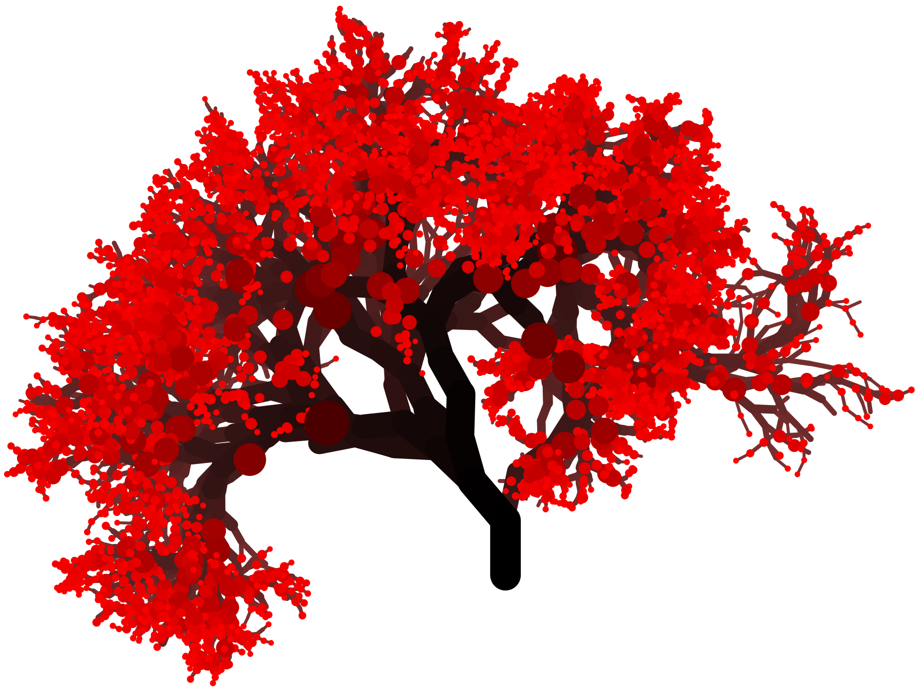
Internship report
Roma Tre University

Spread and Recovery of Infections on Galton-Watson Trees



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Notations

In this report, $\mathbb{N}$ refers to the set of non-negative integers and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For any $(a, b) \in \mathbb{N}^2$ we denote $[\![a, b]\!] := \mathbb{N} \cap [a, b]$. For any set X, we denote by $\#X \in \mathbb{N} \cup \{+\infty\}$ its carnality. Finally for any pair of reel functions f, g defined for all $x > 0$ sufficiently large, we write $f(x) \sim g(x)$ when $x \to +\infty$ if $\lim_{x\to+\infty} f(x)/g(x) = 1$.

Introduction

The original model, called "first passage percolation in hostile environment" (FPPHE), was introduced by Sidoravicius and Stauffer in [SS19] as an auxiliary model to understand properties of "multi-particles diffusion limited aggregation", which is notoriously challenging. Besides this paper, there have been several works studying the properties of FPPHE, such as [FS22b], [FS22a], [CS21a] and [CS21b]. In the original model one starts an FPP process of rate 1 from a fixed reference vertex and, as it spreads on the given graph, it activates seeds of a second type of FPP process, which spreads at some fixed positive rate. Thus, these two types of FPP compete for space, and the resulting cluster of occupied vertices can show a behavior that depends on the structural properties of the underlying graph. In FPPHE vertices that are occupied by one of the types would not change type in time, hence a natural question is to understand what happens when one relaxes this assumption. This was our initial motivation for introducing and studying a simplified model of spatial epidemics with recovery.

The process we study consists of a first passage percolation model in a random environment with an additional recovery system. It can be interpreted as the spread of a disease and it is closely related to the models studied in [CD88]. The general construction is summarized in the following definition.

Definition D (Construction of the process)**.** *First consider a* ***random graph*** *with a fixed countable set of vertices* $\mathbb{V}$*, a fixed* ***origin*** $o \in \mathbb{V}$ *and a* ***random set of edges*** $E \subset \mathbb{V}^2$*. We denote by* G *the set of accessible vertices from the origin, which implicitly has a graph structure induced by* E*. Next we consider a collection* $(T_e)_{e\in\mathbb{V}^2}$ *of i.i.d.* $\mathrm{Exp}(1)$ *random variables (taken independent of* E*), which we interpret as* ***passage times****. These times are attached to the edges of* G *by restriction. For all* $v \in \mathbb{V}$ *we define* τ_v *as the* ***reaching time*** *of* v *from the origin, i.e. the infimum travel time of finite paths in* E *joining* o *to* v *(with the convention* $\inf \emptyset := +\infty$*). For all* $t \in \mathbb{R}$ *we define the set* A_t *of vertices that are reached from the origin within time* t *by:*

$$A_t := \{v \in G : \tau_v \leq t\}$$

Then we fix a ***recovery rate*** $\gamma > 0$ *and we consider a collection* $(C_v)_{v\in\mathbb{V}}$ *of i.i.d.* $\mathrm{Exp}(\gamma)$ *random variables (taken independent of all the previous randomness), which we interpret as* ***recovery clocks****. For all* $t \geq 0$ *we define the set* R_t *of* ***red*** *(or recovering) vertices at time* t *by:*

$$R_t := \{v \in A_t : t < \tau_v + C_v\}$$

Finally for all $t \in \mathbb{R}$ *we define* H_t *as the* ***largest red path size*** *at time* t*, i.e. the maximal size of a directed path in* R_t*, and* M_t *as the* ***largest red cluster size*** *at time* t*, i.e. the maximal size of a weakly connected region in* R_t*. An example is given in Figure 1.*

The natural problem arising from this process is to understand the asymptotic behavior of both H_t and M_t when $t \to +\infty$. Our aim is to study the case where G is a Galton-Watson tree with a branching distribution of finite mean and $\gamma > 0$ is arbitrary. Since all vertices eventually recover, the asymptotic behavior of the process is only of interest when G is infinite. We will then restrict ourselves to the degenerate critical and super-critical cases. In the first section we focus on the degenerate critical case where G is a deterministic infinite line and provide sharp asymptotic bounds that are summarized in the following theorem.

Theorem I. *When* G *is a deterministic infinite line, almost surely we have:*

$$\limsup_{t\to+\infty} \frac{H_t \log\log t}{\log t} = 1 \quad \textit{and} \quad \liminf_{t\to+\infty} H_t = 0$$

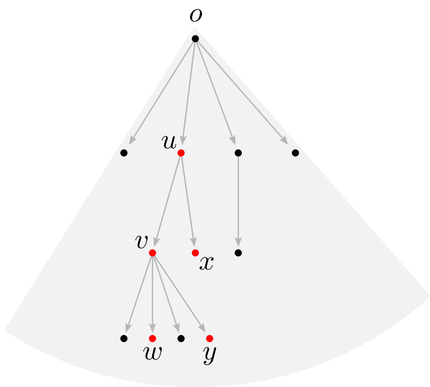

Figure 1: Illustration of a possible configuration for A_t at time $t > 0$ when G is a directed tree. The arrows represent the edges in $E \cap A_t^2$ and the vertices of A_t are represented by dots, colored in red if they belong to R_t. We have $v \in A_t$ for example, which means that $t \geq \tau_v = T_{(o,u)} + T_{(u,v)}$ and $t < \tau_v + C_v$. In this configuration H_t is reached for the directed red paths (u, v, w) and (u, v, y) of size 3 and M_t reached for the weakly connected red region $\{u, v, w, x, y\}$ of size 5.

In the second section we focus on the case where G is a super-critical Galton-Watson tree and provide asymptotic lower bounds, summarized in the following theorem.

Theorem II. *When G is Galton-Watson tree with a super-critical branching distribution of finite mean μ, then almost surely on the event $\{\#G = +\infty\}$ we have:*

$$\liminf_{t\to+\infty} \frac{H_t \log t}{t} \geq \mu - 1 \quad \text{and} \quad \liminf_{t\to+\infty} \frac{M_t \log\log t}{t} \geq \mu - 1$$

1 The degenerate critical case

In this section, G is an infinite deterministic line and $\gamma > 0$ is arbitrary. We assume $\mathbb{V} = \mathbb{N}^*$, $o = 1$ and $E = \{(n, n+1),\ n \geq 1\}$. To simplify the notation for all $n \geq 2$ we set $T_n := T_{(n-1,n)}$. Since a cluster is a path in G, for all $t \geq 0$ we have $M_t = H_t$ and thus we have only one quantity to study.

1.1 The asymptotic upper bound

The goal of this subsection is to find a non-decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that almost surely, we have:

$$\limsup_{t\to+\infty} \frac{H_t}{h(t)} = 1$$

We start by a lemma that will be useful several times.

