Around the limiting fluctuations of diverse determinantal point processes

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Abstract

We study in this report the fluctuations arising from linked random matrix models and free fermions models. More precisely, we show that for linear statistics of eigenvalues arising from the Elliptic Ginibre Ensemble, a uniform Central Limit Theorem arises, with the variance being the sum of the H^1 norm of the test function on the ellipse $\mathcal{E}_\tau$ and of the $H^{1/2}$ norm of the test function on the boundary of the same ellipse. We also study the covariance structure of a harmonic oscillator model of free fermions, which we show to correspond to a weighted $H^{1/2}$ norm on the bulk.

Contents

1 Introduction and main results

1.1 Random matrix theory

Random Matrix Theory is a very dynamical field of study nowadays. The basic idea is to take some family of random variable $(X_{ij})_{i,j\in\mathbb{N}^*}$ with some hypothesis on the law of the variables and their respective independance and to study the distribution of the eigenvalues of the matrix $\frac{1}{f(N)}\mathbf{\Pi}_N\mathbf{X}\mathbf{\Pi}_N$, where $f(N)$ is a certain normalization (which usually turns out to be $f(N)=\sqrt{N}$), and the $N\times N$-sized matrix $\mathbf{\Pi}_N\mathbf{X}\mathbf{\Pi}_N$

is defined $(\mathbf{\Pi}_N \mathbf{X} \mathbf{\Pi}_N)_{ij} = X_{ij}$ for $1 \leq i, j \leq N$ ($\mathbf{\Pi}_N$ will be seen later as a projection operator).

A first model that might come to mind would be when the (X_{ij}) are iid (independant, identically distributed) zero mean complex random variables with variance 1. This model has been already widely studied, for instance in [3], and as $N \to \infty$, the eigenvalues concentrate uniformly on the unit disk in the complex plane $\mathbb{D} = \{z, |z| \leq 1\}$.

Another standard model that might come to mind, is when the matrix is Hermitian : X_{ij} are iid mean zero complex random variables of variance 1 for $i \leq j$ and for $i \neq j$, $X_{ij} = \overline{X_{ji}}$. Then, the eigenvalues of $\frac{1}{f(N)} \mathbf{\Pi}_N \mathbf{X} \mathbf{\Pi}_N$ are real (the matrix is hermitian) and as $N \to \infty$, they tend to be distributed according with the semi-circular law $\frac{1}{2\pi}\sqrt{4 - x^2}\mathbb{1}_{|x| \leq 2}$.

There are a lot of further question that can be studied. The first part of what we are going to study concerns the special case when the X_{ij} are furthermore gaussian. What will be of interest to us is the fluctuations behind these behaviors previouly mentionned. Let us make that more precise.

Perhaps the most standard model in random matrix theory is actually the Gaussian Unitary Ensemble (GUE), which describes the process of the eigenvalues of a conveniently normalised $N \times N$ Hermitian matrix with independant and standard complex gaussian entries in the upper-right triangle. This model has been widely studied in the literature already [2]. Our point of interest will be linked to the fluctuations of the linear statistics of this process : denoting $\lambda_i^{N,1}$ the points (eigenvalues) of the process, in [12] it has been proven that for a polynomial $f : \mathbb{R} \to \mathbb{R}$ the following convergence in law of the linear statistics holds

$$\sum_{i=1}^{N} f(\lambda_i^{N,1}) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^{N,1})\right) \xrightarrow[N \to \infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2_{f,\mathrm{GUE}}) \tag{1}$$

where, denoting $F(\theta) = f(2\cos\theta)$,

$$\sigma^2_{f,\mathrm{GUE}} = \frac{1}{2}\sum_{k \in \mathbb{Z}} |k| \widehat{F}(k) \widehat{F}(-k) = \frac{1}{8\pi^2} \int_{[-2,2]^2} \left| \frac{f(x) - f(y)}{x - y} \right|^2 \frac{4 - xy}{\sqrt{4 - x^2}\sqrt{4 - y^2}} dx dy.$$

There are several generalisation possibles of this model and the fluctuations of some of such generalisations are what are of interest here.

1.2 Determinantal point process

Another way to look at the GUE is to see it as a random point process : we study only the eigenvalues of the matrix, we therefore end up only with a random set of N points in $\mathbb{R}$. What makes the GUE easier to study is that it's a particular kind of point process : a determinantal point process. That means that for borel sets $B_1, \ldots, B_N \subset \mathbb{R}$, we have

$$\mathbb{E}\left(\mathbb{1}_{\lambda_1^N \in B_1, \ldots \lambda_N^N \in B_N}\right) = \int_{B_1 \times \cdots \times B_N} \left(\det_{1 \leq i,j \leq N} K_N(x_i, x_j) \right) dx_1 \ldots dx_N,$$

which is the general form for any determinantal point process, and in the case of the GUE, the kernel $K_N(x, y)$ is

$$K_N(x, y) = \sum_{j=0}^{N-1} \frac{1}{2^j j!} H_j\left(\sqrt{\frac{N}{2}} x\right) H_j\left(\sqrt{\frac{N}{2}} y\right) \frac{N}{\pi} e^{-N\frac{x^2+y^2}{2}}$$

where the H_j are the Hermite polynomials $H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}$. The reason we look at this model in this way is that determinantal point processes tend to have a nice expression for observables of linear statistics, as already widely seen in [12]. For instance, taking $f : \mathbb{R} \to \mathbb{R}$, the expectancy of the linear statistics can be written as

$$\mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^N)\right) = \int_{\mathbb{R}} f(x) K_N(x,x) dx,$$

and the variance can also be written

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} f(\lambda_i^N) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^N)\right)\right)^2\right] = \int_{\mathbb{R}} f(x)^2 K_N(x,x) dx - \int_{\mathbb{R}^2} f(x) f(y) K_N(x,y)^2 dx dy.$$

Some more general formulas will be used for the study of the fluctuations, and notably for their cumulants. For a more complete introduction and general facts on determinantal point processes, we refer to [17].

1.3 The Elliptic Ginibre Ensemble

For the first part then, we study an interpolation between the GUE and another classical model : the Ginibre ensemble, which is the process of the eigenvalues of the matrix described in the first part of the introduction, where all the X_{ij} are independant, and we assume furthermore that they are standard gaussian. To define the interpolation model, called the elliptic Ginibre ensemble, let us take $\tau \in [0,1]$ and H_1 and H_2 two matrices sampled independently from the GUE. The elliptic Ginibre ensemble is the process of the eigenvalues of the random matrix $\sqrt{\frac{1+\tau}{2}} H_1 + i\sqrt{\frac{1-\tau}{2}} H_2$. We recover the GUE for $\tau = 1$ and the Ginibre ensemble for $\tau = 0$. Another way of seeing it is that for H sampled from the τ-elliptic Ginibre ensemble, $\mathbb{E}(H_{ij} H_{ji}) = \frac{\tau}{N}$. It has already been widely studied in [10] and the process asymptotically concentrates uniformly in an ellipse $\mathcal{E}_\tau := \left\{ z \in \mathbb{C}, \frac{(\mathrm{Re} z)^2}{(1+\tau)^2} + \frac{(\mathrm{Im} z)^2}{(1-\tau)^2} \leq 1 \right\}$.

We will be interested here by the fluctuations of the linear statistics of the process ; we will follow the reasoning of [13], and we will show a resembling generalization of the following main theorem of [13] valid for the Ginibre ensemble :

Theorem 1. *If f grows at most exponentially fast at infinity and is differentiable in $(1+\varepsilon)\mathbb{D}$ for some $\varepsilon > 0$, we have the Central Limit Theorem*

$$\sum_{i=1}^{N} f(\lambda_i^{N,0}) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^{N,0})\right) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma_{f,\mathbb{D}}^2 + \sigma_{f,\partial\mathbb{D}}^2)$$

where

$$\sigma_{f,\mathbb{D}}^2 := \frac{1}{4\pi}\int_{\mathbb{D}} |\nabla f|^2 \quad \text{and} \quad \sigma_{f,\partial\mathbb{D}}^2 := \frac{1}{2}\sum_{k\in\mathbb{Z}} |k| \left|\widehat{f}(k)\right|^2 \tag{2}$$

and the Fourier transform is the usual Fourier transform in $\partial\mathbb{D}$.

