Ecole Normale Supérieure; University of California, Los Angeles

Hamilton-Jacobi-Bellman equations on probability simplex

Jason Tridon

Master's internship report, March-July 2023

Internship carried out under the supervision of Wilfrid Gangbo

Contents

This research internship closed up my first year of master's degree at the ENS. I did this internship under the direction of Wilfrid Gangbo at the UCLA department of mathematics. I started working from France in March 2023, but didn't reach the US until April due to delays in obtaining my visa. This internship was an opportunity to find out the operation of a research laboratory at an American university. It was a pleasure to talk to many scholars from this institution, but not only. In particular, I attended the joint analysis seminar with Caltech organized by T. Tao and A. Palmer, which introduced me to many fields of research that are active in mathematics today.

I'm extremely grateful to Pr. Gangbo for his warm welcome and support and mathematical accompaniement during these few months.

1 Introduction

The theory of Mean Field Games (MFG) offers a formal framework to analyze differential games with a large number of players. Each player interacts and make decisions taking account the other agent's decisions, their behavior having a very little influence on the overall system. The theory has been introduced independently by J.-M Lasry and P.-L. Lions in a series of article [9], [10], [11] and by Caines-Huang-Malhamé [2] in the engineering literature. Under some assumptions, mean field games are characterized by a forward-backward system: a Fokker-Plack equation evolving forward in time describing the density of the players and a Hamilton-Jacobi-Bellman (HJB) equation evolving backward in time that governs the optimal path for each agent. Somehow, the information given by these two equations is contained in the so-called master equation. For further details about MFG, the reader can refer to [3].

Let's take a heuristic look at how MFG are related to HJB equations. Let us consider a player that moves in $\mathbb{R}^d$. We assume that each position has a cost $\mathcal{U}(t,x)$ and that moving has also a cost $L(x,x',m)$ that depends of the position and of the density of the other players. Given a time $T>0$, a player wish to minimize the cost of his trajectory and reach the position x_T at the time $t=T$, in other words he wants to minimize the quantity

$$\int_0^T L(x,x',m) + \mathcal{U}(0,x(0))$$

where $x(T) = x_T$. A key point that lies at the heart of MFG theory is the dynamic programming principle: if we are given $0 < t_0 < T$ and that we know the solution of the problem for the final coordinates (x_0, t_0), we get the final cost more easily.

$$\mathcal{U}(T, x_T) = \inf \int_{t_0}^{T} L(x, x', m) + \mathcal{U}(t_0, x_0),$$

It becomes interesting if we apply the same approach with time t_0 and $t_0 + dt$. Under very good assumptions, we get

$$\partial_t \mathcal{U}(t_0, x_0) = \inf_x L(x, x', m) - x'(t_0)\partial_x \mathcal{U}(t_0, x_0)$$

Let's define now the Legendre transform which will be useful in the text.

Definition (Legendre transform). *Let $f : \mathbb{R}^d \to \mathbb{R}$. We denote $f^* : \mathbb{R}^d \to \mathbb{R}$ the Legendre transform of f by*

$$f^*(p) = \sup_{v \in \mathbb{R}^d} (v, p) - f(v)$$

If we come back to our problem and note H the Legendre transform of L, we get

$$\partial_t \mathcal{U} = H(\partial_x \mathcal{U}(t_0, x_0))$$

We can compare this equation with (1.1) wich is the subjet of this text. The term $\mathcal{G}$ is nothing but a term that has been separated from the Hamiltonian for technical convenience. The term $\Delta_{\text{inv}}\mathcal{U}$ corresponds to a random noise that we add to the model.

The following text falls within the finite state MFG. The study of finite state spaces is not only interesting in its own right but also plays a part in the numerical analysis of more general games. Developments can be found here [7] and the master equation with Wright-Fisher common noise have been studied in [1], but very little is known about HJB equations on such spaces. Some works on Hamilton-Jacobi equations were carried out before the introduction of MFGs [8] but the latter have contributed to a renewed interest in the field.

We'll be working on the space of probability simplex endowed by a Monge-Kantorovich metric. The study of optimal control in the space of probability measure is abunding (see [5], [6], [12]). This space is also provided with a differentiable structure, so that we can define a notion of viscosity solution. A definition of viscosity solution in the Wasserstein space was proposed in [6].

