AROUND THE LANDSCAPE FUNCTION OF SCHRÖDINGER OPERATORS

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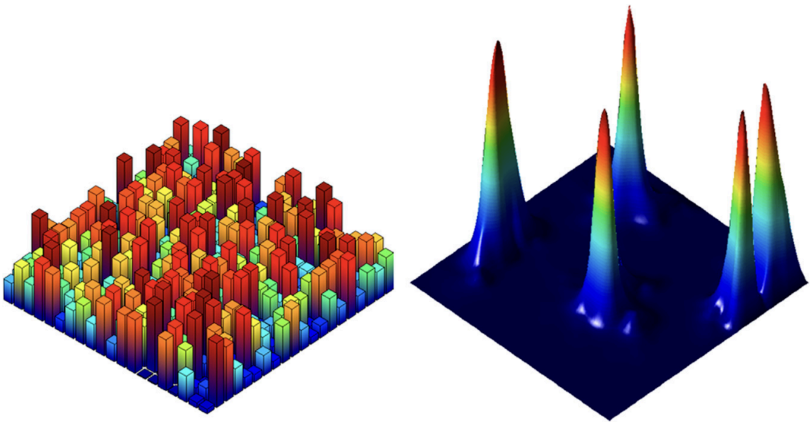
ABSTRACT. During my internship, I studied some aspects of the Localization Landscape Theory for certain linear elliptic operators. The main idea of this theory is that some localization and spectral properties can be acquired through the study of the solution of a single PDE, called the landscape function, which acts for instance as a confining potential at small energies. In the case of a continuous Schrödinger equation with Anderson-type potential, I looked at a numerically-based conjecture relating the first eigenvalues and the first ordered local maxima of the landscape function of the assocatied Schrödinger operator. I tried to extend results concerning the groundstate in 1D, both to higher dimension and several eigenstates.



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1. Context of the internship and interactions

I did my internship from February to the end of July, in the Analysis and Partial Differential Equations group of the Department of Mathematics at Uppsala University, Sweden. I worked under the supervision of Professor Kaj Nyström, who specializes mostly on parabolic and elliptic PDEs and their applications.

Since my internship was quite long, I initially worked on some other topic which I do not include in this report for sake of conciseness. This initial project regarded the study of operators arising from the addition of ”data” in a least square manner to the energy functional of certain PDEs. It was quite interesting but pretty experimental. Questions related to localization appeared during this work which is why we got interested in the theory we then worked on, which I present here.

On both of those topics, I consulted my supervisor around once a week for much useful advices and was pretty much free of choosing what reading material to use or what questions to tackle, which was challenging but very stimulating. Also the choice of working on a specific topic on which he does not really specialize made this quite interesting for the both of us. Unfortunately, for around the last two months of the internship these meetings could not happen due to personal issues on the side of my supervisor, but I still had enough work so it did not affect much of my experience of research. I also consulted during this time Professor Antoine Gloria (UPMC) online, who is working on this topic for ideas of further work directions.

Apart from personal work, I went to the Essén lectures, a week of lectures organized annually directed towards a general audience of PhD students. This year's topic was weighted extrapolation and the lecturer was Professor Tuomas Hytönen (Helsinki). It was really interesting though not related to the topics I studied. I also went weekly to the seminars of the analysis department as well as some of the probability department, and some PhD defenses.

I also of course interacted with PhD students and some Post-Doctoral students, which were very friendly and welcoming. It also gave me the opportunity to learn more about life as a researcher and their personal experiences of research and teaching.

2. General introduction to the subject

Disclaimer : I present here the ideas of the topic rather unformally, and in enough details to understand the specific question I wish to expose. For further details and more precise statements, see the following sections.

The notion of localization : It has been noticed experimentally that many *irregular* physical systems (in the context of optics, mechanics, acoustics, quantum physics...) exhibit very specific vibrating properties which do not appear in the case of smooth, homogeneous systems. In particular, some of the stationary waves tend to be extremely spatially concentrated. Irregular here can either be understood by *spatially inhomogenous* in terms of physical properties, or *disordered* in terms of geometry, with rough edges for instance. This phenomenon emerging in both of these set-ups is called localization, and it raised much attention for perspectives of applications. It also led to many interesting theoretical developments, which we adress thereafter in the case of quantum systems, from the mathematical viewpoint.

Anderson's localization : Consider a particle in the presence of a positive potential V. According to Schrödinger's equation, its stationary quantum states are the eigenfunctions of the Hamiltonian operator : $H = -\Delta + V$ where Δ is a suitable version of the Laplacian. In the case of a spatially inhomogeneous potential V, one observes what is called *Anderson's localization* : for strong enough inhomogeneity, some eigenfunctions are *strongly localized*, that is, most of their mass lies in small subregions of space, and their amplitude decays exponentially away from those regions. Historically, a lot of work has been done in the case of a discrete Laplacian (the *Anderson model*, introduced in 1958) which describes well certain situations, and many mathematically rigorous statements were established.

Localization Landscape Theory. Much more recently, a new tool to approach these questions numerically and theoretically has been introduced by Filoche and Mayboroda, in [4]. Consider a bounded open $\Omega \subseteq \mathbb{R}^d$, and a non-negative and bounded potential $V : \mathbb{R}^d \to \mathbb{R}$, possibly 0 almost everywhere. The object of interest is the Landscape function, defined as the solution (in the classical sense in the case of a regular domain) of the PDE :

$$Hu = -\Delta u + Vu = \mathbf{1} \ \ \& \ u_{|\partial\Omega} = 0 \tag{2.1}$$

that is, the solution of the *non-homogeneous problem* with constant source term and Dirichlet boundary values condition. Basic methods can be employed to show existence and uniqueness of a solution, as well as $u > 0$ inside the domain. It turns out that this is a powerful tool to study the Dirichlet *eigenvalue problem*, as outlined by the following simple result :

Theorem 2.1. *Let φ be a solution to $H\varphi = \lambda\varphi$, vanishing at the boundary, for some $\lambda > 0$. Then we have, almost everywhere, the pointwise bound :*

$$|\varphi(x)| \leq \lambda \|\varphi\|_\infty u(x)$$

The basic meaning is that smallness of u controls smallness of eigenfunctions of H, and the strength of control decreases with the energy of the mode, λ. Physical intuition and the classical limit (removing the kinetic term $-\Delta$) leads to thinking simply that at small energies, eigenfunctions should be small where the potential is large, but it turns out that the effective potential $W = \frac{1}{u}$ provides preciser information : it encodes V but not locally, and the geometry of the domain. Put in simple words, it is not easy to guess where localization of modes occurs with a plot of V, but it is with a plot of u. This is best highlighted by numerics, as can be seen in figure 1. There are rigorous results towards these properties as we will see later.