We will actually show two fluctuations theorems. The first one is a direct generalisation of 1 and of the result 1 given in [13] : defining the Fourier transform in $\partial\mathcal{E}_\tau$,

$$\widehat{f}^{(\partial\mathcal{E}_\tau)}(k) := \frac{1}{2\pi(1-\tau^2)} \int_{\partial\mathcal{E}_\tau} f(z) \left(\frac{\mathrm{Re} z}{1+\tau} - i\frac{\mathrm{Im} z}{1-\tau}\right)^k d\lambda^{(\partial\mathcal{E}_\tau)}, \tag{3}$$

where $\lambda^{(\partial\mathcal{E}_\tau)}$ is the Lebesgue measure on $\partial\mathcal{E}_\tau$, the following two expressions will be contributing to the variance of the limiting distribution

$$\sigma_{f,\mathcal{E}_\tau}^2 := \frac{1}{4\pi}\int_{\mathcal{E}_\tau} |\nabla f|^2 \quad \text{and} \quad \sigma_{f,\partial\mathcal{E}_\tau}^2 := \frac{1}{2}\sum_{k\in\mathbb{Z}} |k| \left|\widehat{f}^{(\partial\mathcal{E}_\tau)}(k)\right|^2 \tag{4}$$

so denoting $\lambda_i^{N,\tau}$ the points of the process, we can now state the first main result of the study :

Theorem 2. *For $\tau \in [0,1)$, if f grows at most exponentially fast at infinity and is differentiable in $(1+\varepsilon)\mathcal{E}_\tau$ for some $\varepsilon > 0$, we have the Central Limit Theorem*

$$\sum_{i=1}^{N} f(\lambda_i^{N,\tau}) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^{N,\tau})\right) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2_{f,\mathcal{E}_\tau} + \sigma^2_{f,\partial\mathcal{E}_\tau}).$$

In regards to the result 1 in [13], this is not surprising. We end up with the same covariance structure, showing that the process behaves as the sum of a H^1 noise on the ellipse $\mathcal{E}_\tau$ and an independant $H^{1/2}$ noise on the boundary of the ellipse $\partial\mathcal{E}_\tau$. Assuming more regularity on f, we also show that we can have uniformity in the convergence, eliminating any chance of things happening for the fluctuations in a weak non-hermicity regime.

Theorem 3. *If f grows at most exponentially fast at infinity and is differentiable in $\bigcup_{\tau\in[0,1]} (1+\varepsilon)\mathcal{E}_\tau$ for some $\varepsilon > 0$, we have the uniform Central Limit Theorem*

$$\sum_{i=1}^{N} f(\lambda_i^{N,\tau}) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^{N,\tau})\right) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\mathcal{E}_\tau} + \sigma^2_{\partial\mathcal{E}_\tau}) \quad \text{uniformly in } \tau \in [0,1]. \tag{5}$$

where $\sigma^2_{\mathcal{E}_\tau} + \sigma^2_{\partial\mathcal{E}_\tau}$ gets replaced by $\sigma^2_{f,GUE}$ when $\tau = 1$.

Here, the uniformity in the convergence in law 5 means that for any continuous and bounded $\phi : \mathbb{R} \to \mathbb{R}$, and denoting $(Z^\tau)_{\tau\in[0,1]}$ a family of random variable respectively of law $\mathcal{N}(0, \sigma^2_{\mathcal{E}_\tau} + \sigma^2_{\partial\mathcal{E}_\tau})$, we have

$$\mathbb{E}\left(\phi\left(\sum_{i=1}^{N} f(\lambda_i^{N,\tau}) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^{N,\tau})\right)\right)\right) \xrightarrow[N\to\infty]{} \mathbb{E}\left(\phi\left(Z^\tau\right)\right) \quad \text{uniformly in } \tau \in [0,1].$$

It therefore means that we have the same result if we take τ depending on N. Indeed, 5 implies that taking $\tau_N \in [0,1]$ a converging sequence, converging to some $\tau_\infty \in [0,1]$, we have the convergence in law

$$\sum_{i=1}^{N} f(\lambda_i^{N,\tau_N}) - \mathbb{E}\left(\sum_{i=1}^{N} f(\lambda_i^{N,\tau_N})\right) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\mathcal{E}_{\tau_\infty}} + \sigma^2_{\partial\mathcal{E}_{\tau_\infty}}). \tag{6}$$

The uniformity is particularly interesting in this case because it has already been seen that some weak non-hermiticity phenomenon happen : the idea is to make τ depend on N and take $\tau_N \in [0,1]$ a sequence converging to 1, at some particular speed, usually of the form $\tau_N = 1 - \frac{\kappa}{N^\gamma}$, and then some phenomenon happen, different from the case where τ does not depend on N. Some examples of such phenomenon may be found in [1]. But in the case of that Central Limit Theorem, this proves that there is no such phenomenon that appears : making τ depending on N does not change the result, in particular in any weak non hermiticity regime $\tau_N \underset{N\to\infty}{\to} 1$.

1.4 A free Fermions model in n dimensions

As a second part of the study, we studied the fluctuations of the linear statistics of the positions of free fermions with quadratic potential in $\mathbb{R}^n$ when $n \geq 2$.

Let us first write the stationary Schrödinger equation in $\mathbb{R}^n$

$$(-\hbar^2\Delta + V(x))\psi(x) = \varepsilon\psi(x),$$

and we take the case of a quadratic potential $V(x) = x \cdot \mathbf{O}x$ where $\mathbf{O}$ may be any positive orthogonal matrix. Diagonalizing $\mathbf{O}$, we may suppose without loss of generality that $\mathbf{O} = \mathcal{T}^2 = \mathrm{Diag}(\tau_1^2, \dots, \tau_n^2)$ where $\tau_1, \dots \tau_n > 0$. For our study, we will fix such a $\vec{\tau} = (\tau_1, \dots, \tau_n)$ and we assume that there exists $T > 0$ such that $T\vec{\tau} \in \mathbb{N}^n$. We will see that this condition actually appears naturally.
We study non-interacting particles that obey the Schrödinger equation with some quadratic potential with energy less than a fixed $\mu > 0$ called the Fermi energy, and that obey the Pauli exclusion principle : fermions. Such particles form a random point process. This point process is in fact determinantal. Introducing the Hermite functions $\psi_k(x) := \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_k(x)$ which are orthogonal with respect to the Lebesgue measure on $\mathbb{R}$, we define the eigenfunctions

$$\psi_{\vec{k}}^{\hbar,\vec{\tau}}(x) := \sqrt{\frac{\hbar^n}{\tau_1 \dots \tau_n}} \psi_{k_1}\left(\sqrt{\frac{\tau_1}{\hbar}} x_1\right) \dots \psi_{k_n}\left(\sqrt{\frac{\tau_n}{\hbar}} x_n\right)$$

which are eigenvector of the operator $-\hbar^2 \Delta + V$ with eigenvalue

$$\varepsilon(\vec{k}) := \left((2k_1+1)\tau_1 + \dots + (2k_n+1)\tau_n\right)\hbar. \tag{7}$$

Denoting $N := \mathrm{Card}\left\{\vec{k}, \varepsilon(\vec{k}) \le \mu\right\}$, these free fermions particles with energy less than μ are described by a random point process in $(\mathbb{R}^n)^N$ of probability density

$$\mathbb{P}_N(x_1, \dots, x_N) = \frac{1}{N!} \left| \det_{\varepsilon(\vec{k}) \le \mu, 1 \le i \le N} (\psi_{\vec{k}}^{\hbar,\vec{\tau}}(x_i)) \right|^2 = \frac{1}{N!} \det_{N \times N} \left(\sum_{\varepsilon(\vec{k}) \le \mu} \psi_{\vec{k}}^{\hbar,\vec{\tau}}(x_i) \psi_{\vec{k}}^{\hbar,\vec{\tau}}(x_j) \right),$$

which is a determinantal point process of kernel

$$K_\hbar(x,y) = \sum_{\varepsilon(\vec{k}) \le \mu} \psi_{\vec{k}}^{\hbar,\vec{\tau}}(x) \psi_{\vec{k}}^{\hbar,\vec{\tau}}(y). \tag{8}$$

These particles tend to concentrate in the ellipse

$$\mathcal{E}_{\vec{\tau},\mu} := \left\{x, x \cdot \mathcal{T}^2 x \le \mu\right\},$$

according to a limiting distribution which has the following density with respect to the Lebesgue measure on $\mathcal{E}_{\vec{\tau},\mu}$: $\frac{1}{Z}\left(\mu - x \cdot \mathcal{T}^2 x\right)^{\frac{n}{2}}$, where Z is a normalizing constant [5].
We are interested by the behaviour of the fluctuations of the process when $\hbar \to 0$: $\hbar$ of course plays the role of the Planck constant, which is very small.
Taking $f : \mathbb{R}^n \to \mathbb{R}$ nice, say of regularity $\mathcal{C}^1$ and compactly supported, denoting x_i the positions associated with the fermions, N the number of particles, which is equivalent to $\frac{\mu^n}{2^n \tau_1 \dots \tau_n \hbar^n n!}$, and

$$X_\hbar(f) := \sum_{i=1}^N f(x_i)$$

the linear statistics it has already been proven for its fluctuations in [15] that

$$\frac{X_\hbar(f) - \mathbb{E}(X_\hbar(f))}{\sqrt{\mathrm{Var}(X_\hbar(f))}} \xrightarrow[\hbar \to 0]{\mathcal{L}} \mathcal{N}(0,1) \quad \text{if} \quad \mathrm{Var}(X_\hbar(f)) \underset{\hbar \to 0}{\longrightarrow} \infty. \tag{9}$$

What we will be doing here is then only to estimate $\mathrm{Var}(X_\hbar(f))$, and we will a fortiori see that we do have the estimate $\mathrm{Var}(X_\hbar(f)) \underset{\hbar \to 0}{\longrightarrow} \infty$ so that the convergence 9 is true, but we will also expose a finer estimate, and different interesting expressions of that estimate.

