

Categories of filtrations in homological algebra

Marius BERTHOUMIEUX

Under the supervision of

Jacques DARNÉ

Université Catholique de Louvain-la-Neuve

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Introduction

Category theory aims at generalizing constructions done in specific contexts and drawing parallels between a vast array of domains of mathematics. The part of it that we will be interested in this work, is homological algebra. The different notions of homological algebra all try to give a framework in which objects behave "as in a certain algebraic notion". One of the simplest example, and also one of the more restrictive is the case of abelian categories, which try to mimic the behaviour of modules over a given ring, introduced as early as 1950 by Buchsbaum [1] and developed by Grothendieck [2]. The Freyd-Mitchell embedding theorem, which can be found in [3, Theorem 7.34], show that they encapsulate modules well.

The context of this work, however, are semi-abelian categories, introduced in [4] and homological categories, introduced in [7]. Their aim is to provide a larger context, which encapsulates groups, for the case of semi-abelian, and which gives a minimal framework for doing homology in the case of homological.

One of the many aspects of homological algebras is that it creates a hierarchy of properties that a category might verify (for example abelian implies semi-abelian implies homological), and constantly adds new properties to this hierarchy. This allows for certain results to transfer from a context to another, sometimes non-trivially.

For general category theory, we direct the reader towards [10, 9], for abelian towards [3] and for semi-abelian and homological towards [7]. We also strongly encourage the reader to visit <https://ncatlab.org>, which is a very complete resource on category theory.

In this work we will prove some results of transfer of structure from a category to categories of filtrations in it (with possibly some additional properties). We start by presenting in details the context of our work in section 1, then we recall a result from Tomas Everaert [5] on categories of regular epimorphisms in 2. We then extend this result to filtrations in section 3 and to central filtrations in section 4 . We also look at the simpler case of monomorphisms in the appendix, section B.

This internship at the Catholic university of Louvain-La-Neuve was the occasion for me to discover what it meant to do research in category theory, and more specifically homological algebra. I had the occasion to work with multiple people in the domain, most notably my supervisor Jacques Darné who was at the time a postdoc at UCL and Tim Van der Linden, a senior researcher there. I presented my work in front of the junior researchers at UCL (Phd students and postdocs) towards the end of my stay there. On top of what is presented there, I attended a reading group on model categories (a type of categories which aim at generalizing homotopy) supervised by Nadja Egner, a Phd student, and in which I spoke multiple times. The culminating point of my internship was the week long CT2023 conference, which was the largest category theory conference of the year. I would like to thank all the people I mentioned as well as the rest of the IRMP (institut de recherche en mathématiques et physique) and the category theory team.

1 Basic definitions

The goal of this section is to give some definitions and some basic results for people that are not familiar with the notions presented in this work. We still assume some knowledge of general category theory, for which we direct the reader towards [8, 9]. We will still mention that a diagram is just a functor, a cone is a natural transformation from a constant functor to a diagram and a limit cone is a terminal object in cones on a given diagram.

Convention. Pullback diagrams are indicated, where possible, by a corner inside the square on the limit object, and if we take the pullback of f and g , the morphisms of the cone will be denoted by f^*g and g^*f as in the following diagram:

$$\begin{array}{ccc} A \times_C B & \xrightarrow{g^*f} & A \\ f^*g \downarrow \lrcorner & & \downarrow g \\ A & \xrightarrow{f} & B. \end{array}$$

Moreover, unless stated otherwise, all limits and colimits considered in this work are finite.

First let us give all the definitions required to know what a homological category is. They can all be found in [7], in their relevant sections.

Definition 1.1 (regular epimorphisms and kernel pairs). *We say that a morphism p is a regular epimorphism when it is the coequalizer of some pair of morphisms f, g . That is, every u such that $uf = ug$ can uniquely be written $u = vp$ and $pf = pg$. We will denote them by double headed arrows \twoheadrightarrow in the rest of this work.*

The kernel pair of some morphism $f : A \rightarrow B$ is defined as the pair (π_1, π_2) given by the pullback of f along itself (see diagram (1.2)).

$$\begin{array}{ccc} R[f] & \xrightarrow{\pi_2} & A \\ \downarrow \lrcorner \pi_1 & & \downarrow f \\ A & \xrightarrow{f} & B. \end{array} \tag{Diagram 1.2}$$

Remark. A regular epimorphism is then always the coequalizer of its kernel pair, when it exists. A proof of this fact can be found in [7, lemma A.4.8].

Note that regular epimorphisms are always epimorphisms, as the name suggests, but that the converse is not always true, even in categories with a lot of structure. For example in the category of topological groups,

which is homological (see definition 1.9), an epimorphism is just a continuous surjection that is a morphism of groups, while a regular epimorphism has the additional property that the topology on the codomain must be the quotient topology.

Proposition 1.3. *Let \mathcal{A} be a category with finite limits. In the following commutative diagram, where f is a regular epimorphism and i is a monomorphism, there exists a necessarily unique lift h such that $hf = u$ and $ih = v$:*

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ f \downarrow & \nearrow h & \downarrow i \\ \cdot & \xrightarrow{v} & \cdot \end{array}$$

Proof. See [7, A.4.3, A.4.4 & A.4.9] □

Definition 1.4 (regular category). *A category \mathcal{A} is regular when:*

1. *it has finite limits,*
2. *it has coequalizers of kernel pairs,*
3. *regular epimorphisms are pullback-stable (i.e. if f and g are morphisms in \mathcal{A} with the same codomain, and f is a regular epimorphism, then so is g^*f).*

Remark. In regular categories, every morphism a can be written uniquely up to isomorphism as the composition of a regular epimorphism followed by a monomorphism. This is called the regular epi-mono factorization. The monomorphic part is called the image of a , denoted $im(a)$ and the regular part is computed by taking the coequalizer of the kernel pair of a .

Here is an extra result on pullbacks of regular epimorphisms in a regular category that we will need later:

Proposition 1.5 ([13, A.1.4.5]). *Let \mathcal{A} be a regular category. The pullback square of two regular epimorphisms in \mathcal{A} is also a pushout.*

Lemma 1.6 ([7, lemma A.5.9]). *Let \mathcal{A} be a category with kernel pairs. In the diagram (1.7) in \mathcal{A} , if the right square is a pullback, then so are the two squares given by the morphisms of the kernel pairs (the morphism $R[f]$ is obtained from the universal property of kernel pairs to make the two left squares commute).*

$$\begin{array}{ccccc} & & v & & \\ & & \curvearrowright & & \\ & & \phi & & \\ & & \searrow & & \\ & & & & \\ u & & & & \\ \downarrow & & & & \\ R[f_0] & \xrightarrow{\pi_0} & A_0 & \xrightarrow{f_0} & B_0 \\ \downarrow R[f] & \xrightarrow{\pi'_0} & \downarrow a & & \downarrow b \\ R[f_1] & \xrightarrow{\pi_1} & A_1 & \xrightarrow{f_1} & B_1 \\ & \xrightarrow{\pi'_1} & & & \end{array} \quad \text{(Diagram 1.7)}$$

One can find a proof of these two results in the references. We give in the appendix A a complete proof that is closer to our case for proposition 1.5 and that is somewhat more elementary for lemma 1.6.

We will be mainly interested in pointed categories, which are categories that have an object that is both initial and final, called the zero object and denoted 0 (or $0_{\mathcal{A}}$ when the notation is ambiguous). In this case, we get the following definition for protomodular categories:

Definition 1.8 (Pointed protomodular category). *A category \mathcal{A} is pointed protomodular when it is pointed and the split short five lemma holds in \mathcal{A} . This means that in the following diagram, where the rows are exact (i.e. k is a kernel of p and p is a cokernel of k), p (resp. p') is a split epimorphism with splitting s (resp. s') and $bk = k'a, cp = p'b, bs = s'c$, if a and c are isomorphisms, then so is b .*

$$\begin{array}{ccccccc}
0 & \longrightarrow & K[p] & \xleftarrow{k} & A & \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} & B & \longrightarrow & 0 \\
& & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & K[p'] & \xleftarrow{k'} & A' & \begin{array}{c} \xrightarrow{p'} \\ \xleftarrow{s'} \end{array} & B' & \longrightarrow & 0
\end{array}$$

Finally, we get:

Definition 1.9 (homological category). *A category \mathcal{A} is homological when it is pointed protomodular and regular.*

A consequence of that definition is that many of the usual diagram lemmas are true in any homological category. These include the five lemma, the nine lemma, the snake lemma, etc. (see [7] for proofs and more examples). We will use some of these results in the rest of this paper.

The last definition that we need before we finally get to the definition of semi abelian categories is that of an exact category:

Definition 1.10 (Exact category). *A regular category \mathcal{A} is exact when, given an equivalence relation on X (i.e. a monomorphic pair of morphisms $r_1, r_2 : R \rightarrow X$, or equivalently $R \hookrightarrow X \times X$, with certain conditions to describe transitivity, reflexivity and symmetry) then this relation is a kernel pair of some morphism.*

Definition 1.11 (Semi-abelian category). *A category \mathcal{A} is semi-abelian if it is homological, exact and it has binary coproducts.*

Example 1.12. The prime example of semi-abelian category, and the reason the notion was created in the first place, is the category Gr of groups (not necessarily abelian).

Other examples arise with classical algebraic structures, such as Lie algebras, rings without unit, associative algebras, and more generally any category of varieties of algebra of a theory that includes the theory of groups and has only one operation of arity 0 (otherwise it is non pointed).

A non-example is given by topological groups, which are homological but not semi-abelian. This is because an equivalence relation is just given by an equivalence relation in groups, whereas a kernel pair has additional requirements on the topology on the relation. We will see later (section 2) that the category of regular epimorphisms in groups also falls under this category. For more examples see [7].

Proposition 1.13. *Let \mathcal{A} be a semi-abelian category. The image of a normal monomorphism (the kernel of some morphism) in \mathcal{A} by a regular epimorphism is a normal monomorphism. This means exactly that in diagram (1.14), if u and v are regular epimorphisms and f is a normal monomorphism, then if g is a monomorphism (i.e. if it is the image of vf), it is also normal.*

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow u & & \downarrow v \\
B & \xrightarrow{g} & Y
\end{array} \tag{1.14}$$

In fact this can be used to give another characterisation of semi-abelian categories, see [11] for the characterisation and the proof.

The next topic of interest in this section are (co)reflective subcategories:

Definition 1.15 (Reflective subcategory). *Let \mathcal{A} be a category. A full subcategory \mathcal{B} of \mathcal{A} is called reflective if the inclusion functor \mathbf{U} has a left adjoint \mathbf{R} , called the reflector, as in the following diagram: $\mathcal{B} \begin{array}{c} \xleftarrow{\mathbf{R}} \\ \perp \\ \xrightarrow{\mathbf{U}} \end{array} \mathcal{A}$.*

We also have the dual notion of coreflective subcategories, which are defined by a coreflector, which is right adjoint to the inclusion.

We will need to know how to compute limits and colimits in these subcategories.

Proposition 1.16 ([8, proposition 4.5.15]). *Let \mathbf{R} be a reflector of \mathcal{A} on \mathcal{B} . Let $D : I \rightarrow \mathcal{B}$. If $\mathbf{U}D$ has a colimit L in \mathcal{A} then $\mathbf{R}(L)$ is a colimit of D in \mathcal{B} (note that this implies $\forall A \in \mathcal{B}, \mathbf{R}(\mathbf{U}A) \cong A$). A limit L of D exists in \mathcal{B} if and only a limit L' of $\mathbf{U}D$ exists in \mathcal{A} , and if it does then $\mathbf{U}L \cong L'$.*

Proof. See [8, proposition 4.5.15] or appendix A □

Lastly, this is just a concept needed for a general definition of filtrations later on in section 4:

Definition 1.17 (Partially ordered set seen as a category). *Let (S, \leq) be a partially ordered set. We define a category, also denoted S , with objects the elements of S as a set and with a single morphism from a to b if and only if $a \leq b$ in S , none otherwise (identities are then given by reflexivity and composition by transitivity).*

Proposition 1.18. *Let \mathcal{A} be a pointed category and S be a partially ordered set as a category. Limits and colimits in $[S, \mathcal{A}]$ are always computed pointwise, that is if $(L \xrightarrow{\psi_i} D(i))_{i \in I}$ is a limit cone in $[S, \mathcal{A}]$, then for every s in S , $(L(s) \xrightarrow{\psi_{i,s}} D(i)(s))_{i \in I}$ is one in \mathcal{A} .*

Proof. See appendix A. □

2 The category of regular epimorphisms

Following the work of T. Everaert in [5, section 2.6], we will first take a look at the category $\text{Reg } \mathcal{A}$ of regular epimorphisms in \mathcal{A} a category. The result given in Everaert's thesis [5, theorem 2.6.3] (Theorem 2.14 here) that will be of interest to us is that the category $\text{Reg } \mathcal{A}$ is homological with some additional properties when \mathcal{A} is.

