Quiver representations

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1 Introduction

This text is devoted to formulating an important problem of the theory of representations of quivers with underlying graph being a Dynkin diagram.

The representation theory of the quivers with underlying graph being a simply-laced (an ADE-type) Dynkin diagram is well studied and the structure of the corresponding categories is simple and well-known. However for the other Dynkin diagrams there are several questions that arise.

One of such questions is inverting quantum Cartan matrices. Those are objects closely connected to semisimple Lie algebras and thus to the Dynkin diagrams. In their work of 2016 (see [1]), Hernandez and Leclerc have given a way to invert quantum Cartan matrices in a case of simply-laced Dynkin diagrams using an Auslander-Reiten quiver of a category of representations of the corresponding quiver. In 2020, Fujita and Oh presented a combinatorial construction of a quiver that generalises in some sence an Auslander-Reiten quiver for a non-simply laced case that is useful n answering to some questions such as inverting quantum Cartan matrices (see [2]). However its categorical sense is yet unknown, which is the main question of my research.

The goal of this text is in the simplest terms possible present the essential part of the construction of Fujita and Oh, giving all the preliminary notions on the way, and to describe some basics of an Auslander-Reiten theory to introduce the reader to the notion of an Auslander-Reiten quiver of a Krull-Schmidt category. We will avoid most of the proofs, for the reason of them being technical, irrelevant or requiring some more advanced notions that we do not present.

The first section will be devoted to the basic notions concerning root systems and the action of Weyl group on them as this part of the root system theory plays an important role in the construction. In the second section we start by presenting the combinatorial Auslander-Reiten quiver construction in a simply-laced case, as it helps visualise what is happening in general. We describe the general construction afterwards. We finish the second section by giving the equivalent construction which is more logical in the categorical sense. The third section will be devoted to the very basic notions of the Auslander-Reiten theory, introducing the notion of a Krull-Schmidt category and an Auslander-Reiten quiver. We end by defining quiver representations and stating the theorem of Happel that the construction given has a categorical sense in the simply-laced case.

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2 Preliminary information

2.1 Root systems

We start by giving the reader the basic definitions of the theory of root systems and weight lattices as well as discuss their connections to Dynkin diagrams and semisimple lie algebras.

Definition 2.1.1. Let E be a finite dimensional euclidean vector space. A finite set $\Phi \subset E$ of vectors is called a *root system* if the following holds:

– The elements of Φ span E;

– For every $\alpha \in \Phi$ the only scalar multiples of α in Φ are α and $-\alpha$;

– For every $\alpha \in \Phi$ the set Φ is stable under reflection through the hyperplane H_α perpendicular to α;

– For α, β in Φ the projection of β on α is an integer or a half-integer multiple of α.

The elements of Φ are called *roots*. A root system Φ is called *reducible* if Φ can be represented as a union of proper subsets $\Phi = \Phi_1 \cup \Phi_2$ such that for every root $\alpha \in \Phi_1$ is perpendicular to every $\beta \in \Phi_2$. Otherwise Φ is called *irreducible*.

Definition 2.1.2. Given a root system Φ over a euclidean space E, one can choose a set of *positive roots* $\Phi^+ \subset \Phi$ such that

– For every $\alpha \in \Phi$ exactly one of the roots α and $-\alpha$ is in Φ^+;

– For every α, β in Φ^+ such that $\alpha + \beta$ is in Φ we have $\alpha + \beta \in \Phi^+$.

A root $\alpha \in \Phi^+$ is called *simple* if it can't be represented a a sum of other elements in Φ^+. The set of all simple roots is called a *base* of Φ.

Remark 2.1.3. The set of positive roots is not unique. One can note that $\Phi^+ \subset \Phi$ is a set of positive roots if and only if there exists a hyperplane $H \subset E$ that doesn't contain any element of Φ such that the elements of Φ^+ are exactly the elements of Φ lying on a fixed side of H. From now on given a root system Φ we fix a set of its positive roots Φ^+.

Given a vector $v \in E$ we will denote by H_v the hyperplane perpendicular to v in E and by s_v the reflection with respect to H_v.

Definition 2.1.4. Given a root system Φ we can define its *Weyl group* as a group generated by reflections s_α for $\alpha \in \Phi$. If $\alpha \in \Phi^+$ is a simple root, the reflection s_α will be called a *simple reflection.*

By the definition of a root system, a Weyl group acts on its root system by automorphisms. We will see later, that we can define a Weyl group as generated by simple reflections only.

The following construction, given a root system, gives a finite graph that we will later see to be a Dynkin diagram.

