The Rationality problem

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Acknowledgments I would like to thank John Christian Ottem, for his supervision throughout this internship. He taught me a lot, about algebraic geometry, but also about research in general. Every time he pointed me to parts of articles, it led me to learn about objects, and methods of algebraic geometry and topology I did not know. Our interactions have been very enriching for me.

I would also like to thank all the researchers, PHD students and students of the 11th floor of the mathematics building of the university of Oslo, who made me feel welcome.

Outline of my internship My internship took place in the University of Oslo, in Norway. I started this internship knowing almost nothing about algebraic geometry. Thus, I devoted a large part of my time in learning a lot about this field of mathematics. I have learned about varieties, sheaves, schemes, cohomology groups... To do it, I have followed two courses of Algebraic Geometry given by my supervisor, John Christian Ottem, and I have read a lot. Then, I have studied articles about the Rationality problem, which I present in this report. These articles, and the interactions with my supervisor led me to learn a lot.
Moreover, I went to the Algebra seminar of the department, approximately hold every two weeks. I interacted a lot with Edvard Asknes, a PHD student in tropical geometry who I shared an office with. I have also joined a learning seminar about toric and logarithmic geometry that took place every weeks, and given a talk for it. It was organized by Felix Thimm, and most of the participants were either master's students, or PHD students. I finally went to the Abel lecture, which was a day of conferences organized by the Abel prize committee.

Warning The goal of this report is to present a classic counterexample of a problem that I have been studying during my internship, found by Artin and Mumford in 1972. To get to it, I have learned a lot of notions of algebraic geometry. I tried my best to make this report understandable by the student I was before doing this internship. Thus, I did not detail certain technical terms and constructions, to concentrate on what seemed to me the most important to present the counterexample. Therefore, some of the proof that I present are not as rigorous as I would like them to be.

1 Introduction

1.1 Introduction to projective varieties

We denote by $\mathbb{A}^n$ the set $\mathbb{C}^n$ and by $\mathbb{P}^n$ the projective space, which is the set of lines of $\mathbb{C}^{n+1}$ passing through the origin, equipped with Zariski topology defined as follows :

Definition 1.1. If $S \subset \mathbb{C}[X_0, \cdots, X_n]$ is a homogeneous ideal, then its zero set is

$$Z_+(S) = \{x \in \mathbb{P}^n, \text{ s.t. } \forall f \in S, f(x) = 0\}$$

They constitute the closed sets of a topology on $\mathbb{P}^n$, called the Zariski topology. For instance, the ideal $(X_0, \cdots, X_n)$ gives us that the empty set is closed set.

When $S = (h)$ is principal, $X = Z_+(h)$ is called a hypersurface, and we call $d = \deg(h)$ the degree of X.

We get the Zariski topology on a subset of $\mathbb{P}^n$ by taking the induced topology.

Definition 1.2. A set $X \subset \mathbb{P}^n$ is called irreducible if it can not be written as the union of two proper closed subsets. That is, if $X = F_1 \cup F_2$, with F_1 and F_2 closed, then $F_1 = X$ or $F_2 = X$.

Equivalently, X is irreducible if and only if any two non-empty opens have a non empty intersection.

This shows that the Zariski topology is really different from the Euclidean. For instance, it is far from being Hausdorff.

Example 1.1. $\mathbb{P}^n$ is irreducible with the Zariski topology. Indeed, if their exist $S, Z \subset \mathbb{C}[X_0, \cdots, X_n]$ ideals such that $\mathbb{P}^n = Z_+(S) \cup Z_+(Z)$, let $f \in S, g \in Z$ non constants. Then

$\mathbb{P}^n = Z_+(S) \cup Z_+(Z) \subset Z_+(f) \cup Z_+(g) = Z_+(fg)$. So the polynomial fg cancels all the points of $\mathbb{P}^n$, so all the points of $\mathbb{A}^{n+1}$. We deduce that fg is the zero polynomial. But f and g are non constant, so it is absurd. $\mathbb{P}^n$ is irreducible.

Proposition 1.1. Open non empty subsets of an irreducible topological space are irreducible and dense.

Definition 1.3. An projective variety is an irreducible closed subset of $\mathbb{P}^n$. We call quasi-projective variety an open set of a projective variety.

Example 1.2. $\mathbb{P}^n, Z_+(X^2 - Y) \subset \mathbb{P}^2$ are projective varieties, every open sets of $\mathbb{P}^n$ are quasi-projective varieties. For f a homogeneous irreducible polynomial of $\mathbb{C}[X_0, \cdots, X_n]$, then $Z_+(f)$ is irreducible.

Now, we want to be able to talk about morphisms between projective varieties, and even between open sets of projective varieties. Let's first remark that non constant polynomials are not well defined on open sets of $\mathbb{P}^n$. Indeed, for $x \in \mathbb{P}^n, t \in \mathbb{K}$, it has to have the same value in x and in tx. But fraction of homogeneous polynomials of same degrees are well defined on some open sets. For instance, $\frac{X_1}{X_0}$ is well defined on $\mathbb{P}^n \backslash Z_+(X_0)$.

Definition 1.4. Let X be a projective variety, and $U \subset X$ a quasi-projective variety. We say that a function $f : U \longrightarrow \mathbb{K}$ is regular at $x \in U$ if there exists a open neighborhood $x \in V \subset U$ of x such that on V, $f = \frac{a}{b}$, a and $b \neq 0$ being homogeneous polynomials of same degree. We denote

$$\mathcal{O}_X(U) = \{f : U \longrightarrow \mathbb{K}, \text{ regular at all points of } U\}$$

Definition 1.5. Let X, Y be two projective varieties. Let $X_0 \subset X$ and $Y_0 \subset Y$ be open sets. A morphism $\phi : X_0 \longrightarrow Y_0$ is a continuous map such that

$$\forall U \subset Y_0 \text{ open}, \ \forall f \in \mathcal{O}_Y(U), f \circ \phi \in \mathcal{O}_X(\phi^{-1}(U))$$

In this case, we denote $\phi^* : f \in \mathcal{O}_X(U) \longrightarrow f \circ \phi \in O_X(\phi^{-1}(U))$ An isomorphism is a morphism that has an inverse morphism. When X and Y are isomorphic, we write $X \cong Y$.

Remark 1.1. The affine space $\mathbb{A}^n$ is also equipped with the Zariski topology defined, as in the projective case by the closed sets : for $a \subset \mathbb{K}[X_1, \cdots, X_n]$ an ideal,

$$Z(a) = \left\{x \in \mathbb{A}^n \text{ such that } \forall f \in a, f(x) = 0\right\}$$

Then, as we did in the projective case, we can define affine varieties as irreducible closed subset of $\mathbb{A}^n$. Finally, varieties are spaces that locally look like affine varieties. To explain properly what it means, we would need to define several other notions. For the sake of clarity and conciseness, we choose not to give more details for this definition.

