathrm{d}\lambda$ *et que* $\lambda_{|TL}$ *est non seulement fermée, mais aussi exacte, on dit que le lagrangien* $L$ *est* ***exact****.*

Les transformations entre variétés symplectiques qui préservent les formes symplectiques se nomment **symplectomorphismes**, mais celles qui nous intéressent le plus ici vérifient une propriété plus forte :

**Définition 1.6.**

1. *Une* ***isotopie hamiltonienne*** *de* $(M, \omega)$ *est une famille* $\phi_H^t$ *de difféomorphismes de* $M$ *qui vérifie l'équation différentielle*

$$\partial_t \phi_H^t = X_t \circ \phi_H^t,$$

*où* $X_t$ *est un champ vectoriel tel que*

$$\iota_{X_t}\omega = \omega(X_t, -) = \mathrm{d}H_t.$$

2. *Un* ***difféomorphisme hamiltonien*** *est un difféomorphisme* $\phi$ *de* $M$ *tel qu'il existe une isotopie hamiltonienne* $\phi_H^t$ *de* $(M, \omega)$ *telle que* $\phi = \phi_H^1$*.*

La qualité de lagrangien (exact) est préservée par ce type de transformations :

**Proposition 1.7.** *Si* $L$ *est un lagrangien de* $(M, \omega)$*, et si* $\phi_H^t$ *est une isotopie hamiltonienne de* $M$*, alors* $\phi_H^1(L)$ *est un lagrangien de* $M$*. Si de plus* $M$ *et* $L$ *sont exacts, alors le lagrangien* $\phi_H^1(L)$ *est exact lui aussi.*

Je vais à présent introduire la théorie de l'homologie de Floer, qui est centrale dans cet article. La langue est maintenant l'anglais car il s'agit d'extraits de mon article agencés différemment.

For an exact symplectic manifold $(M, \mathrm{d}\lambda)$ and two transverse exact Lagrangian submanifolds $L_0, L_1 \subseteq M$, the mod 2 Floer complex is defined by

$$CF(L_0, L_1) = \mathbb{Z}_2\langle L_0 \pitchfork L_1 \rangle,$$

with differential

$$\partial x = \sum_{y \in L_0 \pitchfork L_1} n(x,y)y,$$

where $n(x,y)$ is the cardinal of $\mathcal{M}(x,y)$ modulo 2, $\mathcal{M}(x,y)$ being the set of rigid holomorphic strips in $M$ joining $x$ to $y$ with boundary on $L_0$ and $L_1$.

A **pseudo-holomorphic strip** in $M$ joining $x$ to $y$ with boundary on $L_0$ and $L_1$ is a smooth function

$$u \colon \mathbb{R} \times [0,1] \longrightarrow M$$

such that $\mathrm{d}u \circ j = J \circ \mathrm{d}u$, where $j$ is the canonical complex structure on $\mathbb{C}$ and $J$ is a fixed $\omega$-compatible almost complex structure on $M$, i.e such that $g(v,w) \coloneqq \omega(v,Jw)$ defines a Riemannian metric. In addition, $u$ must satisfy the boundary conditions $u(\mathbb{R} \times \{0\}) \subseteq L_0$, $u(\mathbb{R} \times \{1\}) \subseteq L_1$, and $u(s,r) \underset{s \to +\infty}{\longrightarrow} x$, $u(s,r) \underset{s \to -\infty}{\longrightarrow} y$ for all $r \in [0,1]$.

*Rigid* means that the strip is transversely cut out as a solution, and that its expected dimension is zero. We will not expand further on this, but instead use the well-known fact that when the symplectic manifold is two-dimensional, this is equivalent to the strip being an immersion up to and including the boundary, with convex corners. We will abbreviate the name of the elements of $\mathcal{M}(x,y)$ by *strips*.

Floer proved that $\partial^2 = 0$, so there is a well-defined **Floer homology**

$$HF_*(L_0, L_1) = \operatorname{Ker} \partial / \operatorname{Im} \partial .$$

Floer also proved that for a Hamiltonian diffeomorphism $\phi$ of $M$, there is an isomorphism

$$HF_*(L_0, L_1) \simeq HF_*(L_0, \phi(L_1)),$$

and that if $M$ is an exact symplectic manifold, with $L_0$ an exact Lagrangian and $\phi \in \mathrm{Ham}(M)$ such that $L_0 \pitchfork \phi(L_0)$, we have an isomorphism

$$HF_*(L_0, \phi(L_0)) \simeq H_*(L_0; \mathbb{Z}_2)$$

where the right hand side is the singular homology of $L_0$ with $\mathbb{Z}_2$ coefficients.

## 1.5 Modified Floer complex and main results

Let $(M, \omega = \mathrm{d}\lambda)$ be an exact symplectic manifold of dimension 2, and $L \subseteq M$ a compact exact Lagrangian submanifold of $M$. Let $h \in M \setminus L$ be any point, which will be referred as the *hole*. Let $M' \coloneqq M \setminus \{h\}$ and $\omega' \coloneqq \omega_{|\Lambda^2 TM'}$.

Then, $(M', \omega')$ is an exact symplectic manifold as well, and $L$ is still an exact Lagrangian of $(M', \omega')$.

Finally, let $L_0$ be another compact exact Lagrangian of $M'$ such that $L \pitchfork L_0$. Let $L_1$ be a Lagrangian that is Hamiltonian isotopic to $L_0$ by a Hamiltonian isotopy supported inside $M'$, and such that $L \pitchfork L_1$. We can then choose a Hamiltonian isotopy $\phi_t := \phi_H^t \colon [0,1] \times M' \to M'$ such that $L_t := \phi_t(L_0)$ is transverse to $L$ for all times $t \in [0,1]$ except for a finite set $\{t_1, \dots, t_n\}$, with $0 < t_1 < \cdots < t_n < 1$. That is a well-know fact that such Hamiltonian isotopies always exist for any pair of Hamiltonian isotopic Lagrangians.

The Lagrangian $L$ is exact, so we can choose a primitive $f$ of $\lambda|_{TL}$. For the family $L_t,\ t \in [0,1]$, it follows from Cartan's formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^* \lambda) = \mathrm{d}(\phi_t^* H_t) + \mathrm{d}(\phi_t^* \lambda(X_t))$$

that $L_t$ is a family of exact Lagrangian submanifolds with continuously varying primitives $f_t$ of $\lambda|_{TL_t}$. We then define the **action** function $\ell_t \colon L \cap L_t \to \mathbb{R}$ by

$$\ell_t(x) := f_t(x) - f(x).$$

For the times $t \in [0,1] \setminus \{t_1, \dots, t_n\}$ of transverse intersection, $L \pitchfork L_t \subseteq L$ is a 0-submanifold of $L$, hence a finite set since $L$ is compact.

Let $\beta \in \{0,1\}$, and let $M_1 := M$ and $M_0 := M'$. For $\beta \in \{0,1\}$ and $t \in [0,1] \setminus \{t_1, \dots, t_n\}$, let $C_\beta(t) := CF(L, L_t; M_\beta) := (C(t), \partial_\beta(t))$ be the Floer complex associated with the two transverse Lagrangians $L$ and $L_t$, seen as Lagrangian submanifolds of $M_\beta$.