We mention first that this is a generalisation of the GUE as when $n=1$, we recover the GUE (but then the estimate $\mathrm{Var}(X_\hbar(f)) \underset{\hbar\to 0}{\longrightarrow} \infty$ is false).
Our first result is interesting because the estimate seems rather general, and not bound to the special form of the potential V.
We introduce it now : let $\phi : [0,\infty) \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the flow of the Hamiltonian evolution associated with the Hamiltonian

$$\mathcal{H}(x,\xi) = \xi^2 + V(x) = \xi^2 + x \cdot \mathcal{T}^2 x. \tag{10}$$

Writing $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$ the projection onto the first n coordinates, we have the following CLT :

Theorem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}$. If f grows at most exponentially fast at infinity and is differentiable in $(1+\varepsilon)\mathcal{E}_{\vec{\tau},\mu}$ for some $\varepsilon > 0$, we have the following Central Limit Theorem :*

$$\hbar^{\frac{n-1}{2}} \left(X_\hbar(f) - \mathbb{E}(X_\hbar(f))\right) \underset{\hbar\to 0}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0,\sigma_f^2). \tag{11}$$

where

$$\sigma_f^2 = \frac{4}{(2\pi)^{n+1}} \int_{\mathcal{H}(x_0,\xi_0)=\mu} \int_0^\infty \left| \frac{f\left(\pi\left(\phi(t,x_0,\xi_0)\right)\right) - f\left(\pi\left(\phi(0,x_0,\xi_0)\right)\right)}{t} \right|^2 dt dx_0 d\xi_0. \tag{12}$$

Remark 1. *Our assumption that there exists $T > 0$ such that $T\vec{\tau} \in \mathbb{N}^n$ corresponds to the cases when the flow is periodic. Taking the smallest $T > 0$ satisfying the last condition, the flow is πT periodic, and we can also express σ_f^2 as an integral over half the cycles of the flow:*

$$\sigma_f^2 = \frac{1}{(2\pi)^{n+1}} \int_{\mathcal{H}(x_0,\xi_0)=\mu} \frac{1}{T^2} \int_0^{\frac{\pi T}{2}} \left| \frac{f\left(\pi\left(\phi(t,x_0,\xi_0)\right)\right) - f\left(\pi\left(\phi(0,x_0,\xi_0)\right)\right)}{\sin\left(\frac{t}{2T}\right)} \right|^2 dt dx_0 d\xi_0.$$

Dropping the condition on $\vec{\tau}$ would probably result in the same estimate.
For general potential V, the question is more complicated; first because the flow of the Hamiltonian might not even be defined for all $t \in \mathbb{R}_+$. However, under reasonable assumptions on V – for instance that V be of regularity C^1 and $V(x) \underset{|x|\to\infty}{\longrightarrow} \infty$ – the flow is defined for all $t \in \mathbb{R}_+$ using notably Cauchy-Lipschitz's Theorem. Then we might expect the same limit:

$$\mathrm{Var}(X_\hbar(f)) \underset{\hbar\to 0}{\sim} \frac{4}{\hbar^{n-1}(2\pi)^{n+1}} \int_{\mathcal{H}(x_0,\xi_0)=\mu} \int_0^\infty \left| \frac{f\left(\pi\left(\phi(t,x_0,\xi_0)\right)\right) - f\left(\pi\left(\phi(0,x_0,\xi_0)\right)\right)}{t} \right|^2 dt dx_0 d\xi_0.$$

The case $n=1$ has already been treated for general potential V in a recent article of Delporte and Lambert [6], but it is somehow different from the dimensions $n \geq 2$ because in dimension 1, the flow is periodic and there is only one orbit for the flow of the Hamiltonian evolution, under a natural connectivity assumption of the subset $\{V \leq \mu\}$.

The second estimate shows the limiting variance as a weighted $H^{1/2}$ norm on the ellipse $\mathcal{E}_{\vec{\tau},\mu}$:

Theorem 5. *Let $f : \mathbb{R}^n \to \mathbb{R}$. If f grows at most exponentially fast at infinity and is differentiable in $(1+\varepsilon)\mathcal{E}_{\vec{\tau},\mu}$ for some $\varepsilon > 0$, we have the following Central Limit Theorem :*

$$\hbar^{\frac{n-1}{2}} \left(X_\hbar(f) - \mathbb{E}(X_\hbar(f))\right) \underset{\hbar\to 0}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0,\sigma_f^2). \tag{13}$$

with

$$\sigma_f^2 = \int_{\mathcal{E}_{\vec{\tau},\mu}^2} \frac{|f(x) - f(y)|^2}{|x-y|^{n+1}} \rho_{\vec{\tau}}(x,y) dx dy \tag{14}$$

where $\rho_{\vec{\tau}}(x,y)$ is a non-negative weight function which is non zero and non singular on the diagonal $\Delta_{\vec{\tau}} = \left\{(z,z), z \cdot \mathcal{T}^2 z \leq \mu\right\}$.

In the case $\vec{\tau} = (1, \dots, 1)$, we find an explicit form for $\rho_{\vec{\tau}}$, using the scaled variables $a = \frac{x}{\sqrt{\mu}}$, $b = \frac{y}{\sqrt{\mu}}$:

$$\rho_{(1,\dots,1)}(x,y) = \mathbb{1}_{(x\cdot y)^2+\mu^2 \geq \mu(|x|^2+|y|^2)} \frac{2^{n-1}}{\sqrt{n}(2\pi)^{n+1}} \frac{\mu \cdot \mu^{\frac{n-1}{2}}}{\mu 2^n |a+b|^{n-1}\sqrt{(a\cdot b)^2 - |a|^2 - |b|^2 + 1}}$$
$$\times \Bigg[\left(1 - (a\cdot b - \sqrt{(a\cdot b)^2 - |a|^2 - |b|^2 + 1})\right) \left(\sqrt{|a|^2 + |b|^2 - 2(a\cdot b)^2 + 2(a\cdot b)\sqrt{(a\cdot b)^2 - |a|^2 - |b|^2 + 1}}\right)^{n-1}$$
$$+ \left(1 - (a\cdot b + \sqrt{(a\cdot b)^2 - |a|^2 - |b|^2 + 1})\right) \left(\sqrt{|a|^2 + |b|^2 - 2(a\cdot b)^2 - 2(a\cdot b)\sqrt{(a\cdot b)^2 - |a|^2 - |b|^2 + 1}}\right)^{n-1} \Bigg]. \tag{15}$$

We could also expect an estimate of the type of 14 for more general potential V, because we also expect a $H^{1/2}$ norm to appear for general V.

1.5 Contents of this report

Here, we will only show the polynomial version of Theorems 2 and 3, although all of the results have been shown in their whole generality during the internship. Proofs for all the results should appear in an article to come.

2 Fluctuations of the Elliptic Ginibre Ensemble

2.1 Generalities on the Elliptic Ginibre Ensemble

The first goal of this paper is to study the fluctuations of the elliptic Ginibre ensemble, as advertised in the introduction. Let's explicit this process by first expliciting the normalization in the GUE : a random matrix H of size $N \times N$ is said to be sampled from the GUE if it is hermitian, and the variables in the upper right triangle are independent centered complex gaussian random variables of variance $\frac{1}{N}$, which is to say that their probability distribution function is $\frac{N}{\pi}e^{-N|z|^2}$. Taking a matrix sampled from the GUE that way also amounts to taking H randomly from the probability measure on the space of $N\times N$-sized hermitian matrices which has density with respect to the Lebesgue measure proportional to $e^{-N\frac{\operatorname{Tr} H^2}{2}}$.
Then, taking H_1 and H_2 independantly randomly sampled from the GUE, the elliptic Ginibre ensemble with parameter τ is the process of the random eigenvalues of the random matrix $\sqrt{\frac{1+\tau}{2}}H_1 + i\sqrt{\frac{1-\tau}{2}}H_2$. This process is in fact a determinantal point process [10, 14] with kernel

$$K_N^\tau(z,z') = \sum_{j=0}^{N-1} \frac{\tau^j}{2^j j! \sqrt{\pi}} H_j\left(\sqrt{\frac{N}{2\tau}}z\right) \overline{H_j\left(\sqrt{\frac{N}{2\tau}}z'\right)} \sqrt{\mu_N^\tau(z)\mu_N^\tau(z')},$$

where the H_j are the Hermite polynomials defined by $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$, and the weight functions μ supported on $\mathbb{C}$ when $\tau < 1$ and on $\mathbb{R}$ when $\tau = 1$ are defined by

$$\mu_N^\tau(z) := \frac{Ne^{-N\left(\frac{(\operatorname{Re} z)^2}{1+\tau} + \frac{(\operatorname{Im} z)^2}{1-\tau}\right)}}{\sqrt{\pi(1-\tau^2)}} \quad \text{and} \quad \mu_N^1(x) := \sqrt{N}e^{-N\frac{x^2}{2}}.$$

We mention the weight function μ when $\tau = 1$ but we will not detail the case $\tau = 1$ in this part of the Elliptic Ginibre ensemble. It is already available in the literature as mentioned in the introduction, and it is also actually a direct consequence of the study of the free fermions model that we mentionned in the introduction. All the computations done in this part do work in the case $\tau = 1$ but a small change of notations is always needed for this degenerate case. This will not endanger the uniformity in Theorem 3 because adding just the case $\tau = 1$ to the uniform cases $\tau \in [0, 1)$ does not affect the uniformity.