The goal of this report is to present the Artin-Mumford counter-example that happens to be a projective variety, so that is what we will focus on. Also, as their name suggests, projective varieties are varieties. But we will still write some definitions and propositions using the word variety when there are more general.

Definition 1.6. Let X be a (projective) variety, and $p \in X$. Locally around p, X can be embedded as an affine variety in $\mathbb{A}^n$, so we suppose $X = Z(f_1, \cdots, f_n)$. We define the Jacobian of X at p as $J_p = \left(\frac{\partial f_i(p)}{\partial x_j}\right)_{i,j}$. The tangent space of X at p is

$$T_p(X) = \left\{Y = (y_1, \cdots, y_n) \text{ s.t. } J_p.Y = 0\right\}$$

We say that X is "non singular" at p if $\text{rk}(J_p) = n - \dim(X)$. It exactly says that the dimension of the tangent space at p is the dimension of X. Otherwise, p is called a singular point of X. The smooth locus of X is defined as the subset of points on which X is non singular. X is smooth if it has no singularity. Finally, $T_p(X)$ is a vector space, so we can define the projectivized tangent space of X at p as the set of lines of $T_p(X)$ passing through the origin.

1.2 The Rationality problem

Some useful vocabulary Let $\mathcal{H}$ be a property, that can be verified or not by the elements x of a set $\mathcal{E}$. We say that a *general* $x \in \mathcal{E}$ verifies $\mathcal{H}$ if there exists an open set $U \subset \mathcal{E}$ such that all elements of U satisfy $\mathcal{H}$. This is particularly useful in sets equipped with the Zariski topology, in which open sets are dense. In this case, if a general x verifies a property, it means that "almost" every elements satisfy the property.

For instance, let $\mathcal{H}$: "$x_0 \neq 0$" on $\mathbb{P}^n$. Then, with $U = \mathbb{P}^n \backslash Z_+(X_0)$, a general $x \in \mathbb{P}^n$ satisfies $\mathcal{H}$. This vocabulary allows us to fix a "general" object, and allow it to verify properties along our proofs, as long as they are still "general" enough, ie satisfied by an open subset.

As finite intersection of open sets are open, we can do this trick for any finite number of properties. Finally, if a property is true on a intersection of countably many open sets, we will say that a *very general* x satisfy the property.

Definition 1.7. Let X, Y be two (quasi-projective) varieties. A rational map $\phi : X \dashrightarrow Y$ is a morphism from an open set $U \subset X$ to Y. We say that ϕ is dominant if its image is dense in Y. Finally, ϕ is birational if it induces a isomorphism from an open set of X to an open set of Y.

We say that X is *unirational* if there exist $n \in \mathbb{N}$, and $\phi : \mathbb{P}^n \dashrightarrow X$ a dominant rational map.

X is *rational* if there exists a such birational ϕ.

Remark 1.2. Remember that an open set in a (projective) variety is dense, so it makes sense to study birational maps, we can think about them as "almost isomorphism".

Remark 1.3. In the definition of unirationality, we can assume $n = \dim(X)$. Indeed, if $\phi : \mathbb{P}^n \dashrightarrow X$ is a dominant rational map, then there exists a linear subspace of $\mathbb{P}^n$ of dimension $\dim(X)$ on which the restriction of ϕ is still dominant.

Remark 1.4. A quick way to define the dimension of a (quasi-)projective variety is to define $\dim(X) = \dim(T_p(X))$ for $p \in X$ general. We have basic properties that for $U \subset V, \dim(U) \leq \dim(V)$, we have $\dim(\mathbb{P}^n) = n$, and that, as for vector spaces, "imposing an equation loses a dimension", i.e. if $f \in \mathbb{C}[X_0, \cdots, X_n]$ homogeneous non zero, $\dim(Z_+(f)) = n - 1$. Moreover, if $d, r, n \in \mathbb{N}$ are such that $r + d < n$, and $V \subset \mathbb{P}^n$ is a variety of dimension d, then a general variety $X \subset \mathbb{P}^n$ of dimension r satisfies $X \cap V = \varnothing$.

Actually, a smooth variety is a manifold. We work aver the complex field here, so when thinking about a variety of dimension n, we can think of a manifold of dimension $2n$ over $\mathbb{R}$, with additional structures.

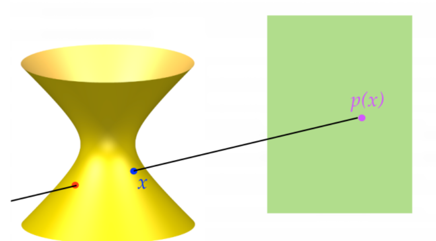
The rationality problem : Is every unirational variety rational ?

Remark 1.5. Between unirationality and rationality, there are many notions. The only other that I have been studying during my internship is the stable-rationality. X is stably rational if there exists $n \in \mathbb{N}$ such that $X \times \mathbb{P}^n$ is rational. We have rational $\Longrightarrow$ stably-rational $\Longrightarrow$ unirational. The Artin-Mumford counter-example that we will study in this report is actually not even stably-rational.

Some history : This problem appears in a note by J. Lüroth, where he proves the positive answer in dimension 1. In the last decade of the 19^{th} century, the theory of algebraic surfaces is developed, and G. Castelnuovo answered positively to the problem in dimension 2. It then become natural to ask what append in higher dimension. Some attempts of counter-example were made in the beginning of the 20^{th} century, but it is only in 1971-1972 that three indisputable counter-examples in dimension 3 appeared. Each of them is based on a different method, the authors developed different birational invariants. But unfortunately, they only apply on specific cases. Recently, C. Voisin largely extended the range of a new method, called decomposition of the diagonal, and lead to many other counter-examples.

Example 1.3. J.Lüroth actually proved that the only unirational curve is $\mathbb{P}^1$.

In dimension 2, for instance, the cubic surface $Z(X_0^3 + \cdots + X_3^3)$ in $\mathbb{A}^3$, the sphere, and more generally quadrics in $\mathbb{A}^3$ are rational. Indeed, let H be a quadric in $\mathbb{A}^3$, choose an arbitrary point y in the quadric, and a plane given by X_0 constant. For all $x \neq y$ in the quadric, the line passing through x and y intersect the plane in exactly one point $p(x)$. This gives a isomorphism between $H \backslash \{y\}$ and $\mathbb{C}^2$, so H is rational.



The Goal of this report is to explain one of the first three counter-examples, the one developed by Artin and Mumford in 1972.

1.3 A example of a non rational curve

Let the curve $X = Z(y^2 - x(x-1)(x-2)) \subset \mathbb{A}^2$. We have $\dim(X) = 1$ (we impose an equation in $\mathbb{A}^2$, which is a space of dimension 2). We define $\mathbb{C}(X) := \mathbb{C}(x,y)/(y^2 - x(x-1)(x-2))$, called the function field of X. We can see them as the rational fractions that are defined on some open subset of X. With this thought, the function field of $\mathbb{P}^1$ is defined as $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(t)$.