Let us begin with a proper definition of $\text{Reg } \mathcal{A}$:

Definition 2.1. *Let \mathcal{A} be a category. The category of arrows of \mathcal{A} , denoted $\text{Arr } \mathcal{A}$, has for objects morphisms in \mathcal{A} . A morphism $f = (f_0, f_1) : a \rightarrow b$ in $\text{Arr } \mathcal{A}$ is given by a commutative square:*

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ \downarrow a & & \downarrow b \\ A_1 & \xrightarrow{f_1} & B_1. \end{array}$$

The category $\text{Reg } \mathcal{A}$ is then defined as the full subcategory of $\text{Arr } \mathcal{A}$ whose objects are the regular epimorphisms.

Example. For the case of groups we have that $\text{Reg}(\text{Gr})$ is equivalent to the category \mathcal{C} whose objects are pairs (G, N) where G is a group and N is a normal subgroup of G . A morphism $(G, N) \rightarrow (G', N')$ in \mathcal{C} is then just a group morphism $f : G \rightarrow G'$ such that $f(N) \subset N'$. The equivalence between \mathcal{C} and $\text{Reg}(\text{Gr})$ is given by $(G, N) \mapsto (G \twoheadrightarrow G/N)$.

Remark 2.2. [5, example 2.6.4] One might notice in the last example that $\text{Reg}(\text{Gr}) \cong \mathcal{C}$ is in fact not semi-abelian. Indeed \mathcal{C} is not exact. For example, if A is a non trivial abelian group, then $(A \times A, \Delta)$ is an equivalence relation on (A, A) (where Δ denotes the diagonal). It is not, however a kernel pair since otherwise it would be the kernel pair of its coequaliser, but its coequaliser is given by $(A, A) \rightarrow (0, 0)$ where 0 is the trivial group. Thus we cannot hope in general that the category of regular epimorphisms in a semi-abelian category is semi-abelian, but theorem 2.14 will imply that it is homological.

Our first goal when proving theorem 2.14 is to understand limits, colimits and regular epimorphisms in $\text{Reg } \mathcal{A}$. For limits and colimits, we begin by noticing that $\text{Arr } \mathcal{A}$ is just a functor category $\text{Cat}(\mathbf{2}, \mathcal{A})$, where $\mathbf{2}$ is the category with two objects 0 and 1 , identity morphisms and a single morphism from 0 to 1 . Thus, $\text{Arr } \mathcal{A}$ has limits and colimits of the same shape as \mathcal{A} and they are computed pointwise.

Convention. In the rest of this work we will denote the domain (resp. codomain) of a morphism in \mathcal{A} , seen as an object of $\text{Arr } \mathcal{A}$ or $\text{Reg } \mathcal{A}$, by the corresponding uppercase letter indexed by 0 (resp. 1). As for morphisms in those categories, they will either be seen as pairs of morphisms in \mathcal{A} and denoted in parenthesis, or as natural transformations, with the use of indices. For example, if a is an object in $\text{Arr } \mathcal{A}$ we denote by A_0 its domain and A_1 its codomain and if we have a commutative diagram in \mathcal{A} :

$$\begin{array}{ccc} A_0 & \xrightarrow{g} & B_0 \\ \downarrow a & & \downarrow b \\ A_1 & \xrightarrow{h} & B_1, \end{array}$$

then if we set $f = (g, h)$, we have $f_0 = g$ and $f_1 = h$.

We get the following structure on $\text{Reg } \mathcal{A}$:

Proposition 2.3. *Let \mathcal{A} be a regular category. The map $\mathbf{R} : \text{Ob}(\text{Arr } \mathcal{A}) \rightarrow \text{Ob}(\text{Reg } \mathcal{A})$ which sends a morphism a to the regular part of its regular epi-mono factorization has a canonical extension to a functor. Moreover it is a coreflector of $\text{Arr } \mathcal{A}$ onto the full subcategory $\text{Reg } \mathcal{A}$.*

The counit ϵ of this adjunction is given for all $a \in \text{Arr } \mathcal{A}$ by $\epsilon_a = (id_{A_0}, im(a))$ (see diagram (2.4)). We shall omit the index from the notation ϵ_a when it is clear which object it should be.

$$\begin{array}{ccc} A_0 & \xrightarrow{id_{A_0}} & A_0 \\ \downarrow \mathbf{R} a & & \downarrow a \\ \mathbf{R} A_1 & \xrightarrow{im(a)} & A_1 \end{array} \quad (\text{Diagram 2.4})$$

Remark. To define \mathbf{R} we have made a choice of factorization for every morphism. This will not matter much since this choice is unique up to unique isomorphism.

Corollary 2.5. *Let \mathcal{A} be a regular category. The category $\text{Reg } \mathcal{A}$ has all small limits. Moreover, if $d : I \rightarrow \text{Reg } \mathcal{A}$ is a diagram and $(l \xrightarrow{\phi_i} d(i))_{i \in I}$ a limit cone for d in $\text{Arr } \mathcal{A}$, then:*

- it has a limit cone in $\text{Reg } \mathcal{A}$ given by $(\mathbf{R}l \xrightarrow{\phi_i \circ \epsilon_l} d(i))_{i \in I}$,
- it has a colimit in $\text{Reg } \mathcal{A}$ exactly when it has one in $\text{Arr } \mathcal{A}$ and when it does, it is the same in both categories. Moreover, if \mathcal{A} is pointed, then it is always computed pointwise.

Proof. This is a consequence of propositions 1.16, 2.3 and 1.18. \square

proof of proposition 2.3. Let $a \in \text{Reg } \mathcal{A}$, $b \in \text{Arr } \mathcal{A}$. We first construct the bijection $\text{Reg } \mathcal{A}(a, \mathbf{R}(b)) \cong \text{Arr } \mathcal{A}(a, b)$. First take $\tau : \text{Reg } \mathcal{A}(a, \mathbf{R}(b)) \rightarrow \text{Arr } \mathcal{A}(a, b)$ to be postcomposition by ϵ_b . This already implies that ϵ is the counit, provided we prove the rest of the result. Let now $f : a \rightarrow b$, we want to obtain a morphism $\omega f : a \rightarrow \mathbf{R}(b)$ such that $\omega \tau$ and $\tau \omega$ are the identity. This morphism ωf is defined by $\omega f_0 = f_0$ and ωf_1 is the lift in the following diagram:

$$\begin{array}{ccc} A_0 & \xrightarrow{(\mathbf{R}b)f_0} & \mathbf{R}(B)_1 \\ a \downarrow & \nearrow \omega f_1 & \downarrow \text{im}(b) \\ A_1 & \xrightarrow{f_1} & B_1. \end{array}$$

The fact that $\tau \omega f = f$ is now a direct consequence of the definitions of ω and τ . We also have $\omega \tau = id$ since if $f = \tau g$ for some $g : a \rightarrow \mathbf{R}b$, then $f_1 = \text{im}(b)g_1$ and g_1 is an appropriate lift in the previous diagram.

Next we define, for $f : a \rightarrow b$ in $\text{Arr } \mathcal{A}$, $\mathbf{R}(f) = \omega(f\epsilon_a)$. To check that this defines a functor, we need to verify that for $a \xrightarrow{g} b \xrightarrow{f} c$ in $\text{Arr } \mathcal{A}$, $\mathbf{R}(f)\mathbf{R}(g) = \mathbf{R}(fg)$, the preservation of identities is clear. Let, then, g and f be such morphisms. The condition is equivalent to $\tau(\mathbf{R}(f)\mathbf{R}(g)) = \tau\mathbf{R}(fg)$. But $\tau\mathbf{R}(fg) = fg\epsilon_a$ by definition and $\tau(\mathbf{R}(f)\mathbf{R}(g)) = \epsilon_c\mathbf{R}(f)\mathbf{R}(g) = f\epsilon_b\mathbf{R}(g) = fg\epsilon_a$.

The last properties to check are the naturality conditions on τ and ω . Let $f : a \rightarrow b$ be a morphism in $\text{Reg } \mathcal{A}$, $c \in \text{Arr } \mathcal{A}$. For every $g : b \rightarrow \mathbf{R}c$, we have $(\tau g)f = \epsilon_c g f = \tau(gf)$. The other naturality condition, which we prove on τ is just the fact that ϵ is a natural transformation. Notice, then, that for $f : a \rightarrow b$ in $\text{Arr } \mathcal{A}$, $\epsilon_b\mathbf{R}(f) = \epsilon_b\omega(f\epsilon_a) = \tau(\omega(f\epsilon_a)) = f\epsilon_a$, which concludes the proof. \square

Remark. The second half of this proof is a consequence of a more general result, saying that for a functor F , if $\text{Hom}(X, F(-))$ is representable by some $G(X)$ for every X then G uniquely extends to a functor, which is left adjoint to F (see [9, corollary 2, section IV.1]).

Let us now give a characterization of regular epimorphisms in $\text{Reg } \mathcal{A}$ that will be needed for the main result:

Proposition 2.6. *A morphism $(f_0, f_1) : a \rightarrow b$ in $\text{Reg } \mathcal{A}$ is a regular epimorphism exactly when f_0 and f_1 are (as morphisms in \mathcal{A}), and when the commutative square of the morphism is a pushout in \mathcal{A} .*

Proof. First suppose that f is a coequalizer in $\text{Reg } \mathcal{A}$. We can write it as the coequalizer of its kernel pair, which exists and is computed as in corollary 2.5. We obtain a diagram of the following shape, where π_0, π'_0 (resp. π_1, π'_1) is the kernel pair of f_0 (resp. f_1):

$$\begin{array}{ccccc} R[f_0] & \xrightarrow{\pi_0} & A_0 & \xrightarrow{f_0} & B_0 \\ & \searrow \pi'_0 & \downarrow a & & \downarrow b \\ R'[f_1] & \xrightarrow{j} & R[f_1] & \xrightarrow{\pi_1} & A_1 & \xrightarrow{f_1} & B_1. \end{array} \quad (\text{Diagram 2.7})$$

Let u, v be morphisms such that $ua = vf_0$. Let us rewrite it in this diagram, in which we see (vf_0, u) as a morphism in $\text{Reg } \mathcal{A}$:

$$\begin{array}{ccccc}
 A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{v} & C \\
 \downarrow a & & \downarrow b & & \parallel \\
 A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{\phi_1} & C \\
 & \searrow u & & &
 \end{array}
 \quad (\text{Diagram 2.8})$$

We know that vf_0 coequalizes π_0 and π'_0 . To see that u coequalizes $\pi_1 j$ and $\pi'_1 j$ we can show that $u\pi_1 j r = u\pi'_1 j r$, since r is epimorphic. But then:

$$u\pi_1 j r = ua\pi_0 = vf_0\pi_0 = vf_0\pi'_0 = ua\pi'_0 = u\pi'_1 j r.$$

Since, then, (vf_0, u) coequalizes $(\pi_0, \pi_1 j)$ and $(\pi'_0, \pi'_1 j)$ in $\text{Reg } \mathcal{A}$, we get a factorization ϕ of (vf_0, u) through f . But ϕ_0 is equal to v since f_0 is epimorphic. The morphism ϕ_1 gives us the desired factorization for the pushout, which is unique since f_1 is epimorphic.