It is a consequence of Stokes' formula and of the exactness of the Lagrangians that

$$\ell_t(y) > \ell_t(x) \tag{1}$$

whenever there is a strip joining $x$ to $y$ in $M$.

Since strips are orientation preserving immersions, they have a preferred behaviour near their inputs and outputs, as shown in Fig. 1: if the strip lies on a blue corner 2 or 4, it means that the strips begins at this point; otherwise, it should end at this point.

We shall now define the modified Floer complex which is central to this article: let

$$D(t) := CF_h(L, L_t) := \bigoplus_{x \in L \cap L_t} \mathbb{Z}_2[T] \cdot x$$

be the complex with differential $\partial_h(t)$ defined on the generators by

$$\partial_h(t) x = \sum_{y \in L \cap L_t} N_t(x,y) y.$$

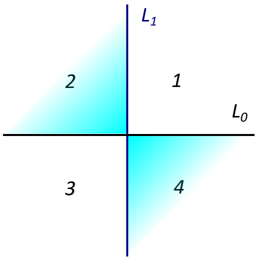

Figure 1: Convention for the orientation of the holomorphic strips

The polynomials $N_t(x, y)$ are defined by

$$N_t(x,y) = \sum_{S \in \mathcal{M}_t^1(x,y)} T^{(S,h)} \mod 2,$$

where $\mathcal{M}_t^1(x, y)$ is the set of rigid holomorphic strips in $M$ joining $x$ to $y$ with boundary on $L$ and $L_t$, modulo holomorphic reparametrization. $(S, h)$ denotes the algebraic intersection number between the strip $S$ and the hole $h$. As the pseudo-holomorphic strips are orientation-preserving immersions, the algebraic intersection number is non-negative and then coincides with $|S^{-1}(\{h\})|$: therefore, we only have to count the number of times that the strip passes over $h$.

Again, these modules are indeed chain complexes: the fact that $\partial_h(t)^2 = 0$ is a consequence of the proof of $\partial_1(t)^2 = 0$, and it will not be further detailed in this article. In short, since pseudo-holomorphic strips are orientation-preserving, the algebraic intersection number $|S^{-1}(\{h\})|$ does not change in the 1-parameter families of index 2 strips.

We extend the definition of the action function $\ell_t$ for every element of $D(t)$:

$$\ell_t\left(\sum_{i=1}^{p} \lambda_i x_i\right) := \inf\{\ell_t(x_i) \mid 1 \leq i \leq p,\ \lambda_i \notin T \cdot \mathbb{Z}_2[T]\},$$

where $(x_1, \ldots, x_p)$ is the list of the generators of $D(t)$ and $\lambda_1, \ldots, \lambda_p \in \mathbb{Z}_2[T]$. We use here the convention $\inf \varnothing = +\infty$.

For simplicity, we will denote $\langle -, - \rangle$ the canonical inner products on $C(t)$ and $D(t)$: for $x, y \in L \cap L_t$, $\langle x, y \rangle = \delta_{x,y}$.

The fact that the strips have positive energy proves that $\partial_h(t)$ is action-increasing: for such a complex $C$, we get a filtration by the subcomplexes $C_s = \{x \in C \colon \ell(x) > s\}$. It is then possible to talk about the **barcode** of the Floer complex, and some *spectral invariants* can be computed thanks to this barcode, such as the boundary depth or the spectral norm that go back to the work of Viterbo in [Vit92]. In short, the **spectral norm** of the pair $(L, L_1)$ can be defined here as the largest difference between the levels of two semi-infinite bars in the full barcode of $(CF(L, L_1), \partial, \ell)$. This barcode will be properly defined in the next section.

Viterbo has conjectured in [Vit08] (Conjecture 1) that for any torus $(T^n, g)$ equipped with a Riemannian metric, and for any Hamiltonian perturbation $\phi^1_H(L)$ of $L$ inside its 1-codisk bundle $D^*_g L$, the spectral norm $\gamma(L, \phi^1_H(L))$ is bounded. Shelukhin had confirmed this conjecture in his articles [She18] and [She19] for certain classes of manifolds, and in particular he solved the case $n = 1$.

Here, we investigate the possibility of extending the conjecture not only to codisk bundles, but to more exotic exact symplectic manifolds with boundary. Dimitroglou Rizell has proved in [Dim22] that the conjecture neither holds for immersed Legendrians, nor for exact Lagrangians in some exact sympletic manifolds as simple as a punctured torus.

In this article, we extend the result by showing that, if an exact Lagrangian inside an exact two-dimensional symplectic manifold satisfies a bound on its spectral norm, then the same is true if one deforms the symplectic manifold by removing a point. This result relies on the following theorem, whose proof will be the main goal of this article.

**Theorem 1.8.** *Let $(M, \mathrm{d}\lambda)$ be an exact two-dimensional symplectic manifold, $L_0$ and $L_1$ two transverse compact exact Lagrangian sumbanifolds of $M$, $h \in M \setminus (L_0 \cup L_1)$ and $\phi$ a Hamiltonian diffeomorphism of $M \setminus \{h\}$ such that $\phi(L_1) \pitchfork L_0 \subseteq M \setminus \{h\}$.*

*Suppose there is an action-preserving chain-isomorphism*

$$CF_h(L_0, L_1; M) \xrightarrow{\sim} CF(L_0, L_1; M \setminus \{h\}) \underset{\mathbb{Z}_2}{\otimes} \mathbb{Z}_2[T].$$

*Then, there is an action-preserving chain-isomorphism*

$$CF_h(L_0, \phi(L_1); M) \xrightarrow{\sim} CF(L_0, \phi(L_1); M \setminus \{h\}) \underset{\mathbb{Z}_2}{\otimes} \mathbb{Z}_2[T].$$

*Therefore, $CF_h(L_0, \phi(L_1); M)$ has a well defined barcode, which moreover coincides with the barcodes of both complexes $CF(L_0, \phi(L_1); M \setminus \{h\})$ and $CF(L_0, \phi(L_1); M)$.*

The assumptions of the above theorem are automatically satisfied when $L_1$ is a small Hamiltonian perturbation of $L_0$. Namely, in that case, all Floer strips can be assumed to be disjoint from the point $h$. The claimed generalization of Viterbo's conjecture is the following corollary:

**Corollary 1.9.** *Let $(M, \mathrm{d}\lambda)$ be an exact two-dimensional symplectic manifold, and let $L_0 \subseteq M$ be a compact exact Lagrangian submanifold of $M$. Let $L_1$ be a small Hamiltonian perturbation of $L_0$ such that $L_0 \pitchfork L_1$. Suppose that for every Hamiltonian diffeomorphism $\phi$ of $M$ such that $L_0 \pitchfork \phi(L_1)$, the spectral norm $\gamma(L_0, \phi(L_1); M)$ in $M$ is bounded by a constant independent of $\phi$.*

*Let $h \in M \setminus (L_0 \cup L_1)$ be chosen sufficiently far away from the support of the Hamiltonian isotopy that takes $L_0$ to $L_1$, and let $M' := M \setminus \{h\}$. $(M', \mathrm{d}\lambda)$ is also an exact symplectic manifold and $L_0, L_1 \subseteq M'$ are two Hamiltonian isotopic exact Lagrangians of $M'$.*

*Then, for every Hamiltonian diffeomorphism $\phi$ of $M'$ such that $L_0 \pitchfork \phi(L_1)$, the spectral norm $\gamma(L_0, \phi(L_1); M')$ in $M'$ is also bounded by the same constant as above.*

A concrete example of a symplectic surface $M$ where the above corollary can be applied is the following:

Let $M$ be the 1-codisk bundle of the circle: $M = D^*\mathbb{S}^1 = \mathbb{S}^1 \times (-1, 1)$. Let $L := \mathbb{S}^1 \times \{0\}$ be its zero section, and let $h \in M \setminus L$. If $\phi$ is a Hamiltonian diffeomorphism of $M \setminus \{h\}$, then the spectral norm $\gamma(L, \phi(L); M \setminus \{h\})$ in $M \setminus \{h\}$ is bounded by the same constant as for Hamiltonian diffeomorphisms in $M$ found in Shelukhin's paper [She18].