Lemma 1.1. If X is rational, then its function field $\mathbb{C}(X)$ is isomorphic to the function field of $\mathbb{P}^1$, ie $\mathbb{C}(X) \cong \mathbb{C}(t)$ as $\mathbb{C}$-algebra.

Sketch of proof. If X is rational, by the remark 1.3, we can assume that there exists a birational map $\phi : \mathbb{P}^1 \dashrightarrow X$. Let $\psi : X \dashrightarrow \mathbb{P}^1$ its rational inverse. Then, $\phi^* : f \in \mathbb{C}(X) \longrightarrow f \circ \phi \in \mathbb{C}(\mathbb{P}^1)$ is an homomorphism of $\mathbb{C}$- algebras, of inverse ψ^*. □

Proposition 1.2. X is not rational.

Proof. Let's contradict the previous lemma. We have $\mathbb{C}(X) = \mathbb{C}(x,y)/(y^2 - x(x-1)(x-2))$. Suppose that there exists a isomorphism of $\mathbb{C}-$algebra $\psi : \mathbb{C}(X) \longrightarrow \mathbb{C}(t)$. For $P = t - a, a \in \mathbb{C}$ in $\mathbb{C}(t)$, we denote by ν_P the $P-$adic valuation. It is defined as the following : for $Q \in \mathbb{C}(t)$, write $Q = (t-a)^n \frac{A}{B}$, with $A, B \in \mathbb{C}[t]$ such that $(t-a) \nmid A, B$. We define $\nu_P(Q) = n$. Then, for $P = t - a$,

$$\nu_P(\psi(x(x-1)(x-2))) = \nu_P(\psi(x)) + \nu_P(\psi(x-1)) + \nu_P(\psi(x-2)) = \nu_P(\psi(x)) + 2\min(\nu_P(\psi(x)), 0))$$

But in $\mathbb{C}(X)$, we have $x(x-1)(x-2) = y^2$. So $\nu_P(\psi(x(x-1)(x-2))) = 2\nu_P(\psi(y))$. We obtained

$$\nu_P(\psi(x)) = 2\nu_P(\psi(y)) - 2\min(\nu_P(\psi(x), 0))$$

So for all irreducible polynomial $P \in \mathbb{C}(t)$, $\nu_P(\psi(x))$ is even. So $\psi(x)$ is a square in $\mathbb{C}(t)$, so x is a square in $\mathbb{C}(X)$.

We can then write $x = (p(x) + yq(x))^2$, and by expanding the expression, and using $y^2 = x(x-1)(x-2)$, we obtain $q = 0$ or $p = 0$. Seeing both cases in $\mathbb{C}(x)$, we obtain a contradiction. □

2 Prerequisite to the construction

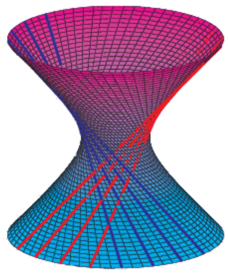
2.1 Proprieties of quadric surfaces in $\mathbb{P}^3$

Let $\mathcal{P}$ be the projective space parametrizing quadric surfaces in $\mathbb{P}^3$, such a quadric is of the form $Z_+(F)$, where $F = \sum\limits_{0 \le i,j \le 3} a_{i,j} X_i X_j$, with $(a_{i,j})_{i,j}$ a 4×4 projective symmetric matrix. So $\mathcal{P} \cong \mathbb{P}^9$ with coordinates $(a_{00} : \cdots : a_{33})$. Let $\mathcal{G} = G(2,4)$ be Grassmannian manifold parametrizing lines in $\mathbb{P}^3$. $\mathcal{G}$ has dimension 4 (a general line can be written as $\{[s : t : as+bt : cs+dt], a, b, c, d \in \mathbb{C}\}$). If H is a fixed quadric, we call set of lines of H the set $\mathcal{G}_H := \{l \in \mathcal{G}$ s.t. l represent a line in $H\}$.

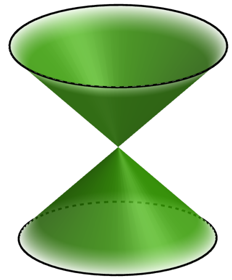
Definition 2.1. Let $H = Z_+(F) \subset \mathbb{P}^3$, with $F = \sum\limits_{i,j} a_{i,j} X_i X_j$, with $A = (a_{i,j})_{i,j}$ a symmetric matrix. The rank of H is defined as the rank of the matrix A. We say that H is smooth if A is invertible.

Let $H = Z_+(F)$ be a smooth quadric in $\mathbb{P}^3$, with $F = \sum\limits_{i,j} a_{i,j} X_i X_j$. We can choose coordinates on $\mathbb{P}^3$ such that H is the image of the Segre embedding $i : ((a_0 : a_1), (b_0 : b_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow (a_0 b_0 : \cdots : a_1 b_1)$. Indeed, $\mathrm{Im}(i) = Z_+(X_0X_3 - X_1X_2)$. So $Y = (y_0 : \cdots : y_3) \in \mathrm{Im}(i) \iff YDY^T = 0$, with $D = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$. Now, $rk(A) = rk(D)$, and A and D are symetrical. So there exists P an orthogonal matrix such that $PAP^T = D$. Now, $X \in Z_+(F) \iff Y = XP^T \in \mathrm{Im}(i)$. We even showed that two quadrics of same rank are the same, up to changes of coordinates. We just proved the following proposition :

Proposition 2.1. Any smooth quadric H is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The quadric surfaces of rank 3 and 2 in $\mathbb{P}^3$ are respectively isomorphic to $Z_+(X_0^2 + X_1^2 + X_2^2)$ and $Z_+(X_0^2 + X_1^2)$



Rank 4



Rank 3

Rank 2

Let Π be a subset of $\mathcal{P}$. Let

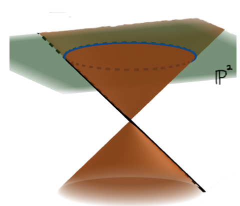
$$I = \{(q,l) \in \Pi \times \mathcal{G} \text{ s.t. } l \text{ is a line of q}\}$$

I is called the incidence correspondence in $\Pi \times \mathcal{G}$. Let $p : I \longrightarrow \Pi, q : I \longrightarrow \mathcal{G}$ be the associated projections. For $H \in \mathcal{G}$, we see that $p^{-1}(H) \cong \mathcal{G}_H$ the set of lines in H.

In $\mathbb{P}^1 \times \mathbb{P}^1$, the lines are only of the two following form : $L_{(a_0:a_1)} = (a_0 : a_1) \times \mathbb{P}^1$, and $L_{(b_0:b_1)} = \mathbb{P}^1 \times (b_0 : b_1)$.

Moreover,if H is the zero locus of $y_0^2 + y_1^2$ in $\mathbb{P}^3$, then $H = \{(\pm i y_1 : y_1 : y_2 : y_3), (y1 : y_2 : y_3) \in \mathbb{P}^2\} := H_+ \cup H_-$. H_+ and H_- are isomorphic to $\mathbb{P}^2$, and there intersection is a line. Then, the set of lines of H is connected.