Weyl's Law, non-asymptotic eigenvalue counting. A spectral quantity of interest for operators with discrete spectrum is the integrated density of states, $N(\lambda)$, which counts the number of eigenvalues with multiplicity that are less than λ.

The asymptotics for large energies of this quantity are governed by Weyl's Law :

$$N(\lambda) \sim (2\pi)^{-d} \int_{|\xi|^2 + V(x) < \lambda} d\xi dx \tag{2.2}$$

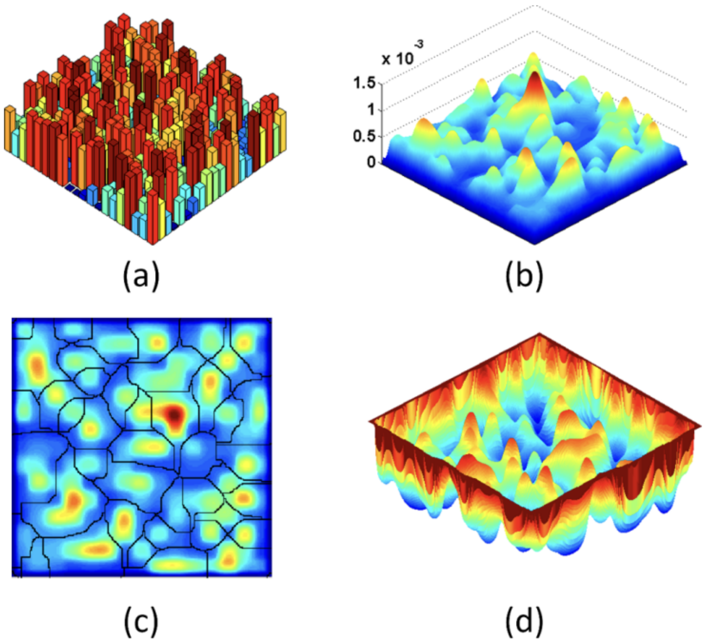

Figure 1. Plots from [2] : (a) 3D representation of a 2D disordered potential V ; (b) 3D representation of the landscape u ; (c) The various localization regions that are determined by the landscape function (d) Effective localization potential $W = u^{-1}$ The localization subregions outlined in (c) are also the basins of W.

It can be shown that we also have the same result when replacing V by $W = \frac{1}{u}$. However, there is strong numerical evidence that for the effective potential W, the approximation is accurate even at small energies, see [1]. Some rigorous results in that direction are obtained in [5]. Later in this report, we adress other features regarding the computation of small eigenvalues using u for certain random potentials, which is also quite interesting and surprising .

3. Some ideas of the Localization Landscape Theory

We start here by stating more rigorously some of the things already discussed, as well as proving some things and introducing some other interesting features of the theory.

3.1. **Basic results.**

3.1.1. *The Landscape function of an elliptic operator.* We give here the set-up regarding some "vibrating" system to which the theory applies. In reality, the main estimates hold in a more general setting but we restrict ourself for the sake of simplicity here. For a reference on those basics, see [6].

Boundary value problems for a certain type of operators. Let Ω be a bounded open set of $\mathbb{R}^d$ and denote by L a divergence form elliptic operator with bounded measurable coefficients :

$$L = -\operatorname{div} A(x)\nabla + V(x) \tag{3.1}$$

Where $A(x) = (a_{ij}(x))_{1\leq i,j\leq d}$, $x \in \Omega$ is elliptic real symmetric with bounded measurable coefficients, that is, for some $c > 0$:

$$a_{ji} = a_{ij} \in L^\infty(\Omega),\ \forall i,j \in \{1,\cdots,d\} \text{ and } A(x)\xi.\xi = \sum_{1\leq i,j\leq d} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \forall x \in \mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d$$

And $V \in L^\infty(\Omega)$ is non-negative.
The action of L is to be understood as usual in the weak sense, that is :

$$\int_\Omega Lu(x)v(x)dx = \int_\Omega A(x)\nabla u(x).\nabla v(x) + V(x)u(x)v(x)dx \tag{3.2}$$

for $u, v \in H_0^1(\Omega)$, the Sobolev space given by the completion of smooth functions vanishing at the boundary, $C_0^\infty(\Omega)$ with respect to the L^2 norm of the gradient :

$$\|u\|_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2$$

The Lax-Milgram Lemma solves the boundary value problems of the form $Lu = f$, $u \in H_0^1(\Omega)$, that is :

$$\int_\Omega A(x)\nabla u(x).\nabla v(x) + V(x)u(x)v(x)dx = \int_\Omega f(x)v(x)dx,\ \forall v \in H_0^1(\Omega)$$

where f belongs to the dual $H^{-1}(\Omega)$ of $H_0^1(\Omega)$. The lemma implies existence and uniqueness : see for instance [3] for a reference.
For domains with enough regularity, weak solutions can also be understood in the strong sense, as twice differentiable functions with vanishing limits at the boundary.

The associated eigenvalue problem. One can now consider, for some real parameter $\lambda \in \mathbb{R}$, the eigenvalue problem :

$$L\varphi = \lambda\varphi,\ \varphi \in H_0^1(\Omega) \tag{3.3}$$

When a non-trivial weak solution exists, we call λ an eigenvalue and the solution φ an eigenfunction. Under the assumptions detailed here, (under which L is self-adjoint), it can be shown that the eigenvalues of L form a sequence of positive numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty$ and the eigenfunctions form a Hilbert basis of $L^2(\Omega)$. These results can again be found in [3].
Thanks to these facts, the linear time dependent equations associated to L ($\partial_t + L = 0$, $i\partial_t + L = 0$, $\partial_t^2 + L = 0$, $\cdots$) can be studied using the decomposition along the eigenfunctions, and gaining knowledge about them is much useful.

The Landscape function, the torsion function. The Landscape function of the operator L, denoted by u, which we introduced earlier with slightly less generality, is defined as the weak solution of :

$$Lu = \mathbf{1},\ u \in H_0^1(\Omega) \tag{3.4}$$

This function has been introduced in [4] as a tool to study mainly spectral properties and localization in a more general set-up of elliptic operators (possibly of higher order, or even fractional operators) satisfying some form of the maximum principle.

Remark 3.1. It should be pointed out that in the simple but already challenging case $L = -\Delta$, u had already been noticed as an object of much interest, which is called in this context the torsion function of the domain Ω. In this case the function u has a simple probabilistic interpretation which we will get into more details later on : $2u(x)$ is the average exit time from Ω for a standard Brownian started at $x \in \Omega$.