Lemma 1.1 (Basic first-passage percolation properties). *Almost surely, as $n \to +\infty$ and $t \to +\infty$ we have:*

$$\tau_n \sim n \quad \textit{and} \quad \#A_t \sim t$$

Proof. For all $n \geq 1$, we have by definition:

$$\tau_n = \sum_{k=2}^{n} T_k$$

Therefore by the law of large numbers $\tau_n \sim n$ almost surely as $n \to +\infty$. Next remark that $\#A_t \geq n$ for all $t \geq \tau_n$ and thus $\#A_t \to +\infty$ as $t \to +\infty$. We deduce then that almost surely as $t \to +\infty$ we have:

$$\tau_{\#A_t} \sim \#A_t \sim \#A_t + 1 \sim \tau_{\#A_t+1}$$

Finally it implies the result since for all $t \in \mathbb{R}_+$:

$$\tau_{\#A_t} \leq t \leq \tau_{\#A_t+1}$$

■

Next we introduce a new quantity that will simplify the problem.

Definition 1.2 (Tail red clusters). *For any $n \geq 1$ we denote by $\hat{H}_n$ the size of the tail red cluster at time τ_n, defined by:*

$$\hat{H}_n := \max\{m \in [\![1, n]\!] : [\![n - m + 1, n]\!] \subset R_{\tau_n}\}$$

Proposition 1.3 (Red clusters are maximal when they appear on the tail). *Let $h : \mathbb{N}^* \to \mathbb{R}_+^*$ be a non-decreasing function, then:*

$$\limsup_{n \to +\infty} \frac{\hat{H}_n}{h(n)} = \limsup_{t \to +\infty} \frac{H_t}{h(\#A_t)} \quad \textit{a.s.}$$

Proof. ($\leq$): By construction $\hat{H}_n \leq H_{\tau_n}$ and $\#A_{\tau_n} = n$. Moreover $\tau_n \to +\infty$ by Proposition 1.1, therefore:

$$L := \limsup_{n \to +\infty} \frac{\hat{H}_n}{h(n)} \leq \limsup_{n \to +\infty} \frac{H_{\tau_n}}{h(\#A_{\tau_n})} \leq \limsup_{t \to +\infty} \frac{H_t}{h(\#A_t)} =: L'$$

($\geq$): If $L' = 0$, the result is clear since $L \geq 0$. We can assume $L' \in (0, +\infty]$. Let $l \in (0, L')$ and $n_0 \geq 1$. Define:

$$t_0 := \tau_{n_0} + \max_{i \in [\![1, n_0]\!]} C_i$$

By definition of the $\limsup$, there exists $t \geq t_0$ such that:

$$\frac{H_t}{h(\#A_t)} \geq l$$

Now consider $N \in [\![1, \#A_t]\!]$ the rightmost vertex of a maximising cluster for H_t. By construction (see Figure 2) we have $\hat{H}_N \geq H_t$ and $t \geq \tau_N$. Moreover since $t \geq t_0$ we must have $N \geq n_0$ by definition of t_0. Finally using the fact that both h and $t \mapsto \#A_t$ are non-decreasing we get:

$$\hat{H}_N \geq lh(\#A_t) \geq lh(\#A_{\tau_N}) = lh(N)$$

In summary, we have just shown that:

$$\limsup_{n \to +\infty} \frac{\hat{H}_n}{h(n)} \geq l \quad \text{a.s.}$$

By making $l \to L'$ we get $L \geq L'$. ■

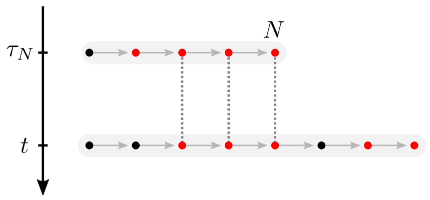

Figure 2: Illustration of the reasoning for the proof of Proposition 1.3. Possible configuration for A_t and A_{τ_N} where N is the rightmost vertex of a maximising cluster for H_t.

Remark that in the right-hand side of the equality stated by Proposition 1.3, the denominator is not a deterministic function of t as we would wish. We will see later that actually fixing a specific function we will be able to get rid of the randomness using the second almost sure asymptotic equivalence stated by Lemma 1.1. The next step is thus to find a non-decreasing function $h : \mathbb{N}^* \to \mathbb{R}_+^*$ such that we have:

$$\limsup_{n \to +\infty} \frac{\hat{H}_n}{h(n)} = 1$$

This new problem is easier since the $\limsup$ is now taken on a countable sequence, which will allow us to apply Borel-Cantelli lemmas. Moreover we can compute explicitly the distribution of $\hat{H}_n$ for all $n \geq 1$, the next lemma gives the formula.

Lemma 1.4 (Tail red clusters distribution)**.** *For all $n \geq 1$ and $m \in [\![0, n]\!]$ we have:*

$$\mathbb{P}(\hat{H}_n \geq m) = \left[\prod_{k=0}^{m-1} (1 + k\gamma)\right]^{-1} =: \Pi(m)$$

Proof. Let's remark that we have:

$$\{\hat{H}_n \geq m\} = \bigcap_{j=n-m+1}^{n} \left\{ C_j > \sum_{k=j+1}^{n} T_k \right\}$$

Therefore, using the laws and the independence of all $(C_k)_{k\geq 1}$ and $(T_k)_{k\geq 1}$ we can compute the probability:

$$\begin{aligned}
\mathbb{P}(\hat{H}_n \geq m) = \mathbb{E}[\mathbb{P}(\hat{H}_n \geq m \mid (T_k)_{k\geq 1})] &= \mathbb{E}\left[\prod_{j=n-m+1}^{n} \exp\left(-\gamma \sum_{k=j+1}^{m} T_k\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\gamma \sum_{k=1}^{m-1} kT_{n-m+1+k}\right)\right] \\
&= \prod_{k=1}^{m-1} \int_0^{+\infty} e^{-k\gamma x} e^{-x} \mathrm{d}x \\
&= \left[\prod_{k=1}^{m-1} (1+k\gamma)\right]^{-1}
\end{aligned}$$

■

Now let's introduce the usual gamma function Γ, which realizes an increasing bijection $[2, +\infty) \to [1, +\infty)$ of inverse Γ^{-1}. Notice that by the fundamental properties of Γ, for all $m \geq 0$ we can write:

$$\Pi(m) = \gamma^{-m} \frac{\Gamma(\gamma^{-1})}{\Gamma(m+\gamma^{-1})}$$

Using Stirling's approximation we deduce that $\log \Pi(m) \sim -\log \Gamma(m)$ as $m \to +\infty$. Since we want to apply Borel-Cantelli lemmas, we can guess from here that Γ^{-1} is a good candidate for the function h. Before getting into the proper proof, we recall a classic fact that we will use several times.

Lemma 1.5 (Asymptotic behavior of Gamma function ratios)**.** *Let $\delta > 0$, we have:*

$$\frac{\Gamma(x+\delta)}{\Gamma(x)} \sim x^{\delta} \quad \text{as} \quad x \to +\infty$$

Proof. By Stirling's approximation. ■

Theorem 1.6 (Asymptotic upper bound for the tail cluster size)**.** *Almost surely, we have:*

$$\Psi := \limsup_{n\to+\infty} \frac{\hat{H}_n}{\Gamma^{-1}(n)} = 1$$

Proof. The idea is to show that $\mathbb{P}(\Psi > r) = 0$ for all $r > 1$ and $\mathbb{P}(\Psi \geq r) = 1$ for all $r < 1$ using the two Borel-Cantelli lemmas.

($\leq$) Let $r > 1$. Denote $m_n := \lceil r\Gamma^{-1}(n) \rceil$ for all $n \geq 1$ and $c_m := \#\{j \geq 1 \mid m = m_j\}$ for all $m \geq 0$. Using Lemma 1.4 and grouping together equal terms we have:

$$\sum_{n\geq 1} \mathbb{P}[\hat{H}_n \geq r\Gamma^{-1}(n)] = \sum_{n\geq 1} \Pi(m_n)\mathbb{1}_{m_n\leq n} \leq \sum_{m\geq 0} c_m \Pi(m) \tag{1}$$

Now remark that for all $m \geq 2r + 1$ and $j \in \mathbb{N}^*$ using that Γ is increasing, we have:

$$m_j = m \iff m - 1 < r\Gamma^{-1}(j) \leq m \iff \Gamma\left(\frac{m-1}{r}\right) < j \leq \Gamma\left(\frac{m}{r}\right)$$

We deduce then using Lemma 1.5 that:

$$c_m \sim \Gamma\left(\frac{m}{r}\right) - \Gamma\left(\frac{m-1}{r}\right) \sim \Gamma\left(\frac{m}{r}\right) \quad \text{as} \quad m \to +\infty$$

Now since $r > 1$, using Lemma 1.5 again, we can then proceed to a ratio test to prove that the series in (1) converge:

$$\frac{c_{m+1}\Pi(m+1)}{c_m\Pi(m)} \sim \frac{\Gamma\left[\frac{m+1}{r}\right]}{\Gamma\left[\frac{m}{r}\right]\gamma m} \sim \frac{m^{\frac{1}{r}-1}}{\gamma r^{\frac{1}{r}}} \to 0 \quad \text{when} \quad m \to +\infty$$

Applying the first Borel-Cantelli lemma we get then that almost surely only finitely many events of the sequence $(\{\hat{H}_n \geq r\Gamma^{-1}(n)\})_{n\geq 1}$ will be realized, which leads to $\mathbb{P}(\Psi > r) = 0$ and finally $\Psi \leq 1$ almost surely by letting $r \to 1$.