Finally, if H is the zero locus of $y_0^2 + y_1^2 + y_2^2$ in $\mathbb{P}^3$, then H is the cone over a quadric in $\mathbb{P}^2$. Then, all the lines that are included in H goes through the point $(0 : 0 : 0 : 1)$ (because otherwise, they would map to a line via the projection through $(0 : 0 : 0 : 1)$, see the draw below). For $x \in H \backslash \{(0 : 0 : 0 : 1)\}$, we denote L_x the line included in H that contain x (and $(0 : 0 : 0 : 1)$). Then, take $x, y \in H \backslash \{(0 : 0 : 0 : 1)\}$, and a continuous map $\phi : [0 : 1] \longrightarrow H$ such that $\phi(0) = x, \phi(1) = y$. Then, ϕ induces a continuous map from L_x to L_y with $t \in [0, 1] \longrightarrow L_{\phi(t)}$. So the set of line of H is connected.



We obtain the following property:

Proposition 2.2. If $H \in \mathcal{P}$ is smooth, then $p^{-1}(H)$ is two disjoint copies of $\mathbb{P}^1$. Otherwise, $p^{-1}(H)$ is connected.

2.2 Some useful observations

For $1 \leq i \leq 4$, let $W_i \subset \mathcal{P}$ denote the closed set of quadrics of rank $\leq i$.

We work over the field $\mathbb{C}$, and we consider a general three dimensional linear system $\Pi \cong \mathbb{P}^3 \subset \mathcal{P}$. More explicitly, it can be written as $\Pi = \left\{a_0 q_0 + \cdots + a_3 q_3, (a_0 : \cdots : a_3) \in \mathbb{P}^3\right\}$ where $q_0, \cdots, q_3$ are homogeneous polynomials of degree 2.

Proposition 2.3. We have $\dim(W_1) = 3, \dim(W_2) = 6$ and $\dim(W_3) = 8$.

Sketch of proof. A quadric is determined by a 4×4 projective symmetric matrix. Moreover, if M is a matrix of rank 1, its rows $L_0, \cdots L_4$ are multiples of the first, i.e. $L_i = c_i L_0$, with $c_i \in \mathbb{C}$, and if M is symmetric, the first line gives us the first column, so we have the c_i. Then, by associating M to its first row, we have shown $W_1 \cong \mathbb{P}^3$. We can write similar arguments to see which coefficients determine symmetric matrices of rank 2 and 3, so we obtain the proposition. □

Proposition 2.4. A general three dimensional linear system $\Pi \subset \mathcal{P}$ satisfy:
(i) Π is based point free, meaning that no $x \in \mathbb{P}^3$ is canceled by all quadrics of Π.
(ii) $S = \Pi \cap W_3$ is a surface of degree 4, $\Pi \cap W_2$ is a finite set of 10 points, and $\Pi \cap W_1 = \varnothing$
(iii) The singular points of S are exactly those of $\Pi \cap W_2$, and for $x \in \Pi \cap W_2$, the projectivized tangent space to S at x is a smooth conic in $\mathbb{P}^2$.

Sketch of proof (i) For any quadric $q \in \mathcal{G}$, $\dim(Z_+(q)) = 2$, so for $q_0, \cdots, q_3$ general, we have $Z_+(q_0) \cap \cdots \cap Z_+(q_3) = \varnothing$.

(ii) We have $\dim(W_1) = 3$, and $\dim(\mathcal{P}) = 9$. So for a general 3-dimensional linear system $\Pi \subset \mathcal{P}$, we have $\Pi \cap W_1 = \varnothing$. The same way, $\dim(W_2) = 6$, so for a general Π, we have $\dim(W_2 \cap \Pi) = 0$, therefore $W_2 \cap \Pi$ is a finite set, and we can show that it is in general a set of 10 elements. Finally, S is the zero set of the determinant of matrices of size 4×4, which is a polynomial of degree 4, in Π.

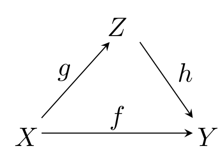
From now on, we take a general $\Pi \cong \mathbb{P}^3$ which satisfies the properties of the proposition above.

3 The Artin-Mumford counter-example, construction

3.1 The Stein factorization

We consider our general three dimensional linear system of quadrics Π constructed in the previous section. We need to add two constructions to obtain our counter-example.

Proposition 3.1 (Stein factorization)**.** Let $f : X \longrightarrow Y$ a morphism of projective varieties. Then, f factors as

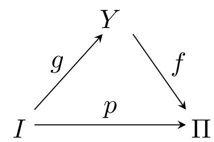


where :

- Z is a projective variety
- g has connected fibers
- h is a finite map

In our case, $\mathcal{G}$ and $\Pi \cong \mathbb{P}^3$ are projective varieties, so $\Pi \times \mathcal{G}$ is projective, and I is closed in $\Pi \times \mathcal{G}$. So I and Π are projective varieties.

By applying the Stein factorization to p, we obtain a factorization of p as :



Moreover, as we saw in proposition 2.2, if $q \in \Pi$ is a quadric, then $p^{-1}(q) = \{(q, l), l \text{ line in q }\}$ is either two disjoint copies of $\mathbb{P}^1$ (when q is smooth), or a connected set. For these reasons, we can think about Y as $\{(q, [l]), (q, l) \in I\}$ where $[l]$ is the class of l under the equivalence relation

$$l \sim l' \iff l \text{ and } l' \text{ are in the same connected component in } p^{-1}(q)$$

Seen this way, g is the map that send (q, l) to $(q, [l])$, and f is the projection on the first coordinate. Therefore, for $h \in \Pi \cap W_3$, $f^{-1}(h)$ is a point, and otherwise, $f^{-1}(h)$ has exactly two elements. So Y is called the double cover of Π along $\Pi \cap W_3$.

By proposition 2.4, Π has exactly 10 points of singularity, given by $\Pi \cap W_2$. For reasons that we will not explain, this imply that the singularity points of Y are exactly the set $f^{-1}(\Pi \cap W_2)$ which is a set of 10 points. Moreover, the point (iii) of proposition 2.4 implies that the projectivized tangent spaces of Y at them are smooth quadrics, so there are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ by proposition 2.1.

Remark 3.1. The structure of variety of the double cover is not very clear as we defined it above. Knowing about schemes, we can make it clearer by taking an other definition of the double cover, with no use of the Stein factorization. To detail the paragraph above, we need this other point of view, so we will not do it here.

3.2 Blow-up construction

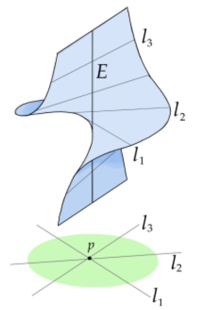
We need a final construction to get to the counter-example. For reasons of space, we will not detailed the construction of blow-ups, but we will explain it briefly. We let the interested readers to refer to [6].