Under the assumptions we chose here, L satisfies indeed a strong maximum principle and u is positive everywhere inside Ω.

Control of eigenfunctions. The following, easy result shows that u can be used to gain insight about solutions of the eigenvalue problem 3.3:

Theorem 3.2. *Let φ be a solution to $L\varphi = \lambda\varphi$, $\varphi \in H_0^1(\Omega)$, for some $\lambda > 0$.*
Then φ is bounded and we have, almost everywhere, the pointwise bound :

$$|\varphi(x)| \leq \lambda \|\varphi\|_\infty u(x)$$

This is usually proved with the Green function of L, but the following proof shows it holds based solely on linearity and the maximum principle. Usual methods show boundedness of φ, so we can set $w_1 = \lambda\|\varphi\|_\infty u - \varphi, w_2 = \lambda\|\varphi\|_\infty u + \varphi$.
One has $Lw_i = \lambda(\|\varphi\|_\infty \pm \varphi) \geq 0$ and thus by the maximum principle $w_i \geq 0$ for $i = 1, 2$.

Basic properties of Landscape functions. We give here some basics regarding the effect of changes in V or Ω on the Landscape function u. To make things simpler assume potentials are now defined on the whole space, as well as A which shall remain fixed.
Denote, extending to zero outside Ω, by $u(\Omega, V)$ the landscape function of $L = -\operatorname{div} A(x)\nabla + V(x)$, where V is any non-negative, measurable and bounded function, A is as previously and Ω is any bounded open. With these notations, we have the following useful properties :

Theorem 3.3. *Domain monotonicity of Landscape functions.*
Let V be fixed, and Ω, Ω' two bounded open sets such that $\Omega \subseteq \Omega'$.
Then one has pointwise :

$$u(\Omega, V) \leq u(\Omega', V)$$

Theorem 3.4. *Potential monotonicity of Landscape functions.*
Let Ω be fixed, and V, V' two non-negative, bounded measurable functions such that $V \leq V'$.
Then one has pointwise :

$$u(\Omega, V) \geq u(\Omega, V')$$

Both of these are consequences from the maximum principle. They are useful to give some nice geometric bounds such as, in the case of a vanishing potential :

$$\frac{\operatorname{dist}(x, \partial\Omega)^2}{2d} \leq u(\Omega)(x) \leq \frac{\operatorname{diam}(\Omega)^2 - (\operatorname{diam}(\Omega) - 2\operatorname{dist}(x, \partial\Omega))^2}{8d} \tag{3.5}$$

We use here the fact that in the case of a ball centered at 0 of radius r, the Landscape function is explicit and given by : $u(x) = \frac{r^2 - |x|^2}{2d}$ inside of the ball, 0 outside.

Localization of eigenfunctions in subdomains. At first glance and without additional assumptions, the bound obtained can seem quite weak. However, the following result from [6] shows that if an eigenfunction is small on the boundary of a subdomain of Ω, than it either has little mass inside of the subdomain or it has an eigenvalue close to one of said subdomain.
The rigorous statement is the following, back with our initial assumptions and notations :

Theorem 3.5. *Let φ be a solution to $L\varphi = \lambda\varphi$, $\varphi \in H^1_0(\Omega)$, for some $\lambda > 0$. Let $D \subseteq \Omega$ be an open set, and v be a solution to $Lv = 0$, $v - \varphi \in H^1_0(D)$. Denote by $\mathcal{S}_L(D)$ be the set of the eigenvalues of L on the domain D.*
Then the following holds :

$$\int_D \varphi^2 \leq \Big(1 + \frac{\lambda}{d(\lambda, \mathcal{S}_L(D))}\Big)^2 \int_D v^2$$

One should note that the term $\frac{\lambda}{d(\lambda,\mathcal{S}_L(D))}$ can blow up to infinity if $\lambda \in \mathcal{S}_L(D)$. If λ is nearly in $\mathcal{S}_L(D)$, than it can be that $\int_D \varphi^2$ is quite large, but in that case φ will typically be small on other subdomains because of their typically different eigenvalues.
This can be used together with the previous result to get the following corollary :

$$\int_D \varphi^2 \leq \Big(1 + \frac{\lambda}{d(\lambda, \mathcal{S}_L(D))}\Big)^2 \lambda^2 |D| \|\varphi\|_\infty \|u_{|\partial D}\|_\infty \tag{3.6}$$

The idea here is that if u is small on a surface enclosing a subdomain, if λ is not close to $\mathcal{S}_L(D)$, than the eigenfunctions associated to λ cannot have much mass in said subdomain : in this sense, u can determine subdomains of localization for eigenfunctions and exclude them from certain subdomains as well.
This type of localization is called weak, as opposed to strong localization, which can be also observed through u as we will see later on.

3.1.2. *The effective potential.* The basic results introduced showed before that indeed u can give insights about the localization of eigenfunctions of L. In fact, we even have that $W = \frac{1}{u}$ can be interpreted as regularized confining potential. This idea revolves around the following :

Theorem 3.6. *Effective equation.*
Let φ be a solution to $L\varphi = \lambda\varphi$, $\varphi \in H^1_0(\Omega)$, for some $\lambda > 0$. Set $\varphi = u\psi$, and denote by $W = \frac{1}{u}$ the effective potential associated to L.
Then we have in the weak sense :

$$-\frac{1}{u(x)^2}\operatorname{div}(A(x)u(x)^2\nabla\psi(x)) + W(x)\psi(x) = \lambda\psi(x)$$

This construction here replaces the initial potential V by a possibly simpler, effective potential W, but the price to pay is that the differential term becomes more complicated.
Another related identity highlighting the same idea is the following :

$$\int_\Omega u(x)^2 A(x)\nabla\Big(\frac{f}{u}\Big)(x).\Big(\frac{f}{u}\Big)(x)dx + \int_\Omega W(x)f(x)^2 = \int_\Omega Lf(x)f(x)dx$$

for all $f \in H^1_0(\Omega)$ Both of these results are simple consequences of the product rule for weak derivatives, and of course of the definition of u. Stated alone, they only highlight the intuitive ideas one could have of the significance of u. However, there are both rigorous results and numerical evidence showing that this interpretation is useful as we will see in the next part.

3.2. **A practical and theoretical tool.**

The set-up of Schrödinger operators. As we saw beforehand, one can deduce from the construction above that the Landscape function can explain weak localization, as well as having an inverse appearing as a potential term for effective equations, for some type of elliptic operators.
Now we restrict ourselves to the case $A(x) = I$, that is, to the case of usual Schrödinger operators. In this context, L is traditionally called the Hamiltonian and denoted by H. For these operators, we will see that W can be used to replace V in some important results, and that this replacement can be much useful.