We introduce the functions

$$\phi_n^{N,\tau}(z) := \tau^{\frac{n}{2}} \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n\left(\sqrt{\frac{N}{2\tau}} z\right) \sqrt{\mu_N^\tau(z)} \tag{16}$$

such that the kernel takes the simplified expression

$$K_N^\tau(z, z') = \sum_{j=0}^{N-1} \phi_j^{N,\tau}(z) \overline{\phi_j^{N,\tau}(z')}. \tag{17}$$

The Hermite polynomials satisfy an orthogonality relation on $\mathbb{C}$ which matches the symetry of the elliptic weight [8] : $\frac{1}{\sqrt{\pi(1-\tau^2)}} \frac{\tau^n}{2^n n! \sqrt{\pi}} \int H_n\left(\frac{z}{\sqrt{2\tau}}\right) \overline{H_m\left(\frac{z}{\sqrt{2\tau}}\right)} \mu_1^\tau(z) dz = \delta_{nm}$, where the integral is over $\mathbb{C}$ when $\tau < 1$. This orthogonality relation implies by a change of variables the orthogonality of the scaled and normalized polynomials :

$$\int_{\mathbb{C}} \phi_n^{N,\tau}(z) \overline{\phi_m^{N,\tau}(z)} dz = \delta_{nm}. \tag{18}$$

With $\lambda_i^{N,\tau}$ denoting the random eigenvalues of the elliptic ensemble, and for $f : \mathbb{C} \to \mathbb{R}$, we use the notation for the linear statistics

$$X_N^\tau(g) := \sum_{i=1}^N f(\lambda_i^{N,\tau}).$$

We already know that this process concentrates as $N \to \infty$ in the ellipse $\mathcal{E}_\tau := \left\{ z \in \mathbb{C}, \frac{(\text{Re}z)^2}{(1+\tau)^2} + \frac{(\text{Im}z)^2}{(1-\tau)^2} \leq 1 \right\}$: almost surely, for, say, continuous f, $\frac{X_N^\tau(f)}{N} \underset{N\to\infty}{\to} \frac{1}{\pi(1-\tau^2)} \int_{\mathcal{E}_\tau} f$. We study here the fluctuations within that convergence and that is the essence of Theorems 2 and 3 : we study the asymptotic behaviour of the random variable $X_N^\tau - \mathbb{E}(X_N^\tau)$. Usually, when showing such a Central Limit Theorem, a normalisation in terms of powers of N is needed for something non trivial to appear, but in the case of Random Matrix Theory, interesting behaviour usually appear without normalisation, and this is the case here.

Thereafter, we give the proof of the following theorem, which is the polynomial version of Theorems 2 and 3:

Theorem 6. *For test functions f polynomial in z and $\overline{z}$, we have the Central Limit Theorem*

$$\sum_{i=1}^N f(\lambda_i^\tau) - \mathbb{E}\left(\sum_{i=1}^N f(\lambda_i^\tau)\right) \underset{N\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0, \sigma_{f,\mathcal{E}_\tau}^2 + \sigma_{f,\partial\mathcal{E}_\tau}^2) \quad \textit{uniformly in } \tau,$$

where the variance structure given by the $\sigma_{f,\cdot}$ is given in the introduction in equation 4.

Here we finish by giving a small outline of the proof.
The first step is to use a recursion relations for the version of the Hermite polynomials 16 to express the cumulants $\mathcal{C}_n(X_N^\tau(g))$ of the random variable X_N^τ when g is a polynomial in $(z, \overline{z})$. These expressions are amenable to asymptotic analysis and we will show that the cumulants vanish for $n \geq 3$ in the limit $N \to \infty$, uniformly in τ.
Then, to extend the result for general test functions f, we would show a bound of the type $\text{Var}(X_N^\tau(f)) \leq C\|\nabla f\|_{L^\infty((1+\varepsilon)\mathcal{E}_\tau)} + \underset{N\to\infty}{o}(1)$ where C is a universal constant, a fortiori independent of N, τ and f, and use the density of the polynomials given by Stone-Weierstrass's Theorem to finally prove Theorems 2 and 3, but that won't be done in this report, only the previous part will be shown here.

2.2 Computation of the cumulants

In this section and the following, we will be as announced showing the previous Theorem 6.
The proof of this theorem is made in two steps. The first step is to compute the cumulants of the random variable $X_N^\tau - \mathbb{E}(X_N^\tau)$, the second step will be to check that the variance is the one given in the theorem, and the final move will be to gather these results to have the uniform convergence in law.

This first step will be to show that the cumulants $\mathcal{C}_n(X_N^\tau(f))$ of the random variable $X_N^\tau(f)$ converge to 0 as $N \to \infty$ for $n \geq 3$, uniformly in τ. Let us first recall the formal definition of the cumulants: they are defined by the expansion $\log \mathbb{E}(e^{itX}) = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \mathcal{C}_n(X)$ when it is well-defined (which is the case in our study). Let us mention that under reasonable assumption on the random variable, the cumulants have all the information of the random variable ([11], Lemma 4.8.), and that is why it seems reasonable to study the cumulants of the random variable. Here, the reason for using the cumulants, and not the moments for example, is that to show convergence in law to a gaussian random variable, it is enough to show the convergence of the $n \geq 3$–cumulants to 0. Furthermore, for a determinantal point process, the cumulants have an easy form [4]:

$$\mathcal{C}_n(X_N^\tau(f)) = \sum_{m=1}^{n-1} \frac{(-1)^{m-1}}{m} \sum_{k_1+\cdots+k_m=n,\ k_i \geq 1} \frac{k!}{k_1! \dots k_m!} \int_{\mathbb{C}} \left(\prod_{l=1}^{m} (g(z_i))^{k_l} K_N^\tau(z_l, z_{l+1}) \right) dz_1 \dots dz_m.$$

The first step of the proof is as announced this Proposition :

Proposition 7. *For polynomial test function f, we have the following convergence of the cumulants : for $n \geq 3$,*

$$\mathcal{C}_n(X_N^\tau(f)) \underset{N\to\infty}{\to} 0 \quad \textit{uniformly in } \tau.$$

Before going to the proof of that proposition, we first state an important combinatorial result that we will be using in the proof : Soshnikov's "Main Combinatorial Lemma" (MCL), which is the following result due to [16] :

Lemma 8 (MCL)**.** *For any $x \in \mathbb{R}^n$ such that $x_1 + \cdots + x_n = 0$ and for any $n \geq 3$ we have*

$$\sum_{m=1}^{n-1} \frac{(-1)^{m-1}}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \max\left\{0, \sum_{\sigma(i)\leq 1} x_i, \sum_{\sigma(i)\leq 2} x_i, \dots, \sum_{\sigma(i)\leq m-1} x_i \right\} = 0,$$

where $\sum_{\sigma:[n]\twoheadrightarrow[m]}$ *is a summation over all σ which are surjections from $[n] := \{1, \dots, n\}$ to $[m]$.*

Proof of Proposition 7. We first rewrite the cumulants in the form

$$\mathcal{C}_n(X_N^\tau(f)) = \sum_{m=1}^{n} \frac{(-1)^m}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \int_{\mathbb{C}} \prod_{l=1}^{m} \left(\prod_{\sigma(k)=l} f(z_l) \right) K_N^\tau(z_l, z_{l+1}) dz_l.$$

We then use the joint cumulants

$$\mathcal{C}_n(X_N^\tau(f_1), \dots, X_N^\tau(f_n)) := \sum_{m=1}^{n} \frac{(-1)^m}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \int_{\mathbb{C}} \prod_{l=1}^{m} \left(\prod_{\sigma(k)=l} f_k(z_l) \right) K_N^\tau(z_l, z_{l+1}) dz_l. \tag{19}$$

which verify $\mathcal{C}_n(X_N^\tau(f)) = \mathcal{C}_n(X_N^\tau(f), \dots, X_N^\tau(f))$.