We define $Y_0 := Y \setminus f^{-1}(\Pi \cap W_2)$ the smooth locus of Y.

Proposition 3.2. There exists a smooth projective variety V, and a morphism $b : V \longrightarrow Y$ such that b induces an isomorphism from $b^{-1}(Y_0)$ onto Y_0, and for each $x \in f^{-1}(\Pi \cap W_2), b^{-1}(x)$ is isomorphic to the projectivized tangent cone at x (which is a smooth conic of $\mathbb{P}^3$ by the previous section). V is called the blow-up of Y along X. We set $V_0 = b^{-1}(Y_0)$, and $E = b^{-1}(f^{-1}(\Pi \cap W_2))$, called the exceptional divisor. Therefore, $V = V_0 \sqcup E$ with $V_0 \cong Y_0$, and $E = E_1 \sqcup \cdots \sqcup E_{10}$, with each $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$.

V is the Artin-Mumford counter-example for the rationality problem.

Example 3.1 (of blow-ups in a easier case)**.** To give an idea of what a blow-up is, let's look at the blow-up of $\mathbb{A}^2$ along $(0, 0)$. It is a subspace of $\mathbb{A}^2 \times \mathbb{P}^1$. We replace $(0, 0)$ by the projectivized tangent cone at $(0, 0)$, which is $\mathbb{P}^1$. The picture looks like the following :



The following diagram synthesize what we just constructed.

$$\begin{array}{ccc} & V & \\ & \downarrow b & \\ & Y & \\ {}^{g}\nearrow & & \searrow^{f} \\ I & \xrightarrow{\quad p \quad} & \Pi \end{array}$$

4 The Artin-Mumford example is a counterexample in dim 3

4.1 Unirationality

Proposition 4.1. The projective variety V is unirational.

Sketch of proof. Let $Y_0 = Y \setminus \{\text{the 10 points of singularity}\}$. Then, Y_0 is an open subset of Y, and isomorphic to $b^{-1}(Y_0)$. So V and Y are birational. Then, $g : I \longrightarrow Y$ is surjective, so it suffices to prove that I is unirational. We will give a sketch of this proof.

Let $q : (H, l) \in I \longrightarrow l \in \mathcal{G}$ be the second projection. Then, q induces a birational map. Indeed:

1^{st} step : For Π general, there exist $l \in \mathcal{G}$ such that $q^{-1}(l)$ is a point. Indeed, let $l \in \mathcal{G}$ be a line. Then $\{H \in \mathcal{P}$ s.t. l is in H$\}$ is of dimension 6. Indeed, l is in H if and only if H contains three points of l, so it imposes three equations. Therefore, the dimension of this variety is $9 - 3 = 6$, so its intersection with a general Π of dimension 3 is of dimension 0.

2^{nd} step : The previous intersection is actually a point, and there is an open neighborhood U of l in which we have the same property. Then, q induces an isomorphism between $q^{-1}(U) \subset I$ and $U \subset \mathcal{G}$. So there exist a birational map $\mathbb{P}^4 \dashrightarrow I$. So I is rational, which proves that V is unirational. □

4.2 Properties of homology and cohomology groups

Sketch of Definition of homology/cohomology During my internship, I have learned different notions of homology and cohomology. Here, I need to use the singular homology and cohomology groups, to use the fact that the torsion of the third cohomology group is a birational invariant. For space reasons, I will not define them, but I will still be using them. I choose to only give the properties that we need. For any topological space X, we can associate two sequences of groups called respectively (singular) holomogy, and cohomology groups, and denoted by $H_i(X,\mathbb{Z})$ and $H^i(X,\mathbb{Z})$ for $i \in \mathbb{N}$. We let the interested readers to refer for instance to the chapter 7 of [5], or chapter 3 of [7].

Proposition 4.2. Let X and Y two topological spaces, and $f : X \longrightarrow Y$ a continuous map.

- For all $i \in \mathbb{N}$, f induces two morphisms $f_* : H_i(X,\mathbb{Z}) \longrightarrow H_i(Y,\mathbb{Z})$ and $f^* : H^i(Y,\mathbb{Z}) \longrightarrow H^i(X,\mathbb{Z})$.

- If X and Y are homeomorphic, then $H^i(X,\mathbb{Z}) \cong H^i(Y,\mathbb{Z})$ for all $i \in \mathbb{Z}$.

- (Poincaré duality) If X be a smooth variety of dimension $2n$ over $\mathbb{R}$, then, for all $0 \leq p \leq 2n$
$$H^p(X,\mathbb{Z}) \cong H_{2n-p}(X,\mathbb{Z})$$

- (The Künneth formula) Let X, Y be smooth (projective) varieties, then, *modulo torsion*, we have
$$H^p(X \times Y,\mathbb{Z}) \cong \bigoplus_{i+j=p} H^i(X,\mathbb{Z}) \otimes H^j(Y,\mathbb{Z})$$
where the tensor is taken over $\mathbb{Z}$.

Proposition 4.3. If X is a smooth (projective) variety of dimension n, then $H_{2n}(X,\mathbb{Z}) \cong \mathbb{Z}$

Proof. With the Poincaré duality, we have $H_{2n}(X,\mathbb{Z}) \cong H^0(X,\mathbb{Z})$, and here, because X is connected, $H^0(X,\mathbb{Z}) = \mathbb{Z}$. □

Definition 4.1. Let X be a smooth variety of dimension n, and $W \subset X$ be a smooth subvariety of dimension d. We denote $i : W \longrightarrow X$ the inclusion. Using the previous proposition, we can associate $[W] \in H_{2d}(W,\mathbb{Z})$ by taking the canonical generator. Then, $i_*([W])$ defines an element in $H_{2d}(X,\mathbb{Z})$, that we also denote $[W]$. Finally, by Poincaré duality, we can see this element in $H^{2n-2d}(X,\mathbb{Z})$.

Moreover, we write $H^*(X,\mathbb{Z}) = \bigoplus_{i\in\mathbb{N}} H^i(X,\mathbb{Z})$ as a graded ring, where for $x \in H^i(X,\mathbb{Z})$, we set $\deg(x) = i$. One can define a product, called the cup product, that makes this group a graded ring. For $\alpha \in H^i(X,\mathbb{Z})$, $\beta \in H^j(X,\mathbb{Z})$, $\alpha.\beta \in H^{i+j}(X,\mathbb{Z})$. It is strongly connected to intersection of varieties. Indeed, let X be a variety, and $W, Z \subset X$ be subvarieties of X of respective codimension i and j, which intersect transversely. Then, seen in cohomology, $[W].[Z] = [W \cap Z] \in H^{i+j}(X,\mathbb{Z})$. We will use this construction in the proof of lemma 4.1

Proposition 4.4. (See theorem 3.19 in [7]) As graded ring, setting $\deg(x) = 2$, we have :
$$H^*(\mathbb{P}^n,\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$$

Proposition 4.5. For every topological space W, $H^1(W,\mathbb{Z})$ has no torsion. This is a consequence of the Universal Coefficient Theorem. For more details, see for instance section 3.1 in [7].