More precisely, we consider the HJB equation

$$\partial_t \mathcal{U}(t, \mu) + \mathcal{H}(\mu, \nabla_{\mathcal{W}} \mathcal{U}(t, \mu)) - \mathcal{G}(\mu) = \frac{\hbar}{2} \Delta_{ind} \mathcal{U}(\mu) \tag{1.1}$$

for a certain class of Hamiltonians. The games are governed by the data $\mathcal{H}$ and $\mathcal{G}$, with Δ_{ind} being an individual noise. We present a viscosity solution to this equation under the form of an optimal control problem (2.3).

2 Definition and notations

In all that follows, $G = (V, E, \omega)$ denote a connected, simple, with no self-loops graph of vertices $V = 1, \ldots, n$ and edges E, with a weighted metric $\omega = (\omega_{ij})$. It is given by a n by n symmetric matrix with entries ω_{ij} such that $\omega_{ij} > 0$ if $(i, j) \in E$. We denote by $\mathcal{P}(G)$ the probability simplex

$$\left\{ \rho \in [0, 1]^n \,\middle|\, \sum_1^n \rho_i = 1 \right\}$$

The interior of $\mathcal{P}(G)$ is denoted by $\mathcal{P}_0(G)$. For a vertex i of the graph G, we define the set of the neighbours of i by $N(i) := \{j \in E \mid w_{ij} > 0\}$. We define some operators on the graph: the discrete gradient ∇_G and the discrete divergence div_σ:

$$\forall \phi \in \mathbb{R}^n,\ (\nabla_G \phi)_{ij} = \left(\sqrt{w_{ij}}(\phi_i - \phi_j)\right)_{ij} \text{ and } \forall v \in \mathbb{S}^{n\times n},\ \text{div}_\sigma(v)_i = \sum_{j\in V} g_{ij}(\sigma)\sqrt{w_{ij}} v_{ji}$$

Definition (G-divergence of vector field). *The divergence operator associates to any vector field m on G is defined by*

$$\forall m \in \mathbb{S}^{n\times n},\ \nabla G \cdot m = \left(\sum_{j\in N(i)} \sqrt{w_{ij}} m_{ji}\right)_{i\in V}$$

We fix a class of functions $L_{ij} \in C\Big([0,1]\times\mathbb{R}\Big) \cap C^\infty\Big((0,1)\times\mathbb{R}\Big)$, two real numbers $\lambda_1, \lambda_2 \in \mathbb{R}_+^2$ and $\kappa > 1$ such that $\lambda_1 a|b|^\kappa \leq L_{ij}(a,b) \leq \lambda_2 a|b|^\kappa$ for all $(i,j)\in E$.

We set

$$l_{ij}(a,\beta) = \begin{cases} L_{ij}\Big(a, \dfrac{\beta}{a}\Big), & \text{if } \quad a \neq 0; \\ 0, & \text{if } \quad a = \beta = 0; \\ +\infty, & \text{if } \quad a = 0, \beta \neq 0. \end{cases} \tag{2.1}$$

We assume that $(a,\beta) \mapsto l_{ij}(a,\beta)$ is convex and that for any β, $a \mapsto l_{ij}(a,\beta)$ is monotone non-increasing.

Let $g : [0,+\infty)\times[0,+\infty) \to \mathbb{R}_+$ that satisfies the following conditions:

- g is continuous on $[0,+\infty)\times[0,+\infty)$ and is of class C^∞ on $(0,\infty)\times(0,\infty)$
- $g(r,s) = g(s,r)$ for any $s, r \in \mathbb{R}_+$
- $g(r,s) > 0$ for any $r, s \in (0,\infty)$
- $g(\lambda r, \lambda s) = \lambda g(r,s)$ for any $\lambda, s, r \in (0,\infty)$
- g is concave

g is used to define an equivalence relation $\sim$ on the set of antisymmetric matrices $\mathbb{S}^{n\times n}$:

$$\forall (v,v') \in \mathbb{S}^{n\times n}, \forall \sigma \in \mathcal{P}(G) \text{ we have } v \sim v' \Leftrightarrow \forall (i,j)\in E, g_{ij}(\sigma)(v_{ij} - v'_{ij}) = 0$$

Finally, we assume that $\int_0^{+\infty} \frac{\mathrm{d}r}{\sqrt{g(r,1-r)}^\beta} < +\infty$ for all $\beta > 0$ to ensure that the square κ-Monge-Kantorovitch metric is defined on $\mathcal{P}(G)$. This hypothesis will be developed in the appendix.