Taking $f(z)=\sum_{0\le i,j\le N_0}\alpha_{i,j}z^i\overline{z}^j$ to be a polynomial in z and $\overline{z}$, and as $\mathcal{C}_n$ is n-linear, we reduce the study to the expressions of the form $\mathcal{C}_n(X_N^\tau(z^{\alpha_1}\overline{z}^{\beta_1}),\dots,X_N^\tau(z^{\alpha_n}\overline{z}^{\beta_n}))$, and more precisely, the proof reduces to showing that $\mathcal{C}_n(X_N^\tau(z^{\alpha_1}\overline{z}^{\beta_1}),\dots,X_N^\tau(z^{\alpha_n}\overline{z}^{\beta_n}))\underset{N\to\infty}{\to}0$ uniformly in τ.

Now, the recurrence relation $zH_n(z)=\frac{1}{2}H_{n+1}(z)+nH_{n-1}(z)$ for the Hermite polynomials [18] implies the following recursion relation for the orthonormal functions ϕ :

$$z\phi_n^{N,\tau}(z)=\sqrt{\frac{n+1}{N}}\phi_{n+1}^{N,\tau}(z)+\tau\sqrt{\frac{n}{N}}\phi_{n-1}^{N,\tau}(z)\,. \tag{20}$$

Therefore, we introduce the recurrence matrix defined by $\mathbf{J}_{N,\tau}=\mathbf{A}+\tau\mathbf{A}^*$, where for all $k\in\mathbb{N}$, $\mathbf{A}e_k=\sqrt{\frac{k+1}{N}}e_{k+1}$ and $\mathbf{A}^*e_k=\sqrt{\frac{k}{N}}e_{k-1}$ ($(e_k)_{k\in\mathbb{N}}$ denotes the canonical basis of $l^2(\mathbb{N})$).

We can also see $\mathbf{J}_{N,\tau}$ as an infinite matrix, indexed by $\mathbb{N}$:

$$\mathbf{J}_{N,\tau}=\begin{pmatrix} 0 & \sqrt{\frac{1}{N}}\tau & 0 & 0 & \dots \\ \sqrt{\frac{1}{N}} & 0 & \sqrt{\frac{2}{N}}\tau & 0 & \ddots \\ 0 & \sqrt{\frac{2}{N}} & 0 & \sqrt{\frac{3}{N}}\tau & \ddots \\ 0 & 0 & \sqrt{\frac{3}{N}} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{21}$$

We define finally the operator $\mathbf{\Pi}_k : l^2(\mathbb{N})\to l^2(\mathbb{N})$ as the orthogonal projector onto $\mathrm{Vect}(e_0,\dots,e_{k-1})$.

We define for $\vec{m}\in\mathbb{N}^n$ and $\sigma:[n]\twoheadrightarrow[m]$, $\sigma\vec{m}\in\mathbb{N}^m$ by

$$(\sigma m)_l := \sum_{\sigma(k)=l} m_k.$$

Now we get on to the computation of $\mathcal{C}_n(X_N^\tau(z^{\alpha_1}\overline{z}^{\beta_1}),\dots,X_N^\tau(z^{\alpha_n}\overline{z}^{\beta_n}))$.
The expression of the kernel 17, the orthogonality relation 18 and the recurrence relation 20, we can rewrite the integral in the definition of $\mathcal{C}_n(X_N^\tau(f_1),\dots,X_N^\tau(f_n))$ 19

$$\int_{\mathbb{C}}\prod_{l=1}^m\left(\prod_{\sigma(k)=l} z_l^{\alpha_k}\overline{z_l}^{\beta_k}\right)K_N^\tau(z_l,z_{l+1})dz_l=\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{(\sigma\beta)_1}\mathbf{J}_{N,\tau}^{(\sigma\alpha)_1}\mathbf{\Pi}_N\dots\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{(\sigma\beta)_m}\mathbf{J}_{N,\tau}^{(\sigma\alpha)_m}\mathbf{\Pi}_N\right).$$

Then, we can isolate two terms by getting rid of the projection operator in the middle of the trace :

$$\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{(\sigma\beta)_1}\mathbf{J}_{N,\tau}^{(\sigma\alpha)_1}\mathbf{\Pi}_N\dots\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{(\sigma\beta)_m}\mathbf{J}_{N,\tau}^{(\sigma\alpha)_m}\mathbf{\Pi}_N\right)=\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\sigma\beta_1}\mathbf{J}_{N,\tau}^{\sigma\alpha_1}\dots(\mathbf{J}_{N,\tau}^*)^{\sigma\beta_m}\mathbf{J}_{N,\tau}^{\sigma\alpha_m}\mathbf{\Pi}_N\right)$$
$$-\sum_{j=0}^{N-1}\sum_{j_1,\dots,j_{m-1}\in\mathbb{N}}[(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}]_{jj_1}[(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}]_{j_1j_2}\dots[(\mathbf{J}_{N,\tau}^*)^{\beta_m}\mathbf{J}_{N,\tau}^{\alpha_m}]_{j_{m-1}j}\mathbb{1}_{\max\{j_1,\dots,j_{m-1}\}\ge N}. \tag{22}$$

We then give a nice expression for each of these terms by substituting $\vec{a}=\sigma\vec{\alpha}$ and $\vec{b}=\sigma\vec{\beta}$, in the two following lemmas, which will allow us to conclude the proof of Proposition 7.

Lemma 9. *We have the following expression for the first term :*

$$\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{b_1}\mathbf{J}_{N,\tau}^{a_1}(\mathbf{J}_{N,\tau}^*)^{b_2}\mathbf{J}_{N,\tau}^{a_2}\dots(\mathbf{J}_{N,\tau}^*)^{b_m}\mathbf{J}_{N,\tau}^{a_m}\mathbf{\Pi}_N\right) = \mathrm{Tr}\left(\mathbf{\Pi}_N\mathbf{J}_{N,\tau}^{|\vec{a}|}(\mathbf{J}_{N,\tau}^*)^{|\vec{b}|}\mathbf{\Pi}_N\right) + \gamma(\vec{a},\vec{b})\frac{1-\tau^2}{N}\mathrm{Tr}\left(\mathbf{\Pi}_N\mathbf{J}_{N,\tau}^{|\vec{a}|-1}(\mathbf{J}_{N,\tau}^*)^{|\vec{b}|-1}\mathbf{\Pi}_N\right) + O\left(\frac{1}{N}\right), \quad (23)$$

where $|\vec{a}| = \sum_{i=1}^{m} a_i$, $\gamma(\vec{a},\vec{b}) = a_1b_1 + a_2(b_1+b_2) + \dots + a_m(b_1+\dots+b_m)$ *and the* $O\left(\frac{1}{N}\right)$ *is uniform in* τ*.*

Proof of Lemma 9. We first highlight that we have the following commutator relation, which comes from the definition 21 of $\mathbf{J}_{N,\tau}$:

$$[\mathbf{J}_{N,\tau}^*, \mathbf{J}_{N,\tau}] = \mathbf{J}_{N,\tau}^*\mathbf{J}_{N,\tau} - \mathbf{J}_{N,\tau}\mathbf{J}_{N,\tau}^* = \frac{1-\tau^2}{N}\mathbf{I}. \quad (24)$$

Then, using this relation to commute successively the $\mathbf{J}_{N,\tau}$ and $\mathbf{J}_{N,\tau}^*$, and using the standard notation $|\vec{a}| \wedge |\vec{b}| = \min\left\{|\vec{a}|, |\vec{b}|\right\}$, we deduce

$$\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{b_1}\mathbf{J}_{N,\tau}^{a_1}\dots(\mathbf{J}_{N,\tau}^*)^{b_m}\mathbf{J}_{N,\tau}^{a_m}\mathbf{\Pi}_N\right) = \sum_{i=0}^{|\vec{a}|\wedge|\vec{b}|}\gamma_i(\vec{a},\vec{b})\left(\frac{1-\tau^2}{N}\right)^i\mathrm{Tr}\left(\mathbf{\Pi}_N\mathbf{J}_{N,\tau}^{|\vec{a}|-i}(\mathbf{J}_{N,\tau}^*)^{|\vec{b}|-i}\mathbf{\Pi}_N\right) \quad (25)$$

where $\gamma_i(\vec{a},\vec{b}) \in \mathbb{N}$ depends only on the $\vec{a}$ and $\vec{b}$, but not on τ and N, and we get $\gamma_0 = 1$ and

$$\gamma_1(\vec{a},\vec{b}) = a_1b_1 + a_2(b_1+b_2) + \dots + a_m(b_1+\dots+b_m).$$