($\geq$) Let $r \in (0, 1)$ and fix $s \in (1, \frac{1}{r})$. For all $n \geq 1$ denote $k_n := \lfloor n^s \rfloor$ and $w_n := \lceil r\Gamma^{-1}(k_n) \rceil$. Since Γ^{-1} is asymptotically smaller than log, we can see that by construction:

$$\frac{w_{n+1}}{k_{n+1} - k_n} \sim \frac{r\Gamma^{-1}(n^s)}{sn^{s-1}} \to 0 \quad \text{as} \quad n \to +\infty$$

Since we have also $w_n \leq n$ for all $n \geq 1$ big enough we deduce that there exists $n_0 \geq 1$ such that for all $n \geq n_0$ we have $w_n \leq n$ and $k_{n+1} - w_{n+1} > k_n$. By construction it implies that the events $(\{\hat{H}_{k_n} \geq w_n\})_{n\geq n_0}$ are independent. For all $m \geq 0$, denote $d_m := \#\{j \geq n_0 \mid m = w_j\}$. As in the first part of the proof, using using lemma 1.4 we get:

$$\sum_{n\geq n_0} \mathbb{P}(H_{k_n} \geq r\Gamma^{-1}(k_n)) = \sum_{m\geq 0} d_m \Pi(m) \tag{2}$$

Finally remark that proceeding like before we have $d_m \sim \Gamma(m/r)^{1/s}$ as $m \to +\infty$ and thus since $rs > 1$, the series of (2) diverges by ratio test:

$$\frac{d_{m+1}\Pi(m+1)}{d_m\Pi(m)} \sim \frac{m^{\frac{1}{rs}-1}}{\gamma r^{\frac{1}{rs}}} \to +\infty \quad \text{as} \quad m \to +\infty$$

Therefore applying the second Borel-Cantelli lemma we get the almost sure existence of an infinite sub-sequence of realized events in $(\{\hat{H}_{k_n} \geq w_n\})_{n\geq n_0}$, which clearly implies $\mathbb{P}[\Psi \geq r] = 1$. We then have the result by letting $r \to 1$. ■

Finally, we recall a classical result on the behavior of Γ^{-1} and we sum up everything we have seen to get the result we wanted.

Lemma 1.7 (Asymptotic for Γ^{-1}). *As $t \to +\infty$ we have:*

$$\Gamma^{-1}(t) \sim \frac{\log t}{\log \log t}$$

Proof. By Stirling's approximation we know that when $x \to +\infty$ we have:

$$\log \Gamma(x) \sim x \log x \quad \text{and} \quad \log \log \Gamma(x) \sim \log x$$

Taking the ratio and making the substitution $x = \Gamma^{-1}(t)$ we deduce the result. ∎

Corollary 1.8 (Asymptotic upper bound). *Almost surely, we have:*

$$\limsup_{t \to +\infty} \frac{H_t \log \log t}{\log t} = \limsup_{t \to +\infty} \frac{H_t}{\Gamma^{-1}(\# A_t)} = 1$$

Proof. The second inequality is given by Theorem 1.6 and Proposition 1.3 since Γ^{-1} is increasing. Finally since $\# A_t \sim t$ almost surely as $t \to +\infty$ by Lemma 1.1, we get the result applying Lemma 1.7. ∎

1.2 The asymptotic lower bound

The goal of this subsection is to show that almost surely:

$$\liminf_{t \to +\infty} H_t = 0$$

To start, we introduce some notation.

Definition 1.9 (Complete recovery probabilities). *First, for any $n \geq 0$ and $v \in [\![1, n]\!]$, we denote:*

$$E_v^n := \left\{ C_v > \sum_{k=v+1}^{n+1} T_k \right\}$$

Then we define the complete recovery probability of order n by:

$$\nu_n := 1 - \mathbb{P}\left[\bigcup_{v=1}^{n} E_v^n \right]$$

By construction for all $n \geq 0$ and $v \in [\![1, n]\!]$, we have $E_v^n = \{v \in R_{\tau_{n+1}}\}$. Therefore, for all $n \geq 0$ we can can write:

$$\nu_n = \mathbb{P}(R_{\tau_{n+1}} = \{n+1\})$$

This is why we call ν_n the complete recovery probability of order n. Next we want to compute these probabilities. The following lemma gives an explicit formula.

Lemma 1.10 (Complete recovery probabilities formula)**.** *For all $n \geq 0$ and $l \geq 0$ we denote:*

$$N_l^n := \left\{ \mathbf{x} \in {\mathbb{N}^*}^l : \sum_{k=1}^{n} \mathbf{x}_k \leq n \right\} \text{ and } S_l(n) := \sum_{\mathbf{x} \in N_l^n} \prod_{k=1}^{l} (1 + k\gamma)^{-\mathbf{x}_k}$$

Then for all $n \geq 0$ we have:

$$\nu_n = \sum_{l=0}^{n} (-1)^l S_l(n) \tag{3}$$

Proof. Let $n \geq 0$, by applying the inclusion-exclusion principle we get:

$$\nu_n = \sum_{A \subset [\![1,n]\!]} (-1)^{\# A} \Pi_A \tag{4}$$

$$\text{where} \quad \Pi_A := \mathbb{P}\left[\bigcap_{v \in A} E_v^n \right] \quad \text{for all } A \subset [\![1,n]\!]$$

Now fix $A \subset [\![1,n]\!]$. Denote $l := \# A$ and $(a_i)_{i \in [\![1,l]\!]}$ the elements of A in increasing order. Let's define $a_{l+1} := n+1$ for convenience. Using the law of the variables, independence, conditional expectancy and splitting sums, we can compute Π_A:

$$\begin{aligned}
\Pi_A &= \mathbb{E}\left[\prod_{j=1}^{l} \exp\left(-\gamma \sum_{k=a_j+1}^{n+1} T_k \right) \right] \\
&= \mathbb{E}\left[\exp\left(-\gamma \sum_{j=1}^{l} \left[j \sum_{k=a_j+1}^{a_{j+1}} T_k \right] \right) \right] \\
&= \prod_{j=1}^{l} \prod_{k=a_j+1}^{a_{j+1}} \int_{\mathbb{R}_+} e^{-\gamma j t} e^{-t} \mathrm{d}t \\
&= \prod_{j=1}^{l} (1 + j\gamma)^{-(a_{j+1} - a_j)}
\end{aligned}$$

Next, for all $l \geq 0$ let's denote:

$$\mathbf{\Psi}_l : N_l^n \to \mathcal{P}_l([\![1,n]\!]), \ \mathbf{x} \mapsto \left\{ n + 1 - \sum_{i=1}^{k} \mathbf{x}_k, \ k \in [\![1,l]\!] \right\}$$

By the construction we have that for all $l \geq 0$ and $\mathbf{x} \in N_l^n$:

$$\Pi_{\boldsymbol{\Psi}_l(\mathbf{x})} = \prod_{k=1}^{n} (1 + k\gamma)^{-\mathbf{x}_k}$$

Finally, since $\boldsymbol{\Psi}_l$ defines a one-to-one correspondence, we obtain the result by replacing $A \subset [\![1, n]\!]$ by $\boldsymbol{\Psi}_l(\mathbf{x})$ in (4), for $l \in [\![0, n]\!]$ and $\mathbf{x} \in N_l^n$. ■

Now it turns out that using the formulas given by Proposition 1.10, we can show that $(\nu_n)_{n\geq 1}$ converges towards an explicit positive number. It is the key point of this subsection.