Let us now assume that f_0 and f_1 are regular epimorphisms and that the square defining f is a pushout. We will show that f is the coequalizer of its kernel pair. We use the same notations as in the first part of the proof. Let now u be a morphism from a to c in $\text{Reg } \mathcal{A}$ that coequalizes $(\pi_0, \pi_1 j)$ and $(\pi'_0, \pi'_1 j)$ as in diagram (2.9).

$$\begin{array}{ccccccc}
 & & & & u_0 & & \\
 & & & & \curvearrowright & & \\
 R[f_0] & \xrightarrow{\pi_0} & A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{\phi_0} & C_0 \\
 \downarrow r & & \downarrow a & & \downarrow b & & \downarrow c \\
 R'[f_1] & \xrightarrow{j} & R[f_1] & \xrightarrow{\pi_1} & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{\phi_1} & C_1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & u_1 & & & &
 \end{array}
 \quad (\text{Diagram 2.9})$$

Since f_0 is regular, it is the coequalizer of its kernel pair π_0, π'_0 . Since u_0 coequalizes this kernel pair, we get the factorization ϕ_0 making the top triangle in diagram (2.9) commute. But now we have $c\phi_0 f_0 = cu_0 = u_1 a$ and we can use the pushout property to get ϕ_1 which makes the rightmost square commute (i.e. (ϕ_0, ϕ_1) is a morphism in $\text{Reg } \mathcal{A}$) and the bottom triangle too. This gives us a factorization of u through f . It is unique since f_0 and f_1 are epimorphisms and thus, f is a coequalizer in $\text{Reg } \mathcal{A}$. \square

One final step, and part of the proof of theorem 2.14 is given by the two following results:

Proposition 2.10. *Let \mathcal{A} be a homological category. Kernels of split epimorphisms in $\text{Reg } \mathcal{A}$ are computed as in $\text{Arr } \mathcal{A}$.*

Corollary 2.11. *Let \mathcal{A} be a homological category. The category $\text{Reg } \mathcal{A}$ is pointed protomodular.*

Proof. The object id_0 is initial and terminal in $\text{Reg } \mathcal{A}$ so that $\text{Reg } \mathcal{A}$ is pointed. Split epimorphisms, isomorphisms and kernels of split epimorphisms in $\text{Reg } \mathcal{A}$ are the same as in $\text{Arr } \mathcal{A}$, since $\text{Reg } \mathcal{A}$ is a full subcategory and due to proposition 2.10. Thus the split short five lemma in $\text{Reg } \mathcal{A}$ follows as soon as we prove it in $\text{Arr } \mathcal{A}$.

Let us, then, look at diagram (2.12), with short exact rows in $\text{Arr } \mathcal{A}$ and suppose f and h are isomorphisms.

$$\begin{array}{ccccccc}
0 & \longrightarrow & k & \xrightarrow{i} & a & \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} & b & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & k' & \xrightarrow{i'} & a' & \begin{array}{c} \xrightarrow{p'} \\ \xleftarrow{s'} \end{array} & b' & \longrightarrow & 0
\end{array} \tag{Diagram 2.12}$$

Since isomorphisms, split epimorphisms and kernels are defined pointwise in $\text{Arr } \mathcal{A}$, we can apply the split short five lemma in \mathcal{A} to this diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K[p_0] & \xrightarrow{i_0} & A_0 & \begin{array}{c} \xrightarrow{p_0} \\ \xleftarrow{s_0} \end{array} & B_0 & \longrightarrow & 0 \\
& & \downarrow f_0 & & \downarrow g_0 & & \downarrow h_0 & & \\
0 & \longrightarrow & K[p'_0] & \xrightarrow{i'_0} & A'_0 & \begin{array}{c} \xrightarrow{p'_0} \\ \xleftarrow{s'_0} \end{array} & B'_0 & \longrightarrow & 0.
\end{array}$$

We get that g_0 is an isomorphism. Similar reasoning shows that g_1 is an isomorphism too, from which we get that g is an isomorphism in $\text{Arr } \mathcal{A}$, which concludes the proof. \square

proof of proposition 2.10. Take a split epimorphism $p : a \rightarrow b$ with splitting s in $\text{Reg } \mathcal{A}$ and build diagram (2.13). First define $K[a]$ and $K[b]$ as the appropriate kernels in \mathcal{A} . This gives a split epimorphism $p' : K[a] \rightarrow K[b]$ with splitting s' that make both squares with vertices $K[a], K[b], A_0$ and B_0 commute. Thus, if we take K', K_0 and K_1 to be the kernels of p', p_0 and p_1 respectively, we get a diagram with short exact rows and the two right columns exact too. This is due to the fact that in homological categories, regular epimorphisms are cokernels (see [11]). Using the nine lemma in \mathcal{A} we get that the left column is a short exact sequence which implies that the morphism $K_0 \rightarrow K_1$ is a regular epimorphism. Using the characterization of limits given by corollary 2.5, we get the fact that it is both the kernel of p in $\text{Arr } \mathcal{A}$ and in $\text{Reg } \mathcal{A}$.

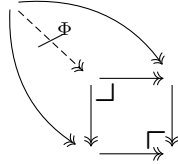
$$\begin{array}{ccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K[a] & \begin{array}{c} \xrightarrow{p'} \\ \xleftarrow{s'} \end{array} & K[b] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & A_0 & \begin{array}{c} \xrightarrow{p_0} \\ \xleftarrow{s_0} \end{array} & B_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & A_1 & \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{s_1} \end{array} & B_1 & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array} \tag{Diagram 2.13}$$

\square

We now give the conditions for the main result to work:

Theorem 2.14 ([5], theorem 2.6.3). *Let \mathcal{A} be a homological category. If it has coequalizers and satisfies property \mathbf{P} , then $\text{Reg } \mathcal{A}$ is homological with coequalizers and satisfies \mathbf{P} as well.*

Property P: In every commutative diagram of the following shape, where the outer part is a pushout, the inner part a pullback, and all plain arrows are regular epimorphisms, the factorization Φ is an epimorphism and every pullback of it is as well (we say that it is a pullback-stable epimorphism).

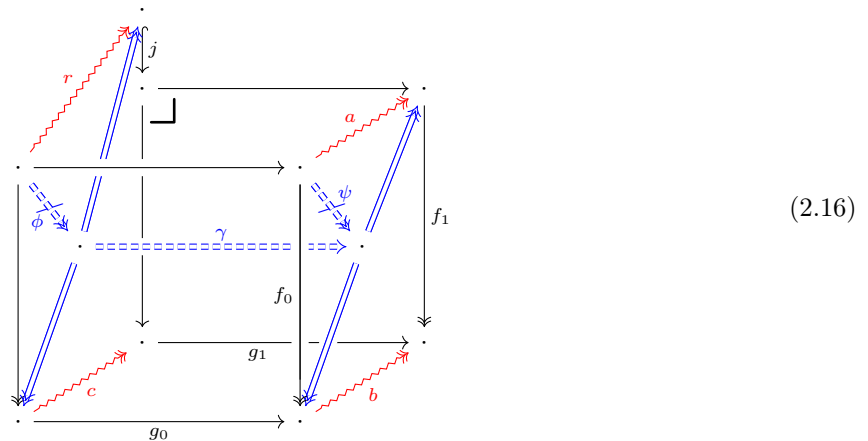


Corollary 2.15. If \mathcal{A} is a semi-abelian category then $\text{Reg } \mathcal{A}$, as well as any number of iterations of taking the category of regular epimorphisms starting from \mathcal{A} , is homological.

Proof. \mathcal{A} satisfies the conditions of theorem 2.14. It has coequalizers (see [7, theorem 5.1.3]) and property **P** is a consequence of the fact that \mathcal{A} is regular and of [6, theorem 5.7]. \square

proof of theorem 2.14. The existence of all finite limits and of coequalizers is implied by their existence in \mathcal{A} and proposition 2.5. The pointed protomodularity of $\text{Reg } \mathcal{A}$ is given by corollary 2.11.

To prove that a pullback of a regular epimorphism in $\text{Reg } \mathcal{A}$ is still a regular epimorphism we are going to use the criterion given by proposition 2.6 along with property **P**.



Let us start by explaining diagram (2.16), which will be the main tool of our proof. First, the morphisms which we see as objects of $\text{Reg } \mathcal{A}$ are denoted by red squiggly arrows. To obtain this diagram we first construct the outer frame as the pullback diagram of f and g . We then take the blue double plain arrows to be the pullbacks in the faces of the cube. We define ϕ and ψ to be the factorization of the faces through their respective pullback. Lastly γ is obtained as the factorization of $g_0 \circ c^*(g^*f)_1$ and $(f_1^*g_1) \circ j \circ (g^*f)_1^*c$ (notice here the difference between pullbacks in \mathcal{A} and in $\text{Reg } \mathcal{A}$). This factorization exists since:

$$b \circ g_0 \circ c^*(g^*f)_1 = g_1 \circ c \circ c^*(g^*f)_1 = g_1 \circ (g^*f)_1 \circ (g^*f)_1^*c = f_1 \circ (f_1^*g_1) \circ j \circ (g^*f)_1^*c.$$

First, we want to prove that ϕ is epimorphic. Using property **P**, we know that ψ is a pullback-stable epimorphism. Thus, if we show that the square given by the dashed blue double arrows and $(f^*g)_0$ is a pullback, we get the result.

Let $u : U \rightarrow C_0, u' : U \rightarrow R_1$ and $v : U \rightarrow A_0$ be such that $\gamma \circ \langle u, u' \rangle = \psi \circ v$ where $\langle u, u' \rangle$ denotes the factorization of u and u' through $C_0 \times_{C_1} R_1$ (we suppose that $c \circ u = (g^*f)_1 \circ u'$). We want to exhibit a unique factorization of the pair $\langle u, u' \rangle, v$ through $\phi, (f^*g)_0$. Notice first that:

$$\begin{aligned} f_0 \circ v &= b^*f_1 \circ \psi \circ v \\ &= b^*f_1 \circ \gamma \circ \langle u, u' \rangle \\ &= g_0 \circ c^*(g^*f)_1 \circ \langle u, u' \rangle \\ &= g_0 \circ u. \end{aligned}$$

Using the front face pullback, this gives us a unique factorization Δ with $(g^*f)_0 \circ \Delta = u$ and $(f^*g)_0 \circ \Delta = v$. These are the unicity and two out of the three commutativity conditions that we need to show.

The last one ($r \circ \Delta = u'$) is equivalent to having both

$$\text{i) } f_1^*g_1 \circ j \circ r \circ \Delta = f_1^*g_1 \circ j \circ u' \text{ and ii) } g_1^*f_1 \circ j \circ r \circ \Delta = g_1^*f_1 \circ j \circ u',$$

since j is a monomorphism and the pullback morphisms form a monomorphic pair. Using $u' = (g^*f)_1^*c \circ \langle u, u' \rangle$ along with the commutativity of the top face of the cube and the definition of γ , we get that i) is equivalent to $f_1^*b \circ \gamma \circ \langle u, u' \rangle = a \circ (f^*g)_0 \circ \Delta$. But now we have $f_1^*b \circ \gamma \circ \langle u, u' \rangle = f_1^*b \circ \psi \circ v = a \circ v = a \circ (f^*g)_0 \circ \Delta$. Equation ii) also simplifies, this time to $c \circ (g^*f)_0 \circ \Delta = c \circ u$, using the commutativity of both squares inside the left face of diagram (2.16), and it is also true. Thus ϕ is indeed an epimorphism.

Let us now show that the left face of diagram (2.16) is a pushout. Notice first that since ϕ is epimorphic, it is equivalent for a pair u, v of morphism with same codomain to have $u \circ r = v \circ (g^*f)_0$ and to have $u \circ (g^*f)_1^*c = v \circ c^*(g^*f)_1$. Thus we have that the outer square of the left face of diagram (2.16) is a pushout if and only if the inner square is. But then the inner square is a pushout due to proposition 1.5, since it is a pullback of regular epimorphisms in a regular category.