Definition (The κ-Monge-Kantorivitch norm). *The κ Monge-Kantorovitch metric between ρ^0 and ρ^1 is defined by*

$$\mathcal{W}_\kappa^\kappa(\rho^0,\rho^1) := \inf_{(\rho,v)} \left\{ \int_0^1 \sum_{(i,j)\in E} g_{ij}(\rho)|v_{ij}|^\kappa \mathrm{d}t \text{ such that } \dot\rho + \text{div}_\rho(v) = 0, \rho(0) = \rho^0, \rho(1) = \rho^1 \right\}$$

Definition (The quotient space $\mathbb{H}_\sigma$). *We denote the quotient space $\mathbb{S}^{n\times n}/\sim$ as being $\mathbb{H}_\sigma$ and endow this one with an inner product and a discrete norm as follows:*

$$(v,v')_\sigma = \frac{1}{2}\sum_{(i,j)\in E} g_{ij}(\sigma) v_{ij} v'_{ij} \text{ and } ||v||_\sigma = \sqrt{(v,v)_\sigma}$$

If $\phi \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n\times n}$, we have the integration by parts formula

$$(\nabla_G \phi, v)_\sigma = -(\phi, \text{div}_\sigma(v)) \tag{2.2}$$

Definition (The tangent space $T_\rho\mathcal{P}(G)$). *We define $T_\rho\mathcal{P}(G)$ as the closure of the range of ∇_G in $\mathbb{H}_\sigma$ and we denote by π_ρ the projection onto $T_\rho\mathcal{P}(G)$.*

We define here a differentiable structures on the space $\mathcal{P}(G)^{\mathbb{R}}$:

Definition (Wasserstein gradient). *Let $\mathcal{F} : \mathcal{P}(G) \to \mathbb{R}$ and $\rho \in \mathcal{P}(G)$.*

1. *We say that $\mathcal{F}$ is $\mathcal{W}$-differentiable at ρ if there exist $v \in T_\rho\mathcal{P}(G)$ and $C > 0$ such that: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\bar{\rho} \in \mathcal{P}(G)$ and $\bar{v} \in T_\rho\mathcal{P}(G)$ then*

$$||\bar{\rho} - \rho||_{l_1} \leq \delta \Rightarrow |\mathcal{F}(\bar{\rho}) - \mathcal{F}(\rho) - (\bar{v}, v)_\rho| \leq \epsilon\mathcal{W}_2(\bar{\rho}, \rho) + C||\bar{\rho} - \rho + \mathrm{div}_\rho(\bar{v})||_{l_1}$$

   If v exists, it is uniquely determined as an element of the quotient space $T_\rho\mathcal{P}(G)$ and we denote $v := \nabla_\mathcal{W}\mathcal{F}(\rho)$

2. *We write $\mathcal{F} \in C^1(\mathcal{P}_0(G), \mathcal{W})$ if $\mathcal{F}$ is $\mathcal{W}$-differentiable everywhere on $\mathcal{P}_0(G)$ and its Wasserstein gradient $\nabla_\mathcal{W}\mathcal{F}$ is continuous on $\mathcal{P}_0(G)$.*

Definition (Fréchet derivate). *Let $\mathcal{F} : \mathcal{P}(G) \to \mathbb{R}$ and $\rho \in \mathcal{P}(G)$.*

1. *We say that $\mathcal{F}$ has a Fréchet derivate at ρ if there exist $p \in \mathbb{R}^n$ such that*

$$\sum_{i=1}^n p_i = 0 \text{ and } \lim_{s\to 0^+} \frac{\mathcal{F}((\infty - \int)\rho + \int\bar{\rho} - \mathcal{F}(\rho)}{s} =< p, \bar{\rho} - \rho >, \forall\bar{\rho} \in \mathcal{P}(G)$$

   If $\mathcal{F}$ has a Fréchet derivate at ρ, then it is uniquely determined and we write $v := \frac{\delta\mathcal{F}}{\delta\rho}(\rho)$

2. *We write $\mathcal{F} \in C^1(\mathcal{P}_0(G), l_2)$ if $\mathcal{F}$ has a continuous Fréchet derivate on $\mathcal{P}_0(G)$.*

The previous definitions are interrelated: if $\mathcal{F}$ has both a Fréchet derivate and a Wasserstein gradient at ρ, we have

$$\nabla_\mathcal{W}\mathcal{F}(\rho) = \nabla_G\left(\frac{\delta\mathcal{F}}{\delta\rho}(\rho)\right)$$