# 2 The barcode of a piecewise continuous family of complexes

We are going to define the main features of the theory of barcodes as presented in the second part of [DS18].

We will only consider finitely generated free chain complexes, but will often omit this precision and just write complexes.

In this whole section, $k$ is a field, and $R$ denotes either $k$ or $k[T]$. All the chain complexes will be $R$-modules.

## 2.1 Filtered chain complexes

**Definition 2.1.** *A **filtered chain complex** is a (finitely generated and free) chain complex $(C, \partial)$ over the ring $R$ with a function $\ell\colon C \to \mathbb{R} \cup \{+\infty\}$ called **action** such that*

- $\ell(x) = +\infty$ *if and only if* $\begin{cases} x = 0 \text{ if } R = k, \\ x \in T \cdot C \text{ if } R = k[T], \end{cases}$

- $\forall x \in C,\ \forall \lambda \in R \setminus T \cdot k[T],\ \ell(\lambda x) = \ell(x)$,
- $\forall x, y \in C,\ \ell(x+y) \geq \min\{\ell(x), \ell(y)\}$,
- $\forall x \in C,\ \ell(\partial\, x) \geq \ell(x)$.

As the name suggests, these complexes are filtered. Namely, there is a natural filtration by the subcomplexes $C_{>s} \coloneqq \{x \in C \colon \ell(x) > s\}$, for $s \in \mathbb{R}$, i.e these subcomplexes verify $C_{>s} \subseteq C_{>s'}$ for $s > s'$, and $C = \bigcup_{s \in \mathbb{R}} C_{>s}$. The ring $R$ being a PID, these subcomplexes are free too.

Such a complex has a distinguished class of bases, called **compatible bases**: these are the bases $(a_1, \ldots, a_n)$ of $C$ such that

$$\forall (\lambda_1, \ldots, \lambda_n) \in R^n, \quad \ell\left(\sum_{i=1}^{n} \lambda_i a_i\right) = \inf_{\lambda_i \neq 0} \ell(a_i).$$

This definition is compatible with our convention $\inf \varnothing = +\infty$.

**Lemma 2.2.** *Any filtered complex possesses a compatible basis.*

*Proof.* As presented in [DS18], one can consider the quotient modules

$$Q_{s,\varepsilon} \coloneqq C_{>s-\varepsilon} / C_{>s}$$

where $C_{>s} \coloneqq \{x \in C \colon \ell(x) > s\}$ is a subcomplex of $C$. There is only a finite number of values of $s$ for which the quotients $Q_{s,\varepsilon}$ are non-zero. Moreover, they are free by the axioms of the filtered chain complex. To see this, note that since $R$ is a PID it suffices to verify that the quotient is torsion free. This follows since scalar multiplication by the elements of $R$ is action-increasing. For $\varepsilon$ small enough, taking a basis of each $Q_{s,\varepsilon}$, and choosing representatives of their classes yields a compatible basis for $C$. ☐

**Definition 2.3.** *A* ***piecewise continuous filtered chain complex*** *is a family of filtered chain complexes* $(C(t), \partial_t, \ell_t)$, $t \in [0,1]$, *with a finite collection of times* $0 < t_1 < \cdots < t_n < 1$, *such that the following conditions hold:*

1. *for* $t, t' \in (t_i, t_{i+1})$, $(C(t), \partial_t)$ *is chain-isomorphic to* $(C(t'), \partial_{t'})$ *through a preferred isomorphism* $\Phi_{t,t'} \colon C(t) \xrightarrow{\sim} C(t')$ *that preserves compatible bases, with continuous action* $\ell_t$ *in the following sense:*

$$\forall x \in C(t), \quad \ell_{t'}(\Phi_{t,t'}(x)) \xrightarrow[t' \to t]{} \ell_t(x).$$

*Moreover, the family is functorial, meaning that if* $t, t', t'' \in (t_i, t_{i+1})$, *we have*

$$\Phi_{t,t''} = \Phi_{t',t''} \circ \Phi_{t,t'}$$

2. *for each $t_i$ with $i \in \{1, \dots, n\}$, one of the following* ***simple bifurcations*** *occur:*

   - ***Birth:*** *there is a 2-dimensional $R$-complex $(S, \partial, \ell_t)$, such that $\ell_t$ is defined to satisfy $\ell_t(c) = \ell_t(d)$ and to be independent of time (note that this differential is thus not strictly action-increasing), such that $(c, d)$ is a compatible basis and $\partial c = d$, and a family of chain-isomorphisms that send compatible bases to compatible bases*

     $$(C(t_i - \varepsilon), \partial_{t_i-\varepsilon}, \ell_{t_i-\varepsilon}) \oplus (S, \partial, \ell_{t_i-\varepsilon}) \xrightarrow{f_\varepsilon} (C(t_i + \varepsilon), \partial_{t_i+\varepsilon}, \ell_{t_i+\varepsilon})$$

     *for $\varepsilon > 0$ small enough, such that*

     $$\forall x \neq 0, \;\; \ell_{t_i+\varepsilon}(f_\varepsilon(x)) - \ell_{t_i-\varepsilon}(x) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

     *Finally, the above preferred isomorphism $\Phi_{t,t'}$ extends to $(t_{i-1}, t_i]$.*
   - ***Death:*** *the family $C(-t)$ has a birth at $-t_i$.*
   - ***Handle-slide:*** *there is a family of vector spaces isomorphisms $H_i(\varepsilon)$ from $(C(t_i - \varepsilon), \partial_{t_i-\varepsilon}, \ell_{t_i-\varepsilon})$ to $(C(t_i + \varepsilon), \partial_{t_i+\varepsilon}, \ell_{t_i+\varepsilon})$ that send compatible bases to compatible bases, for $\varepsilon > 0$ small enough. Moreover, they satisfy the equation*

     $$H_i(\varepsilon) \circ \partial_{t_i-\varepsilon} = \partial_{t_i+\varepsilon} \circ H_i(\varepsilon),$$

     *and $H_i(\varepsilon)$ is represented by an upper-triangular matrix in a compatible basis ordered by decreasing action level. Finally, we have*

     $$\forall x \neq 0, \;\; \ell_{t_i+\varepsilon}(H_\varepsilon(x)) - \ell_{t_i-\varepsilon}(x) \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

     *and the isomorphism $\Phi_{t,t'}$ extends to $(t_{i-1}, t_i]$.*

This terminology comes from the study of the Morse complex. In our case, the handle-slide never comes without a birth/death, as we will see later. However, for higher dimensions, these bifurcations may happen independently.