3.2.1. *Eigenvalue counting.*

Weyl's law. For sake of simplicity, and due to limited knowledge on my part, we present the following rather informally. In the introduction, we already mentioned the celebrated Weyl's law, which can be stated for instance for Schrödinger operators with purely discrete spectrum. One has, if we denote by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \to +\infty$ the eigenvalues of $H = -\Delta + V$:

$$N(\lambda) = \#\{j \in \mathbb{N}_{>0} \mid \lambda_j \leq \lambda\} \sim (2\pi)^{-d} \int_{|\xi|^2 + V(x) < \lambda} d\xi dx \tag{3.7}$$

In the case of a bounded domain and a finite potential, this reduces easily to the asymptotics :

$$N(\lambda) = \#\{j \in \mathbb{N}_{>0} \mid \lambda_j \leq \lambda\} \sim (2\pi)^{-d} \lambda^{d/2} |\Omega| |B_{\mathbb{R}^d}(0,1)| \tag{3.8}$$

However, the basic physical interpretation of the law indicates that the asymptotics of 3.7 should be better.
Indeed, very roughly, eigenfunctions with eigenvalue less than λ fully occupy the volume of $\{|\xi|^2 + V(x) < \lambda\}$, and by the uncertainty principle, they are all roughly supported in phase space in disjoint cubes of volume $\simeq (2\pi)^d$ (we took throughout the report the convention $\hbar = 1$, but this interpretation is more intuitive in the semi-classical limit $\hbar \to 0$, though the high energy limit $\lambda \to +\infty$ is similar in this context).
Even though taking V into consideration instead of just looking at the volume of the domain is better, this approach is still flawed, for instance in the case of disordered potentials, because the right-handside only depends on the volume of the level-sets of V, and not its geometrical structure. The following example precises this idea :

Disordered potentials ; Bernoulli potentials. Consider the simple case where Ω is the open cube $(0,1)^d$ in $\mathbb{R}^d$, and that V is a piece constant potential, defined to take on tiles of the shape $k/N + (0, 1/N)^d$ for $k \in \mathbb{Z}^d$, either the value 0 or $K \in \mathbb{R}_{\geq}0$, with $N \in \mathbb{N}$.
Assume additionally that V is the realization of a random variable, with the value on each tile being taken i.i.d. according to Bernoulli variables with parameter p. We call V a Bernoulli random potential.
For such potentials, the volume of the level-sets is expected to be roughly deterministic, at least in the large N regime. The fact that regions where V takes the value K are well distributed inside the domain should affect the non-asymptotic behaviour of the IDS, while the Weyl law states that in the asymptotic regime, the situation is roughly the same as in the case where there are only two tiles.
In this context, using W instead of V yields good results numerically. Intuitively, W is a smoothed out version of V and thus the knowledge of its level sets contain some information about how V is distributed spatially.

Numerical results. The following plots in figure 2 are taken from [1] and showcase the fact that Weyl's law with the effective potential gives an impressive non-asymptotic approximation in the case of several disordered random potentials, here in the case of dimension $d = 1$. As can be seen with plot 2 and 4 of figure 2 the spatial arrangment of the potential does of course matter in terms of non-asymptotic spectral properties. Also, it can be seen that W takes into account this spatial arrangment and exhibits completely different behaviour in both cases. Good approximation is also achieved in the case of correlated potentials and uniform i.i.d. potentials.
The figure 3 shows that similar properties can be observed in dimension $d = 2$: replacing V by W yields better approximation in the non-asymptotic regime. However, both are not as predictive as in dimension 1.

3.2.2. *Agmon estimates.* Another instance showing the use of the effective potential in the case of disordered systems are Agmon estimates. The following is stated more formally in [1].

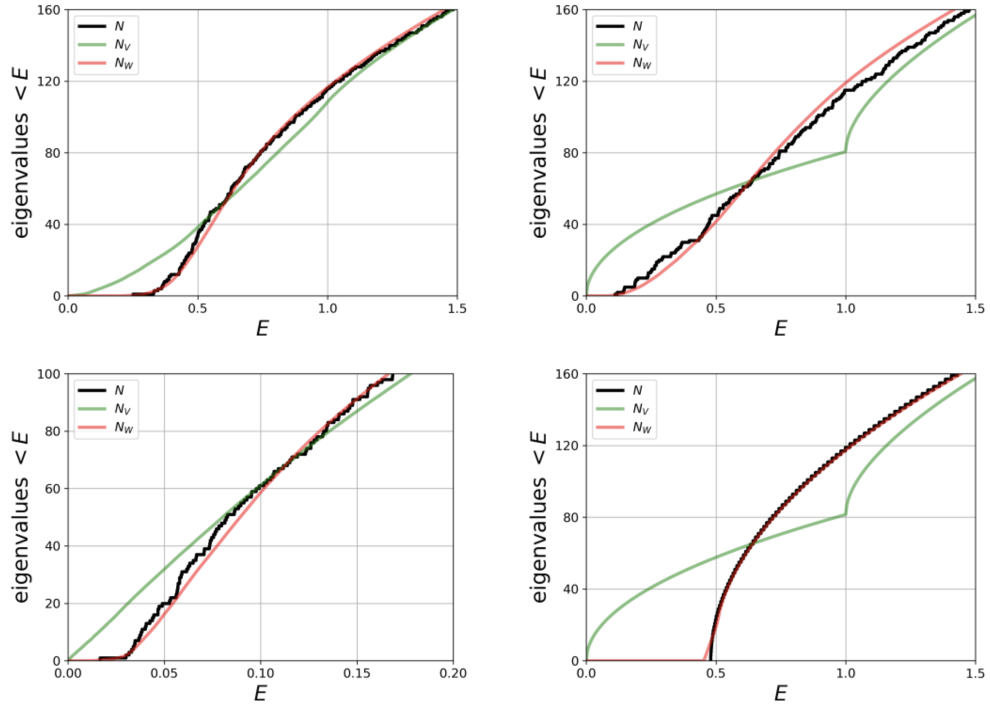

FIGURE 2. Plots from [1] : The eigenvalue counting function N, the Weyl's law approximation N_V , and the effective Weyl's law approximation N_W for some potentials in one dimension. Top row: uniform and Boolean random potentials. Bottom row: correlated and periodic Boolean potential.