Definition (Graph individual noise operator). *If $u : \mathcal{P}(G) \to \mathbb{R}$ is differentiable at $\rho \in \mathcal{P}_0(G)$, the graph individual noise operator $\Delta\mathrm{ind}$ is defined by*

$$\Delta\mathrm{ind}u(p) := (\mathrm{div}_\rho(\nabla_\mathcal{W}u(\rho), \log\rho)$$

In light of (3.2), the noise operator can be extend to the whole set $\mathcal{P}(G)$. We now define two Lagrangians $\mathcal{L}, \bar{\mathcal{L}} : \mathcal{P}(G) \times \mathbb{S}^{n\times n}$ by

$$\mathcal{L}(\sigma, v) = \frac{1}{2}\sum_{(i,j)\in E} L_{ij}(g_{ij}(\sigma), v_{ij}) \text{ and } \bar{\mathcal{L}}(\sigma, m) = \frac{1}{2}\sum_{(i,j)\in E} l_{ij}(g_{ij}(\sigma), m_{ij}).$$

and the Hamiltonian $\mathcal{H}$ by

$$\mathcal{H}(\sigma, p) = \sup_{v\in\mathbb{S}^{n\times n}} \{(v, p)_\sigma - \mathcal{L}(\sigma, v)\} = \sup_{m\in\mathbb{S}^{n\times n}} \left\{(m, p) - \bar{\mathcal{L}}(\sigma, m)\right\}$$

Finally, we consider $\mathcal{G}, \mathcal{U}_0 : \mathcal{P}(G) \to \mathbb{R}$ convex, upper bounded. We aim to demonstrate that, under certain assumptions, the following minimization problem is a solution to the equation (1.1).

$$\mathcal{U}(t, \mu) = \inf_{(\sigma,v)\in\mathcal{C}(t,\mu)} \left\{\int_0^t [\mathcal{L}(\sigma_s, v_s) + \mathcal{G}(\sigma_s)]\, ds + \mathcal{U}_0(\sigma_0)\right\} \tag{2.3}$$

where

$$\mathcal{C}(t, \mu) := \{(\sigma, v)|\dot{\sigma} + \mathrm{div}_\sigma(v + \frac{\hbar}{2}\nabla_G \log\sigma) = 0 \text{ at the weak sense }, \sigma_t = \mu, \sigma \in \mathcal{W}^{1,\kappa}([0, t], \mathcal{P}(G)), v \text{ Borel}\}$$

3 Preliminaries

3.1 Hypothesis

By definition, $\nabla_G \log \sigma$ is well defined in the interior of $\mathcal{P}(G)$ and is given by

$$\nabla_G \log \sigma = \left(\sqrt{w_{ij}}(\log \sigma_i - \log \sigma_j)\right)_{(i,j)\in E}$$

and so

$$\forall i \in \{1,\dots,n\}, \mathrm{div}_\sigma(\nabla_G \log \sigma)_i = \sum_{j\in N(i)} g_{ij}(\sigma)\sqrt{w_{ij}}(\log \sigma_j - \log \sigma_i) \tag{3.1}$$

With the aim of giving a meaning to (2.3), a first challenge is to extend this expression to the entire set $\mathcal{P}(G)$. In order to do so, we assume this condition holds throughout the rest of the analysis:

$$\frac{\log s - \log t}{s-t} g(s,t) \tag{3.2}$$

has a unique continuous extension on $[0,1]^2$ that we'll note $\bar{g}$. We denote in the manuscript

$$\mathrm{d}(\sigma) := (\mathrm{d}_{ij}(\sigma))_{i,j} := \left(g_{ij}(\sigma)\sqrt{w_{ij}}(\log \sigma_i - \log \sigma_j)\right)_{i,j} = \left(\bar{g}_{ij}(\sigma)\sqrt{w_{ij}}(\sigma_i - \sigma_j)\right)_{i,j}$$

3.2 Problem reformulation

In this paragraph, we aim to demonstrate an alternative formulation of $\mathcal{U}$.