Construction 2.1.5. Let Φ be a root system. Let Δ be the set of its simple roots. The set Δ will enumerate the vertices of our graph. We construct the edges by the following rule:

– Two vertices are connected by a single edge if the angle between the following roots is equal to $120°$;

– Two vertices are connected by a double edge if the angle between the following roots is equal to $135°$ with a arrow pointing towards the shorter root;

– Two vertices are connected by a triple edge if the angle between the following roots is equal to $150°$ with a arrow pointing towards the shorter root;

– Two vertices are not connected if the angle between the following roots is equal to $90°$.

Remark 2.1.6. Note that as for every α, $\beta \in \Phi$ the projection of β on α is an integer or half-integer multiple of α, the construction above is well-defined.

The following theorem is going to be important in our later work, however we state it here without proof.

Theorem 2.1.7. *Construction 2.1.5 gives a one-to-one correspondence between connected Dynkin diagrams and irreducible root systems up to isomorphism.*

2.2 Weyl group and Weyl chambers

In this section we continue to develop a root system theory, talking about the structure of a Weyl group using its action on a set of Weyl chambers.

Definition 2.2.1. Given a root system Φ over a euclidean space E, we define its *Weyl chambers* as connected components of the complement in E of the set of hyperplanes H_α perpendicular to roots $\alpha \in \Phi$.

Remark 2.2.2. Every Weyl chamber is a convex cone and its boundary is a finite union of faces of codimension 1. For convenience, given a Weyl chamber C, we will call the hyperplanes containing its faces of codimension 1 its *walls*.

Note that as the Weyl group of Φ acts on E by affine transformations and preserves Φ, it also preserves the set of hyperplanes $\mathcal{H}$ perpendicular to elements of Φ, and so it acts on the set of its Weyl chambers.

Theorem 2.2.3. *The action of a Weyl group on a set of Weyl chambers given above is transitive and free.*

We will not give a detailed proof of this result, however the idea of the proof of transitivity will be useful to us later. Before giving it we consider the following definition.

Definition 2.2.4. Given two Weyl chambers C and C' we call a finite sequence of Weyl chambers $C = C_0, C_1, \ldots, C_k = C'$ a *path* between C and C' if C_i and C_{i+1} are adjacent for every i, meaning that their common face is of codimension 1.

The following lemma will be left without proof.

Lemma 2.2.5. *Every two Weyl chambers C and C' are connected by a path.*

Note that the proof of transitivity part of Theorem 2.2.3 is now immediate. We can take a Weyl chamber C to the adjacent one reflecting with respect to the hyperplane containing their common face. By Lemma 2.2.8 we can take any Weyl chamber C to another Weyl chamber C' in k reflections, where k is the length of a path between C and C'.

The following result will be left without proof as well.

Lemma 2.2.6. *Let Φ be a root system over a euclidean space E. Given a set of positive roots $\Phi^+ \subset \Phi$, there exists a Weyl chamber C_+ such that its walls are exactly the hyperplanes perpendicular to simple roots of Φ.*

Remark 2.2.7. The Weyl chamber C_+ defined in Lemma 2.2.6 is exactly the set of all vectors $v \in E$ such that the hyperplane H_v perpendicular to v separates Φ^+ from $\Phi^- = \Phi \setminus \Phi^+$ (see Remark 2.1.3).

Now we are ready to prove the principal result of this section, giving a construction that is going to be useful to us in our further work.

Proposition 2.2.8. *Consider an element w of a Weyl group $\mathcal{W}$ that takes C_+ to some Weyl chamber C. Then there exists a path of length k between C_+ and C if and only if there exists an expression $w = s_{\alpha_1} \ldots s_{\alpha_k}$ of w as a product of k simple reflections.*

Proof. We will prove the proposition by the induction on k. The case of $k = 1$ is straightforward: as the walls of C_+ are exactly the hyperplanes perpendicular to simple roots, any simple reflection takes the Weyl chamber C_+ to an adjacent one, and conversely C^+ can be taken to any adjacent

Weyl chamber by a simple reflection. Note that here and throughout the whole proof we use that the action of $\mathcal{W}$ on the set of Weyl chambers is free.

Now suppose that we have proved the statement for some $k \in \mathbb{N}$. Consider a path $C_+ = C_0$, C_1, ..., $C_{k+1} = C$ of length $k+1$ between C_+ and some Weyl chamber C. By induction, the element $w' \in \mathcal{W}$ that takes C_+ to C_k can be written as a product of k simple reflections $w' = s_{\alpha_1} \dots s_{\alpha_k}$. Let H_α be the hyperplane containing the common face of C_k and C_{k+1}. Then $w'^{-1}(H_\alpha)$ contains a face of C_+, so it can be written as $H_{\alpha_{k+1}}$, where α_{k+1} is a simple root. Note that the reflection s_α with respect to $H_\alpha = w'(H_{\alpha_{k+1}})$ can be written as $w' s_{\alpha_{k+1}} w'^{-1}$, so

$$C = s_\alpha w'(C_+) = w' s_{\alpha_{k+1}} w'^{-1} w'(C_+) = w' s_{\alpha_{k+1}}(C_+) = s_{\alpha_1} \dots s_{\alpha_{k+1}}(C_+).$$