Proposition 1.11 (Limit of the complete recovery probabilities). *The sequence $(\nu_n)_{n\geq 1}$ converges towards a positive number, more precisely:*

$$\lim_{n\to+\infty} \nu_n = e^{-1/\gamma}$$

Proof. Let $l \geq 0$. Since it is a series of positive terms, we have $S_l(n) \to S_l^\infty$ as $n \to +\infty$ where:

$$S_l^\infty := \sum_{\mathbf{x}\in\mathbb{N}^{*n}} \prod_{k=1}^{l} (1 + k\gamma)^{-\mathbf{x}_k} = \prod_{k=1}^{l} \sum_{j=1}^{+\infty} \left(\frac{1}{1 + k\gamma}\right)^j = \prod_{k=1}^{l} \frac{1}{k\gamma} = \frac{1}{l!\gamma^l}$$

Now remark that for all $n \geq 0$ we have $S_l(n) \leq S_l^\infty$, moreover:

$$\sum_{l=0}^{+\infty} S_l^\infty = e^{1/\gamma} < +\infty$$

There fore, using Lemma 1.10 and by applying the dominated convergence theorem with respect to the counting measure, we deduce the result writing:

$$\lim_{n\to+\infty} \nu_n = \lim_{n\to+\infty} \sum_{l=0}^{+\infty} (-1)^l S_l(n) = \sum_{l=0}^{+\infty} \frac{(-1/\gamma)^l}{l!} = e^{-1/\gamma}$$

■

Finally, we use this positive limit to deduce the following theorem which will imply directly the result we want.

Theorem 1.12 (Complete recovery happens infinitely often almost surely). *Almost surely we have:*

$$\liminf_{n\to+\infty} H_{\tau_n} = 1$$

Proof. Since for all $n \geq 1$ we have $H_{\tau_n} \in \mathbb{N}^*$, we just need to show that for all $n_0 \geq 1$ there exists almost surely $N \geq n_0$ such that $H_{\tau_N} = 1$. Fix $n_0 \geq 1$ and $\delta \in (0, e^{-1/\gamma})$. For any $n \geq 1$, define:

$$\mathcal{F}_n := \sigma[(C_j)_{1\leq j\leq n}, (T_j)_{2\leq j\leq n}]$$

Next, inductively we construct an increasing sequence $(N_k)_{k\geq 0}$ of random indices bigger than n_0 such that for all $k \geq 0$ and $n \geq 1$ we have $\{N_k = n\} \in \mathcal{F}_n$ and:

$$\mathbb{P}[H_{\tau_{N_{k+1}}} > 1 \mid \mathcal{F}_n]\mathbb{1}_{N_k=n} \leq (1-\delta)\mathbb{1}_{N_k=n} \tag{5}$$

Let's define $N_0 := n_0$, the initialisation is verified since it is a deterministic variable. Now fix $k \geq 0$ and assume that the sequence is constructed up to index k. Let $n \geq 1$. Denote $C_n^* := \max_{1\leq j\leq n} C_j$. Recall the notations of Definition 1.9 and for all $m > n$ define:

$$K_m := \bigcup_{v=n+1}^{m-1} E_v^{m-1} \quad \text{and} \quad W_m := \sum_{v=n+1}^{m} T_v$$

By construction, for all $m > n$ we have:

$$\{H_{\tau_m} > 1\} \subset \{C_n^* > W_m\} \cup K_m$$

Now observe that $(W_m)_{m>n}$ is independent of $\mathcal{F}_n$ and goes almost surely to infinity as $m \to +\infty$ by the law of large numbers. Since C_n^* is $\mathcal{F}_n$-measurable, it follows that almost surely:

$$\mathbb{P}(C_n^* > W_m \mid \mathcal{F}_n) \to 0 \quad \text{when} \quad m \to +\infty$$

Moreover by shifting invariance of the law of the passage times and clocks, we can see that $\mathbb{P}(K_m) = 1 - \nu_{m-n-1}$ for all $m > n$. Therefore since K_m is independent of $\mathcal{F}_n$ for all $m > n$, summing up we get that, almost surely:

$$\begin{aligned} P_m &:= \mathbb{P}(H_{\tau_m} > 1 \mid \mathcal{F}_n) \\ &\leq \mathbb{P}(C_n^* > W_m \mid \mathcal{F}_n) + 1 - \nu_{m-n} \to 1 - e^{-1/\gamma} \quad \text{when} \quad m \to +\infty \end{aligned}$$

Thus, we get that there exists an $\mathcal{F}_n$-measurable random variable $N_{k+1}^{(n)} > n$ such that:

$$P_{N_{k+1}^{(n)}} \leq 1 - \delta \quad \text{a.s.}$$

Finally, we set:

$$N_{k+1} := \sum_{n\geq 1} N_{k+1}^{(n)} \mathbb{1}_{N_k=n}$$

By construction (5) is verified. Moreover for all $l \geq 1$ we have:

$$\{N_{k+1} = l\} = \bigcup_{j=1}^{l} \{N_k = j \ \wedge \ N_{k+1}^{(j)} = l\} \in \mathcal{F}_l$$

We have thus constructed $(N_k)_{k\geq 1}$. Now for all $k \geq 0$ denote $Q_k := \{H_{\tau_{N_k}} > 1\}$ and for all integers $p \geq 0$ define:

$$\Lambda_p := \bigcap_{k=0}^{p} Q_k$$

For all $p \geq 0$ fixed, we have:

$$\mathbb{P}(\Lambda_{p+1}) = \sum_{n\geq 1} \mathbb{P}(\Lambda_p \ \cap \ \{N_p = n\} \ \cap \ Q_{p+1})$$

Then observe that for all $n \geq 1$ by construction $\Lambda_p \cap \{N_p = n\}$ is $\mathcal{F}_n$-measurable and $\mathbb{P}(Q_{p+1} \mid \mathcal{F}_n)\mathbb{1}_{N_p=n} \leq (1-\delta)\mathbb{1}_{N_p=n}$. Thus using conditional expectancy for each term of the sum we get:

$$\mathbb{P}(\Lambda_{p+1}) \leq \sum_{n\geq 1} \mathbb{E}[\mathbb{1}_{\Lambda_p}\mathbb{1}_{N_p=n}(1-\delta)] = (1-\delta)\mathbb{P}(\Lambda_p)$$

Therefore, we deduce:

$$\mathbb{P}\left[\bigcap_{k\geq 0} Q_k\right] = \lim_{p\to+\infty} \mathbb{P}(\Lambda_p) \leq \lim_{p\to+\infty} \mathbb{P}(\Lambda_0)(1-\delta)^p = 0$$

This tells us exactly that there exists almost surely $N \geq n_0$ such that $H_{\tau_N} = 1$. Since this holds for any $n_0 \geq 1$ we get the result. ∎

Corollary 1.13 (Asymptotic lower bound)**.** *Almost surely, we have:*

$$\liminf_{t\to+\infty} H_t = 0$$

Proof. First observe that the probability that two reaching or recovery times match is 0. Then almost exists for all $n \geq 1$ there exists $\epsilon_n \in (0,1)$ such that:

$$H_{\tau_n-\epsilon_n} = H_{\tau_n} - 1$$

Finally taking the lim inf as $n \to +\infty$ on both sides, since $\tau_n \to +\infty$ by Lemma 1.1, the result follows from Theorem 1.12. ∎

Since $H_t \in \mathbb{N}$ for all $t \in \mathbb{R}_+$, one implication of Corollary 1.13 is that almost surely for any function $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ we have:

$$\liminf_{t\to+\infty} \frac{H_t}{h(t)} = 0$$

Thus it is almost surely impossible to give an equivalent for H_t as $t \to +\infty$. To summarize, in the case where G is an infinite line, we have sharp bounds for the asymptotic behavior of $(H_t)_{t\in\mathbb{R}_+}$ which do not depend on γ. Theorem I is obtained by summing up Corollaries 1.8 and 1.13.