Note that Condition (1) implies that even though the differential remains the same in $C(t)$, the actions of its elements change, and so does the filtered isomorphism class of $C(t)$.

Finally, one should note that the convention for the choice of $C(t_i)$ themselves makes the function $t \mapsto \dim C(t)$ lower semicontinuous.

## 2.2 Barannikov decompositions

The most well-studied tool for understanding chain complexes is their homology: since $\partial^2 = 0$, the submodule of the **boundaries** $B = \operatorname{Im} \partial$ is included in the submodule of the **cycles** $Z = \operatorname{Ker} \partial$, and we can consider the quotient module $H(C, \partial) \coloneqq Z/B$, which is called the **homology** of $C$. As we have the action $\ell$, we can even define the **persistent homology groups** $H(C_{>s}, \partial)$ for $s \in \mathbb{R}$. In the next subsection we will construct the barcode of a filtered complex, that will turn out to be a useful invariant for visualizing a filtered complex. We will define the barcode using the so-called Barannikov decomposition, which was first considered in [Bar94].

**Definition 2.4.** *A **Barannikov basis** of a filtered complex $(C, \partial)$ is a compatible basis $(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m)$ such that:*

- *for $1 \leq i \leq n$, $\partial a_i = b_i$,*
- *$(b_1, \dots, b_n)$ is a basis of $B$,*
- *$(b_1, \dots, b_n, c_1, \dots, c_m)$ is a basis of $Z$.*

A Barannikov basis allows one to decompose the complex into more simple 1- and 2-dimensional subcomplexes: if $(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m)$ is Barannikov, then

$$(C, \partial) = \bigoplus_{i=1}^{n} (Ra_i \oplus Rb_i, \partial_i) \oplus \bigoplus_{j=1}^{m} (Rc_j, 0)$$

where $\partial_i$ is the restriction of $\partial$ to $Ra_i \oplus Rb_i$. This decomposition allows one to easily compute the homology of $(C_{>s}, \partial)$: it is freely generated by the cycles $c_j$ such that $\ell(c_j) > s$, but also by the boundaries $b_i$ such that $\ell(b_i) > s$ and $\ell(a_i) \leq s$, as in $C_{>s}$ the chain $a_i$ does not exist yet.

Therefore, a necessary condition for a complex $C$ to possess a Barannikov basis is that all the homolgy groups $H(C_{>s}, \partial)$ are free $R$-modules. However this condition is not always the case if $R$ is not a field, as shown by the example below.

**Example.** Let $R = k[T]$, and let $\partial$ be the $R$-linear map on $C = Ra \oplus Rb$ that satisfies $\partial a = Tb$ and $\partial b = 0$. Then $b$ is a cycle but not a boundary, although $Tb$ is a boundary. Thus, $H(C, \partial)$ has torsion and is not a free $R$-module.

**Lemma 2.5** (Barannikov)**.**

1. *Any finite-dimensional filtered chain complex over $k$ has a Barannikov decomposition.*
2. *For a finite-dimensional filtered chain complex over $R$ with a compatible basis $(x_1, \dots, x_n)$ such that $\ell(x_1) < \dots < \ell(x_n)$, any two Barannikov bases (if they exist) are related by an upper-triangular change of basis matrix.*

The proof of the previous statement can be found in [Bar94]. In particular, its first statement guarantees the existence of a Barannikov basis on the complex $C(t)$ for every $t \in [0,1]$. Nevertheless, if $R$ is not a field, then a Barannikov basis does not necessarily exist, as shown in the above example. Moreover, the second statement is not sufficient for us since we need to handle the cases where there is redundancy in the action levels.

Let us first reformulate the existence of the Barannikov decomposition in the case of $k[T]$:

**Proposition 2.6.** *Let $R = k[T]$, and let $(C_T, \partial_T, \ell_T)$ be a filtered chain complex over $R$. Then, $(C_T, \partial_T, \ell_T)$ has a Barannikov basis if and only if there exists a filtered chain complex $(C, \partial, \ell)$ over $k$ such that there is an action-preserving chain-isomorphism*

$$(C_T, \partial_T, \ell_T) \xrightarrow{\sim} (C, \partial, \ell) \underset{k}{\otimes} k[T].$$

*Proof.* If $B = (a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m)$ is a Barannikov basis of the filtered chain complex $(C_T, \partial_T, \ell_T)$, then we define the formal $k$-module

$$C \coloneqq k\langle B \rangle,$$

which we equip with the $k$-linear differential $\partial \colon C \to C$ such that $\partial\, a_i = b_i$, $\partial\, b_i = 0$ and $\partial\, c_j = 0$, and the action $\ell$ such that $B$ is a compatible basis of $(C, \partial, \ell)$ and $\ell(x) = \ell_T(x)$ for each element $x$ of the basis $B$. It is now obvious that $(C_T, \partial_T, \ell_T)$ and $(C, \partial, \ell) \underset{k}{\otimes} k[T]$ are isomorphic as filtered chain-complexes over $k[T]$.

Conversely, if we have an action-preserving chain-isomorphism

$$(C, \partial, \ell) \underset{k}{\otimes} k[T] \xrightarrow{f} (C_T, \partial_T, \ell_T),$$

we can find a Barannikov basis $B$ for $(C, \partial, \ell)$ thanks to Lemma 2.5, and taking $f(B) \subseteq C_T$ we get a Barannikov basis for $(C_T, \partial_T, \ell_T)$. $\square$

## 2.3 Towards the barcode

The next definition identifies the class of filtered chain-complexes we will be working with.

**Definition 2.7.** *A filtered chain complex of $R$-modules $(C, \partial, \ell)$ satisfying one of the equivalent conditions in Proposition 2.6 is called a* ***standard*** *complex.*

The following proposition will provide a uniqueness result for the Barannikov basis of a standard complex.

**Proposition 2.8.** *If $(C_T, \partial_T, \ell_T)$ is a standard filtered $k[T]$-complex, there is a filtered $k$-complex $(C, \partial, \ell)$, which is unique up to action-preserving chain-isomorphism, such that*

$$(C_T, \partial_T, \ell_T) \simeq (C, \partial, \ell) \underset{k}{\otimes} k[T].$$

*Moreover, the form of the Barannikov decomposition is unique for a standard $k[T]$-complex.*

*Proof.* $(C_T\, \partial_T, \ell_T)$ begin standard, we can choose a filtered $k$-complex $(C, \partial, \ell)$ such that we have an action-preserving chain-isomorphism

$$(C_T, \partial_T, \ell_T) \xrightarrow{f} (C, \partial) \underset{k}{\otimes} k[T].$$

We then define the following $k$-vector space:

$$\tilde{C} \coloneqq C_T / \langle T \rangle,$$

where $\langle T \rangle$ denotes the submodule generated by $T$. Let $p \colon C_T \longrightarrow \tilde{C}$ be the natural projection. $\tilde{C}$ naturally comes with a differential $\tilde{\partial}$ such that

$$\forall x \in C_T,\ \tilde{\partial}(p(x)) = p(\partial\, x).$$

Lastly, we define an action $\tilde{\ell}$ on $\tilde{C}$ by

$$\forall x \in C_T,\ \tilde{\ell}(p(x)) = \ell_T(x).$$

The axioms of the action for a $k[T]$-complex guarantee that $\tilde{\ell}$ indeed defines an action.