Conversely, if $C = s_{\alpha_1} \dots s_{\alpha_{k+1}}(C_+)$, by induction the chamber $C' = s_{\alpha_1} \dots s_{\alpha_k}(C_+)$ can be connected to C_+ by a path of length k. It remains to prove that C' is adjacent to C. As α_{k+1} is a simple root, the hyperplane $H_{\alpha_{k+1}}$ is a wall of C_+ and so $s_{\alpha_1} \dots s_{\alpha_k}(H_{\alpha_{k+1}})$ is a wall of C'. By the computations given above, the map taking C_+ to the reflection of C' with respect to $s_{\alpha_1} \dots s_{\alpha_k}(H_{\alpha_{k+1}})$ is exactly $s_{\alpha_1} \dots s_{\alpha_{k+1}}$, but $s_{\alpha_1} \dots s_{\alpha_{k+1}}(C_+) = C$, so C' is adjacent to C. $\square$

Remark 2.2.9. We will say that a path between C_+ and C *corresponds* to a decomposition $w = s_{\alpha_1} \dots s_{\alpha_k} \in \mathcal{W}$ if it is acquired by the construction given above. It will be useful to us that the hyperplanes H_1, ..., H_k containing the common faces of the adjacent Weyl chambers in the path corresponding to $w = s_{\alpha_1} \dots s_{\alpha_k}$ are given by $H_i = H_{s_{\alpha_1} \dots s_{\alpha_{i-1}}(\alpha_i)}$.

Corollary 2.2.10. *The Weyl group is generated by simple reflections.*

3 Construction of combinatorial Auslander-Reiten quiver

This section is devoted to a combinatorial construction developed by Fujita and Oh in [2] to generalise the notion of an Auslander-Reiten quiver for a simply-laced Dynkin diagram to a more general case. In the first part we will give the combinatorial construction for the case of a simply-laced Dynkin diagrams only, in the second part we introduce the notion of Q-datum to generalise it to the case of any Dynkin diagram, and the third part is devoted to presenting an equivalent construction that illustrates the connection of the quiver constructed to the Auslander-Reiten quiver of a category $\mathrm{Rep}(Q)$ as well as with the Auslander-Reiten quiver of its derived category.

3.1 Combinatorial Auslander-Reiten quiver in a simply laced case

The following several definition aim to define a notion of a commutational class of sequences of simple roots.

Definition 3.1.1. Given a root system Φ we denote by *a word* a sequence of simple roots $(i_0, i_1, \dots, i_k)$.

Definition 3.1.2. We call *a permutation* a transformation that takes a word $(i_0, \dots, i_j, i_{j+1}, \dots i_k)$ to $(i_0, \dots, i_{j+1}, i_j, \dots i_k)$ for perpendicular i_j, i_{j+1}. Two words are called *commutational equivalent* if they can be acquired one from another by a sequence of permutations. We denote by $[i]$ a commutational class of a word i.

Definition 3.1.3. Let $\mathcal{W}$ be a Weyl group for a root system Φ. Given $w \in \mathcal{W}$, a word $(i_0, \dots, i_k)$ is called a *reduced word* for w if $w = s_{i_0} s_{i_1} \dots s_{i_k}$ and k is the smallest integer for which it is possible. The integer k is called the *length* of w.

Remark 3.1.4. Note that by Proposition 2.2.8 the length of a word $w \in \mathcal{W}$ is equal to the number of hyperplanes H_α, $\alpha \in \Phi^+$ separating C^+ from $w(C^+)$.

The following proposition is straight-forward in view of Remark 3.1.4

Proposition 3.1.5. *There exists a unique element w_0 of $\mathcal{W}$ of maximal length. The length of w_0 is equal to $|\Phi^+|$.*

Proof. By Remark 3.1.4 there is no word of length greater then $|\Phi^+|$. Consider an element that takes C^+ to $-C^+$. Note that every hyperplane H_α, $\alpha \in \Phi^+$ is separating C^+ from $-C^+$ and $-C^+$ is the only Weyl chamber with this quality. Using the transitivity of the action of $\mathcal{W}$ on the set of Weyl chambers we get the result. □

We are mainly interested in commutational classes for reduced words for w_0 for the reasons we will see soon.

Lemma 3.1.6. *If i is a reduced word for w_0 then every element of $[i]$ is a reduced word for w_0.*

Proof. Note that if roots i_j, i_{j+1} are perpendicular, then the corresponding symmetries s_j, s_{j+1} commute. □

The following construction, given a commutational class for the longest element w_0, presents a partial ordering on the set of positive roots.