Proposition. *We can express $\mathcal{U}$ as follows:*

$$\mathcal{U}(t,\mu) = \inf_{(\sigma,m)\in\bar{\mathcal{C}}(t,\mu)} \left\{ \int_0^t \left(\bar{\mathcal{L}}(\sigma_s, m_s) + \mathcal{G}(\sigma_s)\right) ds + \mathcal{U}_0(\sigma_0) \right\},\ \forall (t,\mu) \in \mathbb{R}^+ \times \mathcal{P}_0(G), \tag{3.3}$$

where

$$\bar{\mathcal{C}}(t,\mu) := \{(\sigma,m) | \dot{\sigma} + \nabla_G \cdot m + \frac{\hbar}{2}\mathrm{div}_\sigma(\nabla_G \log \sigma) = 0 \textit{ at the weak sense }, \sigma_t = \mu, \sigma \in \mathcal{W}^{1,\kappa}([0,t],\mathcal{P}(G)), m \textit{ Borel}\}$$

Proof. Let $(t,\mu) \in \mathbb{R}^+ \times \mathcal{P}_0(G)$. The set $\mathcal{C}(t,\mu)$ is non empty and $\mathcal{U}(t,\mu) < +\infty$. Indeed, we can consider $\sigma = \mu$ constant and $v = -\nabla_G \log \mu$.

For $(\sigma, v) \in \mathcal{C}(t,\mu)$, set $m \in \mathbb{S}^{n\times n}$ measurable by

$$m_{ij}(t) = g_{ij}(\sigma(t))v_{ij}(t) \tag{3.4}$$

We verify that $\nabla_G \cdot m = \mathrm{div}_\sigma(v)$ so $(\sigma, m) \in \bar{\mathcal{C}}(t,\mu)$. We also have the equality $\int_0^t \mathcal{L}(\sigma, v) = \int_0^t \bar{\mathcal{L}}(\sigma, m)$. Conversely if $m \in \mathbb{S}^{n\times n}$ Borel and satisfies $\int_0^t \bar{\mathcal{L}}(\sigma,m) < +\infty$, we can define v

$$v_{ij} = \begin{cases} \frac{m_{ij}}{g_{ij}(\sigma)} & \text{if } g_{ij}(\sigma) \neq 0 \\ 0 & \text{if } g_{ij}(\sigma) = 0 \end{cases}$$

In that case, (2.1) ensures that for $(i,j) \in E$,

$$l_{ij}(g_{ij}(\sigma), m_{ij}) = \begin{cases} L_{ij}(g_{ij}(\sigma), v_{ij}) & \text{if } g_{ij}(\sigma) \neq 0 \\ L_{ij}(g_{ij}(\sigma), v_{ij}) & \text{if } g_{ij}(\sigma) = 0 \text{ and } m_{ij} = 0 \\ +\infty & \text{if } g_{ij}(\sigma) = 0 \text{ and } m_{ij} \neq 0 \end{cases}$$

in such a way that $\{s \in [0,t] \mid \bar{\mathcal{L}}(\sigma,m) \neq \mathcal{L}(\sigma,v)\}$ and $\{t \mid m_{ij}(t) \neq 0, g_{ij}(\sigma)(t) = 0\}$ have measure 0. Thus, $\nabla_G \cdot m = \mathrm{div}_\sigma(v)$ almost everywhere and $\int_0^t \mathcal{L}(\sigma,v) = \int_0^t \bar{\mathcal{L}}(\sigma,m)$. We can finally conclude from the above that $(\sigma,v) \in \mathcal{C}(t,\mu)$ and that

$$\mathcal{U}(t,\mu) = \inf_{(\sigma,m)\in\bar{\mathcal{C}}(t,\mu)} \left\{ \int_0^t \left[\bar{\mathcal{L}}(\sigma_s, m_s) + \mathcal{G}(\sigma_s)\right] ds + \mathcal{U}_0(\sigma_0) \right\},\ \forall (t,\mu) \in \mathbb{R}^+ \times \mathcal{P}_0(G)$$

□

We denote in the following $\bar{F}(\sigma,m) := \int_0^t \left(\bar{\mathcal{L}}(\sigma,m)+\mathcal{G}(\sigma)\right)ds + \mathcal{U}_0(\sigma)$ and $F(\sigma,v) := \int_0^t \left(\mathcal{L}(\sigma,v)+\mathcal{G}(\sigma)\right)ds + \mathcal{U}_0(\sigma)$.

3.3 Minimizer

The interest of the previous reformulation lies in the fact that we now need to minimize a convex function.

Proposition. *$\bar{\mathcal{L}}$ is convex and lower semi-continuous*

Proof. Consider $(a_n,\beta_n) \subset ([0,1]\times\mathbb{R})^n$ a sequence that converges to (a,β). We want to verify that $\liminf l_{ij}(a_n,\beta_n) \geq l_{ij}(a,\beta)$. If $a=0$, this is the case by (1.4). Else, we can assume that $a_n > 0$ for all $n\in\mathbb{N}$ and use the continuity of L_{ij} on $(0,1]\times\mathbb{R}$. This shows the lower semicontinuity of l_{ij}. Since g is continuous, we get that $(\sigma,m)\mapsto l_{ij}(g_{ij}(\sigma),m_{ij})$ is lower semicontinuous by composition.