It is obvious that $f(\langle T \rangle) = T \cdot C$, so the quotient map $\tilde{f}$ is a linear bijection

$$(\tilde{C}, \tilde{\partial}, \tilde{\ell}) \xrightarrow{\tilde{f}} ((C, \partial, \ell) \underset{k}{\otimes} k[T])/T \cdot C \simeq (C, \partial, \ell),$$

and the fact that $\tilde{f}$ is an action-preserving chain-map can be verified easily. This shows that $(C, \partial, \ell)$ is chain-isomorphic to $(\tilde{C}, \tilde{\partial}, \tilde{\ell})$, the latter filtered complex being independent of the choice of $(C, \partial, \ell)$, thus concluding the proof of the first statement. The last statement follows from the first bullet point of Lemma 2.5, since a Barannikov basis for $C$ is sent to a Barannikov basis for $\tilde{C}$ under the projection map. □

In view of the above proposition, the form of the Barannikov decomposition of a filtered $R$-complex is uniquely determined by its filtered isomorphism class. Therefore, the following definition makes sense.

**Definition 2.9.**

1. *A* ***barcode*** *is a finite (multi-)set of semi-closed intervals of the form $(e, s]$, called bars, where $s \in \mathbb{R}$ is the starting point, and $e \in \mathbb{R} \cup \{-\infty\}$ is the endpoint of the bar.*

2. *The barcode of a standard complex $(C, \partial, \ell)$, denoted by $\mathcal{B}(C, \partial, \ell)$, is defined as the barcode with the bars $(\ell(a_i), \ell(b_i)]$ and $(-\infty, \ell(c_j)]$, where $(a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m)$ is a Barannikov basis of $(C, \partial, \ell)$.*

# 3 Invariance of $CF_h(L, L_t)$ under Hamiltonian isotopies

Let us remind ourselves of the family of chain complexes $(D(t), \partial_h(t))$: we have

$$D(t) = C_h(L_0, L_t; M) = \bigoplus_{x \in L \cap L_t} \mathbb{Z}_2[T] \cdot x,$$

with

$$\langle \partial_h(t)(x), y \rangle = \sum_{S \in \mathcal{M}_t(x,y)} T^{(S,h)}$$

being defined for $t \in [0, 1]$ such that $L \pitchfork L_t$. Here, $\mathcal{M}_t(x, y)$ denotes the $t$-dependent set of rigid holomorphic strips in $M$ joining $x$ to $y$ and $(S, h)$ the algebraic intersection number of the strip $S$ and the point $h$. Moreover, we define an action $\ell_t$ for the complex $D(t)$ by

$$\ell_t\left(\sum_{i=1}^{p} \lambda_i x_i\right) := \inf\{\ell_t(x_i) \mid 1 \leq i \leq p,\ \lambda_i \notin T \cdot \mathbb{Z}_2[T]\},$$

where $(x_1, \dots, x_p)$ is the list of the generators of $D(t)$. One should note that the differential $\partial_h(t)$, as well as $\partial_\beta(t)$ for $\beta \in \{0, 1\}$, is strictly action-increasing with this action $\ell_t$. Indeed, it is a consequence of the definition of $\ell_t$ and Equation (1).

We denote $\{t_1 < \cdots < t_n\}$ the set of all times $t$ in which $L$ is not transverse to $L_t$. Remember that the complex $D(t)$ is not well-defined at these moments. However, we get a filtered vector space with a canonical basis that consists of the transverse intersection points (thus disregarding the non-transverse intersections). Denote

$$\Delta t = \inf_{1 \leq i < j \leq n} t_j - t_i.$$

Our first milestone for the proof of Theorem 1.8 is the following statement.

**Theorem 3.1.** *After a slight continuous perturbation of the actions of the canonical basis elements of the family $D(t)$ of complexes, one obtains a family $\tilde{D}(t)$ of complexes which is a piecewise continuous family of filtered chain complexes, where the canonical identification of basis elements of $D(t)$ and $\tilde{D}(t)$ moreover gives an action-preserving chain-isomorphism away from some small neighborhood of the non-transverse moments $t_1, \ldots, t_n$.*

The goal of this part is to prove Theorem 3.1. The family $\tilde{D}(t)$ will be defined in Subsection 3.3, and the proof of Theorem 1.8 will be completed in Subsection 3.4.

## 3.1 First observations

We start by breaking down the transformations undergone by the complex $D(t)$ through time variations into elementary steps as pictured in Fig. 2, which describes a birth moment in which precisely two new intersections $c$ and $d$ are born.

The following discussion has been inspired by [Che97]; although the context of Floer homology is much simpler than the differential graded algebra used by Chekanov, there is a deep relationship between these two setups. Indeed, a pair of Lagrangian embeddings lifts to a pair of Legendrian embeddings in the contactization of the symplectic manifold, and the Floer complex of the pair of Lagrangians can then be recovered from Chekanov's DGA of the pair of Legendrians. For more details, the reader can refer to [Dim22].

Though several crossings could happen simultaneously at a time $t_0$, we can let them happen successively by adding a small perturbation $\delta(x,t)$ to the Hamiltonian, with support in $(t_0 - \varepsilon, t_0 + \varepsilon) \times M'$, with $\varepsilon < \Delta t$. Thus, it is correct to assume that the bifurcations are completely described by Fig. 2.

We now consider the case of a birth moment at $t = t_0$, which means that there are two new transverse intersection points $c$ and $d$ for $t > t_0$ as shown in Fig. 2. Denote $\partial_0 := \partial_h(t_0 - \varepsilon)$ and $\partial_1 := \partial_h(t_0 + \varepsilon)$. Let us order

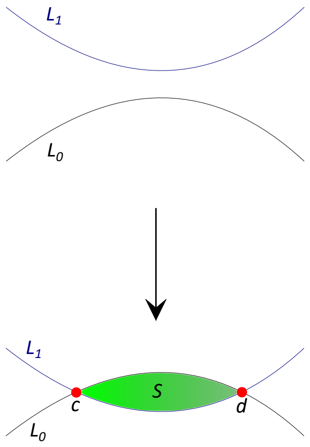

Figure 2: An elementary crossing

the generators of $D(t_0+\varepsilon)$, that is the elements of $L \cap L_{t_0+\varepsilon}$, by decreasing actions:

$$\ell(b_m) \geq \cdots > \ell(b_1) \geq \ell(d) > \ell(c) \geq \ell(a_1) \geq \cdots \geq \ell(a_n).$$

Here we assume that this order does not change between $t_0-\varepsilon$ and $t_0+\varepsilon$, so that we can omit the indices on $\ell$. Again, a small Hamiltonian perturbation allows us to make this assumption without loss of generality.

We first focus on the new born points (Lemma 3.2), but also on the strips that survive through the crossing (Proposition 3.3).