Finally let us prove property **P** in $\text{Reg } \mathcal{A}$. The key argument lies in the fact that, given a colimit cocone $(d(i) \xrightarrow{\phi_i} l)_{i \in I}$ in $\text{Reg } \mathcal{A}$, the part indexed by 0 is a colimit in \mathcal{A} . With that knowledge, a pushout of regular epimorphisms in $\text{Reg } \mathcal{A}$ gives rise, with its 0-indexed face, to a pushout of regular epimorphisms in \mathcal{A} . Moreover, the 0-indexed face of any limit in $\text{Reg } \mathcal{A}$ is the limit of the 0-indexed face of the corresponding diagram (see corollary 2.5). Let $\phi = (\phi_0, \phi_1)$ be the factorization of the pushout square through the pullback inside the square. Then, using the previous result, ϕ_0 is the corresponding factorization in \mathcal{A} for the 0-indexed face. Thus, property **P** implies that ϕ_0 is a pullback stable epimorphism in \mathcal{A} . But, in general, if (f_0, f_1) a morphism in $\text{Reg } \mathcal{A}$ is such that f_0 is an epimorphism in \mathcal{A} , then f_1 is too because $f_1a = bf_0$ is, and thus f is epimorphic in $\text{Reg } \mathcal{A}$. Combining the results, we get **P** for $\text{Reg } \mathcal{A}$. \square

Remark. In the hypothesis of the theorem as it appears in [5], \mathcal{A} is not required to have all coequalizers. We found however that without this hypothesis, the proof of the fact that coequalizers of kernel pairs exist in $\text{Reg } \mathcal{A}$ was incomplete, as it failed to explicitly take into account the fact that kernel pairs in $\text{Reg } \mathcal{A}$ are computed differently as in $\text{Arr } \mathcal{A}$. This means that the kernel pair of a morphism in $\text{Reg } \mathcal{A}$ might not be one in $\text{Arr } \mathcal{A}$ and thus might not have a coequalizer. We were not, however, able to find a counterexample.

3 Regular S-filtrations

The goal of this section is to extend the results of the previous one to the more general notion of filtrations.

Convention. For the remainder of this section we assume that S is a partially ordered set seen as a category (see definition 1.17). We also assume that S has an initial object, denoted 0 , and that $S \neq \{0\}$.

Here, S will play the role of the shape of the filtrations. For more practical applications, for example group filtrations, one might chose $S = \mathbb{N}$ or $S = \mathbb{Z} \cup -\infty$ in the following definition:

Definition 3.1 (regular S -filtrations). *Let \mathcal{A} be a category. The category $\text{Fil}(S, \mathcal{A})$ of regular S -filtrations of \mathcal{A} (or regular filtrations of \mathcal{A} of shape S) is the full subcategory of the functor category $[S, \mathcal{A}]$ whose objects F send all morphisms to regular epimorphisms. We will denote, for $F \in [S, \mathcal{A}]$, $F(a \rightarrow b) = F_a^b$.*

Remark. Whenever \mathcal{A} is regular, the condition on F is equivalent to only asking that for all $a \in S$, F_0^a is a regular epimorphism. Indeed, for $a \leq b \in S$, $F_0^b = F_a^b F_0^a$ implies that F_a^b is a regular epimorphism since \mathcal{A} is regular and F_0^b is a regular epimorphism by hypothesis (see [7, A.5.4]).

Example. Regular epimorphisms are a special case of regular S -filtrations for any S . Indeed, a regular epimorphism $f : A \rightarrow B$ can be represented by $F : S \rightarrow \mathcal{A}$ with $F(0) = A$, for all $a \in S \setminus \{0\}$, $F(a) = B$ and $F(g)$ being either f or id depending on the domain and codomain of g .

We will show that $\text{Reg } \mathcal{A}$ and $\text{Fil}(S, \mathcal{A})$ behave very similarly and most results proved in section 2 can be transferred to, or proved similarly in the context of regular S -filtrations, where $[S, \mathcal{A}]$ plays the role of $\text{Arr } \mathcal{A}$. The first result to be transferred in this context is that $\text{Fil}(S, \mathcal{A})$ is a coreflective subcategory of $[S, \mathcal{A}]$ when \mathcal{A} is regular:

Proposition 3.2. *Let \mathcal{A} be a regular category. For every $F \in [S, \mathcal{A}]$, if we take the regular epi mono factorization of every F_0^a to be $\epsilon_{F,a} \circ \mathbf{R}F_0^a$, $\mathbf{R}F$ extends to a regular S -filtration and $\epsilon_F : \mathbf{R}F \rightarrow F$ is a monomorphism in $[S, \mathcal{A}]$. This defines a functor \mathbf{R} which is right adjoint to the inclusion $\mathbf{U} : \text{Fil}(S, \mathcal{A}) \rightarrow [S, \mathcal{A}]$, and ϵ is the counit of this adjunction.*

Proof. We start by showing that $\mathbf{R}F$ is indeed a regular filtration. We have defined $\mathbf{R}F$ only on morphisms $0 \rightarrow a$. We take $\mathbf{R}F_a^b$ to be the lift in the following diagram:

$$\begin{array}{ccc}
 F(0) & \xrightarrow{\mathbf{R}F_0^b} & \mathbf{R}F(b) \\
 \mathbf{R}F_0^a \downarrow & \nearrow \mathbf{R}F_a^b & \downarrow \epsilon_{F,b} \\
 \mathbf{R}F(a) & \xrightarrow{F_a^b \epsilon_{F,a}} & F(b).
 \end{array}$$

With this definition, $\mathbf{R}F$ is a functor since for any $a \leq b \leq c \in S$, then $\mathbf{R}F_b^c \mathbf{R}F_a^b$ is a lift in the diagram defining $\mathbf{R}F_a^c$ (the fact that $\mathbf{R}F$ preserves identities is obvious).

We want to prove the natural bijection $\text{Fil}(S, \mathcal{A})(F, \mathbf{R}G) = [S, \mathcal{A}](F, G)$ where F is a regular S -filtration in \mathcal{A} and $G : S \rightarrow \mathcal{A}$ is a functor. We take the first direction $\tau : \text{Fil } \mathcal{A}(F, \mathbf{R}G) \rightarrow [S, \mathcal{A}](F, G)$ to be composition with ϵ_G . To construct the inverse bijection, we send α to $\omega(\alpha)$, with $\omega(\alpha)_a$ defined as the lift in the following diagram (in particular $\omega(\alpha)_0 = \alpha_0$):

$$\begin{array}{ccc}
 F0 & \xrightarrow{\mathbf{R}G_0^a \alpha_0} & \mathbf{R}Ga \\
 F_0^a \downarrow & \nearrow \omega(\alpha)_a & \downarrow \epsilon_{G,a} \\
 Fa & \xrightarrow{\alpha_a} & Ga.
 \end{array}$$

We check that $\omega(\alpha)$ is natural. Let $a \leq b \in S$, we want $\omega(\alpha)_b F_a^b = (\mathbf{R}G_a^b)\omega(\alpha)_a$. Since F_0^a is an epimorphism, this is equivalent to $\omega(\alpha)_b F_a^b F_0^a = (\mathbf{R}G_a^b)\omega(\alpha)_a F_0^a$. But by definition of $\omega(\alpha)$ we have:

$$\mathbf{R}G_a^b \omega(\alpha)_a F_0^a = \mathbf{R}G_a^b \mathbf{R}G_0^a \alpha_0 = \mathbf{R}G_0^b \alpha_0 = \omega(\alpha)_b F_a^b F_0^a.$$

Proving that this defines a bijection, that it is natural, along with the definition of \mathbf{R} on morphisms, and proving that ϵ is the counit, is done in the same way as in the proof of proposition 2.3. \square

Remark 3.3. The proof of the naturality of $\omega(\alpha)$ holds for any transformation β between regular S -filtrations that is natural on the morphisms $0 \rightarrow a$. This is used later for proposition 3.5.

The start of this section can be summed up in commutative diagram (3.4), where \mathcal{A} is a regular category.

$$\begin{array}{ccc} \text{Fil}(S, \mathcal{A}) & \begin{array}{c} \xrightarrow{\mathbf{U}} \\ \perp \\ \xleftarrow{\mathbf{R}} \end{array} & [S, \mathcal{A}] \\ \prod_{a \in S} (-)_0^a \downarrow & & \downarrow \prod_{a \in S} (-)_0^a \\ \prod_{a \in S} \text{Reg } \mathcal{A} & \begin{array}{c} \xrightarrow{\mathbf{U}} \\ \perp \\ \xleftarrow{\mathbf{R}} \end{array} & \prod_{a \in S} \text{Arr } \mathcal{A} \end{array} \quad (\text{Diagram 3.4})$$

Proposition 3.5. *Let \mathcal{A} be a regular category. In diagram (3.4), the vertical functors are faithful. Moreover, the left one is full and injective on objects.*

Proof. See appendix A \square

The following results give some additional properties when \mathcal{A} is pointed. Namely, corollary 3.7 says that it reflects and preserves limits and colimits and corollary 3.8 shows that it preserves and reflects regular epimorphisms.

We first give the characterization of limits and colimits in $\text{Fil}(S, \mathcal{A})$:

Corollary 3.6. *Let \mathcal{A} be a regular category. The category $\text{Fil}(S, \mathcal{A})$ has all limits. Moreover, if $D : I \rightarrow \text{Fil}(S, \mathcal{A})$ is a diagram and $(L \xrightarrow{\phi_i} D(i))_{i \in I}$ a limit cone in $[S, \mathcal{A}]$, then:*

- *it has a limit cone in $\text{Fil}(S, \mathcal{A})$ given by $(\mathbf{R}L \xrightarrow{\phi_i \circ \epsilon_L} D(i))_{i \in I}$.*
- *it has a colimit cocone in $\text{Fil}(S, \mathcal{A})$ exactly when it has one in $[S, \mathcal{A}]$ and when it does, it is the same in both categories. Moreover, if \mathcal{A} is pointed, it is always computed pointwise.*

Proof. This is a consequence of propositions 1.16, 3.2 and 1.18. \square

Notice the similarity between this result and corollary 2.5. In fact, when \mathcal{A} is pointed, we can reformulate corollary 3.6:

Corollary 3.7. *Let \mathcal{A} be a pointed regular category, let $D : I \rightarrow \text{Fil}(S, \mathcal{A})$ be a diagram. A cone $(F \xrightarrow{\phi_i} D(i))_{i \in I}$ (resp. a cocone $(D(i) \xrightarrow{\psi_i} G)_{i \in I}$) is a limit cone (resp. a colimit cocone) in $\text{Fil}(S, \mathcal{A})$ exactly when $(F_0^a \xrightarrow{(\phi_{i,0}, \phi_{i,a})} D(i)_0^a)_{i \in I}$ (resp. $(D_0^a \xrightarrow{(\psi_{i,0}, \psi_{i,a})} G_0^a)_{i \in I}$) is one in $\text{Reg } \mathcal{A}$ for all $a \in S$.*

Remark. Note that \mathcal{A} pointed isn't actually needed for limits since \mathcal{A} has all small limits.

Corollary 3.8. *Let \mathcal{A} be a pointed regular category. A morphism α in $\text{Fil}(S, \mathcal{A})$ is a regular epimorphism if and only if for all $a \in S$, (α_0, α_a) is a regular epimorphism in $\text{Reg } \mathcal{A}$.*

Proof. By the previous result, a kernel pair π, π' of α in $\text{Fil}(S, \mathcal{A})$ gives by taking $(\pi_0, \pi_a), (\pi'_0, \pi'_a)$ a kernel pair of all the (α_0, α_a) in $\text{Reg } \mathcal{A}$. We get, then:

$$\begin{aligned} \alpha \text{ is a regular epimorphism} &\Leftrightarrow \alpha \text{ is a coequalizer of } \pi, \pi' \\ &\Leftrightarrow \forall a \in S, (\alpha_0, \alpha_a) \text{ is a coequalizer in } \text{Reg } \mathcal{A} \text{ of } (\pi_0, \pi_a), (\pi'_0, \pi'_a) \quad (\text{coro 3.7}) \\ &\Leftrightarrow \forall a \in S, (\alpha_0, \alpha_a) \text{ is regular in } \text{Reg } \mathcal{A}. \end{aligned}$$

□

This allows us to extend the result of section 2, indeed, we have:

Theorem 3.9. *Let \mathcal{A} be a homological category which satisfies property **P** and has coequalizers. Then $\text{Fil}(S, \mathcal{A})$ is homological, satisfies property **P** and has coequalizers as well.*

Proof. The functor $0 : S \rightarrow \mathcal{A}$ which sends all objects to $0 \in \mathcal{A}$ and all morphisms to id_0 is a zero object in $\text{Fil}(S, \mathcal{A})$. The existence of all small limits and of coequalizers is guaranteed by corollary 3.6 and their existence in \mathcal{A} . The stability of regular epimorphisms under pullbacks is a consequence of corollaries 3.7, 3.8 and theorem 2.14. The split short five lemma can be retrieved from the one in $\text{Reg } \mathcal{A}$ using the left functor in diagram 3.4, since it reflects and preserves split epimorphisms, kernels and isomorphisms thanks to results 3.5 and 3.7. And lastly, property **P** for $\text{Fil}(S, \mathcal{A})$ is proved in the same way as in the proof of theorem 2.14. □

Remark. Some of the work done in that section could have been done with S any small category with an initial object. It isn't, however, more general since any pair of morphisms $u, v : a \rightarrow b$ in S is sent by any F in $\text{Fil}(S, \mathcal{A})$ to the same morphism. Indeed $F(u)F(0 \rightarrow a) = F(0 \rightarrow b) = F(v)F(0 \rightarrow a)$ and $F(0 \rightarrow a)$ is an epimorphism. Similar reasoning shows that morphisms $a \xrightarrow[u]{v} b$ are sent to an inverse pair of isomorphisms.