2 The super-critical case

In this section G is a Galton-Watson tree with supercritical branching distribution of finite mean and the recovery rate $\gamma > 0$ is arbitrary. Recall the notations defined in Definition D. We assume that $\mathbb{V}$ is the set of all finite words on the alphabet $\mathbb{N}$, i.e.:

$$\mathbb{V} = \bigcup_{n\geq 0} \mathbb{N}^n$$

We assume that the origin o is the empty word of $\mathbb{V}$. Let $*$ denote the usual word concatenation operator on $\mathbb{V}$. Let's fix a random variable $B \in \mathbb{N}$ with $1 <$

$\mathbb{E}B < +\infty$ and consider a collection $(B_v)_{v\in\mathbb{V}}$ of i.i.d. copies of B. We assume that E consists of all the edges of the form $(v, v*c)$ for $v \in \mathbb{V}$ and $c \in [\![1, B_v - 1]\!]$, so that every vertex $v \in \mathbb{V}$ has exactly B_v neighbors. By construction G is a Galton-Watson tree with super-critical branching distribution B. In Figure 3 a simulation is presented. We denote $\mathcal{S}$ the event that G is infinite and $q < 1$ the probability that it is finite. To simplify the notations for all $v \in \mathbb{V}$ and $c \in \mathbb{N}$ we set $T_{v*c} := T_{(v,v*c)}$. We define $\alpha := \mathbb{E}B - 1$ and $\mathbb{P}_{\mathcal{S}} := \mathbb{P}[.|\mathcal{S}]$. Unless otherwise stated, we work under the initial probability measure $\mathbb{P}$. The aim of this section is to find two non-decreasing functions $h, g : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that almost surely on $\mathcal{S}$ we have:

$$\liminf_{t\to+\infty} \frac{H_t}{h(t)} \geq 1 \quad \text{and} \quad \liminf_{t\to+\infty} \frac{M_t}{g(t)} \geq 1$$

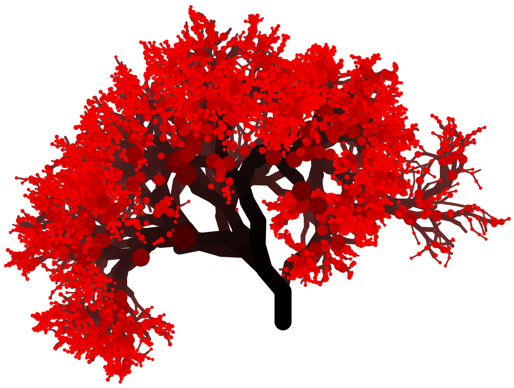

Figure 3: Simulation of A_t with $B \sim \mathrm{Bin}(2, \frac{4}{5})$, $\gamma = \frac{1}{2}$ and $t = 13$. The red dots represent the vertices of R_t and the dark lines represent the edges in $E \cap A_t^2$.

2.1 Useful notations and properties

In this section we do some preliminary work. We start by defining a sequence of crucial stopping times for the process.

Definition 2.1 (Growing times). *For all $n \geq 1$, we define:*

$$\theta_n := \inf(\{t \geq 0 : \#A_t \geq n\} \cup \{+\infty\})$$

Observe that almost surely all the reaching times of vertices in G are distinct and thus by construction for all $n \geq 1$ θ_n coincides with the reaching time of a vertex and$\#A_{\theta_n} = n$, assuming that $\#G \geq n$.

For any $V \subset \mathbb{V}$ we define $\partial^* V$ as the external boundary of V in G, i.e. the set of all vertices of $G \setminus V$ that are the neighbors of a vertex of V. The first goal of this subsection is to understand the behavior of the shape of the process as time grows by providing asymptotic estimates for both θ_n and $\#\partial^* A_{\theta_n}$ as $n \to +\infty$. We start with the following lemma that gives us a useful characterisation for the distribution of $[(\#\partial^* A_{\theta_n})_{n\geq 1}, (\theta_n)_{n\geq 1}]$.

Lemma 2.2 (Step-by-step construction)**.** *Take a sequence $(D_i)_{i\geq 1}$ of i.i.d copies of B and a sequence $(X_i)_{i\geq 1}$ of i.i.d* $\mathrm{Exp}(1)$ *random variables independent of $(D_i)_{i\geq 1}$. Define two new sequences of random variables $(J_n)_{n\geq 0}$ and $(K_n)_{n\geq 1}$ by induction by:*

- $J_0 := 1$
- *for all $n \geq 0$, $J_{n+1} := J_n + (D_{n+1} - 1)\mathbb{1}_{J_n>0}$*
- $K_1 := 0$
- *for all $n \geq 1$, $K_{n+1} := +\infty$ if $J_n = 0$ and $K_{n+1} := K_n + \frac{X_{n+1}}{J_n}$ otherwise*

Then $[(\#\partial^* A_{\theta_n})_{n\geq 1}, (\theta_n)_{n\geq 1}]$ is distributed as $[(J_n)_{n\geq 1}, (K_n)_{n\geq 1}]$.

Proof. Let's show by induction that $\mathcal{W}_n := [(\#\partial^* A_{\theta_k})_{1\leq k\leq n}, (\theta_k)_{1\leq k\leq n}]$ is distributed as $[(J_k)_{1\leq k\leq n}, (D_k)_{1\leq k\leq n}]$ for all $n \geq 1$. The initialisation is verified for $n = 1$ since B_o is distributed as B and $\theta_1 = K_1 = 0$. Next fix $n \geq 1$ and assume the result is true at index n. Let's fix $V, F \subset \mathbb{V}$ such that $V \cap F = \emptyset$, $F \subset \{v * c,\ (v, c) \in V \times \mathbb{N}\}$ and $\#V = n$. We define:

$$\mathcal{C} := \{A_{\theta_n} = V \wedge \partial^* A_{\theta_n} = F\} \quad \text{and} \quad \mathcal{F} := \sigma(T_v,\ v \in V)$$

Observe that on $\mathcal{C}$, $\mathcal{W}_n$ can be written as an $\mathcal{F}$-measurable variable. Moreover using the memoryless property of the exponential distribution we have that conditional on $\mathcal{C}$, $(\tau_v - \theta_n)_{v\in F}$ are i.i.d. Exp(1) random variables independent of $\mathcal{F}$. Therefore conditional on $\mathcal{C}$, considering the next random vertex of F reached by the process, we deduce that $[(\theta_{n+1} - \theta_n)\#F, \#\partial^* A_{\theta_{n+1}} - \#\partial^* A_{\theta_n}]$ is independent of $\mathcal{F}$ and distributed as $(X_{n+1}\#F, D_{n+1} - 1)$. Figure 4 gives an illustration of this step-by-step construction. Finally summing up over all possibilities for V and F, we deduce that the results also holds at index $n + 1$, which conclude the proof. ■

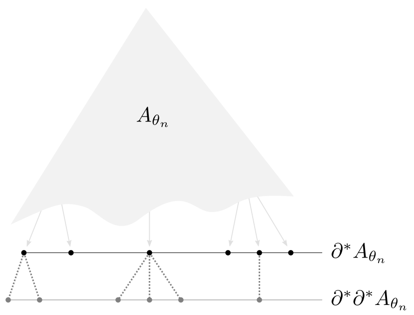

Figure 4: Illustration of the step-by step construction of first-passage percolation in G explained in the proof of Lemma 2.2.

The next proposition follows from the previous lemma and will be useful several times later on.

Proposition 2.3 (Controlled-growth). *Almost surely on $\mathcal{S}$, we have:*

$$\theta_n \sim \frac{\log n}{\alpha} \quad \textit{and} \quad \#\partial^* A_{\theta_n} \sim \alpha n \quad \textit{when} \quad n \to +\infty$$

Proof. First, remark that we can write:

$$\mathcal{S} = \{\forall n \geq 1 \ \#\partial^* A_{\theta_n} > 0\}$$

Therefore, using the construction given by Lemma 2.2 it is enough to show that conditionally on $\mathcal{S}' := \{\forall n \geq 1 \ J_n > 0\}$, we have almost surely:

$$J_n \sim \alpha n \quad \text{and} \quad K_n \sim \frac{\log n}{\alpha} \quad \text{as} \quad n \to +\infty$$

Next remark that on $\mathcal{S}'$, for all $n \geq 1$ we can write:

$$J_n = 1 + \sum_{k=1}^{n} (D_k - 1) \quad \text{and} \quad K_n = \sum_{k=1}^{n-1} \frac{X_k}{J_k}$$

Thus by the law of large number, almost surely on $\mathcal{S}'$ we have $J_n \sim \alpha n$ as $n \to +\infty$. Moreover, by Kolmogorov three series theorem (see, for exemple, [Dur19]), since $\sum_{k\geq 1} \frac{1}{k^2} < +\infty$ and $(X_k - 1)_{k\geq 1}$ is a sequence of i.i.d. centred variables, we have that:

$$\sum_{k=1}^{n-1} \frac{X_k - 1}{k} \quad \text{converges in } \mathbb{R} \text{ almost surely as } n \to +\infty$$

We deduce then that almost surely:

$$\sum_{k=1}^{n-1} \frac{X_k}{k} \sim \log n \quad \text{as} \quad n \to +\infty$$

Finally combining everything we get the result. ∎

Remark. *In fact, the asymptotic behavior of $(\theta_n)_{n\geq 1}$ has been already studied in more details, see [AK67].*

Now we introduce some useful processes coupled to the original one.

Definition 2.4 (Shifted processes). *Let $v \in \mathbb{V}$. We construct the shifted process associated with v by applying Definition D with the same set up as for the original process except for the origin that we set at v, in particular we do not change the random variables. All the quantities obtained are denoted with a superscript (v).*

By construction, for any collection of vertices in $\mathbb{V}$ such that none of them is a prefix of any other, the corresponding shifted processes are independent and distributed as the original one. In fact, there is a natural way of decomposing the process into several independent ones, which is explained in the following proposition.