**Lemma 3.2.** *There is a unique strip joining $c$ to $d$. Moreover, it does not pass through the hole, so we have*

$$\langle \partial_1 \, c, d \rangle = 1.$$

*Proof.* At $t=t_0$, $L$ and $L_t$ are tangent in one point $e$ of action $l$, such that $\ell_t(c), \ell_t(d) \xrightarrow[t\to t_0]{} l$. We can perturb $H$ around $t_0$ so that only a neighborhood $U$ of $e$ is changed over time. The only strip that stays in $U$ is $S$, so any other strip has an area greater than some constant $C>0$ during this process. However, $\ell_t(d)-\ell_t(c) \xrightarrow[t\to t_0]{} 0$, and $\ell_t(d)-\ell_t(c)$ is the area at $t$ of any strip joining $c$ to $d$. Therefore, there cannot be a strip joining $c$ to $d$ other than $S$. □

**Proposition 3.3.** *The strips joining two higher-action generators $b_k$ and $b_l$, and those joining two lower-action generators $a_i$ and $a_j$ are not disturbed by* *the birth. More precisely, we have*

$$\forall k \in \{1,\dots,m\}, \quad \partial_0\, b_k = \partial_1\, b_k. \tag{2}$$

$$\forall i,j \in \{1,\dots,n\}, \quad \langle \partial_1\, a_i, a_j\rangle = \langle \partial_0\, a_i, a_j\rangle. \tag{3}$$

*Proof.* An old strip that disappears after the birth had to pass through the green part in Fig. 3. Thus, it gives birth to two new strips: one of them begins on $c$, and thus ends on some generator $b_k$, and the other one ends on $d$, and thus begins on some point $a_i$.

Therefore, all the strips that disappeared were those joining some $a_i$ to some $b_k$, and those that appear must have endpoints or starting points on $c$ or $d$. This proves the proposition, whose the two identities are direct consequences. □

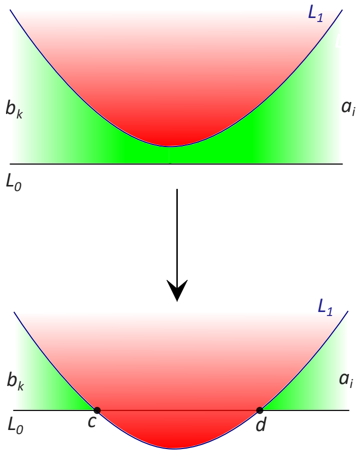

Figure 3: The only old strips that break are the ones joining $a_i$'s to $b_k$'s

## 3.2 The chain-isomorphism $H$

Let us define the linear map

$$H\colon D(t_0-\varepsilon) \oplus \mathbb{Z}_2[T]\langle c,d\rangle \longrightarrow D(t_0+\varepsilon)$$

which satisfies:

- for $1 \le k \le m$, $H(b_k) = b_k$,
- $H(c) = c$ and $H(d) = \partial_1\, c$,

- finally, for $1 \leq i \leq n$, $H(a_i) = a_i + \langle \partial_1 a_i, d\rangle c$.

Setting $\partial_0 c := d$ and $\partial_0 d := 0$ endows $D_0 := D(t_0 - \varepsilon) \oplus \mathbb{Z}_2[T]\langle c, d\rangle$ with a structure of filtered chain-complex. Denote $D_1 := D(t_0 + \varepsilon)$. The function $H$ is the subject of the following proposition:

**Proposition 3.4.** *$H$ is a chain-isomorphism from $(D_0, \partial_0)$ to $(D_1, \partial_1)$.*

Lemma 3.2 ensures that the matrix representing $H$ in the basis $(b_m, \dots, b_1, d, c, a_1, \dots, a_n)$ is upper-triangular with diagonal coefficients 1, hence invertible.

In addition, we need to verify that the following equation holds:

$$\partial_1 \circ H = H \circ \partial_0 \,. \tag{4}$$

At first glance, (4) is true on the generators $b_k,\ 1 \leq k \leq m$ and for $c$ and $d$. Proving it for $a_i,\ 1 \leq i \leq n$, requires some work.

Our first step is the following:

**Proposition 3.5.** *Let $i \in \{1, \dots, n\}$. Then*

$$\langle \partial_1 a_i, c\rangle = \sum_{j=1}^{i-1} \langle \partial_1 a_i, a_j\rangle \langle \partial_1 a_j, d\rangle$$

*Proof.* First, using Proposition 3.3 we see that

$$\partial_1 a_i = \partial_0 a_i + \langle \partial_1 a_i, c\rangle c + \langle \partial_1 a_i, d\rangle d + \sum_{k=1}^{m} \langle \partial_1 a_i, b_k\rangle b_k. \tag{5}$$

We then apply $\partial_1$ to Equation (5) and use $\partial_1^2 = 0$:

$$0 = \partial_1 \partial_0 a_i + \langle \partial_1 a_i, c\rangle \, \partial_1 c + \langle \partial_1 a_i, d\rangle \, \partial_1 d + \sum_{k=1}^{m} \langle \partial_1 a_i, b_k\rangle \, \partial_1 b_k. \tag{6}$$

For $1 \leq j < i$ and $1 \leq k \leq m$, we define

$$\begin{aligned} \lambda_j &:= \langle \partial_1 a_j, c\rangle, \\ \mu_j &:= \langle \partial_1 a_j, d\rangle, \\ \xi_{j,k} &:= \langle \partial_1 a_j, b_k\rangle, \\ \nu_j &:= \langle \partial_0 a_i, a_j\rangle. \end{aligned}$$

Using $\partial_0^2 = 0$ and again Proposition 3.3, the first term of Equation (6) may then be simplified:

$$\begin{aligned}
\partial_1 \partial_0 a_i &= \sum_{j=1}^{i-1} \langle \partial_0 a_i, a_j\rangle \, \partial_1 a_j + \sum_{k=1}^{m} \langle \partial_0 a_i, b_k\rangle \, \partial_1 b_k \\
&= \sum_{j=1}^{i-1} \nu_j \left( \partial_0 a_j + \lambda_j c + \mu_j d + \sum_{k=1}^{m} \xi_{j,k} b_k \right) + \sum_{k=1}^{m} \langle \partial_0 a_i, b_k\rangle \, \partial_0 b_k \\
&= \partial_0^2 a_i + \sum_{j=1}^{i-1} \nu_j \left( \lambda_j c + \mu_j d + \sum_{k=1}^{m} \xi_{j,k} b_k \right) \\
&= \sum_{j=1}^{i-1} \nu_j \left( \lambda_j c + \mu_j d + \sum_{k=1}^{m} \xi_{j,k} b_k \right).
\end{aligned}$$

Equation (6) then becomes:

$$\begin{aligned}
0 = &\sum_{j=1}^{i-1} \nu_j \left( \lambda_j c + \mu_j d + \sum_{k=1}^{m} \xi_{j,k} b_k \right) \\
&+ \langle \partial_1 a_i, c\rangle \, \partial_1 c + \langle \partial_1 a_i, d\rangle \, \partial_1 d + \sum_{k=1}^{m} \langle \partial_1 a_i, b_k\rangle \, \partial_1 b_k.
\end{aligned}$$

Using Lemma 3.2, the projection on $\mathbb{Z}_2[T] \cdot d$ of this equation gives

$$0 = \sum_{j=1}^{i-1} \nu_j \mu_j + \langle \partial_1 a_i, c\rangle.$$

Replacing the $\nu_j$'s and the $\mu_j$'s, and using Equation (3) yields

$$\langle \partial_1 a_i, c\rangle = \sum_{j=1}^{i-1} \langle \partial_1 a_i, a_j\rangle \langle \partial_1 a_j, d\rangle,$$

which proves the proposition. □

In particular, $\langle \partial_1 a_1, c\rangle = 0$.