We finish this section by saying that the theorems of this section and of the previous one can be transferred to normal monomorphisms (i.e. kernels) and to the more classical notion of normal S -filtrations, where instead S has a final object and we look at functors who send every morphism to normal monomorphisms. This is due to the fact that there is an equivalence of category between the category of regular filtrations of shape S and the category of normal filtrations of shape S' which is defined to be S without its initial object and with an added terminal object 1. It is defined for a regular S -filtration F by sending it to G with G_a^1 defined to be the kernel of F_0^a . The other morphisms are constructed using universal properties of kernels. The other direction of the equivalence is defined similarly using cokernels. Indeed, in a homological category, being a regular epimorphism is equivalent to being the cokernel of some morphism (we call these normal epimorphisms). See [11] for the result.

Some of the results could have been proved directly in this context, when \mathcal{A} is homological with coequalizers, since there is a reflector **N** of $\text{Arr } \mathcal{A}$ on $\text{NMon } \mathcal{A}$ the category of normal monomorphisms in \mathcal{A} given by sending f to the kernel of the cokernel of f . We could not, however, find a nice characterization of regular epimorphisms in this category. If \mathcal{A} is semi abelian though, regular epimorphisms are just described pointwise, using proposition 1.13. We will use those results later, in section 4.

4 Central and strongly central filtrations

This section will focus on central filtrations and strongly central filtrations. Here, filtration will be taken using the normal filtrations point of view mentioned at the end of section 3, with shape $\mathbb{N}^{*,op}$.

Before diving into technical definitions, let us start with the case of Gr. Let then G be a normal filtration in groups. Since the $G(n)$ are isomorphic to normal subgroups of $G(1)$ and G sends all morphisms to ones that are isomorphic to subgroup inclusions, we just assume that $G(n)$ are indeed subgroups of $G(1)$. With that, we say that G is central if and only if for all $i \in \mathbb{N}^*$, $[G(i), G(1)] \subset G(i+1)$, where $[-, -]$ denotes the usual commutator of subgroups. We say that G is strongly central if for all $i, j \in \mathbb{N}^*$, $[G(i), G(j)] \subset G(i+j)$.

For the rest of this section we will use the categorical notion of subobjects developed in the appendix in section C. One can skip reading this section by restricting oneself to the case of groups, where subobjects are just subgroups, normal subobjects are normal subgroups, and f_* (resp. f^*) denote the normal subgroup generated by the direct image by f (resp. the preimage). the notation $\text{Sub } \mathcal{A}$ is then set of subgroups of A , and $\text{NSub } \mathcal{A}$ is the set of normal subgroups. Meets and joins for the inclusion in them are defined as usual.

It is worth noting that with those definitions, a normal filtration F in \mathcal{A} can be seen as an object $F = F(1)$ along with a sequence of normal subobjects of F $(F(n) \subset F)_{n \in \mathbb{N}^*}$ such that for all $n \in \mathbb{N}^*$, $F(n+1) \subset F(n)$, giving us the following convention:

Convention. A normal filtration will sometimes be given in the rest of this work by a sequence $(A_n)_{n \in \mathbb{N}^*}$, with $A_n \subset A$, where A is equal to A_1 and the A_n are normal subobjects. We also require that, for every $n \in \mathbb{N}^*$, $A_{n+1} \subset A_n$.

Remark 4.1. This allows to see morphisms of filtrations differently. Indeed, given (A_n) and (B_n) normal filtrations in \mathcal{A} , and $(f_n : A_n \rightarrow B_n)_{n \in \mathbb{N}^*}$ forming a morphism of normal filtrations then, (f_n) can be given only by $f = f_1$, since the inclusions of subobjects are monomorphic and thus f_n is uniquely determined by $f_1 i_n = i'_n f_n$, where i and i' denote the inclusion of subobjects. Conversely, a morphism $f : A_1 \rightarrow B_1$ can be interpreted as a morphism of filtrations as long as $f_*(A_n) \subset B_n$ for every $n \in \mathbb{N}^*$. The morphism f_n is then defined as $f|_{A_n}^{B_n}$.

Now on to a general notion of commutators:

Definition 4.2 (Huq commutator, [12]). *Let \mathcal{A} be a pointed category with finite limits and colimits. Let $A \in \mathcal{A}$ and $X, Y \subset A$. We define the Huq commutator of X and Y , denoted in this work $[X, Y]$, as the kernel of g in the following diagram, where G is the colimit of the outer frame:*

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow & \vdots a & \searrow & \\
 & (0,1) & \downarrow & & \\
 X \times Y & \xrightarrow{h} & G & \xleftarrow{g} & A \\
 & \swarrow & \vdots b & \searrow & \\
 & (0,1) & \uparrow & & \\
 & & Y & &
 \end{array}$$

(Diagram 4.3)

Example. In Gr, this commutator coincides with the normal subgroup generated by the usual commutator. Both are equal when we consider the commutator of two normal subgroups, which is often the case in this work. This justifies the use of the same notation.

By definition it is clear that the Huq commutator of two subobjects of A in \mathcal{A} a category with finite limits and colimits is a normal subobject of A . One can also check that the Huq commutator $\text{Sub}(A) \times \text{Sub}(A) \rightarrow \text{Sub}(A)$ is functorial in both variables (i.e. it is increasing in both variables). The proof of that is similar to the one of lemma 4.6 that we will do later.

The first result we want to give though is the following lemma regarding g , which will be useful in the rest of the section.

Lemma 4.4 ([12]). *Let \mathcal{A} be a homological category with finite colimits and $X, Y \subset A$. The morphism g in diagram (4.3) is a regular epimorphism.*

Proof. See appendix A □

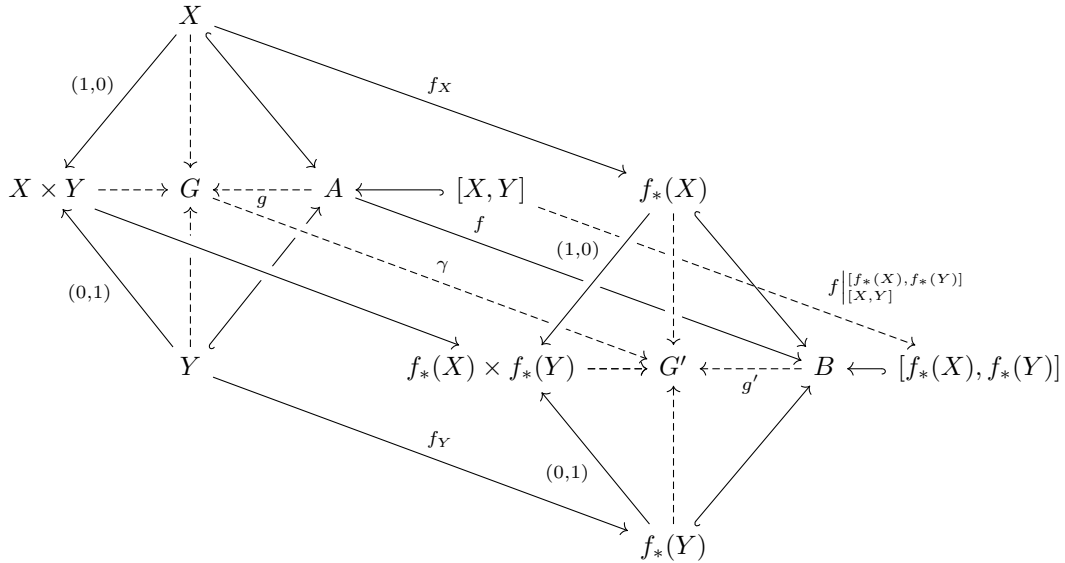
We can extend the definition of central filtrations given in the case of groups to a more general case.

Definition 4.5 (central and strongly central filtrations). *Let \mathcal{A} be a pointed category with finite limits and colimits. A normal filtration $(A_n)_{n \in \mathbb{N}^*}$ in \mathcal{A} is said to be central whenever for all $n \in \mathbb{N}^*$, $[A_n, A] \subset A_{n+1}$. It is said to be strongly central whenever for all $i, j \in \mathbb{N}^*$, $[A_i, A_j] \subset A_{i+j}$. This defines the categories $\text{CFil } \mathcal{A}$ and $\text{SCFil } \mathcal{A}$ as the full subcategories of normal filtrations in \mathcal{A} that are central and strongly central respectively.*

We will see that central and strongly central filtrations are both reflective subcategories of normal filtrations. For that we will need a lemma first on the direct image of commutators.

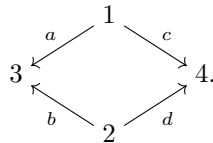
Lemma 4.6. *Let \mathcal{A} be a pointed regular category with finite colimits. Let $f : A \rightarrow B$ in \mathcal{A} and $X, Y \subset A$. We have the following inclusion of normal subobjects: $f_*([X, Y]) \subset [f_*(X), f_*(Y)]$*

Proof. We construct the Huq commutator of X and Y and of $f_*(X)$ and $f_*(Y)$ in diagram (4.7).



(Diagram 4.7)

Let us denote I the category whose objects and morphisms are represented by the following diagram:



Let also $D : I \rightarrow \mathcal{A}$ (resp. D') denote the diagram defining G (resp. G'), and $(\phi_i : D(i) \rightarrow G)_{i \in I}$ (resp. ϕ') denote the corresponding colimit cocone. We then get a natural transformation $\alpha : D \Rightarrow D'$ given by f , its restrictions on X and Y and their product, as in diagram (4.7). The naturality condition for α given by c and d is just definition of restrictions, and the ones for a and b can be gotten by composing with the projections of the product $f_*(X) \times f_*(Y)$. But now we know that $\phi'\alpha$ is a cocone on D , and thus, we get the factorisation $\phi'\alpha = \gamma\phi$, with $\gamma : G \rightarrow G'$. With that, $g'f|_{[X,Y]} = \gamma g|_{[X,Y]} = 0$ since, by definition $g|_{[X,Y]} = 0$. This shows that $f_*([X,Y]) \subset [f_*(X), f_*(Y)]$ since the latter is the kernel of g' . \square

Proposition 4.8. *Let \mathcal{A} be a pointed, regular category with finite colimits. Let $(A_n)_{n \in \mathbb{N}^*}$ be a normal filtration and define recursively $\mathbf{C}(A_n)_{n \in \mathbb{N}^*}$ by $\mathbf{C}A = A$ and for all $n \in \mathbb{N}^*$, $\mathbf{C}A_{n+1} = A_{n+1} \vee [A, \mathbf{C}A_n]$. This defines a reflector of $\mathbf{N}\text{Fil}(\mathcal{A})$ the category of normal filtrations on $\mathbf{C}\text{Fil}\mathcal{A}$. Its unit at $(A_n)_{n \in \mathbb{N}^*}$, η_A is the pointwise inclusion $A_n \subset \mathbf{C}A_n$. Taking $\mathbf{S}\mathbf{C}A = A$ and $\mathbf{S}\mathbf{C}A_{n+1} = A_{n+1} \vee \bigvee_{i=1}^n [\mathbf{S}\mathbf{C}A_i, \mathbf{S}\mathbf{C}A_{n+1-i}]$ defines a reflector of $\mathbf{N}\text{Fil}(\mathcal{A})$ on $\mathbf{S}\mathbf{C}\text{Fil}\mathcal{A}$.*

Proof. We prove this result for central filtrations, as the proof is similar for strongly central filtrations. Also, as before, we will focus on the bijection $\mathbf{C}\text{Fil}\mathcal{A}(\mathbf{C}A, B) = \mathbf{N}\text{Fil}\mathcal{A}(A, B)$ for given $A \in \mathbf{N}\text{Fil}\mathcal{A}, B \in \mathbf{C}\text{Fil}\mathcal{A}$, as the rest of the proof is similar to what is done in the proof of lemma 2.3.