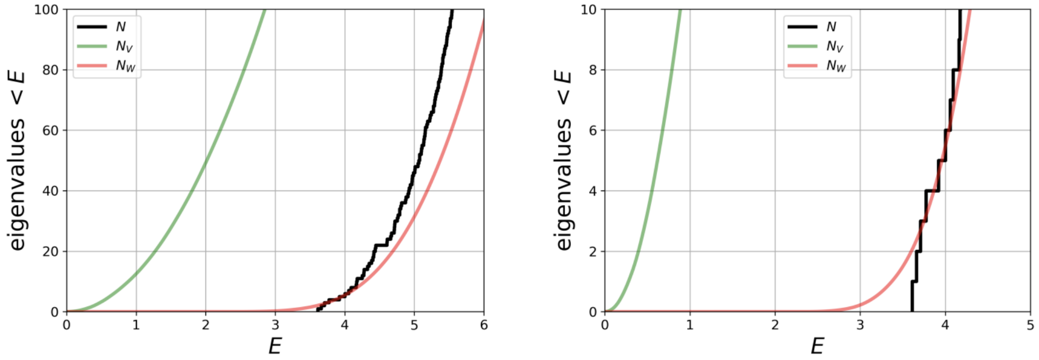

FIGURE 3. Plots from [1] : The eigenvalue counting function N, the Weyl's law approximation N_V , and the effective Weyl's law approximation N_W for a 2D random uniform potential, showing the first 100 eigenvalues on left and restricting to the first 10 on right.

Exponential decay of eigenfunctions. For a real parameter λ, we call $\{x \in \Omega \,|\, V(x) \leq \lambda\}$ the *classical zone* : in classical rather than quantum physics, a particle of energy equal to λ can only live in the classical zone. In the quantum case, the density of state can still be positive outside of this zone, as can be seen with quantum tunneling effects, but should be rather small. Agmon estimates give quantitative estimates regarding how much the density of steady states is decaying away from this zone. The decay is not simply given by the euclidean distance to the classical zone, as the value of the potential is responsible of said decay.

To quantify this idea, one introduces the Agmon weight : $w_\lambda(x) = (V(x) - \lambda)_+$ and uses it to define a degenerate metric in Ω :

$$h_\lambda(x,y) = \inf \int_0^1 \sqrt{w_\lambda(\gamma(t))}|\gamma'(t)|dt \tag{3.9}$$

where the infinum is taken over all path γ going from x to y that are absolutely continuous. This defines a "geodesic distance", but where travel inside of the classical zone is "free", which we call the Agmon distance.

Now denoting by $g_\lambda(x)$ the Agmon distance of x to the classical zone, one can test against the eigenvalue equation $L\varphi = \lambda\varphi$ with a clever choice of test function to get "smallness" (in L^∞ or L^2 depending on the type of result) of $e^{(1-\varepsilon)g_\lambda}\varphi$, for $\varepsilon > 0$.

Basically, this implies that where g is large, eigenfunctions are going to be very small. Now consider again a disordered potential : typically, the classical zone is going to percolate through the whole domain, and those estimates will typically be useless since the function g_λ will remain relatively small everywhere.

The strength of the LLT can apply here : the effective equation looks enough like a regular Schrödinger equation to apply the principles of the proof of Agmon estimates, and allows one to replace g_λ with another distance $g^{\text{eff}}_{\lambda,\delta}$, expressed as the h^{eff}_λ-distance of x to the effective classical zone (with some room left, given by some $\delta > 0$): $\{x \in \Omega \mid \frac{1}{u(x)} \leq \lambda + \delta\}$, where h^{eff}_λ is now less degenerate and given by :

$$h^{\text{eff}}_\lambda(x,y) = \inf \int_0^1 \sqrt{\left(\frac{1}{u(\gamma(t))} - \lambda\right)_+}|\gamma'(t)|dt \tag{3.10}$$

More specifically, the estimate they give is :

$$\int_\omega e^{g^{\text{eff}}_{\lambda,\delta}}\varphi^2 \leq C \int_\Omega \varphi^2$$

where the constant C is only dependant on δ and $\|V\|_\infty$. Sharper estimates with explicit constants were also proven by the same authors in a more general set-up.

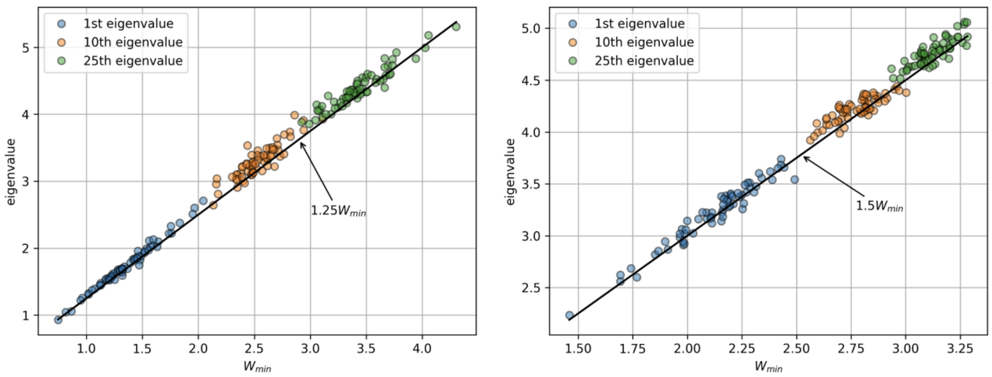

Figure 4. Plots from [1] : the 1st, 10th, and 25th eigenvalues versus the corresponding minima values of W for numerous different realizations of a random potential. The first figure displays 64 realizations of a 1D potential on $[0,256]$ with 256 values selected uniformly i.i.d. from $[0,16]$, while the second figure displays 64 realizations of a 2D potential on $[0,40]\times[0,40]$ with 1 600 random values again chosen uniformly i.i.d. from $[0,16]$.

4. Around a numerical conjecture for small eigenvalues of random Schrödinger operators

The conjecture. Another interesting feature regarding the spectrum that has been noticed numerically by Arnold & al. in [1], can be loosely stated as this :
For an *Anderson type potential*, provided the domain is large, the eigenvalues $0<\lambda_1\le\lambda_2\le\dots$ of H and the decreasingly ordered local maxima of u, $m_1\ge m_2\ge\dots$ satisfy :

$$\lambda_i m_i \approx C(d) \text{ for } i \ll |\Omega| \tag{4.1}$$

where they initially conjectured $C(d)=1+\frac{d}{4}$. It turns out that the correct constant should be $\frac{\mu_d}{2d}$ where μ_d is the principal eigenvalue of the Laplacian on the unit ball.
Rather than eigenvalue counting, if true, this allows for computation of small eigenvalues. What makes this particularly interesting is the fact that the constant is only dimension dependent, and does not appear to depend much on the structure of the random potential (it seems to hold for spatially correlated random variables as well as uncorrelated for instance).
We decided that this specific point could be a good direction of work, as not much had been done yet even though some restrictions of the problem seemed approachable.