Construction 3.1.7. Given a reduced word $(i_0, \dots i_k)$ for w_0 we note that the set of $s_{i_0} s_{i_1} \dots s_{i_j}(i_{j+1})$ is exactly Φ^+ (see, for exemple, the proof of Proposition 2.2.8). So every reduced word i for w_0 gives an order $<_i$ on Φ^+. We define a partial order $\leqslant_{[i]}$ as $\alpha \leqslant_{[i]} \beta$ if $\alpha <_j \beta$ for all $j \in [i]$.

Definition 3.1.8. We say that a quiver Γ is a Hasse quiver of a partial ordering $\leqslant$ on a finite set S if S is the set of vertices of Γ and for every two elements x, y in S there is a arrow from x to y if and only if $x \leqslant y$ and there is no z in S such that $x \leqslant z \leqslant y$.

We denote by $\Gamma_{[i]}$ the Hasse quiver of the partial ordering $\leqslant_{[i]}$ of Φ^+. The following definitions show, that for an oriented Dynkin diagram Q there is a distinguished commutational class of w_0.

Definition 3.1.9. Given an oriented Dynkin diagram Q we call a vertex its *source* if there is no arrow pointing to it. A *reflection* of Q with respect to its source is a new oriented Dynkin diagram Q' acquired from Q by inverting all the arrows parting from the source.

Definition 3.1.10. We call a reduced word $(i_0, \dots, i_k)$ for w_0 compatible with quiver the orientation of a Dynkin diagram Q if for every $j \in \{0, \dots, k\}$ the vertex i_j of Q is a source of $s_0 \dots s_{j-1}(Q)$.

Proposition 3.1.11. *All reduced words compatible with Q form a single commutation class.*

This proposition is simply proved by induction on the number of corresponding elements between two representatives.

We denote the constructed class by $[Q]$. We call $\Gamma_{[Q]}$ a *combinatorial Auslander-Reiten quiver* of Q.

3.2 Q-datum and generalisation of combinatorial Auslander-Reiten quiver

In this section, following the article of Fujita and Oh [2] we introduce a notion of Q-datum and construct a generalisation of the construction above.

We will use a following notation: given a graph Γ we will denote by Γ_0 its vertex set and by Γ_1 the set of its arrows. For two vertices $x, y \in \Gamma_0$ we write $x \sim y$ if and only if x and y are connected by an arrow.

We start by a construction that allows us to acquire a non simply-laced Dynkin diagram from a simply-laced one. Let Δ be a simply-laced Dynkin diagram and σ be its automorphism such that there is no $i \in \Delta_0$ such that $i \sim \sigma(i)$.