Now, since all the entries of $\mathbf{J}_{N,\tau}$ we encounter when computing the traces in 25 are less or equal than some constant C depending only on $\vec{a}$ and $\vec{b}$ (not on N and τ), we have

$$\left|\mathrm{Tr}\left(\mathbf{\Pi}_N\mathbf{J}_{N,\tau}^{|\vec{a}|-i}(\mathbf{J}_{N,\tau}^*)^{|\vec{b}|-i}\mathbf{\Pi}_N\right)\right| = \left|\sum_{k=1}^{N}\left[\mathbf{J}_{N,\tau}^{|\vec{a}|-i}(\mathbf{J}_{N,\tau}^*)^{|\vec{b}|-i}\right]_{kk}\right| \leq N(2\times C)^{|\vec{a}|+|\vec{b}|-2i}. \quad (26)$$

Then, all the $i \geq 2$ terms in 25 are $O\left(\frac{1}{N}\right)$ uniformly in τ, and that concludes the proof of Lemma 9. □

We mention the combinatorial identities

$$\sum_{m=1}^{n}\frac{(-1)^m}{m}\sum_{\sigma:[n]\twoheadrightarrow[m]}1 = 0 \quad \text{and} \quad \sum_{m=1}^{n}\frac{(-1)^m}{m}\sum_{\sigma:[n]\twoheadrightarrow[m]}\gamma(\sigma\vec{\alpha},\sigma\vec{\beta}) = 0. \quad (27)$$

The second expression is proven in the Lemma 8 of [13] while the first one follows from the fact that if $f=1$, the cumulants $\mathcal{C}_n(X_N^\tau(f))$ vanish for $n \geq 2$.
Now, Lemma 9, the fact that $|\sigma\vec{\alpha}| = |\vec{\alpha}|$ does not depend on σ, and the previous combinatorial identities 27 imply that for every $n \geq 1$,

$$\sum_{m=1}^{n}\frac{(-1)^m}{m}\sum_{\sigma:[n]\twoheadrightarrow[m]}\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\sigma\beta_1}\mathbf{J}_{N,\tau}^{\sigma\alpha_1}\dots(\mathbf{J}_{N,\tau}^*)^{\sigma\beta_m}\mathbf{J}_{N,\tau}^{\sigma\alpha_m}\mathbf{\Pi}_N\right) \underset{N\to\infty}{\to} 0 \quad \text{uniformly in } \tau. \quad (28)$$

As for the second term in 22, we first denote it by

$$\varphi_{N,\tau}(\vec{a},\vec{b}) = \sum_{j=0}^{N-1}\sum_{j_1,\dots,j_{m-1}\in\mathbb{N}}[(\mathbf{J}_{N,\tau}^*)^{b_1}\mathbf{J}_{N,\tau}^{a_1}]_{jj_1}\dots[(\mathbf{J}_{N,\tau}^*)^{b_m}\mathbf{J}_{N,\tau}^{a_m}]_{j_{m-1}j}\mathbb{1}_{\max\{j_1,\dots,j_{m-1}\}\geq N}. \quad (29)$$

Lemma 10. *We have the following convergence for the second term :*

$$\sum_{m=1}^{n} \frac{(-1)^m}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \varphi_{N,\tau}(\sigma\vec{\alpha}, \sigma\vec{\beta}) \underset{N\to\infty}{\to} 0 \quad \textit{uniformly in } \tau.$$

Proof of Lemma 10. We first note that $\mathbf{J}_{\infty,\tau} := \mathbf{T} + \tau\mathbf{T}^*$, with $\mathbf{T} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ the left shift operator, is a right-limit of $\mathbf{J}_{N,\tau}^{(N)}$ as defined in [12], which means that for any $i, j \in \mathbb{Z}$, we have the convergence

$$[\mathbf{J}_{N,\tau}]_{N+i,N+j} \underset{N\to\infty}{\to} [\mathbf{J}_{\infty,\tau}]_{ij} \quad \text{uniformly in } \tau.$$

Taking a closer look at 29, we see that there is only a fixed finite number of non zero terms, which leads us to write that

$$\varphi_{N,\tau}(\vec{a}, \vec{b}) \underset{N\to\infty}{\to} \varphi_{\infty,\tau}(\vec{a}, \vec{b}) \quad \text{uniformly in } \tau,$$

where

$$\begin{aligned}
\varphi_{\infty,\tau}(\vec{a}, \vec{b}) &= \sum_{j<0} \sum_{j_1,\ldots,j_{m-1}\in\mathbb{Z}} [(\mathbf{J}_{\infty,\tau}^*)^{b_1}\mathbf{J}_{\infty,\tau}^{a_1}]_{jj_1} \ldots [(\mathbf{J}_{\infty,\tau}^*)^{b_m}\mathbf{J}_{\infty,\tau}^{a_m}]_{j_{m-1}j} \mathbb{1}_{\max\{j_1,\ldots,j_{m-1}\}\geq 0} \\
&= \sum_{j=1}^{\infty} \sum_{j_1,\ldots,j_{m-1}\in\mathbb{Z}} [(\mathbf{J}_{\infty,\tau}^*)^{b_1}\mathbf{J}_{\infty,\tau}^{a_1}]_{0j_1} \ldots [(\mathbf{J}_{\infty,\tau}^*)^{b_m}\mathbf{J}_{\infty,\tau}^{a_m}]_{j_{m-1}0} \mathbb{1}_{\max\{j_1,\ldots,j_{m-1}\}\geq j} \\
&= \sum_{j_1,\ldots,j_{m-1}\in\mathbb{Z}} \max\{0, j_1, \ldots, j_{m-1}\} [(\mathbf{J}_{\infty,\tau}^*)^{b_1}\mathbf{J}_{\infty,\tau}^{a_1}]_{0j_1} \ldots [(\mathbf{J}_{\infty,\tau}^*)^{b_m}\mathbf{J}_{\infty,\tau}^{a_m}]_{j_{m-1}0}. \qquad (30)
\end{aligned}$$

We will now express that sum in terms of paths.

We define the increments of the paths between 0 and 0 of length $|\vec{\alpha}| + |\vec{\beta}|$:

$$\Omega := \left\{ (\omega_1, \ldots, \omega_{|\vec{\alpha}|+|\vec{\beta}|-1}, \omega_{|\vec{\alpha}|+|\vec{\beta}|}) \in \{-1, 1\}, \quad \sum_{i=1}^{|\vec{\alpha}|+|\vec{\beta}|} \omega_i = 0 \right\},$$

and for $\omega \in \Omega$, we also define the path described by ω by, for $0 \leq j \leq |\vec{\alpha}| + |\vec{\beta}|$,

$$\Gamma(\omega)_j = \sum_{i=1}^{j} \omega_i.$$

Then we can write 30 in terms of paths : for $\sigma : [n] \twoheadrightarrow [m]$,

$$\varphi_{\infty,\tau}(\sigma\vec{\alpha}, \sigma\vec{\beta}) = \sum_{\omega\in\Omega} \max\{0, \Gamma(\omega)_{\sigma\alpha_1+\sigma\beta_1}, \ldots, \Gamma(\omega)_{\sigma\alpha_1+\sigma\beta_1+\cdots+\sigma\alpha_{m-1}+\sigma\beta_{m-1}}\}\tau^{n(\omega,\sigma)} \qquad (31)$$

where $n(\omega, \sigma)$ counts the number of times that τ appears in the product associated with the path.
We define the blocks of $\omega \in \Omega$ for $1 \leq j \leq n$, by defining $s_i := \sum_{i=1}^{j-1} (\alpha_i + \beta_i)$ and

$$\omega[j, 1] = (\omega_{s_j+1}, \ldots, \omega_{s_j+\alpha_j}) \quad \text{and} \quad \omega[j, 2] = (\omega_{s_j+\alpha_j+1}, \ldots \omega_{s_j+\alpha_j+\beta_j}). \qquad (32)$$

Denoting by a usual product the concatenation of two uplets, for $\sigma : [n] \twoheadrightarrow [m]$, we rearrange the blocks of some $\omega \in \Omega$ in the following way :

$$\sigma\omega := \left(\prod_{i\in\sigma^{-1}(1)} \omega[i, 1] \right) \times \left(\prod_{i\in\sigma^{-1}(1)} \omega[i, 2] \right) \times \cdots \times \left(\prod_{i\in\sigma^{-1}(m)} \omega[i, 1] \right) \times \left(\prod_{i\in\sigma^{-1}(m)} \omega[i, 2] \right) \qquad (33)$$

where the concatenations in the products $\prod_{i\in\sigma^{-1}(j)}$ are done, say, in ascending order of $i \in \sigma^{-1}(j)$.
The mapping $\omega \in \Omega \mapsto \sigma\omega \in \Omega$ being a bijection, we can rewrite 31

$$\varphi_{\infty,\tau}(\sigma\vec{\alpha}, \sigma\vec{\beta}) = \sum_{\omega\in\Omega} \max\{0, \Gamma(\sigma\omega)_{\sigma\alpha_1+\sigma\beta_1}, \dots, \Gamma(\sigma\omega)_{\sigma\alpha_1+\sigma\beta_1+\cdots+\sigma\alpha_{m-1}+\sigma\beta_{m-1}}\}\tau^{n(\sigma\omega,\sigma)}. \tag{34}$$