Proposition 2.5 (Split property)**.** *There exist an $\sigma(E)$-measurable random set $O \subset \mathbb{V}$ such that under $\mathbb{P}_{\mathcal{S}}$:*

1. *Almost surely we have $O \subset G$, $1 < \#O < +\infty$ and:*

$$\#G \setminus \bigcup_{v \in O} G^{(v)} < +\infty$$

2. *Conditional on O, $(G^{(v)})_{v \in O}$ is a collection of independent random sets distributed as G.*

Proof. For any $v \in \mathbb{V}$, let's define the descendants of v which are the root of an infinite sub-tree in G by:

$$\mathbf{S}_v := \{w \in \partial^* \{v\} : \#G^{(w)} = +\infty\}$$

To start, we want to show that on $\mathcal{S}$ there exists almost surely a vertex $v \in G$ such that $\#\mathbf{S}_v > 1$. To do so, let's define:

$$\hat{\mathcal{S}} := \{\exists v \in G \text{ such that } \#\mathbf{S}_v > 1\} \quad \text{and} \quad \hat{q} := 1 - \mathbb{P}(\hat{\mathcal{S}})$$

First, remark that we have $\hat{q} < 1$ since:

$$\mathbb{P}[\hat{\mathcal{S}}] \geq \mathbb{P}[\#\mathbf{S}_o > 1] \geq \mathbb{P}[B > 1](1 - q)^2 > 0$$

Next observe that reasoning conditionally on B_o, we have:

$$\hat{q} \leq \mathbb{P}\left[\bigcap_{v=0}^{B_0 - 1} \hat{\mathcal{S}}^{(v)}\right] = \mathbb{E}[\hat{q}^B]$$

Therefore since for all $s \in [0, 1]$ we know that $\mathbb{E}[s^B] < s \iff s \in (q, 1)$ we deduce that $\hat{q} \leq q$. Since $\hat{\mathcal{S}} \subset \mathcal{S}$ it shows that on $\mathcal{S}$, $\hat{\mathcal{S}}$ is almost surely verified. Now remark that if u and v are two vertices of G with common prefix $a \notin \{u, v\}$ such that $\#\mathbf{S}_u > 1$ and $\#\mathbf{S}_v \geq 1$, then $\#\mathbf{S}_a > 1$. It follows then that there exists an $\sigma(E)$-measurable random variable $W \in G$ such that almost surely on $\mathcal{S}$, W is the unique vertex of G such that $\#\mathbf{S}_W > 1$ and $\#G \setminus G^{(W)} < +\infty$. We set $O := \mathbf{S}_W$. See Figure 5 for an illustration. It remains to show the second point of the proposition. Let $n \geq 1$ and $V \subset \mathbb{N}^n$ with $\#V > 1$. We define:

$$\mathbf{O} := \left\{\#G \setminus \bigcup_{v \in V} G^{(v)} < +\infty\right\} \cap \{V \subset G\}$$

By construction, we have:

$$\mathcal{S} \cap \{O = V\} = \mathbf{O} \cap \bigcap_{s \in V} \mathcal{S}^{(s)}$$

Finally since $(G^{(v)})_{v \in V}$ are i.i.d. random variables distributed as G and independent of $\mathbf{O}$, we deduce the result. ■

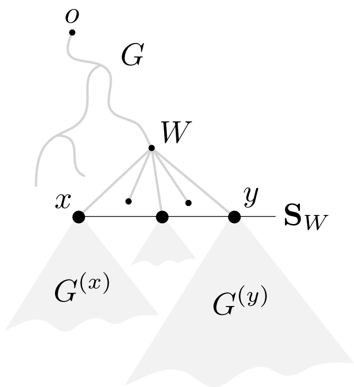

Figure 5: Illustration of the split property.

2.2 General approach for asymptotic lower bounds

Let's fix $(\mathcal{Q}_t)_{t\in\mathbb{R}} \in \{(H_t)_{t\in\mathbb{R}}, (M_t)_{t\in\mathbb{R}}\}$ and recall that $\mathcal{Q}_t = 0$ for all $t < 0$ by construction. The goal of this subsection is to provide a non-trivial sufficient property to characterize asymptotic lower bounds for $\mathcal{Q}_t$ as $t \to +\infty$, i.e. if $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a non-decreasing function we want to get a non-trivial property on f that guarantees that we have:

$$\liminf_{t\to+\infty} \frac{\mathcal{Q}_t}{f(t)} \geq 1 \quad \text{a.s. on } \mathcal{S}$$

To do so, we want to use the Borel-Cantelli lemma and we start by looking for some upper bounds for the probability of $\{\mathcal{Q}_t \leq m\}$ when $(t, m) \in \mathbb{R}_+^2$ are big. The starting point of the reasoning is to say that if we fix $t \in \mathbb{R}_+$, the vertices of R_t are likely to be close to the boundary of A_t. It leads to the following proposition.

Proposition 2.6 (Boundary inequality)**.** *Let T be an* Exp(1) *random variable independent of the process. For all $m \in \mathbb{R}_+$ we define:*

$$\eta(m) := \mathbb{P}[\mathcal{Q}_{1-T} \leq m]$$

Then for all $(t, m, n) \in \mathbb{R}_+^3$, we have:

$$P_m^n(t) := \mathbb{P}[\mathcal{Q}_t \leq m \;\wedge\; \#\partial^* A_{t-1} \geq n] \leq \eta(m)^n$$

Proof. Observe that $\partial^* A_{t-1}$ is almost surely finite since on $\mathcal{S}$ there exists a.s. $n \geq 1$ big enough such that $t - 1 < \theta_n$ by Lemma 2.3. Let $\mathcal{B}_n \subset \mathcal{P}(\mathbb{V})$ denote the set all possible shapes of size greater than n for $\partial^* A_{t-1}$, i.e. the set of all subset $F \subset \mathbb{V}$ such that $\#F \geq n$ and $\mathbb{P}(\partial^* A_{t-1} = F) > 0$. It is a countable set by the previous observation and we have:

$$\begin{aligned}P^n_m(t) &= \sum_{F\in\mathcal{B}_n} \mathbb{P}(\mathcal{Q}_t \le m \ \wedge \ \partial^* A_{t-1} = F)\\ &\le \sum_{F\in\mathcal{B}_n} \mathbb{P}[\forall v\in F \ \mathcal{Q}^{(v)}_{t-\tau_v} \le m \ \wedge \ \partial^* A_{t-1} = F]\end{aligned}$$

Let $F \in \mathcal{B}_n$. By the memoryless property of the exponential distribution, we know that conditional on $\{\partial^* A_{t-1} = F\}$, $(\tau_v - t + 1)_{v\in F}$ are i.i.d. Exp(1) variables. Moreover on $\{\partial^* A_{t-1} = F\}$, the shifted processes corresponding to F are i.i.d., independent of $(\tau_v)_{v\in F}$ and distributed as the initial process. We deduce then the result by writing:

$$\begin{aligned}P^n_m(t) &\le \sum_{F\in\mathcal{B}_n} \mathbb{P}[\mathcal{Q}_{1-T} \le m]^{\#F}\mathbb{P}(\partial^* A_{t-1} = F)\\ &= \eta(m)^n \mathbb{P}(\#\partial^* A_{t-1} \ge n) \le \eta(m)^n\end{aligned}$$

An illustration of the reasoning is given in Figure 6. ■

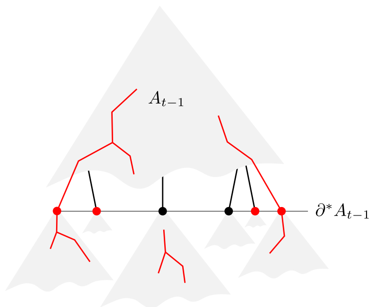

Figure 6: Illustration of the reasoning in the proof of Proposition 2.6. The red lines represent edges connecting vertices of R_t.

Next, we apply the previous proposition together with the Borel-Cantelli lemma for some random sub-sequences of times to deduce the following lemma. Notice that we introduce two different sequences of times in the statement, it is because we will look at some shifted processes later on and it will induce some decays.