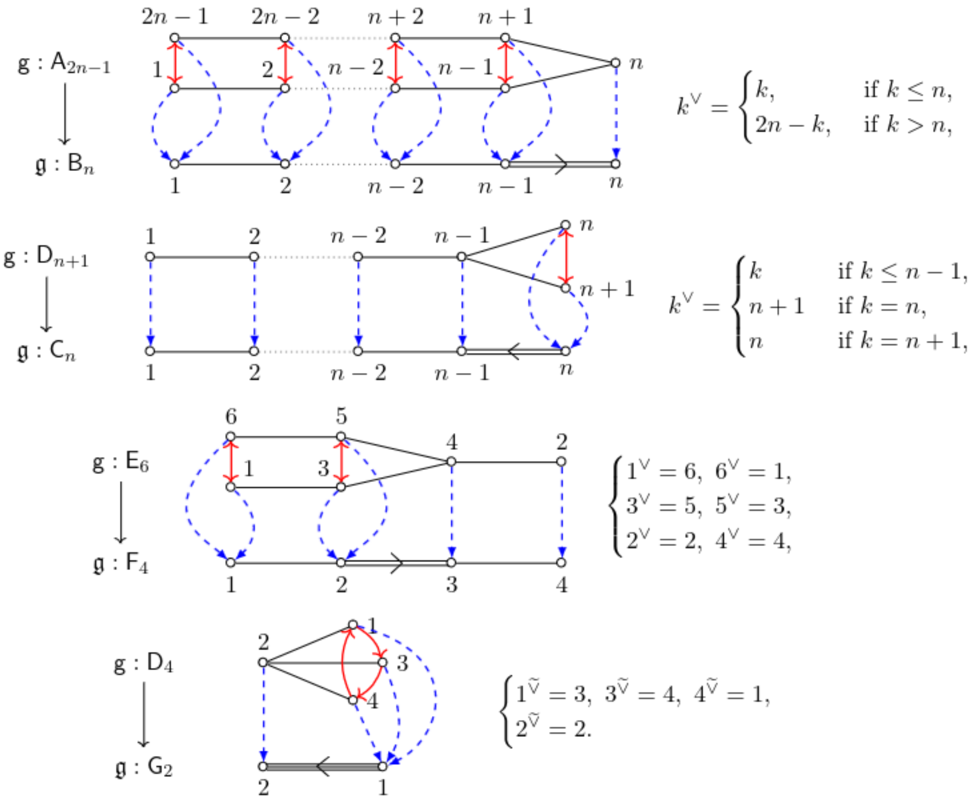


Figure 1

Figure 1 presents all σ with this property. We denote by I the set of σ-orbits in Δ_0 and consider a "quotient quiver" defined as in the figure. Denote by $r \in \{1, 2, 3\}$ the order of σ. For $i \in I$ we define by $d_i \in \{1, r\}$ the number of elements in orbit i.

Definition 3.2.1. Given a simply-laced Dynkin diagram Δ and σ as above, a function $h \colon \Delta_0 \to \mathbb{Z}$ is called a *height function* if the following two conditions hold:

- For every $i \sim j \in \Delta_0$ such that $d_{[i]} = d_{[j]}$ we have $|h(i) - h(j)| = d_{[i]}$.
- For every $[i], [j] \in I$ such that $[i] \sim [j]$ and $d_{[i]} = 1,\ d_{[j]} = r$ there exists a unique $j_0 \in [j]$ such that $|h(i) - h(j_0)| = 1$ and $h(\sigma^l(j_0)) = h(j_0) - 2l$ for every $0 \leq l < r$.

Remark 3.2.2. The height is an analog of orientation in a non-simply-laced case. For example when $\sigma = \mathrm{id}$ we can construct an orientation by putting an arrow from a vertex with a greater height to the one with the smaller. In this case height can also be retrieved from an orientation up to a constant.

A triple $\mathcal{Q} = (\Delta, \sigma, h)$ where h is a height function on (Δ, σ) is called a *Q-datum* for the quotient Dynkin diagram.

The following definition mimicks the definition of a source in a simply-laced case.

Definition 3.2.3. Given a Q-datum $\mathcal{Q} = (\Delta, \sigma, h)$ a vertex $i \in \Delta_0$ is called a *source* of $\mathcal{Q}$ if for every $j \in \Delta_0$ such that $j \sim i$ we have $h(i) > h(j)$.

Construction 3.2.4. Given a Q-datum $\mathcal{Q} = (\Delta, \sigma, h)$ and a vertex $i \in \Delta_0$, we define a function $s_i h \colon \Delta_0 \to \mathbb{Z}$ as $(s_i h)(j) = h(j) - \delta_{i,j} \times 2d_{[j]}$.

Lemma 3.2.5. *In the construction above the function $s_i h$ is a height function if and only if i was a source of $\mathcal{Q}$.*

Given a Q-datum $\mathcal{Q} = (\Delta, \sigma, h)$ and its source i, we denote by $s_i\mathcal{Q}$ the "reflected" Q-datum $(\Delta, \sigma, s_i h)$. In the case of trivial σ this will correspond to the reflection with respect to a source defined in the simply-laced case.

Definition 3.2.6. A sequence $i_0, \dots, i_n$ of elements of Δ_0 is called *adapted* to $\mathcal{Q}$ if i_k is a source of $s_{i_{k-1}} s_{i_{k-2}} \dots s_{i_1} \mathcal{Q}$.

As in the simply-laced case takes place the following theorem.

Theorem 3.2.7. *All adapted reduced words for w_0 form a single commutational class.*

We leave it without proof here.

We denote the corresponding commutation class by $[\mathcal{Q}]$. The quiver $\Gamma_{[\mathcal{Q}]}$ is called a *combinatorial Auslander-Reiten quiver* for a Q-datum $\mathcal{Q}$.

Remark 3.2.8. As the main parts of the construction, the quiver itself for a Q-datum $\mathcal{Q} = (\Delta, \mathrm{id}, h)$ where h corresponds to a chosen orientation is isomorphic to the quiver received by the construction above.

3.3 Repetition quiver and twisted Auslander-Reiten quiver

In this section we give the reader a different, in some sense more natural construction of a combinatorial Auslander-Reiten quiver for a Q-datum Q. It is based on the construction in a simply-laced case that coincides an Auslander-Reiten quiver of a category of representations as a degree 0 quiver of its derived category.