This proposition tells us that the strips joining $a_i$ to $c$ do not matter, since the new coefficients get canceled out by the definition of $H$. Indeed,

$$\begin{aligned}
\langle H(\partial_0 a_i), c\rangle &= \sum_{j<i} \langle \partial_0 a_i, a_j\rangle \langle H(a_j), c\rangle \\
&= \sum_{j<i} \langle \partial_0 a_i, a_j\rangle \langle \partial_1 a_j, d\rangle,
\end{aligned}$$

so by Proposition 3.5,

$$\langle H(\partial_0\, a_i), c\rangle = \langle \partial_1\, a_i, c\rangle. \tag{7}$$

The second step is the following.

**Proposition 3.6.** *Let us denote* $p\colon D(t_0+\varepsilon) \to \mathbb{Z}_2[T]\langle b_m, \dots, b_1, d\rangle$ *the canonical projection. Then,* $p(\partial_1\, c) = \partial_1\, c$ *and*

$$p(\partial_1\, a_i) = p(\partial_0\, a_i) + \langle \partial_1\, a_i, d\rangle\, \partial_1\, c. \tag{8}$$

*Proof.* At time $t' \coloneqq t_0+\varepsilon$, for each strip $S_1$ joining $a_i$ to $d$, and for each strip $S_2$ joining $c$ to $b_k$, there was a strip $S_0$ at $t \coloneqq t_0-\varepsilon$, which disappears at $t = t_0$, joining $a_i$ to $b_k$, which satisfies

$$(S_0, h) = (S_1, h) + (S_2, h).$$

Here, $(S, h)$ denotes the algebraic intersection number, or the number of preimages of $h$ by $S$, as discussed in Subsection 1.5.

Let $\tilde{\mathcal{M}}_t(a_i, b_k)$ be the set of the strips that contribute to the difference $\langle \partial_1\, a_i, b_k\rangle - \langle \partial_0\, a_i, b_k\rangle$:

$$\langle \partial_1\, a_i, b_k\rangle - \langle \partial_0\, a_i, b_k\rangle = \sum_{S_0 \in \tilde{\mathcal{M}}_t(a_i, b_k)} T^{(S_0, h)}.$$

The function $(S_1, S_2) \mapsto S_0$ described above gives rise to a bijection

$$\mathcal{M}_{t'}(a_i, d) \times \mathcal{M}_{t'}(c, b_k) \xrightarrow{\sim} \tilde{\mathcal{M}}_t(a_i, b_k).$$

Therefore, we see that for all $k \in \{1, \dots, m\}$, we have

$$\begin{aligned}
\langle \partial_1\, a_i, b_k\rangle - \langle \partial_0\, a_i, b_k\rangle &= \sum_{S_0 \in \tilde{\mathcal{M}}_t(a_i, b_k)} T^{(S_0, h)} - \langle \partial_0\, a_i, b_k\rangle \\
&= \sum_{(S_1, S_2) \in \mathcal{M}_{t'}(a_i, d) \times \mathcal{M}_{t'}(c, b_k)} T^{(S_1, h) + (S_2, h)} \\
&= \left( \sum_{S_1 \in \mathcal{M}_{t'}(a_i, d)} T^{(S_1, h)} \right) \left( \sum_{S_2 \in \mathcal{M}_{t'}(c, b_k)} T^{(S_2, h)} \right) \\
&= \langle \partial_1\, a_i, d\rangle \langle \partial_1\, c, b_k\rangle.
\end{aligned}$$

By Lemma 3.2, the coefficient in front of $d$ of $\partial_0\, a_i + \langle \partial_1\, a_i, d\rangle\, \partial_1\, c$ is exactly $\langle \partial_1\, a_i, d\rangle$, so combining it with the result of the equation above proves the proposition. This proof is summed up in Fig. 4. ☐

Let us then prove Proposition 3.4:

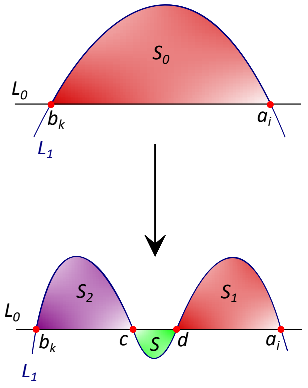

Figure 4: The strip joining $a_i$ to $b_k$ breaks down into three parts

*Proof.* As explained in the discussion after the statement of the proposition, we only have to show that

$$\partial_1 H(a_i) = H(\partial_0 a_i). \tag{9}$$

This will be a consequence of the definition of $H$ and the following equation:

$$\partial_1 a_i = H(\partial_0 a_i) + \langle \partial_1 a_i, d\rangle\, \partial_1 c. \tag{10}$$

Since $p(H(\partial_0 a_i))) = p(\partial_0 a_i)$, we already have by Proposition 3.6

$$p(\partial_1 a_i) = p\Big(H(\partial_0 a_i) + \langle \partial_1 a_i, d\rangle\, \partial_1 c\Big). \tag{11}$$

We then only need to check if the projections on $c$ and on the $a_i$'s of both terms of (10) hold.

The projection on $a_j$ is given by Proposition 3.3:

$$\langle \partial_1 a_i, a_j\rangle = \langle \partial_0 a_i, a_j\rangle = \langle H(\partial_0 a_i), a_j\rangle + \langle \partial_1 a_i, d\rangle\langle \partial_1 c, a_j\rangle, \tag{12}$$

because $\langle \partial_1 c, a_j\rangle = 0$.

Lastly, Equation (7) in the discussion after Proposition 3.5 shows that

$$\langle \partial_1 a_i, c\rangle = \langle H(\partial_0 a_i), c\rangle + \langle \partial_1 a_i, d\rangle\langle \partial_1 c, c\rangle, \tag{13}$$

where $\langle \partial_1 c, c\rangle = 0$.

By the three equations (11), (12) and (13), we see that (10) holds. Therefore, we have proved Proposition 3.4. □

## 3.3 Proof of Theorem 3.1

We are now able to prove Theorem 3.1:

*Proof.* First of all, it is clear that $D(t)$ defines a piecewise continuous complex away from the birth and death moments. By genericity, we may assume that only a finite number of birth/death moments occur, and that only two generators appear/disappear simultaneously.

Let $t_1 < \cdots < t_n$ be the birth/death moments. Denote $t_0 := 0$ and $t_{n+1} := 1$. We will use the time interval

$$\Delta t = \inf_{0 \leq i \leq n} t_{i+1} - t_i.$$

Let $1 \leq i \leq n$; we may assume that there is a birth at $t = t_i$. We are going to define a piecewise continuous family $\tilde{D}(t)$ such that $\tilde{D}(t) = D(t)$ when $t < t_i - \frac{\Delta t}{2}$ or $t > t_i + \frac{\Delta t}{2}$.

Denote $c(t)$ and $d(t)$ the two new-born generators of $D(t)$ at time $t > t_i$, with $\ell_t(c(t)) < \ell_t(d(t))$.