4.3 Non-rationality

Proposition 4.6. If X is a smooth projective variety that is birational to $\mathbb{P}^n$, then $H^3(X,\mathbb{Z})$ is torsion free.

Sketch proof. To write a complete proof, one need to know more about blow-ups. Thus, we will only give a sketch of proof. Let $\phi : \mathbb{P}^n \longrightarrow X$ a birational map, then, by Hironaka's resolution of singularity, there exists a sequence of blow-ups with smooth center $Z_r \longrightarrow \cdots \longrightarrow Z_0 = \mathbb{P}^n$, such that the map $Z_r \longrightarrow X$ induced by ϕ is a morphism. Then, one prove that blowing-up along a smooth center does not change the torsion of H^3, and that $H^3(Z_r,\mathbb{Z})_{\text{tor}} = H^3(X,\mathbb{Z})_{\text{tor}}$. □

The previous proposition gives us a birational invariant. Now, we want V to contradict this invariant, so we want to construct a non zero element in $H^3(V,\mathbb{Z})_{tor}$.

Let

$$\mathcal{H} = \Big\{(a_0 : \cdots : a_3), (x_0 : \cdots : x_3) \in \mathbb{P}^3 \times \mathbb{P}^3 \text{ s.t. } a_0 q_0(x_0, \cdots, x_3) + \cdots + a_3 q_3(x_0, \cdots, x_3) = 0\Big\}$$

$\mathcal{H}$ is called the universal hypersurface of Π. It is a projective variety.

Definition 4.2. Let $\phi : X \longrightarrow Y$ be a morphism. A rational section of ϕ is a pair (s, U), where $U \subset Y$ open, and $s : U \longrightarrow X$ morphism such that $\phi \circ s = id_U$

Lemma 4.1. If $\mathcal{H} \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3$ is the universal hypersurface of Π, then the map $p : \mathcal{H} \longrightarrow \mathbb{P}^3$ induced by the first projection has no rational section.

Proof. Suppose that p has a rational section $s : U \subset \mathbb{P}^3 \longrightarrow \mathcal{H}$. Let W be the closure of $Im(s)$, W is a subvariety of $\mathcal{H}$ of dimension 3. Then, the inclusion $W \hookrightarrow \mathcal{H}$ gives $\mathbb{Z} \cong H_6(W,\mathbb{Z}) \xrightarrow{i_*} H_6(\mathcal{H},\mathbb{Z})$, and the image by i_* of a generator of $H_6(W,\mathbb{Z})$ gives a class $[W] \in H_6(\mathcal{H},\mathbb{Z})$ as explained in definition 4.1. Then, $\mathcal{H}$ has dimension 5 (by the remark 1.4 seing that we have $\dim(\mathbb{P}^3 \times \mathbb{P}^3) = 6$ and we impose an equation), so $\mathcal{H}$ has dimension 10 over $\mathbb{R}$. By the Poincarré duality, $H_6(\mathcal{H},\mathbb{Z}) \cong H^4(\mathcal{H},\mathbb{Z})$. Finally, by the Lefschetz hyperplane theorem, $i^* : H^4(\mathbb{P}^3 \times \mathbb{P}^3,\mathbb{Z}) \longrightarrow H^4(\mathcal{H},\mathbb{Z})$ induced by the inclusion is an isomorphim.

Then, we want to study $H^4(\mathbb{P}^3 \times \mathbb{P}^3,\mathbb{Z})$. By the Künneth formula,

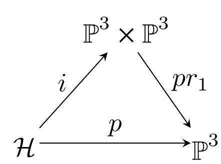
$$H^4(\mathbb{P}^3 \times \mathbb{P}^3,\mathbb{Z}) \cong \bigoplus_{i+j=4} H^i(\mathbb{P}^3,\mathbb{Z}) \otimes H^j(\mathbb{P}^3,\mathbb{Z})$$

But by proposition 4.4 $H^*(\mathbb{P}^3,\mathbb{Z}) \cong \mathbb{Z}[X]/(X^4) = \mathbb{Z} \oplus \mathbb{Z}X \oplus \mathbb{Z}X^2 \oplus \mathbb{Z}X^3$, with $deg(X) = 2$. So the image of X by this isomorphism is in $H^2(\mathbb{P}^3,\mathbb{Z})$. We denote it h. Then, let $pr_{1,2} : \mathbb{P}^3 \times \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ be the first and second projections. Let $h_1 = h \otimes 1 = pr_1^*(h)$ and $h_2 = 1 \otimes h = pr_2^*(h) \in H^2(\mathbb{P}^3 \times \mathbb{P}^3,\mathbb{Z})$. The Künneth formula gives us $H^4(\mathbb{P}^3 \times \mathbb{P}^3,\mathbb{Z}) \cong \mathbb{Z}h_1^2 \oplus \mathbb{Z}h_1 h_2 \oplus \mathbb{Z}h_2^2$. Now, let $D_1 = i^*(h_1)$ and $D_2 = i^*(h_2)$ in $H^2(\mathcal{H},\mathbb{Z})$.

Then,

$$H^4(\mathcal{H},\mathbb{Z}) \cong \mathbb{Z}D_1^2 \oplus \mathbb{Z}D_1 D_2 \oplus \mathbb{Z}D_2^2$$

Actually, the following commutative diagram gives us $D_1 = p^*(h)$.



Now, let's go back to $[W]$. We have shown that there exist $a, b, c \in \mathbb{Z}$ such that

$$[W] = aD_1^2 + bD_1D_2 + cD_2^2 \tag{1}$$

Now, for $x \in \mathbb{P}^3$, let $F = p^{-1}(x)$ be a fiber. $\dim(F) = 2$ (F is isomorphic to a quadric surface), so as for W, we obtain a class $[F] = [p^{-1}(x)] = p^*([x]) \in H_4(\mathcal{H}, \mathbb{Z}) \cong H^6(\mathcal{H}, \mathbb{Z})$. So $[W].[F] \in H^{10}(\mathcal{H}, \mathbb{Z}) \cong \mathbb{Z}$. The intersection $F \cap W$ is a point, because s is a section of p, so we have $[W].[F] = 1$, for all fibers of p. Let's show that this is impossible.

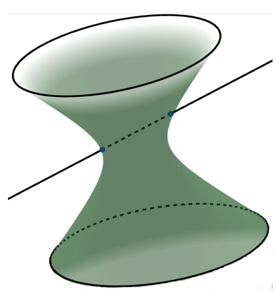
Let $a \in \mathbb{P}^3$. We can write $\{a\}$ as the intersection of 3 hyperplanes in $\mathbb{P}^3$: $a = L_1 \cap L_2 \cap L_3$. Then, $[a] = [L_1 \cap L_2 \cap L_3] = [L_1].[L_2].[L_3]$ in $H^6(\mathbb{P}^3, \mathbb{Z})$, and $\dim(L_i) = 2$. Therefore, for $i = 1, 2, 3$, $[L_i] \in H_4(\mathbb{P}^3, \mathbb{Z}) = H^2(\mathbb{P}^3, \mathbb{Z}) = \mathbb{Z}h$. So there exist $\alpha \in \mathbb{Z}$ such that $[a] = \alpha[h]^3$. Finally, $[F] := [p^{-1}(a)] = p^*([a]) = \alpha D_1^3$.