Definition 3.3.1. Given a Q-datum $\mathcal{Q} = (\Delta, \sigma, h)$ we call a *σ-parity function* a function $\epsilon \colon \Delta_0 \to \mathbb{Z}$ such that the following conditions hold

- $\forall i \in \Delta_0$ we have $0 \leqslant \epsilon(i) < d_i$;
- For any $i, j \in \Delta_0$ such that $[i] = [j]$ we have $\epsilon(i) \equiv \epsilon(j) \mod 2$;
- For any $i \in \Delta_0$ and $l \in \mathbb{Z}$ we have $\epsilon(\sigma^l(i)) \equiv \epsilon(i) - 2l \mod 2$;
- For every $i, j \in \Delta_0$ such that $[i] \sim [j]$ we have $\epsilon(i) \equiv \epsilon(j) + \min(d_{[i]}, d_{[j]}) \mod 2\min(d_{[i]}, d_{[j]})$.

Note that there are only $2r$ possible parity functions.

Remark 3.3.2. Note that the height function h satisfies the last three conditions.

Construction 3.3.3. Given a Q-datum $\mathcal{Q} = (\Delta, \sigma, h)$ and a parity function ϵ we construct a *repetition quiver* $\hat{\Delta}^\sigma$ associated to it as follows: we denote by $\hat{\Delta}_0^\sigma$ the vertex set and by $\hat{\Delta}_1^\sigma$ the arrow set.

$$\hat{\Delta}_0^\sigma = \{(i,p) \in \Delta_0 \times \mathbb{Z} \,|\, p - \epsilon(i) \in 2d_{[i]}\mathbb{Z}\},$$

$$\hat{\Delta}_1^\sigma = \{(i,p) \to (j,s) \,|\, (i,p),\ (j,s) \in \hat{\Delta}_0^\sigma, , [i] \sim [j] s - p = \min(d_{[i]}, d_{[j]})\}.$$

Construction 3.3.4. Denote by ξ the value $\frac{2|\Phi|}{|I|}$. We construct a *twisted AR quiver* as a finite subquiver of $\hat{\Delta}^\sigma$. Its set of vertices is given by

$$(\Gamma_{\mathcal{Q}})_0 = \{(i, h(i) - 2d_{[i]}k) \in \hat{\Delta}_0^\sigma \,|\, 0 \leqslant 2d_{[i]}k < \xi + h(i) - h(i^*)\}.$$

Remark 3.3.5. This definition, though a lot less pretty, is a lot more logical in a sence. In the simply-laced case it is the original construction of an Auslander-Reiten quiver of a category of representations of Q, viewed as a subquiver of an Auslander-Reiten quiver of a derived category of a category of representations of Q, which is exactly the repetition quiver.

Theorem 3.3.6. *There is an isomorphism of quivers* $\Gamma_{\mathcal{Q}} \cong \Gamma_{[\mathcal{Q}]}$.

4 A bit of Auslander-Reiten theory

In this section we remind the reader of the very basic notions of Auslander-Reiten theory in order to define an Auslander-Reiten quiver of a category. The section is inspired by the course [4].

4.1 Basic notions

We start by giving some very basic definitions.

Definition 4.1.1. A category $\mathcal{A}$ is called a *preadditive* if its Hom-sets are abelian groups and the compositions are bilinear.

A preadditive category is called *additive* if it admits finite products.

Remark 4.1.2. It is easy to show that in a preadditive category a product is canonically isomorphic to a coproduct, so the condition of a category being additive can be formulated in terms of coproducts instead.

The notion of an additive category is a very important one as most of the categories we encounter in everyday life (such as abelian groups, vector spaces, coherent sheaves over a variety) are additive. As a basic example here we recommend to keep in mind a category of finite-dimensional representations of a finite-dimensional algebra over a field.

Definition 4.1.3. Let $\mathcal{A}$ be an additive category. An object $X \in \mathcal{A}$ is called *indecomposable* if it cannot be non-trivially decomposed in a direct sum. We use a notation $\mathrm{Ind}(\mathcal{A})$ to define a set of all indecomposable objects.

We are naturally interested in the cases when by describing all indecomposable objects we can deduce the structure of the whole category. This inspires the following definitions.

Definition 4.1.4. A category $\mathcal{A}$ is called *idempotent-complete* if for every object $X \in \mathcal{A}$ every idempotent in $\mathrm{Aut}(X)$ corresponds to a projection on a direct summand.

Definition 4.1.5. An additive category is called *Krull-Schmidt* if every object can decomposed in a finite summ of indecomposables and if the endomorphism ring of every indecomposable object is local.

The two theorems that follow explain, why the definition is given in this way and what the second condition actually means.

Definition 4.1.6. Let k be a field. An preadditive category $\mathcal{A}$ is called k-linear if the Hom-sets form k-vector spaces.

Theorem 4.1.7. *Let $\mathcal{A}$ be a k-linear additive idempotent-complete category such that for every X, $Y \in \mathcal{A}$ we have $\dim_k(\mathrm{Hom}_{\mathcal{A}}(X,Y)) < \infty$. Then $\mathcal{A}$ is a Krull-Schmidt category.*

Remark 4.1.8. Note that this condition is very powerful. It lets us construct a lot of Krull-Schmidt category examples, such as A-mod for a finite-dimensional algebra A or coh-X for a projective variety X over k.