Let us define the family of filtered complexes

$$S := \mathbb{Z}_2[T]\langle c, d\rangle,$$

endowed with the differential $\partial$ that satisfies $\partial c = d$ and $\partial d = 0$, and with constant action $l$ such that $(c, d)$ is a compatible basis and

$$l(c) = l(d) := \lim_{s \to t_i^-} \ell_s(c).$$

Now, let us define the interval

$$I_i := \left(t_i - \frac{\Delta t}{4}, t_i\right],$$

and then the family of complexes

$$(\tilde{D}(t), \tilde{\partial}_h(t), \tilde{\ell}_t) := \begin{cases} (D(t), \partial_h(t), \ell_t) & \text{if } t \notin I_i, \\ (D(t_i - \frac{\Delta t}{4}), \partial_h(t_i - \frac{\Delta t}{4}), \ell_t) \oplus (S, \partial, l) & \text{if } t \in I_i, \end{cases}$$

for $t \in [t_i - \frac{\Delta t}{2}, t_i + \frac{\Delta t}{2}]$.

Unlike $D(t)$, the family $\tilde{D}(t)$ defines a true piecewise continuous family in the sense of Definition 2.3. Indeed:

- $D(t_i)$ is not a filtered chain complex, but $D(t_i - \frac{\Delta t}{4})$ is, and it is chain-isomorphic to every $D(t)$ with $t \in \operatorname{int} I_i$, so the definition makes sense for all $t$, even in the bifurcation times $t_i$ and $t_i - \frac{\Delta t}{4}$;

- for $t < t_i - \frac{\Delta t}{4}$, $t > t_i$ or $t_i - \frac{\Delta t}{4} < t < t_i$, the complexes are chain-isomorphic with continuous action, as seen before;
- at time $t = t_i - \frac{\Delta t}{4}$, the complex $D(t)$ obviously undergoes a birth;
- at time $t = t_i$, the complex $D(t)$ undergoes a handle-slide, via the map $H$ which was proven to be suitable in Proposition (3.4).

Therefore, the theorem is now proved for the family $\tilde{D}(t)$. $\square$

## 3.4 Proof of Theorem 1.8

In order to prove Theorem 1.8, we have to make the link between the complexes $C_\beta(t)$ and $D(t)$. This link is described by the following lemma.

**Lemma 3.7.** *Let* $\beta \in \{0,1\}$. *The evaluation map*

$$\mathrm{ev}_\beta \colon D(t) \longrightarrow C_\beta(t)$$

*sending* $T$ *to* $\beta$ *is a chain-map that is action-preserving.*

*Moreover, if* $D(t)$ *has a Barannikov basis* $B$, *then* $\mathrm{ev}_\beta(B)$ *is a Barannikov basis of* $C_\beta(t)$. *In this case, we have in addition*

$$\mathcal{B}(D(t)) = \mathcal{B}(C_\beta(t)).$$

*Proof.* By definition of $\partial_h(t)$, for each $x \in L \cap L_t$ we have

$$\mathrm{ev}_\beta(\partial_h(t)x) = \partial_\beta(t)x = \partial_\beta(t)\mathrm{ev}_\beta x.$$

Since $\partial_h(t)$ is $\mathbb{Z}_2[T]$-linear and $\mathrm{ev}_\beta \colon \mathbb{Z}_2[T] \to \mathbb{Z}_2$ is a ring homomorphism, this relation even holds for every $x \in D(t)$, so $\mathrm{ev}_\beta$ is a chain-map. It obviously sends the canonical basis of $D(t)$ on the canonical basis of $C_\beta(t)$, so it preserves action. It thus preserves compatible bases.

Let $B = (a_1(T), \dots, a_N(T), \partial_h a_1(T), \dots, \partial_h a_N(T), c_1(T), \dots, c_M(T))$ be a Barannikov basis of $D(t)$. Being a compatible basis, $\mathrm{ev}_\beta(B)$ is a compatible basis of $C_\beta(t)$. Moreover, we have

$$\begin{aligned} &\forall i \in \{1, \dots, N\}, \quad \partial_\beta \, \mathrm{ev}_\beta(a_i(T)) = \mathrm{ev}_\beta(\partial_h \, a_i(T)), \\ &\forall j \in \{1, \dots, M\}, \quad \partial_\beta \, \mathrm{ev}_\beta(c_j(T)) = \mathrm{ev}_\beta(\partial_h \, c_j(T)) = 0. \end{aligned}$$

Therefore, $\mathrm{ev}_\beta(B)$ is a Barannikov basis of $C_\beta(t)$. $\square$

Finally, we use Lemma 3.7 and Theorem 3.1 to prove Theorem 1.8, which is a direct consequence of the following result.:

**Proposition 3.8.** *Suppose $D(0)$ is a standard complex. Then, besides in some small neighborhood of its bifurcation times, $D(t)$ is a family of standard complexes, and its barcode satisfies*

$$\mathcal{B}(D(t)) = \mathcal{B}(C_1(t)) = \mathcal{B}(C_0(t)).$$

*Proof.* Let $\tilde{D}(t)$ be a modified version of $D(t)$ given by Theorem 3.1. Let $0 < t_1 < \cdots < t_n < 1$ be the bifurcation times.

Between two bifurcations, the complexes are canonically chain-isomorphic, so if $\tilde{D}(t)$ is standard for some $t \in (t_i, t_{i+1})$, then all the $\tilde{D}(t')$ for all other $t' \in (t_i, t_{i+1})$ are standard as well.

We use induction over $i$ to prove that all the complexes $\tilde{D}(t)$ are standard. Suppose that for all $t \in [0, t_i)$, $\tilde{D}(t)$ is a standard complex.
Let $B \coloneqq (a_1, \ldots, a_N, \partial_0 a_1, \ldots \partial_0 a_N, c_1, \ldots, c_M)$ be the Barannikov basis of $\tilde{D}(t_i - \varepsilon)$.

Suppose there is a birth at time $t_i$. Then, there is a filtered decomposition

$$\tilde{D}(t_i + \varepsilon) = \tilde{D}(t_i - \varepsilon) \oplus S,$$

and therefore, after a proper reordering $B \sqcup (c, d)$ is the Barannikov basis of $\tilde{D}(t_i + \varepsilon)$.

If there is a death at time $t_i$, by reversing the course of time we end up in the above case.

Suppose there is a handle-slide at time $t_i$. Denote

$$H \colon (\tilde{D}(t_i - \varepsilon), \partial_0) \longrightarrow (\tilde{D}(t_i + \varepsilon), \partial_1)$$

the chain-isomorphism that corresponds to the handle-slide. Thus, $H(B)$ is a Barannikov basis of $(\tilde{D}(t_i + \varepsilon), \partial_1)$. Indeed,

$$\begin{aligned} \forall i \in \{1, \ldots, N\}, \quad & \partial_1 H(a_i) = H(\partial_0 a_i), \\ \forall j \in \{1, \ldots, M\}, \quad & \partial_1 H(c_j) = H(\partial_0 c_j) = 0, \end{aligned}$$

and this proves that for all $t \in [0, t_{i+1})$, $\tilde{D}(t)$ is a standard complex. Moreover, it coincides with $D(t)$ out of a small neighborhood of the bifurcation times.

Finally, we use Lemma 3.7 to see that the barcode of $D(t)$ is $\mathcal{B}(C_1(t))$. $\square$

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