Now, $[W].[F] = \alpha.(aD_1^5 + bD_1^4.D_2 + cD_1^3.D_2^2)$. We compute each terms. There exist $H_1, \cdots, H_4$ four hyperplanes of $\mathbb{P}^3$ of empty intersection. Then, up to an integer coefficient, $D_1^4 = p^*([h]^4) = [p^{-1}(H_1) \cap \cdots \cap p^{-1}(H_4)] = [p^{-1}(H_1 \cap \cdots \cap H_4)] = 0$. We obtain $[W].[F] = \alpha c D_1^3 D_2^2$, for $F = p^{-1}(a)$ an arbitrary fiber.

Now, we have shown that up to an integer coefficient, $D_1^3 = [p^{-1}(a)]$ for any $a \in \mathbb{P}^3$. Denote $q : \mathcal{H} \longrightarrow \mathbb{P}^3$ the second projection. Then, same as for D_1, $D_2 = q^*(h)$. Let M_1, M_2 be two distinct hyperplanes of $\mathbb{P}^3$. Then, as for D_1, up to an integer coefficient, $D_2^2 = [q^{-1}(M_1 \cap M_2)]$, and $M_1 \cap M_2$ is an arbitrary line in $\mathbb{P}^3$, that we denote L, so $D_1^3.D_2^2 = [p^{-1}(a) \cap q^{-1}(L)]$, for arbitrary $a \in \mathbb{P}^3$, and L line in $\mathbb{P}^3$. Moreover, we can explicit

$$p^{-1}(a) \cap q^{-1}(L) = \{(a, x), \text{ s.t. } x \in L \text{ and } a_0q_0(x) + \cdots a_3q_3(x) = 0\}$$

So this set is the intersection of a smooth quadric and a line. Therefore, we can chose a and L such that this intersection is two points.



Then, up to an integer coefficient, $[F].[W] = 2$, so $[F].[W]$ is even, which contradicts $[F].[W] = 1$. So p has no rational section.

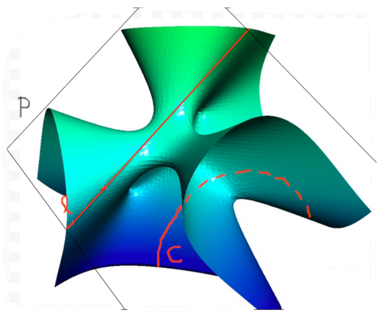
□

Definition 4.3. A $\mathbb{P}^1$-fibration is a morphism of smooth varieties $f : P \longrightarrow X$ such that $\forall x \in X, f^{-1}(x) \cong \mathbb{P}^1$

Example 4.1. Let X be a smooth cubic surface in $\mathbb{P}^3$. Choose a line included in X. For instance, let's suppose that the line l given by $x_0 = x_1 = 0$ is included in X. Then, X is of the form $X = Z_+(x_0q_0 + x_1q_1)$, where $q_0, q_1 \in \mathbb{C}[X_0, X_1, X_2, X_3]$, and $deg(x_0q_0 + x_1q_1) = 3$. X is smooth, so the set $Z_+(x_0, x_1, q_0, q_1)$ is empty. Now, define

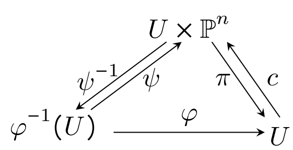
$$\varphi : (x_0 : \cdots : x_1) \in X \backslash l \longrightarrow (x_0 : x_1) \in \mathbb{P}^1$$

Moreover, for points in X where it is defined, we have $\frac{x_0}{x_1} = \frac{q_1}{q_0}$ and $\frac{x_1}{x_0} = \frac{q_0}{q_1}$, so we can extend φ on l by associating, for $x \in l$, $\varphi(x) = (1 : \frac{q_0}{q_1})$ if $q_1 \neq 0$, or $(\frac{q_1}{q_0} : 1)$ otherwise. Let $(a : b) \in \mathbb{P}^1$, $\varphi^{-1}((a : b)) = \{aq_0 - bq_1 = 0\} := C$ is a curve of degree ≤ 2 in $\mathbb{P}^3$. Therefore, C is isomorphic to $\mathbb{P}^1$. So ϕ is a $\mathbb{P}^1$ fibration. Geometrically, if P is the plane containing l and a point in C which is not in l, then $P \cap X = l \cup C$.



Proposition 4.7. A projective bundle is a fiber bundle $\varphi : X \longrightarrow Y$ whose fibers are projective spaces. A projective bundle always have a rational section.

Proof. Indeed, let $x \in Y$. There exist $U \subset Y$ open neighborhood of x and $\psi : \varphi^{-1}(U) \longrightarrow U \times \mathbb{P}^n$ a homeomorphism such that φ agrees with the first projection. Then, we fix $y \in \mathbb{P}^n$ an arbitrary point, and $c : x \in U \longrightarrow (x, y) \in U \times \mathbb{P}^n$. Then, we have the following diagram :



So $s = \psi^{-1} \circ c$ is a rational section of φ □

Recall that we denoted $Y_0 = Y \backslash \{10 \text{ points of singularity of } Y\}$, and $V_0 = b^{-1}(Y_0)$. With $E = b^{-1}(10 \text{ points of singularity of } Y)$ the exceptional divisor, we have $V = V_0 \cup E$.

Definition 4.4. The Brauer group of a variety X is given by

$$Br(X) := \{\mathbb{P}^n - \text{fibration that maps to X}, n \in \mathbb{N}\} \,/\, \{\text{projective bundles}\}$$

For reasons that we will not explain in this report, in our case, we have $Br(Y_0) \cong H^3(Y_0, \mathbb{Z})_{\text{tor}}$. For details, see for instance [4].

Finally, we arrive to the main theorem :

Theorem 4.1. The variety V is not rational, and thus gives a counter-example for the rationality problem in dimension 3.

Proof. By proposition 4.6, it is enough to construct a non zero element in $H^3(V, \mathbb{Z})_{\text{tor}}$. Let $g' = g^{-1}(Y_0) \longrightarrow Y_0$ be the morphism induced by g. We have seen in section 3.1 that its fibers are isomorphic to $\mathbb{P}^1$, and Y_0 and $g^{-1}(Y_0)$ are smooth. So g' is a $\mathbb{P}^1$-fibration. Now, let's prove that g' is not a projective bundle. By proposition 4.7, it suffices to prove that g' has no rational section.