Now, $\sigma\omega$ has been built such that $n(\omega) := n(\sigma\omega, \sigma)$ does not depend on σ, and denoting

$$\kappa_1(\omega) = \omega_{\alpha_1+\beta_1} - 0, \dots, \kappa_n(\omega) = \omega_{\alpha_1+\beta_1+\cdots+\alpha_n+\beta_n} - \omega_{\alpha_1+\beta_1+\cdots+\alpha_{n-1}+\beta_{n-1}},$$

and $|\sigma^{-1}(i)| = \mathrm{Card}(\sigma^{-1}(i))$, we have

$$\Gamma(\sigma\omega)_{\sigma\alpha_1+\sigma\beta_1+\cdots+\sigma\alpha_i+\sigma\beta_i} = \sum_{\sigma(j)\le i} \kappa_j(\omega),$$

so that we finally rewrite 34 into

$$\sum_{m=1}^{n} \frac{(-1)^m}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \varphi_{\infty,\tau}(\sigma\vec{\alpha}, \sigma\vec{\beta}) =$$
$$\sum_{\omega\in\Omega} \tau^{n(\omega)} \sum_{m=1}^{n} \frac{(-1)^m}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \max\left\{0, \sum_{\sigma(i)\le 1} \kappa_i(\omega), \sum_{\sigma(i)\le 2} \kappa_i(\omega), \dots, \sum_{\sigma(i)\le m-1} \kappa_i(\omega)\right\}. \tag{35}$$

Finally, using the Main Combinatorial Lemma of Soshnikov 8, we deduce

$$\sum_{m=1}^{n} \frac{(-1)^m}{m} \sum_{\sigma:[n]\twoheadrightarrow[m]} \varphi_{\infty,\tau}(\sigma\vec{\alpha}, \sigma\vec{\beta}) = 0.$$

□

Finally, Lemma 10 and equation 28 imply Proposition 7.

□

2.3 Computation of the variance

In the light of Proposition 7, the final big step in proving Theorem 6 is to compute the asymptotics of the variance of the linear statistics $\mathrm{Var}(X_N^\tau(f))$. That is the essence of the following proposition.

Proposition 11. *For polynomial in z and $\overline{z}$, $f : \mathbb{C} \to \mathbb{R}$, the variance* $\mathrm{Var}(X_N^\tau(f))$ *of the linear statistics converges as*

$$\mathrm{Var}(X_N^\tau(f)) \underset{N\to\infty}{\to} \frac{1}{4\pi} \int_{\mathcal{E}_\tau} |\nabla f|^2 + \frac{1}{2} \sum_{k\in\mathbb{Z}} |k| \left|\widehat{f}^{(\partial\mathcal{E}_\tau)}(k)\right|^2 \quad \textit{uniformly in } \tau,$$

where $\mathcal{E}_\tau$ is the ellipse $\mathcal{E}_\tau = \left\{z \in \mathbb{C}, \frac{(\mathrm{Re}\ z)^2}{(1+\tau)^2} + \frac{(\mathrm{Im}\ z)^2}{(1-\tau)^2} \le 1\right\}$ and the Fourier transform in $\partial\mathcal{E}_\tau$ is defined in 3.

Proof. As in the proof of Proposition 7, we may reduce the proof to that for the monomials, by showing for $f(z) = z^{\alpha_1}\overline{z}^{\beta_1}$ and $g(z) = z^{\beta_2}\overline{z}^{\alpha_2}$ that

$$\mathrm{Cov}(X_N^\tau(f), X_N^\tau(g)) \underset{N\to\infty}{\to} \frac{1}{\pi} \int_{\mathcal{E}_\tau} \overline{\partial} f \overline{\overline{\partial} g} + \sum_{k>0} |k| \widehat{f}^{(\partial\mathcal{E}_\tau)}(k) \widehat{g}^{(\partial\mathcal{E}_\tau)}(k). \tag{36}$$

As $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, equation 36 does imply Proposition 11. We denote also the unit disk by $\mathbb{D} := \{z \in \mathbb{C}, |z| \le 1\}$. We recall that we denoted $\mathbf{J}_{\infty,\tau} = \mathbf{T} + \tau\mathbf{T}^*$, where $\mathbf{T}$ is the left shift operator.

We mention that $\mathbf{T}$ and $\mathbf{T}^*$ commute. We first use the general formula for the variance of any determinantal point process :

$$\mathrm{Cov}(X_N^\tau(f), X_N^\tau(g)) = \int f(z)\overline{g(z)}K_N^\tau(z,z)dz - \int\int f(z)\overline{g(z')}|K_N^\tau(z,z')|^2 dzdz'.$$

Then we can write the covariance as a trace of the matrix $\mathbf{J}_{N,\tau}$ as in the proof of Proposition 7 :

$$\begin{aligned}\mathrm{Cov}(X_N^\tau(z^{\alpha_1}\overline{z}^{\beta_1}), X_N^\tau(z^{\beta_2}\overline{z}^{\alpha_2})) &= \mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1+\beta_2}\mathbf{J}_{N,\tau}^{\alpha_1+\alpha_2}\mathbf{\Pi}_N\right) - \mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}\mathbf{\Pi}_N\right)\\ &= \mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1+\beta_2}\mathbf{J}_{N,\tau}^{\alpha_1+\alpha_2}\mathbf{\Pi}_N\right) - \mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}\mathbf{\Pi}_N\right).\end{aligned}$$

We continue following the same path by expliciting two terms in the following same way :

$$\begin{aligned}\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}\pi_N\right) &= \mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}\mathbf{\Pi}_N\right)\\ &\quad -\sum_{j=0}^{N}\sum_{j_1\in\mathbb{N}}[(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}]_{jj_1}[(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}]_{j_1j}\mathbb{1}_{j_1\geq N+1}. \quad (37)\end{aligned}$$

So that following the same cummuting idea as in Lemma 9, by using 24 the first term writes

$$\begin{aligned}&\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1+\beta_2}\mathbf{J}_{N,\tau}^{\alpha_1+\alpha_2}\mathbf{\Pi}_N\right) - \mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}\mathbf{\Pi}_N\right)\\ &\qquad = \beta_1\alpha_2\frac{1-\tau^2}{N}\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1+\beta_2-1}\mathbf{J}_{N,\tau}^{\alpha_1+\alpha_2-1}\mathbf{\Pi}_N\right) + O\left(\frac{1}{N}\right), \quad (38)\end{aligned}$$

where the $O\left(\frac{1}{N}\right)$ is uniform in τ. Now, expanding

$$\mathrm{Tr}\left(\mathbf{\Pi}_N(\mathbf{J}_{N,\tau}^*)^{\beta_1+\beta_2-1}\mathbf{J}_{N,\tau}^{\alpha_1+\alpha_2-1}\mathbf{\Pi}_N\right) = \sum_{k=0}^{N-1}[(\mathbf{J}_{N,\tau}^*)^{\beta_1+\beta_2-1}\mathbf{J}_{N,\tau}^{\alpha_1+\alpha_2-1}]_{kk}, \quad (39)$$

and using the definition of $\mathbf{J}_{N,\tau}$ 21, we deduce that the first term 38 converges by Riemannian sums as $N\to\infty$ to