We begin by proving that $\mathbf{C}(A_n)_{n \in \mathbb{N}^*}$ is a filtration, the fact that it is central is clear, as is the fact that the A_n are normal subobjects of A since we take the normal join. So we only need to prove $\mathbf{C}A_{n+1} \subset \mathbf{C}A_n$ which we do by recurrence. For $n = 1$ the result is clear, then suppose it for n arbitrary. We have $[A, \mathbf{C}A_{n+1}] \subset [A, \mathbf{C}A_n] \subset \mathbf{C}A_{n+1}$ by hypothesis and functoriality of the commutator. Then we get $\mathbf{C}A_{n+2} \subset \mathbf{C}A_{n+1}$ since it is the join of two subobjects of $\mathbf{C}A_{n+1}$.

Let, then, $A \in \mathbf{N}\text{Fil}(\mathcal{A})$ and $B \in \mathbf{C}\text{Fil}\mathcal{A}$. We get $\tau : \mathbf{C}\text{Fil}\mathcal{A}(\mathbf{C}A, B) \rightarrow \text{Fil}(A, B)$ by sending f to $f\eta_A$. For $\omega : \text{Fil}(A, B) \rightarrow \mathbf{C}\text{Fil}\mathcal{A}(\mathbf{C}A, B)$, let $f : A \rightarrow B$ be a morphism of normal filtrations. By the previous lemma, we have for all $n \in \mathbb{N}^*$, $f_*([A, \mathbf{C}A_n]) \subset [f_*(A), f_*(\mathbf{C}A_n)]$. With that knowledge, let us prove recursively that for all $n \in \mathbb{N}^*$, $f_*(\mathbf{C}A_n) \subset B_n$. For $n = 1$ this is a direct consequence of $\mathbf{C}A_1 = A_1$. Suppose the result known for $n \in \mathbb{N}^*$, we then have:

$$\begin{aligned}
f_*(\mathbf{C}A_{n+1}) &= f_*(A_{n+1} \vee [A, \mathbf{C}A_n]) \\
&= f_*(A_{n+1}) \vee f_*([A, \mathbf{C}A_n]) && (f_* \text{ preserves joins}) \\
&\subset B_{n+1} \vee [f_*(A), f_*(\mathbf{C}A_n)] \\
&\subset B_{n+1} \vee [B, B_n] && ([-, -] \text{ increases and recursion hypothesis}) \\
&\subset B_{n+1}. && (B \text{ is central})
\end{aligned}$$

Thus, f_1 defines by restriction a morphism $\omega f : \mathbf{C}A \rightarrow B$, such that $\tau\omega$ and $\omega\tau$ are the identity since they both preserve f_1 . \square

With this, along with proposition 1.16 and the reflector of $[\mathbb{N}^{*,op}, \mathcal{A}]$ on $\mathbf{N}\text{Fil}$, we have the existence of all colimits and limits in $\mathbf{C}\text{Fil}\mathcal{A}$, with \mathcal{A} a pointed regular category with finite colimits. Limits are thus computed pointwise and colimits as the centralization of the normalization of the pointwise colimit.

Our last step before proving any sort of transfer of structure from \mathcal{A} to $\mathbf{C}\text{Fil}\mathcal{A}$ will be to characterize regular epimorphisms, for which we need the following result:

Lemma 4.9. *Let \mathcal{A} be a semi abelian category and $f : A \rightarrow B$ be a regular epimorphism in \mathcal{A} . For any two X, Y normal subobjects of A , then $f_*([X, Y]) = [f_*(X), f_*(Y)]$.*

Proof. See appendix A. □

We are now able to characterize regular epimorphisms in central and strongly central filtrations:

Proposition 4.10. *Let \mathcal{A} be a semi-abelian category. A morphism $f : A \rightarrow B$ between (strongly) central filtrations in \mathcal{A} is regular if and only if for all $n \in \mathbb{N}^*$, f_n is regular. That is, when f seen as a morphism $A \rightarrow B$ in \mathcal{A} is regular and for all $n \in \mathbb{N}^*$, $f_*(A_n) = B_n$.*

Proof. First note that the equivalence between the two conditions is just a consequence of proposition 1.13. Now, see that f is a regular epimorphism if and only if it can be written $f = \eta_{B'}g$ where η is defined as in proposition 4.8 and $g : A \rightarrow B'$ is a regular epimorphism in of normal filtrations, so for all $n \in \mathbb{N}^*$, $B'_n = g_*(A_n)$ (here g_* is both the normal and regular image due to proposition 1.13). Using the previous result we get that for all $n \in \mathbb{N}^*$, $[B', B'_n] = [g_*(A), g_*(A_n)] = g_*[A, A_n] \subset g_*(A_{n+1}) = B'_{n+1}$ since A is central. This proves that B' is central, and thus $g = f$ so f is pointwise a regular epimorphism.

Conversely, pointwise regular epimorphisms in $\text{CFil } \mathcal{A}$ are regular epimorphisms in this category because they are the coequalizer of their kernel pair. Indeed, they are the pointwise coequalizer of their kernel pair in $\text{CFil } \mathcal{A}$ and are themselves in $\text{CFil } \mathcal{A}$. The proof for strongly central filtrations is similar to this one. □

This brings us to our last transfer result:

Theorem 4.11. *Let \mathcal{A} be a semi-abelian category. The categories $\text{CFil } \mathcal{A}$ and $\text{SCFil } \mathcal{A}$ are homological with all small colimits.*

Proof. We prove the result for $\text{CFil } \mathcal{A}$, the proof is the same for SCFil . The filtration $(0)_{n \in \mathbb{N}^*}$ is a zero object in $\text{CFil } \mathcal{A}$. Small limits and colimits exist in this category as stated under proposition 4.8. Since pullbacks are defined pointwise, as well as regular epimorphisms, due to the previous result, the pullback of a regular epimorphism is pointwise a regular epimorphism since \mathcal{A} is regular, and so it is a regular epimorphism. Lastly, since split epimorphisms, kernels and isomorphisms are defined pointwise in $\text{CFil } \mathcal{A}$, the proof of the split short five lemma is done just as in corollary 2.11. □

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Proof of proposition 1.16. First let us write, for any $B \in \mathcal{B}, A \in \mathcal{A}$ $\mathcal{B}(\mathbf{R}A, B) \xrightleftharpoons[\omega]{\tau} \mathcal{A}(A, \mathbf{U}B)$ the natural bijections of the adjunction $\mathbf{R} \dashv \mathbf{U}$.

We begin by proving the remark: $\forall A \in \mathcal{B}, \mathbf{R}(\mathbf{U}A) \cong A$. Let $A \in \mathcal{B}$. We show that A and $\mathbf{R}\mathbf{U}A$ represent the same functor. For that let $B \in \mathcal{B}$, we have $\mathcal{B}(A, B) \cong \mathcal{A}(\mathbf{U}A, \mathbf{U}B) \cong \mathcal{B}(\mathbf{R}\mathbf{U}A, B)$ where the first bijection uses the fact that \mathcal{B} is a full subcategory and the second is just ω . In fact, the isomorphism between A and $\mathbf{R}\mathbf{U}A$ is given by the counit of the adjunction, so it is natural. Since \mathbf{R} is a left adjoint, it preserves colimits and if we have a diagram D in \mathcal{B} with C colimit of $\mathbf{U}D$ in \mathcal{A} , then \mathbf{R} is a colimit of $\mathbf{R}\mathbf{U}D$ in \mathcal{B} . But by the remark D and $\mathbf{R}\mathbf{U}D$ are isomorphic so that $\mathbf{R}C$ is also a colimit of D in \mathcal{B} .

Let now $D : I \rightarrow \mathcal{B}$ be a diagram. We wish to prove it has a limit in \mathcal{B} exactly when $\mathbf{U}D$ has one in \mathcal{A} and they one is sent to the other via \mathbf{U} when they exist. Suppose, then, that D has a limit cone Ψ in \mathcal{B} . Since \mathbf{U} is a right adjoint, it preserves limits and $\mathbf{U}\Psi$ a limit of $\mathbf{U}D$ in \mathcal{A} .

Suppose now that $\mathbf{U}D$ has a limit cone $(L \xrightarrow{\psi_i} \mathbf{U}D(i))_{i \in I}$ in \mathcal{A} . We prove that $L \cong \mathbf{U}\mathbf{R}L$. This implies that $\mathbf{U}\mathbf{R}L$ is a limit of $\mathbf{U}D$ in \mathcal{A} and thus $\mathbf{R}L$ is a limit of D in \mathcal{B} since \mathbf{U} is fully faithful. We get a cone on D in \mathcal{B} by taking $\omega(\psi)$. It is a cone since ω is natural in the second variable. We then get the factorization $\mathbf{U}\omega(\psi) = \psi j$ in \mathcal{A} , using the fact that ψ is a limit cone. But then $\mathbf{U}\omega(\psi)\eta_L = \tau\omega\psi = \psi$ so $\psi = \psi j\eta_L$. Since ψ is a limit cone, we get $j\eta_L = id_L$. The last step of this proof is getting the equality $\eta_L j = id_{\mathbf{U}\mathbf{R}L}$. It is equivalent to $\tau f = \tau id_{\mathbf{R}L}$, with f unique such that $\mathbf{U}f = \eta_L j$, since \mathbf{U} is fully faithful. But $\tau id_{\mathbf{R}L} = \eta_L$ by definition and $\tau f = \eta_L j\eta_L = \eta_L$ using the previous equality. \square

Proposition A.6 (proposition 1.18). *Let \mathcal{A} be a pointed category and S a partially ordered set as a category. Limits and colimits in $[S, \mathcal{A}]$ are always computed pointwise, that is if $(L \xrightarrow{\psi_i} D(i))_{i \in I}$ is a limit cone then for every s in S , $(L(s) \xrightarrow{\psi_{i,s}} D(i)(s))_{i \in I}$ is as well.*

Proof of proposition 1.18. Let \mathcal{A} be a pointed category, S a partially ordered set and $(L \xrightarrow{\psi_i} D(i))_{i \in I}$ a limit cone in $[S, \mathcal{A}]$. Let $s \in S$ and $(c \xrightarrow{\phi_i} D(i)(s))_{i \in I}$ a cone. We construct a cone $(C' \xrightarrow{\phi'_i} D(i))_{i \in I}$ by imposing $C'(a) = c$ if $a \geq s$ and $C'(a) = 0$ otherwise. This uniquely determines C' on morphisms ($C'(a \rightarrow b) = id_c$ if $s \leq a$ and $C'(f) = 0$ otherwise) without any issue since if $a \leq b \leq c$ then either $a \geq s$ and $C'(a \rightarrow c) = id_c = C'(b \rightarrow c)C'(a \rightarrow b)$ or $C'(a \rightarrow c) = 0 = C'(b \rightarrow c)C'(a \rightarrow b)$. Moreover we define $\phi'_{i,a} = D(i)(s \rightarrow a)\phi_i$ if $a \geq s$ and $\phi'_{i,a} = 0$ else. This ensures that for every i in I , ϕ'_i is natural. Now the naturality of ϕ' is gotten from the naturality of ϕ and of the morphisms $D(i \rightarrow j)$.