4.1. Numerical evidences and already existing result in dimension 1.

In the paper [1], the following plots from figure 4 can be found. From this experiment and others, they formulated the conjecture above.
Some effort has already been made towards the formulation and a proof of a precise statement (See [7], [8]), but only for the principal eigenvalue and global maximum of u, that is, $i=1$, and in $d=1$ or only as a lower bound (the lower bound holds in all dimension, in the discrete case) . In dimension 1, recall that the constant $\frac{\mu_1}{2}$ is equal to $\frac{\pi^2}{8}$.

Theorem 4.1 (The Landscape Law for the groundstate for $d=1$)**.** *Let ω_j, $j\in\mathbb{N}$ be independent, identically distributed random variables in $\mathbb{R}\cup\{\infty\}$ such that :*

$$\mathbb{P}(\omega_0\ge 0)=1 \textit{ and } 0<\mathbb{P}(\omega_0=0)<1 \tag{4.2}$$

and consider the (random) piecewise constant potential :

$$V_\omega(x)=\sum_{j\in\mathbb{N}}\omega_j\mathbf{1}_{[0,1)}(x-j) \tag{4.3}$$

For a fixed integer n, let $\Omega_n = (0,n)$ *and denote by* $(\lambda_i^{n,\omega})_{i\geq 1}$ *the non-decreasingly ordered eigenvalues of the Schrödinger operator* $H^{n,\omega} = -\Delta + V_\omega$ *with Dirichlet boundary conditions on* Ω_n*, and by* $(m_i^{n,\omega})_{i\geq 1}$ *the decreasingly ordered maxima of the associated landscape function* $u^{n,\omega}$*.*
Then the following holds :

$$\lambda_1^{n,\omega} m_1^{n,\omega} \underset{n\longrightarrow+\infty}{\longrightarrow} \frac{\pi^2}{8} = \frac{\mu_1}{2} \text{ almost surely.} \tag{4.4}$$

Remark 4.2. Here we allow for sake of convenience the value ∞ for the potential : formally, this just means the set $\{V_\omega = \infty\}$ should be removed from the domain, while adding null Dirichlet boundary conditions on its boundary.

The general statement that would be nice to prove goes by replacing $\mathbb{N}$ with $\mathbb{N}^d$, $(0,n)$ with $(0,n)^d$, as well as $\frac{\pi^2}{8}$ with $\frac{\mu_d}{2d}$.
Also, the same should hold in cases where $\mathbb{P}(\omega_0 = 0) \neq 1$ but $\mathbb{P}(\omega_0 < \varepsilon)$ decays not too fast as $\varepsilon \to 0$, as in [8]. We avoided these extensions for the sake of technical simplicity.

Remark 4.3. Clearly, the way we chose to "implement" disorder here is rather arbitrary. First, replacing $\mathbf{1}_{[0,1)^d}$ by any bump-like, compactly supported positive function χ would be reasonable. The same technique should apply in this case without complicating too much the proof, and of course the limiting constant should remain the same.
Another physically relevant construction would be to make the *locations* of the bumps random rather than their heights :

$$V_\omega(x) = \sum_{j\in\mathbb{N}} \chi(x - x_j^\omega)$$

where the sequence $(x_j^\omega)_{j\in\mathbb{N}}$ enumerates a random discrete set of $\mathbb{R}^d$.
For instance, one can take the set associated to a Poisson point cloud of given intensity η, that is, random variables such that the random variables $N_\omega(B) = \#\{j \in \mathbb{N} \mid x_j^\omega \in B\}$ for any measurable set B satisfy :

- for any $n \in \mathbb{N}$ and B measurable set, one has : $\mathbb{P}(N_\omega(B) = n) = \frac{1}{n!} e^{-\eta|B|}(\eta|B|)^n$
- for any disjoint measurable sets $B_1, \cdots, B_k$, the random variables $N_\omega(B_1), \cdots, N_\omega(B_k)$ are independent.

The arising potentials are fairly common in this context, and though technical details might differ, the principles of the proof in dimension 1 should work.
An interesting but difficult direction of work would be to study potentials that are spatially correlated. As evidenced by numerics, the result is expected to hold in this set-up as well.

4.2. **Extension to several eigenvalues in dimension 1.**

I tried to give some generalization of the already present proof of the previous result from [7], theorem 4.1, and leveraging on their arguments, I proved the following :

Theorem 4.4 (The Landscape Law for small eigenvalues for $d = 1$)**.** *Let* $(k_n)_{n\geq 0}$ *be a sequence of integer such that :*

$$\log k_n = o(\log n) \tag{4.5}$$

Under the assumptions of 4.1, and with the same notations, the following holds:

$$\max_{i\leq k_n} |\lambda_i^{n,\omega} m_i^{n,\omega} - \frac{\pi^2}{8}| \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \text{ almost surely.} \tag{4.6}$$

Remark 4.5. For any fixed n, the number of local maxima is finite. However, the condition on k_n implies that the left-hand side is meaningful for n large enough almost surely. Also note that any constant sequence k satisfies condition 4.5, that is, any fixed, finite, number of eigenvalues satisfy the asymptotics.

Essentially, the idea of the proof is that the domain can be split spatially into smaller domains, and as long as we don't split it in too many pieces (hence the condition on $(k_n)_{n\geq 0}$), it is possible to apply the already existing arguments to the subdomains. What makes this work is that for this kind of random potentials, the connected components of $\{V_\omega = 0\}$ are of deterministic and logarithmic size, asymptotically.
When it comes to the ideas of the original proof, the main tools are roughly variational principles for eigenvalues, comparison principles for landscape functions, and the Borel-Cantelli lemma.

Remark 4.6. Restriction to dimension 1 makes the problem way easier in many regards : both eigenfunctions of the Laplacian on an interval and the torsion function of an interval are explicit and simple functions. This makes the comparison tools super useful. Also, the geometric structure arising from taking the potential randomly is much more complicated when $d \geq 2$. For instance, depending on the value of $\mathbb{P}(\omega_0 = 0)$, the connected components of $V_\omega = 0$ might remain either bounded, or form a large unbounded cluster percolating throughout the whole space. Percolation theory studies the phase transition between both of these behaviors : in dimension 1 for a non-trivial parameter, the transition does not occur, and the percolating wells do not appear.