Let $(\sigma_0,\sigma_1)\in\mathcal{P}(G)^2$, $m_0,m_1\in\mathbb{R}^2$ and $t\in[0,1]$. Let us verify the Jensen's inequality for l_{ij}:

$$\begin{aligned} l_{ij}(g_{ij}(t\sigma_0+(1-t)\sigma_1), tm_0+(1-t)m_1) &\leq l_{ij}(tg_{ij}(\sigma_0)+(1-t)g_{ij}(\sigma_1), tm_0+(1-t)m_1) \\ &\leq tl_{ij}(g_{ij}(\sigma_0),m_0)+(1-t)l_{ij}(g_{ij}(\sigma_1),m_1) \end{aligned}$$

Then, $(\sigma,m)\mapsto l_{ij}(g_{ij}(\sigma)$ is convex. Finally, $\bar{\mathcal{L}}$ is a convex lower semicontinuous function as a sum of convex lower semicontinuous functions. □

This theorem is the main result of our section and will be used many times later. We will show that the minimum is reached and that the Hamiltonian is conserved.

Theorem 3.1

1. *$\mathcal{U}(t,\mu)$ is a minimum.*
2. *Taking a minimizer $(\tilde{\sigma},\tilde{m})$, the quantity*

$$\bar{\mathcal{L}}(\tilde{\sigma},\tilde{m}) - (\tilde{m}+\frac{\hbar}{2}\mathrm{d}(\tilde{\sigma}))\cdot\nabla_m\bar{\mathcal{L}}(\tilde{\sigma},\tilde{m})+\mathcal{G}(\tilde{\sigma})$$

is constant almost everywhere.

1.. Let $(t,\mu)\in\mathbb{R}^+\times\mathcal{P}_0(G)$. Let (σ,m) such that $\bar{F}(\sigma,m) < \mathcal{U}(t,\mu)+1$. Since $\mathcal{G}$ and $\mathcal{U}_0$ are upper bounded, there exists $M>0$ such that $\int_0^t \bar{L}(\sigma,m) < M$.

We have

$$\int_0^t |m_{ij}|^\kappa \leq \int_0^t \left(\frac{g_{ij}(\sigma)^{\kappa-1}}{\lambda_1}\right) l_{ij}(g_{ij}(\sigma),m_{ij}) \leq \left(\frac{|g|_\infty^{\kappa-1}}{\lambda_1}\right)\int_0^t l_{ij}(g_{ij}(\sigma),m_{ij})$$

and we know that $\sigma\mapsto \mathrm{div}_\sigma(\nabla_G\log\sigma)$ is bounded on the compact set $\mathcal{P}(\mathcal{G})$ according to the equation (3.1).

The consequence of the equation that links σ and m and those two remarks yield that there exists $S>0$ such that $|\dot{\sigma}|_{L^\kappa}\leq S$. By the Poincaré inequality we can change the value of S to obtain

$$|\sigma|_{W^{1,\kappa}} \leq S \tag{3.5}$$

This shows that we can replace the set of all the admissible couples (σ,m) by a bounded space of $W^{1,\kappa}([0,t],\mathcal{P}(G))\times L^\kappa([0,t],\mathbb{S}^{n\times n})$. Let (σ_n,m_n) a minimizing sequence. By the Banach Alaoglu theroem and the reflexivity of L^κ, even if it means extracting we can assume that

1. σ_n converges weakly to $\sigma\in L^\kappa$
2. $\dot{\sigma}_n$ converges weakly to $\tilde{\sigma}\in L^\kappa$

3. m_n converges weakly to $m \in L^\kappa$

Then, we verify that $\dot{\sigma} = \tilde{\sigma}$ thus showing that (σ, m) is a minimizer of the problem. □

2.. Let $(\tilde{\sigma}, \tilde{m})$ a minimizer of the problem. Let $\phi \in C_c^1(0,t)$ and $u(s) := s+\epsilon\phi(s)$. We have $u(0) = 0, u(t) = t$ and u is a diffeomorphism for a small enough ϵ. Denote $n := u^{-1}$. Let

$$\sigma(s) := \tilde{\sigma}(n(s)), m(s) = \dot{n}(s)\tilde{m}(n(s)) + \frac{\hbar}{2}(\dot{n} - 1)d(\tilde{\sigma}(n(s)))$$