From this we get, since L is a limit, a factorization $\phi' = \psi\gamma$ and we get $\phi = \psi_{-,s}\gamma_s$. Moreover, given any factorization $\phi = \psi_{-,s}f$ we get a factorization $\phi' = \psi\alpha$ with $\alpha_s = f$ by taking $\alpha_a = L(s \rightarrow a)f$ if $a \geq s$ and $\alpha_a = 0$ otherwise. Indeed we have for any i in I and $a \geq s$ in A ,

$$\psi_{i,a}\alpha_a = \psi_{i,a}L(s \rightarrow a)f = D(i)(s \rightarrow a)\psi_{i,s}f = D(i)(s \rightarrow a)\phi_i = \phi'_{i,a},$$

for $a \not\geq s$, the equality is trivial. This association $f \mapsto \alpha$ is injective, thus, since the factorization $\phi' = \psi\gamma$ is the unique one, then $\phi = \psi_{-,s}\gamma_s$ is unique as well, and $\psi_{-,s}$ is a limit cone. \square

Proposition A.7 (proposition 3.5). *Let \mathcal{A} be a regular category. In diagram (A.8), the vertical functors are faithful. Moreover, the left one is full and injective on objects.*

$$\begin{array}{ccc}
& \xrightarrow{\mathbf{U}} & \\
\text{Fil}(S, \mathcal{A}) & \perp & [S, \mathcal{A}] \\
& \xleftarrow{\mathbf{R}} & \\
\prod_{a \in S} (-)_0^a \downarrow & & \downarrow \prod_{a \in S} (-)_0^a \\
\prod_{a \in S} \text{Reg } \mathcal{A} & \perp & \prod_{a \in S} \text{Arr } \mathcal{A} \\
& \xleftarrow{\mathbf{R}} &
\end{array}
\quad (\text{Diagram A.8})$$

Proof of proposition 3.5. The first statement is just saying that a natural transformation α in $[S, \mathcal{A}]$ is uniquely determined by all the α_a . For the injectivity on objects of the left functor, this comes from the fact that F_a^b is uniquely determined by F_0^a and F_0^b using $F_0^b = F_a^b F_0^a$ since F_0^a is epimorphic. For the fullness, let $(\alpha_a : Fa \rightarrow Ga)_{a \in S}$ be such that $\alpha_a F_0^a = G_0^a \alpha_0$ for every a , with F and G regular S -filtrations in \mathcal{A} . We get that α is natural using remark 3.3. \square

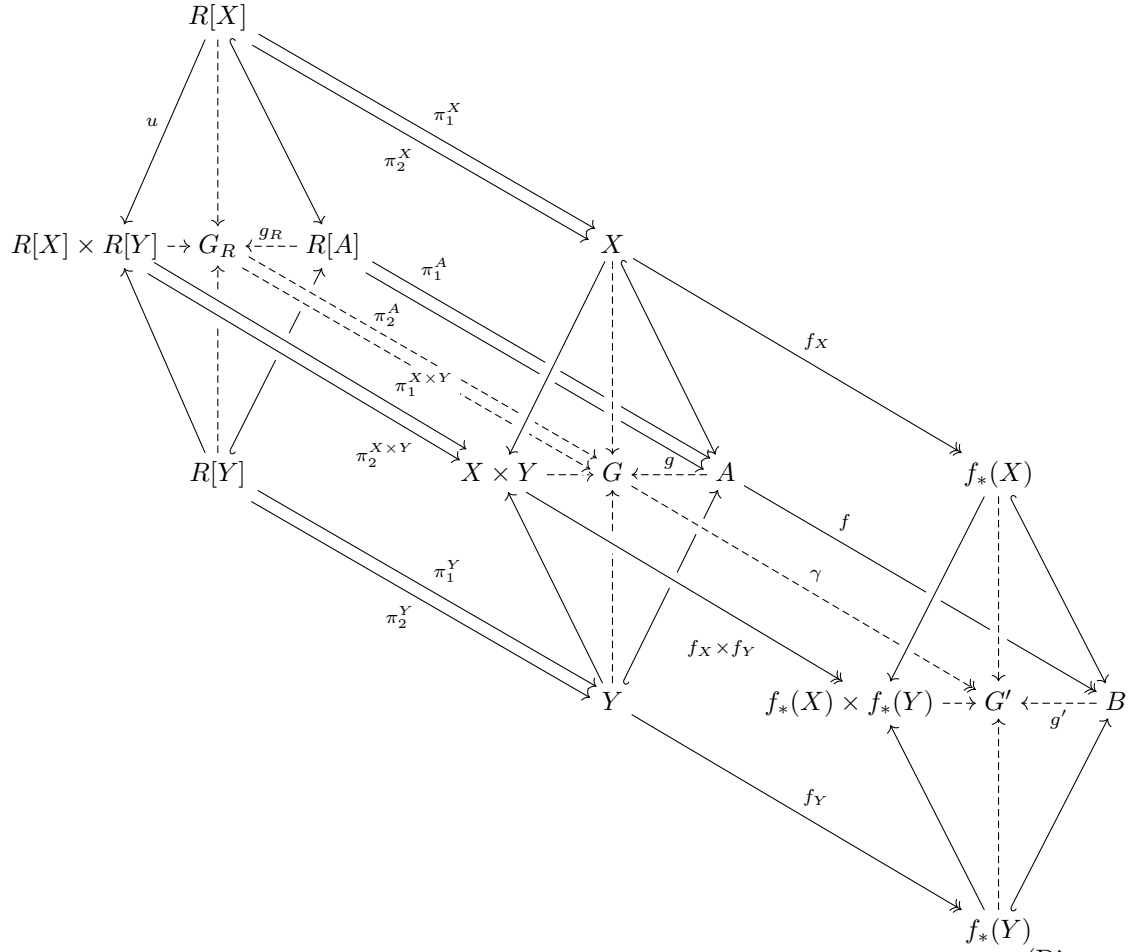
Lemma A.9 ([12], lemma 4.4). *Let \mathcal{A} be a homological category with finite colimits and $X, Y \subset A$. The morphism g in diagram (A.10) is a regular epimorphism.*

$$\begin{array}{ccccc}
& & X & & \\
& \swarrow (0,1) & \vdots a & \searrow & \\
X \times Y & \xrightarrow{h} & G & \xleftarrow{g} & A \\
& \swarrow (0,1) & \vdots b & \searrow & \\
& & Y & &
\end{array}
\quad (\text{Diagram A.10})$$

Proof of lemma 4.4. We begin with the fact that $(0, 1), (1, 0)$ form a strongly epimorphic pair of morphisms. This means that the only subobject of $X \times Y$ they factor through is $X \times Y$ itself, which we reformulate saying they don't factor through any proper subobject of $X \times Y$. This is essentially a consequence of [7, Proposition 3.1.18]. We then use [7, Proposition A.4.18] to see that the colimit cocone family on G is strongly epimorphic. A proof similar to that of [7, Proposition A.4.5] shows that, since $(0, 1), (1, 0)$ and the colimit cocone are strongly epimorphic families, then the family $\{g, a, b, h(0, 1), h(1, 0)\} = \{g, a, b\}$ is strongly epimorphic. But this implies that g is strongly epimorphic hence regular since \mathcal{A} is regular (see [7, Corollary A.5.4]). Indeed, a monomorphism with codomain G jointly factors g, a, b if and only if it factors g since a and b factor through g themselves. \square

Lemma A.11 (4.9). *Let \mathcal{A} be a semi abelian category and $f : A \rightarrow B$ be a regular epimorphism in \mathcal{A} . For any two X, Y normal subobjects of A , then $f_*([X, Y]) = [f_*(X), f_*(Y)]$.*

Proof. Start by taking diagram (A.12), in which the two right diamonds define Huq commutators. In this diagram, define the left diamond to be the pointwise kernel pair of the morphism from the middle diamond to the right diamond. Define the morphisms in it using universal properties of kernel pairs such that all squares between the first and second diamond commute.



(Diagram A.12)

Notice first that the morphism $R[X] \rightarrow R[A]$ is a monomorphism. Indeed, since limits commute with limits, one can obtain the kernel pair of $R[X] \rightarrow R[A]$ by taking the kernel pairs of the inclusions $X \subset A$ and $f_*(X) \subset B$, and taking the kernel pair of the morphism between them. One obtains that id, id is a kernel pair of $R[X] \rightarrow R[A]$ since it is one for $X \rightarrow A$, and thus, $R[X] \rightarrow R[A]$ is a monomorphism (see [10, 5.1.32]). We could also have gotten that result using corollary B.3 in the appendix. Next, we get, thanks to commutativity of limits, that the leftmost object of diagram (A.12) is indeed $R[X] \times R[Y]$ since both are equal to $R[X \times Y]$. Moreover $p_X \pi_1^{X \times Y} = \pi_1^X p_{R[X]}$ where p_X denotes the projection of the product on X , (we also have similar conditions for Y and for π_2). This implies that the morphism $u : R[X] \rightarrow R[X] \times R[Y]$ is indeed $(1, 0)$. To see this, compose it with $p_{R[X]}, p_{R[Y]}$ then with π_1^X, π_2^X and π_1^Y, π_2^Y , which are all monomorphic pairs and use the previous commutativity results along with the commutativity conditions in the definition of u to get $\pi_1^X p_{R[X]} u = p_X(1, 0) \pi_1^X = \pi_1^X = \pi_1^X p_{R[X]}(1, 0)$ and so on. We then define $q_1^G, q_2^G : G_R \rightarrow G$ and $\gamma : G \rightarrow G'$ using universal properties of the colimits (notice that G_R and q_1^G, q_2^G don't necessarily define the kernel pair of γ since colimits don't commute with limits in general).

Since \mathcal{A} is semi-abelian, $f_X = f|_X^{f_*(X)}$ and f_Y are regular epimorphisms (see proposition 1.13), and $f_X \times f_Y$ too, since products of regular epimorphisms are regular in a regular category (see [7, lemma A.5.2]). Thus, the diagram on the right is the coequalizer in $[I, \mathcal{A}]$ of π_1, π_2 , with the coequalizer morphisms being the ones

in diagram (A.12). Since colimits commute with colimits, this shows that γ is the coequalizer of q_1^G and q_2^G . Moreover, since \mathcal{A} is semi abelian, we know that g, g' and g_R are regular epimorphisms due to lemma 4.4.

Take a look now at the cokernel h of the normal inclusion $f_*([X, Y]) \subset B$. We wish to show that $h = \phi g'$ for some ϕ . If we do then $hi = 0$ where $i : [f_*(X), f_*(Y)] \rightarrow B$ is the inclusion. Thus since $f_*([X, Y])$ is the kernel of h , we have $[f_*(X), f_*(Y)] \subset f_*([X, Y])$. That, along lemma 4.6, would prove the result.

Let us then show that h factors through g' . first notice that, if we define j to be the inclusion $[X, Y] \subset A$ then $h f j = h i f \Big|_{[X, Y]}^{f_*([X, Y])} = 0$. Since g is the cokernel of j , we get a factorization $h f = \psi g$. We want to show that ψ factors through γ , thus we will show that ψ coequalizes q_1^G and q_2^G . We have:

$$\psi q_1^G g_R = \psi g \pi_1^A = h f \pi_1^A = h f \pi_2^A = \psi q_2^G g_R,$$

where the first equality is by definition of q_1^G , the second one is the one just above and the third one is because f is the coequalizer of its kernel pair. Since g_R is a regular epimorphism, we do get that $\psi q_1^G = \psi q_2^G$ so we get the factorization $\psi = \phi \gamma$. But now we have $\phi g' f = \phi \gamma g$ by definition of γ , and so $\phi g' f = \psi g = h f$. Again since f is an epimorphism, we get $\phi g' = h$, which concludes the proof. \square

B The case of monomorphisms

We take a quick detour in this section to take a look at the case of monomorphisms. As a lot of results quite similar and/or dual to those in section 2 can be proved. We will only take a look at the case of single monomorphisms, the extension to monomorphic filtrations is done in a somewhat dual way to section 3.