Lemma 2.7 (Asymptotic lower bounds on sub-sequences)**.** *Let $(Y_n)_{n\ge1}$ and $(Z_n)_{n\ge1}$ be two random sequences of times, such that $(Z_n)_{n\ge1}$ is independent of the process and almost surely as $n \to +\infty$ we have:*

$$Y_n \sim \frac{\log n}{\alpha} \quad \textit{and} \quad Z_n \sim \frac{\log n}{\alpha}$$

Let $h : \mathbb{R}_+ \to \mathbb{R}^*_+$ be a non-decreasing function such that for all $x \in \mathbb{R}_+$ big enough we have:

$$\eta \circ h(x) \le \exp(-1/x)$$

Then for all $\beta \in (0, \alpha)$ we have:

$$\liminf_{t \to +\infty} \frac{\mathcal{Q}_{Z_n}}{h(e^{\beta Y_n})} \geq 1 \quad \text{a.s. on } \mathcal{S}$$

Proof. We will use the Borel-Cantelli lemma together with the boundary inequality and the controlled-growth property (respectively Propositions 2.6 and 2.3). Let's fix $y \in (\frac{\beta}{\alpha}, 1)$ and $z \in (y, 1)$. For all $n \geq 1$ we define:

$$\begin{aligned} &B_n := \{\mathcal{Q}_{Z_n} \leq h(e^{\beta Y_n})\} \\ &\text{and} \quad G_n := \{\#\partial^* A_{Z_n - 1} \geq \beta n^z \ \wedge \ h(e^{\beta Y_n}) \leq h(n^y)\} \end{aligned}$$

Using that h non-decreasing and that $(Z_n)_{n\geq 1}$ is independent of the process, we deduce that for all $n \geq 1$, with the notation of Proposition 2.6, we have:

$$\mathbb{P}(B_n \cap G_n) \leq \mathbb{P}[\mathcal{Q}_{Z_n} \leq h(n^{-y}) \ \wedge \ \#\partial^* A_{Z_n - 1} \geq \beta n^z] = \mathbb{E}[P^{\beta n^z}_{h(n^y)}(Z_n)]$$

Thus using Proposition 2.6 and that $\eta \circ h(x) \leq \exp(-1/x)$ for all x big enough we get that for all $n \geq 1$ sufficiently large:

$$\mathbb{P}(B_n \cap G_n) \leq \exp(-\beta n^{z-y})$$

Since by construction $z - y > 0$, it follows that $\sum_{n \geq 1} \mathbb{P}(B_n \cap G_n) < +\infty$. Moreover $\mathbb{P}$ is dominating $\mathbb{P}_{\mathcal{S}}$ and thus applying the Borel-Cantelli we obtain:

$$\mathbb{P}_{\mathcal{S}}\left[\limsup_{n \to +\infty} (B_n \cap G_n)\right] = 0 \tag{6}$$

Now we want get rid of the events $(G_n)_{n\geq 1}$ in this lim sup by showing that their lim inf has probability 1 under $\mathbb{P}_{\mathcal{S}}$. From now on, let's reason under $\mathbb{P}_{\mathcal{S}}$. Since $y/\beta > 1/\alpha$ by construction, using the assumption on the asymptotic behavior of $(Y_n)_{n\geq 1}$ we know that almost surely for all $n \geq 1$ big enough we have:

$$Y_n \leq \frac{y \log n}{\beta}$$

And thus since $t \mapsto h(e^{\beta t})$ is non-decreasing:

$$h(e^{\beta Y_n}) \leq h(n^{-y})$$

Next, using Proposition 2.3 and the assumptions on the asymptotic behavior of $(Z_n)_{n\geq 1}$, we get that:

$$Z_n - 1 \sim \theta_n \sim \frac{\log n}{\alpha} \quad \text{a.s. when} \quad n \to +\infty$$

Therefore since $z \in (0, 1)$, almost surely for all $n, k \geq 1$ big enough we have:

$$Z_n - 1 - \theta_k \geq \frac{z}{\alpha} \log n - \frac{1}{z\alpha} \log k$$

Setting $k = \lfloor n^z \rfloor$, we deduce then that a.s. $Z_n - 1 \geq \theta_{\lfloor n^z \rfloor}$ for all $n \geq 1$ big enough. Now by Proposition 2.3, since $\beta < \alpha$, a.s. for all $n \geq 1$ big enough we have:

$$\#\partial^* A_{\theta_n} \geq \beta(n+1)$$

Which implies that a.s. for all $n \geq 1$ big enough and $t \geq \theta_{\lfloor n^z \rfloor}$ we have $\#\partial^* A_t \geq \beta n^z$. Summing up we obtain that a.s. for all $n \geq 1$ big enough:

$$\#\partial^* A_{Z_n - 1} \geq n^z$$

We just showed that almost surely under $\mathbb{P}_{\mathcal{S}}$, G_n is verified for all $n \geq 1$ big enough. Together with (6) it implies that:

$$\mathbb{P}_{\mathcal{S}}\left[\limsup_{n \to +\infty} B_n\right] = 0$$

Finally by definition we have the result. ∎

Finally we apply the previous lemma together we the the split property (Proposition 2.5) to obtain the following theorem, which conclude the subsection by giving a sufficient characterisation for asymptotic lower bounds.

Theorem 2.8 (Characterization of some asymptotic lower bounds)**.** *Let $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a non-decreasing function such that $\eta \circ h(x) \leq \exp(-1/x)$ for all x big enough. Then for all $\beta \in (0, \alpha)$ we have:*

$$\liminf_{t \to +\infty} \frac{\mathcal{Q}_t}{h(e^{\beta t})} \geq 1 \quad \textit{a.s. on } \mathcal{S}$$

Proof. To begin, let's observe that for any $t > 0$ we have:

$$\mathcal{Q}_t \geq \mathcal{Q}^-_{\theta_{1+\#A_t}} \quad \text{and} \quad h(e^{\beta t}) \leq h\left(e^{\beta \theta_{1+\#A_t}}\right) \quad \text{a.s.}$$

Where the minus sign means we don't count the last vertex reached by the process. Dividing these inequalities and taking the $\liminf$, since $\#A_t \to +\infty$ almost surely as $t \to +\infty$ we get that:

$$\liminf_{t \to +\infty} \frac{\mathcal{Q}_t}{h(e^{\beta t})} \geq \liminf_{n \to +\infty} \frac{\mathcal{Q}^-_{\theta_n}}{h(e^{\beta \theta_n})}$$

Now we already have a special random sub-sequence of time to look at, but it is not independent of the process and thus we can not apply Lemma 2.7 yet. To get around this problem we use the split property given by Proposition 2.5. First, recall from Proposition 2.5 that O is a discrete variable. Therefore it is enough to prove the proposition on the event $\{O = D\}$ for arbitrary $D \subset \mathbb{V}$. Let's fix $D \subset \mathbb{V}$ such that $\mathbb{P}_{\mathcal{S}}(O = D) > 0$, in particular we have $\#D > 1$. From now on we reason under $\mathbb{P}_{\mathcal{S}}$ and conditionally on $\{O = D\}$. For every $u \in D$ fix $u^* \in D$ such that $u^* \neq u$. For all $u \in D$ and $n \geq 1$, define:

$$Y_n^u := \tau_u + \theta_n^{(u)} \quad \text{and} \quad Z_n^u := Y_n^u - \tau_{u^*}$$

By the split property, since a.s. except a finite set all vertices of G belong to $G^{(u)}$ for some $u \in D$, we know that a.s. for all $n \geq 1$ big enough θ_n will be reached for a vertex v in $G^{(u)}$ for some $u \in D$ (see Figure 7). Therefore for all $n \geq 1$ there exists a random vertex $V_n \in D$ and a random index $L_n \geq 1$ such that almost surely:

$$\theta_n = Y_{L_n}^{V_n}$$

By construction we have then:

$$\mathcal{Q}_{\theta_n}^{-} \geq \mathcal{Q}_{Z_{L_n}^{V_n}}^{(V_n^*)}$$

Now observe observe that $L_n \to +\infty$ as $n \to +\infty$ since by definition for all $u \in D$ it is increasing on the sub-sequence of indices n such that $V_n = u$. Therefore we deduce that:

$$\liminf_{n\to+\infty} \frac{\mathcal{Q}_{\theta_n}^{-}}{h(e^{\beta\theta_n})} \geq \min_{u\in D} \liminf_{n\to+\infty} \frac{\mathcal{Q}_{Z_n^u}^{(u^*)}}{h(e^{\beta Y_n^u})} \quad \text{a.s.}$$

Next fix $u \in D$. By construction we have:

$$Z_n^u \sim Y_n^u \sim \theta_n^{(u)} \sim \frac{\log n}{\alpha} \quad \text{a.s. when} \quad n \to +\infty$$

Moreover the shifted process corresponding to u^* is distributed as the original and is independent of $(Z_n^u)_{n\geq 1}$. Therefore applying Lemma 2.7 we get that:

$$\liminf_{n\to+\infty} \frac{\mathcal{Q}_{Z_n^u}^{(u^*)}}{h(e^{\beta Y_n^u})} \geq 1 \quad \text{a.s.}$$

Finally summing up we obtain the result. ■

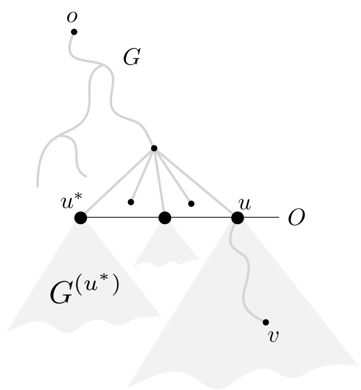

Figure 7: Illustration for the proof of Theorem 2.8.