To prove Theorem 4.1.7 we need two simple lemmas from commutative algebra that will be left without proof.

Lemma 4.1.9. *Let A be a finite-dimensional k-algebra with no non-trivial idempotents. Then every element of A is either invertible or a nilpotent.*

Lemma 4.1.10. *Let A be a ring such that every element of A is either invertible or a nilpotent. Then A is local.*

Proof of Theorem 4.1.7. We prove the existence of a finite decomposition by induction on a dimention $\dim_k(\mathrm{End}(M))$ of a direct summand. Note that if $M = M_1 \oplus M_2$ then $\dim_k(\mathrm{End}(M)) \geqslant 2$, so if $\dim(M) = 1$ then M is indecomposable. Moreover, if $M = M_1 \oplus M_2$, then $\dim_k(\mathrm{End}(M)) > \dim_k(\mathrm{End}(M_1))$ and $\dim_k(\mathrm{End}(M)) > \dim_k(\mathrm{End}(M_2))$.

Now suppose that $M \in \mathcal{A}$ is indecomposable. Then the ring $\mathrm{End}(M)$ is finite-dimensional by assumption and does not contain any idempotents as $\mathcal{A}$ is idempotent-complete. So by Lemma 4.1.9 and Lemma 4.1.10 the ring $\mathrm{End}(M)$ is local. □

4.2 Krull-Schmidt theorem

The following theorem explains why Krull-Schmidt categories are important.

Theorem 4.2.1 (Krull-Schmidt)**.** *The decomposition into indecomposable objects in a Krull-Schmidt category is unique. In other words, given a Krull-Schmidt category $\mathcal{A}$ if for some $M_1, \ldots, M_m, N_1, \ldots, N_n \in \mathrm{Ind}(\mathcal{A})$ we have*

$$M_1 \oplus \ldots \oplus M_m \cong N_1 \oplus \ldots \oplus N_n,$$

then $m = n$ and there exists a substitution $\sigma \in S_m$ such that $N_i = M_{\sigma(i)}$.

As we noticed before, this means that describing irreducible objects and morphisms between them is enough to describe the structure of a category as a whole. We will give the idea of proof of this theorem after some preparation.

Definition 4.2.2. Given an additive category $\mathcal{A}$ the family $\mathcal{I}$ of morphisms is called an *ideal* if the following conditions hold:

1. $\forall X \in \mathcal{A}$ we have $0_X \in \mathcal{I}$.

2. $\forall f, g \in \mathcal{I}$ and for every morphism $h \in \mathcal{A}$ the morphisms $-f$, $f+g$, hf, fh are in $\mathcal{I}$ when defined.

The definition of ideal is very similar to the one we are used to. Let us now define the quotient category.

Definition 4.2.3. We define the new category $\mathcal{A}/\mathcal{I}$ as follows: $\mathrm{Ob}(\mathcal{A}/\mathcal{I}) = \mathrm{Ob}(\mathcal{A})$. And morphisms $\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)/\mathcal{I}(X, Y)$, where $\mathcal{I}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y) \cap \mathcal{I}$.

It is easy to see that the quotient is well-defined. There is also a canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$.
Now we consider a special ideal in a Krull-Schmidt category that will help us to distinguish isomorphic indecomposable objects from non-isomorphic ones.

Definition 4.2.4. Given a Krull-Schmidt category $\mathcal{A}$ its *radical* $R(\mathcal{A})$ is an ideal generated by all morphisms $f\colon X \to Y$ where $X, Y \in \mathrm{Ind}(\mathcal{A})$ and f is not an isomorphism.

We are interested in the structure of $\mathcal{A}/R(\mathcal{A})$. However, to understand it, we need an additional lemma.

Lemma 4.2.5. *Given two indecomposable objects X, $Y \in \mathrm{Ind}(\mathcal{A})$ the set of morphisms $R(X, Y)$ is exactly all non-isomorphisms between X and Y.*

Before the proof we remind the reader of the notions of split monomorphisms and epimorphisms.

Definition 4.2.6. Let $\mathcal{A}$ be an additive category. A morphism $f\colon X \to Y$ is called a *split monomorphism* if it admits a left inverse. Dually, a *split epimorphism* is a morphism that admits a right inverse.