Definition B.1. *Let \mathcal{A} be a category. The category $\text{Mon } \mathcal{A}$ is defined as the subcategory of $\text{Arr } \mathcal{A}$ whose objects are monomorphisms.*

This time, $\text{Mon } \mathcal{A}$ is a reflective subcategory. Indeed:

Proposition B.2. *Let \mathcal{A} be a regular category. If we write the regular epi-mono factorization of $a \in \text{Arr } \mathcal{A} : (\mathbf{M}a)r_a$, \mathbf{M} extends to a functor $\text{Arr } \mathcal{A} \rightarrow \text{Mon } \mathcal{A}$ which is left adjoint to the inclusion functor $\mathbf{U} : \text{Mon } \mathcal{A} \rightarrow \text{Arr } \mathcal{A}$. The unit of this adjunction is given by $\eta_a = (r_a, id_{A_1})$.*

Proof. We will only construct the bijection $\text{Arr } \mathcal{A}(a, \mathbf{U}b) = \text{Mon } \mathcal{A}(\mathbf{M}a, b)$ for $a \in \text{Arr } \mathcal{A}, b \in \text{Mon } \mathcal{A}$. The rest of the proof is very similar to what was already done in propositions 2.3 and 3.2. From right to left, this is just the precomposition with η_a . From left to right we send $f : a \rightarrow b$ to (g, f_1) where g is the following lift:

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ r_a \downarrow & \nearrow g & \downarrow b \\ \mathbf{M}A_0 & \xrightarrow{f_1 \circ \mathbf{M}a} & B_1 \end{array}$$

\square

Again, we have a characterization of limits and colimits thanks to the previous result and proposition 1.16:

Corollary B.3. *Let \mathcal{A} be a regular category. Limits exists in $\text{Mon } \mathcal{A}$ and are computed pointwise. If a colimit of a diagram in $\text{Mon } \mathcal{A}$ exists in $\text{Arr } \mathcal{A}$ then it exists in $\text{Mon } \mathcal{A}$ and can be computed by taking \mathbf{M} of a colimit in $\text{Arr } \mathcal{A}$. In that case, a colimit cocone can be obtained by taking $\eta_c \psi$, where $(\psi_i : d(i) \rightarrow c)_{i \in I}$ is a colimit cocone in $\text{Arr } \mathcal{A}$.*

This leads us to this characterization of regular epimorphisms in $\text{Mon } \mathcal{A}$ in the case where \mathcal{A} is a pointed regular category:

Proposition B.4. *Let \mathcal{A} be a regular category. A morphism f in $\text{Mon } \mathcal{A}$ is a regular epimorphism if and only if f_0 and f_1 are regular epimorphisms in \mathcal{A}*

Proof. If f is pointwise a regular epimorphism then it is the coequalizer of its kernel pair in $\text{Arr } \mathcal{A}$, but its kernel pair there is computed the same as in $\text{Mon } \mathcal{A}$. This implies that the kernel pair of f has a coequalizer in $\text{Mon } \mathcal{A}$, computed by taking $\mathbf{M}f$ which is f , so f is a regular epimorphism in $\text{Mon } \mathcal{A}$.

Reciprocally, if $f : a \rightarrow b$ is a coequalizer in $\text{Mon } \mathcal{A}$, it is the coequalizer of its kernel pair. But we know that its kernel pair, which is computed pointwise, has pointwise a coequalizer, since \mathcal{A} has coequalizers of kernel pairs. Thus f is computed as in corollary B.3 and is equal to $\eta_{b'}g$ where $g : a \rightarrow b'$ is the pointwise coequalizer of the kernel pair of f . This implies that f is pointwise a regular epimorphism since g and $\eta_{b'}$ are, and regular epimorphisms in \mathcal{A} are stable under composition since \mathcal{A} is regular (see [7, A.5.4]). \square

And finally:

Theorem B.5. *If \mathcal{A} is a homological category then so is $\text{Mon } \mathcal{A}$.*

Proof. The morphism id_0 is a zero object in $\text{Mon } \mathcal{A}$. The existence of finite limits is guaranteed by corollary B.3. Since regular epimorphisms are defined pointwise, and pullbacks too, then the pullback of a regular epimorphism in $\text{Mon } \mathcal{A}$ is pointwise a regular epimorphism in \mathcal{A} and is thus a regular epimorphism in $\text{Mon } \mathcal{A}$. Let f be a morphism in $\text{Mon } \mathcal{A}$. Since its kernel pair π, π' is computed pointwise, π_i, π'_i are kernel pairs for $i \in \{0, 1\}$ and thus have coequalizers, implying that the coequalizer of π, π' exists in $\text{Arr } \mathcal{A}$ and thus in $\text{Mon } \mathcal{A}$. For the split short five lemma, split epimorphisms, kernels and isomorphisms are defined pointwise in $\text{Mon } \mathcal{A}$ and so it is done in the same way as in corollary 2.11. \square

Remark. Notice that we didn't need property **P** and the existence of all coequalizers due to the much nicer characterization of regular epimorphisms and the fact that limits are computed in the same way in $\text{Mon } \mathcal{A}$ and in $\text{Arr } \mathcal{A}$. We focus however on the case of regular epimorphisms and normal monomorphisms instead, as we believe they have more applications (compare, for example, filtrations of normal subgroups compared to filtrations of subgroups).

C Subobjects in category theory

Here we introduce the categorical notion of subobjects and prove some results on them

Definition C.1. *[subobjects and normal subobjects] Let \mathcal{A} be a category and $A \in \mathcal{A}$. We define a preorder \subset on the monomorphisms of codomain A by $j \subset i$ if and only if there exists a morphism u in \mathcal{A} such that $j = iu$.*

The category of subobjects of A : $\text{Sub}(A)$ is then defined to be the class of monomorphisms in \mathcal{A} of codomain A , quotiented by the equivalence relation induced by \subset . Morphisms are then determined by the order. We often refer to subobjects by one of their representatives, and even, when that doesn't cause confusion, by the domain of one of their representatives. For example, if $i : C \rightarrow A$ is a monomorphism, we write i or just C instead of $[i]$ to speak of the associated subobject. We will write $X \subset A$ to signify that X is a subobject of A , instead of $X \in \text{Sub}(A)$ (this can be interpreted as $X \subset Id_A$).

When \mathcal{A} is pointed, we define normal subobjects as subobjects who have a representative that is a normal monomorphism (i.e. a kernel). They make up the category of normal subobjects of A , denoted $\text{NSub}(A)$.

Remark. When two monomorphisms i and j in \mathcal{A} a category represent the same subobject of $A \in \mathcal{A}$, then we have $iu = j$ and $jv = i$ for some u, v morphisms in \mathcal{A} . But then $iuv = i$ and $uv = Id$ since i is monomorphic. Doing the same with j shows that $vu = Id$ so that i is isomorphic to j . The isomorphism is also unique since i and j are monomorphic. This shows that the category of subobjects of A is equivalent to the category of monomorphisms of codomain A , so we will often use both notions interchangeably, defining objects that should be monomorphisms as subobjects or vice-versa. The reason that we introduced this notion is that a lot of our constructions yield more naturally subobjects rather than monomorphisms (taking a kernel or an image for example), we just proved that this amounts to practically the same.

Since we have a partially ordered set given by subobjects of A , one natural question that arises is the existence of meets and joins in it. The simplest ones are the finite meets:

Proposition C.2. *Let \mathcal{A} be a category, and $A \in \mathcal{A}$. Meets exist in $\text{Sub } \mathcal{A}$ as soon as \mathcal{A} admits pullbacks and they are given for $i, j \in \text{Sub } \mathcal{A}$ by $i \wedge j = i(i^*j)$*

Proof. This is a consequence of pullback properties. □

For joins we have the following result:

Proposition C.3. *Let \mathcal{A} be a pointed category and $A \in \mathcal{A}$. If \mathcal{A} has pushouts, kernels and cokernels then joins exists in normal subobjects of A . If \mathcal{A} is homological, any two normal subobjects of A have a join in the subobjects of A . Moreover, if \mathcal{A} is also exact, then the join of two normal subobjects of A is a normal subobject too (and thus it defines a join in normal subobjects).*

Proof. For the first case, let i, j be normal subobjects of A and r, p their respective cokernels, then let u be the pushout of r along p . The join of i and j is then taken to be the kernel of up . The join property comes from the fact that we can define the order $p \leq q$ if and only if $p = fq$ for some f , on cokernels of domain A . Taking kernel and cokernels then become order reversing bijections up to equivalence. Lastly the pushout of two cokernels defines meet from the pushout properties.

For the case \mathcal{A} homological, see [7, 4.3.12 & 4.3.15]. □

One last construction is the image and preimage morphisms, which are a generalisation of images and preimages of subgroups:

Definition/proposition C.4. *Let \mathcal{A} be a regular category and $f : A \rightarrow B$ in \mathcal{A} . We have two functors*

$$\text{Sub}(A) \begin{array}{c} \xrightarrow{f_*} \\ \perp \\ \xleftarrow{f^*} \end{array} \text{Sub}(B) \quad \text{with } f_*(i) = \text{im}(fi) \text{ and } f^*(j) = f^*j \text{ for } i \in \text{Sub}(A), j \in \text{Sub}(B). \text{ Moreover, } f_*$$

preserves the joins that exist and f^ preserves meets. We call f_* the image by f and f^* the preimage, or coimage by f .*

Proof. First we see that f_* is a functor (i.e. it is increasing). Indeed, if $a \subset b \subset A$ then write $a = bi$. We get

$$\text{im}(fa) = \text{im}(fb)\text{im}(\mathbf{R}(fb)i) \text{ as in the following diagram: } \begin{array}{ccccc} A'' & \xleftarrow{i} & A' & \xleftarrow{b} & A \\ \downarrow & & \downarrow \mathbf{R}(fb) & & \downarrow f \\ B'' & \xrightarrow{\text{im}(\mathbf{R}(fb)i)} & B' & \xrightarrow{\text{im}(fb)} & B. \end{array} \text{ Thus, } f_*(a) \subset f_*(b) \text{ and } f_*$$

increases.

For f^* , if $a \subset b \subset B$, then write $a = bi$ and ia^*f , f^*a defines a cone on the diagram defining the pullback of f and b . Using the universal property of said pullback, we get a factorization of $f^*(a)$ through $f^*(b)$ and f^* increases.

Let now $a \subset A, b \subset B$. We have the following equivalences:

$$\begin{aligned} f_*a \subset b \text{ in Sub}(B) &\Leftrightarrow \text{there exists } g \text{ such that } fa = bg && \text{(Proposition 1.3)} \\ &\Leftrightarrow a \subset f^*b \text{ in Sub}(A). \end{aligned}$$

This proves the adjunction.

The preservation of meets (resp. joins) is just the fact that those are products (resp. coproducts) in the categorical structure. Thus they are preserved by right adjoints (resp. left adjoints). \square

Convention. Given \mathcal{A} a regular category, $i : A \rightarrow B$ a subobject, and $f : B \rightarrow C$, we denote $f|_A$ the morphism fi , which is called the restriction of f to A . It coincides with the notion of restriction in groups. If moreover $f_*(B) \subset D$ with $D \subset C$, we denote $f|_D$ the factorization of f through D , and we call it corestriction.

When \mathcal{A} is pointed, regular and has finite colimits, we can combine these results with the adjunction $\mathbf{N} \dashv \mathbf{U}$ between the category of normal monomorphisms and the category of monomorphisms. Recall from the end of section 3 that \mathbf{N} is defined as taking the kernel of the cokernel and is well defined and a reflector as soon as kernels and cokernels exist. We get an adjunction $f_* \dashv f^*$ between the category of subobjects of A and the category of normal subobjects of B . On top of that, this adjunction can be restricted to the category of normal subobjects of A since f^* takes its values in it. Indeed, if a and b are morphisms with same codomain, with a a normal monomorphism, then b^*a is the kernel of $\text{coker}(a)b$. This is because a morphism f is such that $\text{coker}(a)bf = 0$ if and only if bf factors uniquely through a which is the cokernel of its kernel. But that is equivalent to f factoring uniquely through b^*a . We then get the same results of preservation of joins and meets. Since we will be mostly interested in normal filtrations, it can be assumed outside this section that, otherwise stated, f_* and f^* denote the normal image and coimage.