$$\begin{aligned}&(1-\tau^2)\beta_1\alpha_2[(\mathbf{J}_{\infty,\tau}^*)^{\beta_1+\beta_2-1}\mathbf{J}_{\infty,\tau}^{\alpha_1+\alpha_2-1}]_{00}\int_0^1 t^{\frac{\alpha_1+\alpha_2+\beta_1+\beta_2-2}{2}}dt\\ &= (1-\tau^2)\beta_1\alpha_2[(\mathbf{T}^*+\tau\mathbf{T})^{\beta_1+\beta_2-1}(\mathbf{T}+\tau\mathbf{T}^*)^{\alpha_1+\alpha_2-1}]_{00}\int_0^1 t^{\frac{\alpha_1+\alpha_2+\beta_1+\beta_2-2}{2}}dt\\ &= (1-\tau^2)\beta_1\alpha_2\oint_{|z|=1}(z+\tau\overline{z})^{\alpha_1+\alpha_2-1}(\overline{z}+\tau z)^{\beta_1+\beta_2-1}\frac{dz}{2i\pi z}\int_0^1 2u^{\alpha_1+\alpha_2+\beta_1+\beta_2-2}udu\\ &= \frac{1}{\pi}(1-\tau^2)\beta_1\alpha_2\int_{\mathbb{D}}(z+\tau\overline{z})^{\alpha_1+\alpha_2-1}(\overline{z}+\tau z)^{\beta_1+\beta_2-1}d^2z\\ &= \frac{1}{\pi}(1-\tau^2)\beta_1\alpha_2\int_{\mathbb{D}}((1+\tau)\mathrm{Re}\ z+i(1-\tau)\mathrm{Im}\ z)^{\alpha_1+\alpha_2-1}\times((1+\tau)\mathrm{Re}\ z-i(1-\tau)\mathrm{Im}\ z)^{\beta_1+\beta_2-1}dxdy\\ &= \frac{1}{\pi}\int_{\mathcal{E}_\tau}\beta_1\alpha_2 z^{\alpha_1+\alpha_2-1}\overline{z}^{\beta_1+\beta_2-1}d^2z \quad \text{by making a linear change of variables on the Re and Im parts}\\ &= \frac{1}{\pi}\int_{\mathcal{E}_\tau}\overline{\partial}(z^{\alpha_1}\overline{z}^{\beta_1})\overline{\overline{\partial}(z^{\beta_2}\overline{z}^{\alpha_2})}d^2z.\end{aligned}$$

This therefore gives the first term in 36 as wanted. As for the second term, we compute

$$\sum_{j=0}^{N}\sum_{j_1\in\mathbb{N}}[(\mathbf{J}_{N,\tau}^*)^{\beta_2}\mathbf{J}_{N,\tau}^{\alpha_2}]_{jj_1}[(\mathbf{J}_{N,\tau}^*)^{\beta_1}\mathbf{J}_{N,\tau}^{\alpha_1}]_{j_1j}\mathbb{1}_{j_1\geq N+1}\underset{N\to\infty}{\to}\sum_{k>0}k[(\mathbf{J}_{\infty,\tau}^*)^{\beta_2}\mathbf{J}_{\infty,\tau}^{\alpha_2}]_{0k}[(\mathbf{J}_{\infty,\tau}^*)^{\beta_1}\mathbf{J}_{\infty,\tau}^{\alpha_1}]_{k0}$$

uniformly in τ. We already see the wanted sum appearing, and we then express the $\mathbf{J}_{\infty,\tau}$ in terms of the shift $\mathbf{T}$ so that the Fourier coefficients will appear naturally :

$$\sum_{k>0} k[(\mathbf{J}_{\infty,\tau}^*)^{\beta_2}\mathbf{J}_{\infty,\tau}^{\alpha_2}]_{0k}[(\mathbf{J}_{\infty,\tau}^*)^{\beta_1}\mathbf{J}_{\infty,\tau}^{\alpha_1}]_{k0} = \sum_{k>0} k\left[(\mathbf{T}^*+\tau\mathbf{T})^{\beta_2}(\mathbf{T}+\tau\mathbf{T}^*)^{\alpha_2}\mathbf{T}^k\right]_{00}\left[(\mathbf{T}^*)^k(\mathbf{T}^*+\tau\mathbf{T})^{\beta_1}(\mathbf{T}+\tau\mathbf{T}^*)^{\alpha_1}\right]_{00},$$

and we finally recover the Fourier coefficients :

$$\begin{aligned}
\left[(\mathbf{T}^*+\tau\mathbf{T})^{\alpha}(\mathbf{T}+\tau\mathbf{T}^*)^{\beta}\mathbf{T}^k\right]_{00} &= \frac{1}{2\pi}\int_{\partial\mathbb{D}}(z+\tau\overline{z})^{\alpha}(\overline{z}+\tau z)^{\beta}\overline{z}^k d\lambda \\
&= \frac{1}{2\pi(1-\tau^2)}\int_{\partial\mathcal{E}_\tau} z^{\alpha}\overline{z}^{\beta}\left(\frac{\mathrm{Re}z}{1+\tau}-i\frac{\mathrm{Im}z}{1-\tau}\right)^k d\lambda^{(\partial\mathcal{E}_\tau)} \\
&= \widehat{z^{\alpha}\overline{z}^{\beta}}^{(\partial\mathcal{E}_\tau)}(k).
\end{aligned}$$

□

Now, to finish the proof of Theorem 6, we only need to link the uniform convergence of the cumulants with the uniform convergence in law. That is done in the following :

Proof of Theorem 6. Starting from Proposition 7 and Proposition 11, this is a straightforward compacity argument. Let us sketch the proof. We denote the centered variable $Z_N^\tau = X_N^\tau(f) - \mathbb{E}(X_N^\tau(f))$.

Let us suppose that we do not have this uniform convergence. Then, denoting Z^τ a sequence of random variables of law $\mathcal{N}(0, \sigma^2_{\mathcal{E}_\tau} + \sigma^2_{\partial\mathcal{E}_\tau})$, there exists $\phi : \mathbb{R} \to \mathbb{R}$ continous and bounded, $\varepsilon > 0$, an increasing sequence of integers (N_j) and a sequence $\tau_j \in [0,1]$, that we may assume without loss of generality convergent to some $\tau_\infty \in [0,1]$, such that for all $j \in \mathbb{N}$,

$$|\mathbb{E}(\phi(Z_{N_j}^{\tau_j})) - \mathbb{E}(\phi(Z^{\tau_j})| \geq \varepsilon. \tag{40}$$

But now, the sequence of random variables $(Z_{N_j}^{\tau_j})$ has bounded second moment, so that by Prokhorov's theorem, it has a subsequence $(Z_{N_{j_i}}^{\tau_{j_i}})$ that converges in law to a certain limit Z_∞. But because of the uniform convergence of the cumulants we get that the cumulants of Z_∞ are the limit of the cumulants of $(Z_{N_{j_i}}^{\tau_{j_i}})$ so that Z_∞ has law $\mathcal{N}(0, \sigma^2_{\mathcal{E}_{\tau_\infty}} + \sigma^2_{\partial\mathcal{E}_{\tau_\infty}})$ and that contradicts the inequality 40. □

A Course of the internship

The proofs showed here are only a small fraction of the work done in these 6 past months.

It might be worth mentionning that at the beginning, our idea of the subject was very far from these random matrix theory topics on which we landed on : we were thinking about studying branching random walks and more precisely, studying fluctuations of the maximum of the variables over a fixed height N, when taking $N \to \infty$, i.e. the asymptotic distribution of the second order term of this maximum, the first order being deterministic. However, after one day of studying existing literature, we found that what we wanted to do was actually already done, so we shifted to that random matrix subject.

Then, the real first part consisted in getting familiar enough with determinantal point processes to be able to work on the question, and to read and understand [13]. Thereafter, it was [12] that was read : the proof presented here gathers ideas from [13] and [12]. The few weeks after the reading of these 2 articles consisted of making the proof presented here.

Then, as is done in [13], I wanted to extend the result presented here to non polynomial test functions with not so much differentiability hypothesis, and that took a lot more time than the proof presented in this report, because it required a lot more initiative than for polynomial test functions, as the reasoning does not really lean on any existing reasoning that we know of. [13] does do a proof of such an extension, but it is not applicable in the elliptic case $\tau \in [0, 1)$. For an outline of the proof, it is basically an extension result with two parts : density result using Stone-Weierstrass' theorem and bound of the variance. The bound is the hardest part, and is again divided in two parts : firstly, in the ellipse, a new (it seems at least) trick is used to handle this part and to avoid an long estimation of the kernel in the bulk, which would it seems be very heavy in computations, consisting on using the results we have for polynomial test function and the differentiability of the test function in the ellipse, and secondly, an estimation of the kernel outside the ellipse, where we just need some exponential decay. We therefore leaned partly on the results of [1] to handle this last computational part.

Then, as there were still a few months left, we tackled a new problem about free fermions. Gaultier made recently a lot of progress in problems about free fermions [6, 7] ; especially, he and Alix Deleporte showed a Central Limit Theorem for one-dimensional free fermions, and he was wondering about how to generalize it in higher dimensions. As a test towards that goal, he then tasked me to study the variance of the linear statistics of free fermions in dimensions $d \geq 2$ but for a harmonic potential, where we have a nice expression in terms of Hermite polynomials of the kernel. I then followed the same path as for the elliptic Ginibre ensemble : first for polynomial test functions using the recursion relations and orthogonality ideas of [12] and then extention for more general test function using the trick and an estimate of the kernel.

Finally, for the free fermions model, we tried to study another question about the behaviour of the variance in the case of non-differentiable functions in the bulk, and especially for indicatrice function on an centered ball strictly included in the ball but we did not have enough time to finish that part.

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