2.3 Application

In this subsection, we put into application the results of the previous subsection to compute explicitly some asymptotic lower bounds.

Lemma 2.9 (Fast red paths inequality)**.** *Let $T \sim \mathrm{Exp}(1)$ be independent of the process and $r \in (0,1)$. Then for all $x > e$ big enough we have:*

$$\mathbb{P}\left[H_{1-T} \leq \frac{r \log x}{\log \log x}\right] \leq e^{-1/x}$$

Proof. Let's fix $x > e$ and define:

$$m_x := \frac{r \log x}{\log \log x} \quad \text{and} \quad p_x : \mathbb{R} \to [0,1],\ t \mapsto \mathbb{P}[H_t \leq m_x]$$

By independence, using conditional expectancy we have:

$$\mathbb{P}[H_{1-T} \leq m_x] = \mathbb{E}[p_x(1-T)] \tag{7}$$

Next, denote:

$$\mathcal{M}(x) := \inf\left\{1 - p_x(t),\ t \in [\log 2, 1]\right\}$$

By splitting the expectancy in (7) into two parts using the event $\{T \leq \log 2\}$, we have:

$$\mathbb{P}[H_{1-T} \leq m_x] \leq \mathbb{P}(T > \log 2) + \mathbb{P}(T \leq \log 2)[1 - \mathcal{M}(m)] = 1 - \frac{\mathcal{M}(x)}{2}$$

Thus since $\log(1+z) \leq z$ for all $z > -1$, taking the log we get:

$$\log \mathbb{P}[H_{1-T} \leq m_x] \leq -\frac{\mathcal{M}(x)}{2} \tag{8}$$

Now for all $n \geq 0$ let $v_n \in \mathbb{V}$ denote the word composed of n zeros. Recall that $(C_v)_{v \in \mathbb{V}}$ corresponds to the recovery clocks. Set $T_o := 0$ for convenience and remark that for all $t \in [\log 2, 1]$, we have:

$$\bigcap_{k=0}^{\lfloor m_x \rfloor} \left\{ C_{v_k} > 1 \ \wedge\ B_{v_k} \geq 1 \ \wedge\ T_{v_k} < \frac{\log 2}{m_x + 1} \right\} \subset \{H_t > m_x\}$$

Therefore computing the probability we get:

$$\mathcal{M}(x) \geq \mathbb{P}[B \geq 1]^{m_x+1} e^{-\gamma(m_x+1)} \left(1 - \exp\left[-\frac{\log 2}{m_x + 1}\right]\right)^{m_x} =: \hat{\mathcal{M}}(x)$$

Next, since $\log(1 - e^{-z}) \sim \log z$ as $z \to 0^+$ and $m_x \to +\infty$ as $x \to +\infty$, we have:

$$\log \hat{\mathcal{M}}(x) \sim -m_x \log m_x \sim -r \log x \quad \text{as} \quad x \to +\infty$$

It follows then that for all $x > e$ big enough we have:

$$\hat{\mathcal{M}}(x) \geq 2 \exp(-\log x)$$

Finally since $\hat{\mathcal{M}}(x) \leq \mathcal{M}(x)$, plugging these inequalities in (8) and then taking the exp, we get the result. ■

Lemma 2.10 (Fast red clusters inequality)**.** *Let* $T \sim \operatorname{Exp}(1)$ *be independent of the process and* $r \in (0,1)$*. Then for all* $x > e^e$ *big enough we have:*

$$\mathbb{P}\left[M_{1-T} \leq \frac{r \log x}{\log\log\log x}\right] \leq e^{-1/x}$$

Proof. We proceed as in the proof of Lemma 2.9. Let's fix $x > e^e$ and define:

$$m'_x := \frac{r \log x}{\log\log\log x}, \quad p'_x : \mathbb{R} \to [0,1] \quad \text{and} \quad t \mapsto \mathbb{P}[M_t \leq m_x]$$

We know that we have:

$$\begin{aligned} &\log \mathbb{P}[M_{1-T} \leq m_x] \leq -\frac{\mathcal{M}'(x)}{2} \\ &\text{where} \quad \mathcal{M}'(x) := \inf\{1 - p'_x(t),\ t \in [\log 2, 1]\} \end{aligned}$$

Now let $V_x \subset \mathbb{V}$ denote the set of the first $1 + \lfloor m'_x \rfloor$ words on the alphabet $\{0,1\}$ in lexicographical order. By construction the length of any word in V_x does not exceed $\log_2(m'_x + 1)$ and for all $t \in [\log 2, 1]$, setting $T_o := 0$ for convenience, we can write:

$$\bigcap_{v \in V_x} \left\{ C_v > 1 \ \wedge\ B_v > 1 \ \wedge\ T_v < \frac{1}{\log(m'_x + 1)} \right\} \subset \{M_t > m'_x\}$$

Therefore computing the probability it implies that:

$$\mathcal{M}'(x) \geq \mathbb{P}[B > 1]^{m'_x + 1} e^{-\gamma(m'_x+1)} \left(1 - \exp\left[-\frac{1}{\log(m'_x + 1)}\right]\right)^{m'_x} =: \hat{\mathcal{M}}'(x)$$

Next, remark that we have:

$$\log \hat{\mathcal{M}}'(x) \sim -m'_x \log\log m'_x \sim -r \log x \quad \text{as} \quad x \to +\infty$$

Finally summing up we get the result. ■

Theorem 2.11 (Asymptotic lower bounds for the process)**.** *Almost surely on* $\mathcal{S}$*, we have:*

$$\liminf_{t \to +\infty} \frac{H_t \log t}{t} \geq \alpha \quad \textit{and} \quad \liminf_{t \to +\infty} \frac{M_t \log\log t}{t} \geq \alpha$$

Proof. Fix $r \in (0,1)$ and define:

$$f : (e, +\infty) \to \mathbb{R}_+^*,\ x \mapsto \frac{r \log x}{\log\log x}$$

This function is increasing since it has a positive derivative and we can construct a non-decreasing function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+^*$ that matches with f for all $x > e$ big enough. By Lemma 2.9, the function h satisfies the assumptions of Theorem 2.8 for $(\mathcal{Q}_t)_{t\in\mathbb{R}} = (H_t)_{t\in\mathbb{R}}$ and thus for all $\beta \in (0, \alpha)$ we have:

$$\liminf_{t\to+\infty} \frac{H_t}{h(e^{\beta t})} \geq 1 \quad \text{a.s. on } \mathcal{S}$$

Simplifying the expression we get:

$$\liminf_{t\to+\infty} \frac{H_t \log t}{t} \geq r\beta \quad \text{a.s. on } \mathcal{S}$$

Then, by making $\beta \to \alpha$ and $r \to 1$ we get the first part of the result. The second part of the result is obtained similarly using Lemma 2.10 and Theorem 2.8 with $(\mathcal{Q}_t)_{t\in\mathbb{R}} = (M_t)_{t\in\mathbb{R}}$. ■

Corollary 2.12. *Almost surely on $\mathcal{S}$ we have:*

$$\liminf_{t\to+\infty} \frac{H_t}{\Gamma^{-1}(\#A_t)} \geq 1$$

Proof. Recall that by Lemma 1.7, we have:

$$\Gamma^{-1}(x) \sim \frac{\log x}{\log\log x} \quad \text{as} \quad x \to +\infty$$

Now observe that almost surely on $\mathcal{S}$, we have $\#A_t \to +\infty$ as $t \to +\infty$ and thus by Proposition 2.3 as $t \to +\infty$ we have:

$$\theta_{\#A_t} \sim \frac{\log \#A_t}{\alpha} \sim \frac{\log(\#A_t + 1)}{\alpha} \sim \theta_{\#A_t}$$

Then since for all $t \in \mathbb{R}$ we have $\theta_{\#A_t} \leq t \leq \theta_{\#A_t+1}$, we deduce then that:

$$\log \#A_t \sim \alpha t \quad \text{a.s. on } \mathcal{S} \text{ as } \quad t \to +\infty$$

Therefore summing up we get:

$$\Gamma^{-1}(\#A_t) \sim \frac{\alpha t}{\log t} \quad \text{a.s. on } \mathcal{S} \text{ as } \quad t \to +\infty$$

Finally the result follows then from Theorem 2.11. ■

Notice that Corollary 2.12 is similar to what we had in the first section in Corollary 1.8. Summarizing, we get Theorem II.

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