We have $\nabla_G \cdot \mathrm{d}(\sigma) = \mathrm{div}_\sigma(\nabla_G \log \sigma)$ and thus, the couple (σ, m) is admissible:

$$\begin{cases} \dot{\sigma}(s) = \dot{n}(s)\dot{\tilde{\sigma}}(n(s)) \\ \nabla_G.(m(s)) = \dot{n}(s)\nabla_G.(\tilde{m}(n(s))) + \frac{\hbar}{2}(\dot{n}(s) - 1)\nabla_G \cdot \mathrm{d}(\tilde{\sigma}(n(s))) \end{cases}$$

Let's compute $\bar{F}(\sigma, m) - \mathcal{U}_0(\sigma)$:

$$\begin{aligned} \int_0^t \left[\bar{\mathcal{L}}(\sigma(s), m(s)) + \mathcal{G}((\sigma(s))\right])ds &= \int_0^t \left[\bar{\mathcal{L}}\left(\tilde{\sigma}(n), \dot{n}\tilde{m}(n) + \frac{\hbar}{2}(\dot{n} - 1)\mathrm{d}(\tilde{\sigma})\right) + \mathcal{G}(\tilde{\sigma}(n))\right] ds \\ &= \int_0^t \dot{u}(y) \left[\bar{\mathcal{L}}\left(\tilde{\sigma}(y), \frac{\tilde{m}(y)}{\dot{u}(y)} + \frac{\hbar}{2}(\frac{1}{\dot{u}} - 1)\mathrm{d}(\tilde{\sigma})\right) + \mathcal{G}(\tilde{\sigma}(y))\right] dy \end{aligned}$$

Then,

$$\int_0^t \left(\bar{\mathcal{L}}(\sigma(s), m(s)) + \mathcal{G}(\sigma(s)\right) ds = \int_0^t (1 + \epsilon\dot{\phi}) \left[\bar{\mathcal{L}}\left(\tilde{\sigma}, (1 - \epsilon\dot{\phi})\tilde{m} - \epsilon\dot{\phi}\frac{\hbar}{2}\nabla_G \cdot \mathrm{d}(\tilde{\sigma}) + \epsilon^2 C(\epsilon, s)\right) + \mathcal{G}(\tilde{\sigma})\right] dy$$

where $(\epsilon, s) \mapsto C(\epsilon, s)$ is a continuous bounded function. By the derivability of L_{ij} on $(0, 1) \times \mathbb{R}$, we get

$$\bar{F}(\sigma, m) - \mathcal{U}_0(\sigma) = \int_0^t \left[\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) + \mathcal{G}(\tilde{\sigma}) + \epsilon\dot{\phi}\left(\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) - \left(\tilde{m} + \frac{\hbar}{2}\mathrm{d}(\tilde{\sigma})\right) \cdot \nabla_m\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) + \mathcal{G}(\tilde{\sigma})\right) + \epsilon^2 C(\epsilon, t)\right] dy$$

where $(\epsilon, s) \mapsto C(\epsilon, s)$ is a new continuous bounded function. By definition of $\mathcal{U}$, the minimum of the right part is reached when $\epsilon = 0$ so its derivate is null at $\epsilon = 0$. This shows that

$$\int_0^t \left[\dot{\phi}\left(\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) - \left(\tilde{m} + \frac{\hbar}{2}\mathrm{d}(\tilde{\sigma})\right) \cdot \nabla_m\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) + \mathcal{G}(\tilde{\sigma})\right)\right] dy = 0$$

This equality holds for all $\phi \in C_c^1(0,t)$ so $\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) - (\tilde{m} + \frac{\hbar}{2}\mathrm{d}(\tilde{\sigma})) \cdot \nabla_m\bar{\mathcal{L}}(\tilde{\sigma}, \tilde{m}) + \mathcal{G}(\tilde{\sigma})$ is constant. □

3.4 Dynamic Programming

Let $(\sigma, v) \in \mathcal{C}(t, \mu)$ and $(\tilde{\sigma}, \tilde{v}) \in \mathcal{C}(t, \mu)$ that minimizes $\mathcal{U}(t, \mu)$ and $\tau \in (0, t)$. By definition of $\mathcal{U}$,

$$\mathcal{U}(\tau, \sigma(\tau)) \leq \mathcal{U}_0(\sigma_0) + \int_0^\tau [\mathcal{L}(\sigma, v) + \mathcal{G}(\sigma)]\, ds.$$