Suppose there exists $s : U \subset Y_0 \longrightarrow g^{-1}(Y_0)$ a rational section of g'. After possibly replacing U by a smaller open subset, we may assume that $U \subset f^{-1}(\Pi \backslash S)$, thus, for each $(q, \bar{l}) \in U$, q has two distinct classes of families of line. Now, if $y = (q, \bar{l}) \in U$, and $\overline{m}$ is the class of the other family of line of q, we write $\sigma(y) = (q, \overline{m})$.

By replacing again U by a smaller open set, we may assume that if $y \in U$, then $\sigma(y) \in U$. Now, for $y = (q, \bar{l}) \in U$, we write $s(y) = (s'(y), L_y)$, where $(s'(y), \overline{L_y}) = (q, \bar{l})$, because s is a section of g'. So $\overline{L_y} \neq \overline{L_{\sigma(y)}}$. So $L_y \cap L_{\sigma(y)}$ is a point, that only depends on q, so we denote it $\phi(q) \in \mathbb{P}^3$, where ϕ is defined on an open set $V \subset \Pi$ (the quadrics that appear in U). Then, $\phi : \Pi \dashrightarrow \mathbb{P}^3$ maps each quadrics of V to a point lying in the quadric. This contradicts the lemma 4.1. So g' has no rational section.

Therefore, we have a non zero element in $Br(Y_0)$. So we obtained $H^3(Y_0, \mathbb{Z})_{\text{tor}} \neq 0$. Then, let $V_0 = b^{-1}(Y_0)$; b induces an isomorphism between V_0 and Y_0. Now, by proposition 4.2 we have a non zero element in $H^3(V_0, \mathbb{Z})_{\text{tor}}$.

Moreover, by properties of the relative cohomology and the Borel-Moore homology (see for instance the Wikipidia page of the Borel-Moore homology for more details), we have the following exact sequence

$$H^3(V, \mathbb{Z}) \longrightarrow H^3(V_0, \mathbb{Z}) \longrightarrow H_2(E, \mathbb{Z})$$

But as we saw in section 3.2, $E = E_1 \sqcup \cdots \sqcup E_{10}$, with $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$. By Poincaré duality and the Kunneth formula, $H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}^2$. So $H_2(E, \mathbb{Z}) \cong \mathbb{Z}^{20}$ has no torsion. So we obtain a non zero element in $H^3(V, \mathbb{Z})_{\text{tor}}$

□

5 A glimpse of the decomposition of the diagonal method

The proof of the non rationality of the Artin-Mumford example uses a birational invariant : the torsion of H^3. In this section, we develop an other birational invariant, which has been introduced in [8] by Claire Voisin. She proved that it can be use to show stable irrationality of many classes of unirational varieties.

Definition 5.1. Let X be a variety of dimension n. Then, let $\Delta_X = \{(x, x) \in X \times X, x \in X\}$ the diagonal, Δ_X is a subvariety of $X \times X$ of dimension n. We say that X has a cohomological decomposition of the diagonal if we can write

$$[\Delta_X] = [\Gamma] + [X \times x]$$

in $H^{2n}(X \times X, \mathbb{Z})$, with $x \in X$ and Γ a cycle supported in $D \times X$ for some closed proper subset $D \subsetneq X$.

Theorem 5.1. Having a cohomological decomposition of the diagonal is a birational property, and $\mathbb{P}^n$ satisfy this property.

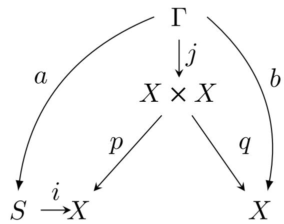
For a proof of this theorem, see [8]. We will only prove the following theorem :

Theorem 5.2. If X has a cohomological decomposition of the diagonal, then $H^3(X, \mathbb{Z})_{\text{tor}} = 0$.

Proof. Let X be a variety of dimension n. Then, Δ_X has dimension n, so we write in $H^{2n}(X \times X, \mathbb{Z}), [\Delta_X] = [\Gamma] + [X \times x]$. In this proof, we suppose Γ smooth, we can take the desingularization of Γ if it is not true, and adapt the proof. Let $p, q : X \times X \longrightarrow X$ be the first and second projections. Now, for $\alpha \in H^{2n}(X \times X, \mathbb{Z})$, we define the following map as the composition:

$$\alpha_3 : H^3(X,\mathbb{Z}) \xrightarrow{q^*} H^3(X\times X,\mathbb{Z}) \xrightarrow{.\alpha} H^{3+2n}(X\times X,\mathbb{Z}) = H_{2n-3}(X\times X,\mathbb{Z}) \xrightarrow{p_*} H_{2n-3}(X,\mathbb{Z}) = H^3(X,\mathbb{Z})$$

Now, if X has a cohomological decomposition of the diagonal, then $[\Delta_X]_3 = [\Gamma]_3 + [X \times x]_3$. Then, one can show that $[\Delta_X]_3$ is the identity, and that $[X \times x]_3 = 0$. Let $\sigma \in H^3(X,\mathbb{Z})$. We have then $\sigma = [\Gamma]_3(\sigma) = p_*(q^*(\sigma).[\Gamma])$. Let $S \subset X$ be a subvariety of dimension $n-1$ such that $\Gamma \subset S \times X$. We look at the following diagram, in which i and j are the inclusions, and a and b are induced by restricting respectively p and q:



Then, following from the definition of the cup product, one can show that for $\sigma \in H^3(X,\mathbb{Z})$, $p_*(q^*(\sigma).[\Gamma]) = i_*(a_*(b^*(\sigma)))$. Moreover, $b^*(\sigma) \in H^3(\Gamma,\mathbb{Z}) \cong H_{2n-3}(\Gamma,\mathbb{Z})$, by Poincaré duality, so $a_*(b^*(\sigma)) \in H_{2n-3}(S,\mathbb{Z}) \cong H^1(S,\mathbb{Z})$, which has no torsion by Proposition 4.5. So if $\sigma \in H^3(X,\mathbb{Z})_{\text{tor}}$, then $a_*(b^*(\sigma)) = p_*(q^*(\sigma).[\Gamma]) = \sigma = 0$. We proved $H^3(X,\mathbb{Z})_{\text{tor}} = 0$

□

But the major recent breakthrough using the cohomological decomposition of the diagonal is the following theorem (see [8]):

Theorem 5.3. Let $(\chi_b)_{b\in\mathcal{B}}$ be a family of projective variety ($\chi \longrightarrow B \ni 0$ is a morphism satisfying some properties, and the χ_b are the fibers). Suppose that χ_b is smooth for $b \neq 0$, and has at worst ordinary quadratic singularity for $b = 0$. Then χ_b has a cohomological decomposition of the diagonal for b very general $\implies$ χ_0 has a cohomological decomposition of the diagonal.

This theorem has huge consequences. Indeed, if one wants to show non stable-rationality for a family of projective varieties, one can construct a "limit" χ_0 that does not admit a cohomological decomposition of its diagonal. This method was for instance used by Voisin to prove that a very general smooth quartic double solid with $k \leq 7$ nodes is not stably rational.

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