Proof of lemma 4.2.5. Let $f \in R(X, Y)$ be a morphism. If $X \not\cong Y$ there is nothing to prove, so consider $X = Y$. Then by the definition of the radical $f = \sum f_i$ where f_i is a composition

$$X \xrightarrow{a_i} X_i \xrightarrow{b_i} Y_i \xrightarrow{c_i} X \ ,$$

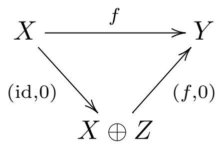
where X_i and Y_i are indecomposable objects and b_i a non-invertible morphism. Suppose that f_i is invertible. Then $b_i a_i$ is a split epimorphism. However note that Y_i is an indecomposable object, so it has no endomorphism idempotents, so $b_i a_i$ is an isomorphism. Applying the same argument to $b_i a_i$ we acquire that b_i is an isomorphism, which is a contradiction. So $f_i \in m_{\mathrm{End}(X)}$ — the only non-trivial ideal of the local ring $\mathrm{End}(X)$, so the sum $f = \sum f_i \in m_{\mathrm{End}(X)}$ so it is not invertible. □

Now the proof of Theorem 4.2.1 is straight-forward.

Proof of Theorem 4.2.1. Consider a quotient category $\mathcal{A}/R(\mathcal{A})$. It is easy to see that all direct sums are preserved, so indecomposable objects in the quotient coincide with indecomposable objects in $\mathcal{A}$. However in $\mathcal{A}/R(\mathcal{A})$ non-isomorphic indecomposable objects become orthogonal. So the decomposition of an object $X \in \mathcal{A}$ into indecomposables is determined uniquely by its Hom-sets $\mathrm{Hom}_{\mathcal{A}/R(\mathcal{A})}(X, Y)$ for all indecomposable objects $Y \in \mathrm{Ind}(\mathcal{A})$. □

4.3 Auslander-Reiten quiver of a category

In what follows we want to define an indecomposable morphism. We want to give some sense to a phrase *a morphism can not be decomposed into a composition.* The phrase does not make any sense on its own, because, even if we forbid the compositions with identity, we can construct a lot of trivial examples as follows. Given any morphism $f\colon X \to Y$ and any object Z we can decompose f as a composition



We want to introduce a way to avoid trivial compositions as above in the following definition.

Definition 4.3.1. Let $\mathcal{A}$ be an additive category. A morphism $f\colon X \to Y$ is called *indecoposable* if the following conditions hold:

1. f is not a split monomorphism;

2. f is not a split epimorphism;

3. $\forall f = gh$ either h is a split monomorphism or g is a split epimorphism.

This definition lets us avoid the cases of invertible morphisms on either side. The following lemma explains what irreducible morphisms are in a Krull-Schmidt category.

Lemma 4.3.2. *Let $\mathcal{A}$ be a Krull-Schmidt category and $X, Y \in \mathrm{Ind}(\mathcal{A})$. Then $f\colon X \to Y$ is indecomposable if and only if $f \in R(\mathcal{A})$ and $f \notin R^2(\mathcal{A})$.*

We will denote by $\mathrm{Ind}(X, Y)$ the set $R(X, Y)/R^2(X, Y)$.

Definition 4.3.3. Given a Krull-Schmidt category $\mathcal{A}$ we define an *Auslander-Reiten quiver* of $\mathcal{A}$: the vertices are classes of isomorphism of indecomposable objects and the number of arrows between two classes $[X]$ and $[Y]$ is the dimention $\dim_k(\mathrm{Ind}(X, Y))$.

4.4 Representations of quivers

Now we can finally move to the category of representations of a given quiver, to explain the main problem of this text.

Sometimes it is convenient to consider a quiver Q as a small category with items being vertices and morphism sets between two objects being the sets of oriented paths between the corresponding vertices. We will denote the corresponding category by $\mathbf{Q}$.

Definition 4.4.1. Given a finite quiver Q we define a *representation* of Q as a functor $R\colon \mathbf{Q} \to \mathrm{Vect}_k$. The morphisms between representations are exactly the natural transformations of functors.

We denote the category of representations by $\mathrm{Rep}_k(Q) = \mathrm{Func}(\mathbf{Q}, \mathrm{Vect}_k)$.

Remark 4.4.2. In other words, a representation of a quiver Q is a set of finite-dimensional vector spaces, one for each vertex, and morphisms, one for each arrow.

The following theorem states that in a simply-laced case the combinatorial Auslander-Reiten quiver is a very meaningful categorical object.

Theorem 4.4.3. *In a case of a simply-laced Dynkin diagram the combinatorial Auslander-Reiten quiver is isomorphic to an Auslander-Reiten quiver of a category* $\mathrm{Rep}_k(Q)$.

However, for all other types of Dynkin diagrams it is not a case. The theorem proved by Gabriel in 1972 states that the only quivers with finite number of isomorphism classes of indecomposable representations are those whose underlying graph is a simply-laced Dynkin diagram. So it is natural to ask, what is the categorical sense in this construction in other cases as it has similar properties.

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