4.3. **Ideas and principles towards a result in higher dimension.**

Here we explain how the arguments we used, together with additional, well-known results, give a strong heuristic in favour of a similar result in higher dimensions. This constitutes a summary of my ideas of what a proof could be. I did not have the time to make this succeed and I probably lack some technical tools.
As explained in dimension 1, the first few eigenfunctions as well as the landscape function tend to be small where the potential is non-zero, and on connected components where it is zero, they are comparable to their analogue in the potential-free case on this restricted domain.
This is particulary useful in dimension 1, where those regions we call potential wells are bounded intervals.
When the parameter $\mathfrak{p} = \mathbb{P}(\omega_0 = 0) > 0$ is smaller than the percolation threshold (which is 1 in $d = 1$), those connected components are of finite size and we roughly only have to compare them to deduce the values of λ_1 and m_1. We build an heuristic in this case, which should leverage on intuitions from the $d = 1$ case.
In this setup, we have to examinate the connected components and compare them. Thus we should consider first the case $V = 0$, on a sufficiently regular open Ω (which would eventually be a finite reunion of squares if we restrict ourselves to potentials as in the result 4.4).
To follow this plan, we give here some additionnal tools to fully explain the heuristic :

Scaling. We have the following scaling properties for the k-th Dirichlet eigenvalues of the Laplacian and the k-th local maxima of torsion functions :

$$\lambda_k(t\Omega) = \frac{\lambda_k(\Omega)}{t^2} \text{ and } m_k(t\Omega) = t^2 m_k(\Omega) \text{ for } t > 0$$

This implies in particular that the product $\lambda_k m_k$ is scale-invariant.
It follows directly that for any non-empty open of finite measure $\Omega \subseteq \mathbb{R}^d$, denoting by Ω^* the open set of volume 1 : $\frac{1}{|\Omega|^{\frac{1}{d}}}\Omega$, we have that :

$$\lambda_k(\Omega) = \frac{\lambda_k(\Omega^*)}{|\Omega|^{\frac{2}{d}}} \text{ and } m_k(\Omega) = |\Omega|^{\frac{2}{d}} m_k(\Omega^*) \text{ for } t > 0 \tag{4.7}$$

Isoperimetric inequalities. Denote now by $\lambda^*(\Omega)$, $m^*(\Omega)$ the quantities $\lambda_1(\Omega^*)$, $m_1(\Omega^*)$. According to the scaling, these quantities are the shape-dependency of the quantities of interest. For connected sets, in dimension 1, these are trivial, since the notion of shape is trivial. In any dimension, they satisfy the following inequalities, called *isoperimetric* for obvious reasons :

Theorem 4.7. *Isoperimetric inequalities for the principal eigenvalue and the maximum of the torsion function.*

For any $d \geq 1$, for any open of finite measure $\Omega \subseteq \mathbb{R}^d$, and for B an open ball,

$$\lambda^*(\Omega) \geq \lambda^*(B) \text{ and } m^*(\Omega) \leq m^*(B)$$

Remark 4.8. We will return to these later on, and explain how both of these inequalities are tightly related by stochastic interpretations.

Remark 4.9. We have that $\lambda^*(B)m^*(B) = \frac{\mu_d}{2d}$: this is precisely the constant from the conjecture.

Disjoint unions. Also, for disjoint open sets $\Omega_1, \cdots, \Omega_j$ we have for $k = 1$:

$$\lambda_1(\Omega_1 \cup \cdots \cup \Omega_j) = \min(\lambda_1(\Omega_1), \cdots, \lambda_1(\Omega_k)) \,,$$

$$m_1(\Omega_1 \cup \cdots \cup \Omega_j) = \max(m_1(\Omega_1), \cdots, m_1(\Omega_k))$$

Also recall the already stated following monotonicity properties, for any open subsets $\Omega \subseteq \Omega'$:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega') \text{ and } m_1(\Omega) \leq m_1(\Omega')$$

Bernoulli random domains. The rough approximation of only looking at low-potential wells can be investigated by looking at the case where $\omega = 0$ or $+\infty$ almost surely, that is, the potential-free case on a "Bernoulli" random domain Ω_ω composed of tiles. In this case, consider the partition $\mathcal{O}_{n,\omega}$ into connected components of the set $\Omega_{n,\omega} = \Omega_\omega \cap (0,n)^d$.
Combining the tools recalled above, one has for any open $V \subseteq \Omega_{n,\omega}$:

$$|V|^{-2/d}\lambda^*(V) \leq \lambda_1^{n,\omega} = \min_{U \in \mathcal{O}_{n,\omega}} \lambda_1(U) \leq \lambda^*(B) \min_{U \in \mathcal{O}_{n,\omega}} |U|^{-2/d} = \lambda^*(B)\Big(\max_{U \in \mathcal{O}_{n,\omega}} |U|\Big)^{-2/d}$$

$$|V|^{2/d}m^*(V) \geq m_1^{n,\omega} = \max_{U \in \mathcal{O}_{n,\omega}} m_1(U) \geq m^*(B) \max_{U \in \mathcal{O}_{n,\omega}} |U|^{2/d} = m^*(B)\Big(\max_{U \in \mathcal{O}_{n,\omega}} |U|\Big)^{2/d}$$

Thus one has :

$$\Big(\frac{\beta_n^\omega}{\alpha_n^\omega}\Big)^{2/d} \leq \frac{2d\lambda_1^{n,\omega}m_1^{n,\omega}}{\mu_d} \leq \Big(\frac{\alpha_n^\omega}{\beta_n^\omega}\Big)^{2/d} \tag{4.8}$$

where $\alpha_n^\omega = \max_{U \in \mathcal{O}_{n,\omega}} |U|$ and β_n^ω is the volume of the largest ball inscribed in $\Omega_{n,\omega}$.
Ideally, one would like to get a result of the type : $\alpha_n^\omega \sim \beta_n^\omega$ almost surely, which would conclude directly in this setup. Unfortunately, this fails to be true : if $\mathfrak{p}$ is large enough, α_n^ω will be quite large (superlinear) with positive probability, since long paths (of volume larger than their length since our tiles are of measure 1) will occur when there is percolation in the quarter plane, while (see [8] for a reference on a similar result) :

$$\beta_n^\omega \sim |B(0,1)| \frac{d \log n}{\log \frac{1}{\mathfrak{p}}} \text{ almost surely.}$$

remains true whatever the (non-trivial) value of $\mathfrak{p}$ is.

Coarse-graining procedure. According to these last remarks, one should seek better bounds. Indeed, simply looking at all the components directly is not sharp enough : connected sets with a dumbell-like shape for instance can have a high measure and be really far from equality in the isoperimetric inequalities. Trying to rule out those opens with thin strips of the discussion could succeed to make things work.
Asymptotic bounds can be shown on the volume of a finite number of shapes occuring in the $V = 0$ set, in the same manner as for the length of wells in $d = 1$. Indeed, it can be shown that for any sequence of integer $(g_n)_{n \geq 0}$ growing sufficiently slowly, and any sequence $(\Omega_i)_{i \geq 1}$ of open subsets, if we denote by $\theta_n^\omega(\Omega)$ the volume of the largest open of the form $x + t\Omega$ contained in $\Omega_{n,\omega}$, we have :

$$\limsup_{n \longrightarrow +\infty} \max_{1 \leq j \leq g_n} \frac{\theta_n^\omega(\Omega_j)}{\log n} \leq \frac{d}{\log \frac{1}{\mathfrak{p}}}$$

This can be done through a Borel-Cantelli argument.
This means that we can look at several shapes at the same time, though not too much, and be sure

that they can only be embedded in $\{V=0\}\cap\Omega_n$ with comparable volume. Then we would like to show that for any $U\in\mathcal{O}_{n,\omega}$ there exists $k_U\leq g_n$ such that

$$\max_{U\in\mathcal{O}_{n,\omega}} |m_1(U)-m_1(\Omega_{k_U})\theta_n(\Omega_{k_U})^{2/d}| = o(\log n)$$

almost surely, as well as a similar principle for λ_1.
If for instance we take $(\Omega_i)_{i\leq g_n}$ to be all connected open sets that are composed of adjacent unit cubes with less than h_n cubes for some integer sequence $(h_n)_{n\geq 0}$, (taking h_n carefully to have g_n growing not too fast) this boils down to looking at a coarse-graining of the potential.

Remark 4.10. Tightly related ideas have already been developped for the principal eigenvalue, in [10], where the method is known as the enlargment of obstacles, and relies on stochastic interpretations. Following these ideas could help formalize the ideas above.

In this set-up, a remarkable fact is that what is intuitive in the small $\mathfrak{p}$ regime holds for all $\mathfrak{p}\in(0,1)$.

4.4. **Stochastic interpretation and isoperimetric inequalities.**

As seen above, the constants that emerge through the scaling, namely, shape-dependency of λ_k and m_k, play an important role.
In [8], one can understand that the lower bound proved by the author ($\liminf \lambda_1^{n,\omega} m_1^{n,\omega} \geq \frac{\mu_d}{2d}$ almost surely :it is done here in the discrete set-up) comes somehow (though in a way not directly similar to what we outlined above) from the isoperimetric inequality for the principal eigenvalue, also called the Faber-Krahn inequality, stating that among domains of fixed (finite) volume, the ball minimizes the principal eigenvalue. A reference for this classical result can be found in [9].
It is also true that the problem of maximizing $m_1(\Omega)$ has the same solutions, as already recalled earlier : see for instance [11] for a proof in a more general set-up.
The fact that both these results are similar is no coincidence : indeed they can be proved together through the following ideas, which we briefly explain here for the sake of fun.

Stochastic interpretation. Both λ_1 and the torsion function, that is, the potential-free landscape function, have stochastic interpretations arising from the interplay between diffusion equations and stochastic processes. See for instance [12] for some basics on this. For a bounded open $\Omega\subseteq\mathbb{R}^d$ and some point $x\in\Omega$, denote by τ_x^Ω the exit time from Ω of a standard Brownian motion started at x. Then one has :

$$\lambda_1(\Omega)=\lim_{t\to+\infty} -\frac{2}{t}\log\mathbb{P}(\tau_x^\Omega>t)$$

and denoting by u^Ω the torsion function of Ω, using Fubini-Tonelli :

$$u^\Omega(x)=1/2\mathbb{E}[\tau_x^\Omega]=1/2\int_0^{+\infty}\mathbb{P}(\tau_x^\Omega>t)dt$$

The first can be found in [13] (it follows easily from the Feynman-Kac formula), and the second follows easily from what is exposed in [12]. Now, we see that both isoperimetric inequalities we presented are consequences of a third one, regarding the *optimal trapping* of a Brownian motion.
For any $t>0$, one has the following : $\mathbb{P}(\tau_x^\Omega>t)\leq\mathbb{P}(\tau_0^{\Omega^*}>t)$. The "physical" meaning is rather intuitive : for a blind-folded prisoner, a spherical prison is the hardest to escape.
Almost-sure continuity of sample paths allows one to reduce the question to finite-dimensional events, namely, for $s_i>0$: $\{B_{s_1}\in\Omega, B_{s_1+s_2}\in\Omega,\cdots,B_{s_1+\cdots+s_n}\in\Omega\}$, for which the probability is explicit and given by an integral of Gaussian densities :

$$\int_{\Omega\times\cdots\times\Omega} g_{s_1}(x_1)g_{s_2}(x_2-x_1)\cdots g_{s_n}(x_n-x_{n-1})dx_1dx_2\cdots dx_n$$

The Riesz-Sobolev inequality. In that set-up, the result follows from a slight generalization (due to Brascamp-Lieb-Luttinger) of the following famous and fundamental integral inequality (see [14]). For $f, g, h : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$,

$$\int f(x)g(y)h(x-y)dxdy \leq \int f^*(x)g^*(y)h^*(x-y)dxdy$$

where φ^* denotes the *symmetric decreasing rearrangement* of $\varphi \in \{f, g, h\}$. See the reference [14] for details on this. Such rearrangement inequalities are keys in many proofs of various isoperimetric inequalities.

5. Some interesting outlooks

There are many interesting outlooks which make this topic exciting I think, some of which I will mention here as a conclusion.

Of course, giving a real proof of what I sketched above for higher dimensions could be nice, and probably not really straightforward to do. If I had more time to spend on this project, trying to fully understand the enlargment of obstacles theory in order to achieve this is something that I would have liked to do.

Another very interesting idea would be to look at the well-studied and very insightful quasi-periodic Mathieu operator : it is a discrete, deterministic Schrödinger operator in dimension 1 which relies on number theoretical principles to model disorder. The discrete Hamiltonian acting on $L^2(\mathbb{Z})$:

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(n\alpha + \theta)u_n$$

where $\theta \in \mathbb{R}$ is the phase, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency and $\lambda \in \mathbb{R}$ is the coupling constant, is known to exhibit a phrase transition in terms of its spectrum with respect to the value of λ. For large enough values of $|\lambda|$, there is exponential localization of eigenfunctions, whereas for small enough coupling, states are delocalized (in fact, the spectrum is purely singular continuous, then purely absolutely continuous. See for instance [15]).

Now one could of course, with a properly adapted definition, introduce a Landscape equation. Then, one could expect a phase transition in terms of existence of a solution to the equation : does it indeed occur ? Does it occur for the same critical value of the parameter λ ? Such questions seem rather difficult, but the theory of this operator is well developped. Additionally, the techniques involved are more in the realms of number theory and dynamical systems than analysis.

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