Then,

$$\inf_{(\sigma,v)} \left\{\mathcal{U}(\tau, \sigma(\tau)) + \int_\tau^t [\mathcal{L}(\sigma, v) + \mathcal{G}(\sigma)]\, ds, \dot{\sigma} + \mathrm{div}_\sigma(v + \nabla_G \log \sigma) = 0, \sigma_t = \mu\right\} \leq \mathcal{U}(t, \mu) \tag{3.6}$$

If we have

$$\mathcal{U}(\tau, \tilde{\sigma}(\tau)) < \mathcal{U}_0(\tilde{\sigma}_0) + \int_0^\tau [\mathcal{L}(\tilde{\sigma}, \tilde{v}) + \mathcal{G}(\tilde{\sigma})]\, ds,$$

we can construct a new path (σ, v) such that $F(\sigma, v) < F(\tilde{\sigma}, \tilde{v})$. This is absurd and $(\tilde{\sigma}, \tilde{v})$ minimizes the left part of (3.6) wich is so an egality.

4 $\mathcal{U}$ is a viscosity solution

Definition (Viscosity solution). *We introduce here the definition of viscosity solution on the Wasserstein space of probability measures.*

- *A fonction* $u \in \mathrm{USC}([0,T) \times \mathcal{P}_0(G))$ *is a viscosity subsolution to (1.1) if* $u(0,\cdot) \leq u_0$ *and for every* $\phi \in C^1((0,T) \times \mathcal{P}_0(G), l_2)$ *such that* $u - \phi$ *has a local maximum at* (t_0, ρ_0)*, we have*

$$\partial_t \varphi(t_0,\rho_0) + \mathcal{H}(\rho_0, \nabla_{\mathcal{W}}\varphi(t_0,\rho_0)) + \mathcal{G}(\rho_0) \leq \mathcal{O}_{\rho_0}(\nabla_{\mathcal{W}}\varphi(t_0,\rho_0)) \tag{4.1}$$

- *A fonction* $u \in \mathrm{LSC}([0,T) \times \mathcal{P}_0(G))$ *is a viscosity supersolution to (1.1) if* $u(0,\cdot) \geq u_0$ *and for every* $\phi \in C^1((0,T) \times \mathcal{P}_0(G), l_2)$ *such that* $u - \phi$ *has a local maximum at* (t_0, ρ_0)*, we have*

$$\partial_t \varphi(t_0,\rho_0) + \mathcal{H}(\rho_0, \nabla_{\mathcal{W}}\varphi(t_0,\rho_0)) + \mathcal{G}(\rho_0) \geq \mathcal{O}_{\rho_0}(\nabla_{\mathcal{W}}\varphi(t_0,\rho_0)) \tag{4.2}$$

The purpose of this section is to show that $\mathcal{U}$ is a viscosity solution of (1.1).

4.1 Viscosity subsolution

First, we need to show that $\mathcal{U}$ is upper semicontinuous. Here, we assume here the following lemma

Proposition. *$\mathcal{U}$ is upper semicontinuous.*

Proof. Let $(\mu_n, t_n) \to (\mu, t)$ and $\epsilon > 0$. There exists $(\sigma, v) \in \mathcal{C}(t,\mu)$ such that

$$\int_0^t F(\sigma, v)d\tau + \mathcal{U}_0(\sigma) \leq \mathcal{U}(t,\mu) + \epsilon$$

Let $p_n \in (0,1)^{\mathbb{N}}$ that we'll precise later, (σ_n, v_n) a geodesic connecting μ and μ_n for the Monge-Kantorovitch metric $\mathcal{W}_\kappa$ and $(\tilde{\sigma}_n, \tilde{v_n})$ defined by

$$\tilde{\sigma}_n(\tau) = \begin{cases} \sigma\left(\frac{t}{p_n t_n}\tau\right) & \text{for } \tau \in [0, p_n t_n] \\ \sigma_n\left(\frac{1}{(1-p_n)t_n}(\tau - p_n t_n)\right) & \text{for } \tau \in [p_n t_n, t_n] \end{cases}$$

$$\tilde{v_n}(\tau) = \begin{cases} \frac{t}{p t_n} v\left(\frac{t}{p_n t_n}\tau\right) & \text{for } \tau \in [0, p_n t_n] \\ \frac{1}{(1-p_n)t_n} v_n\left(\frac{1}{(1-p_n)t_n}(\tau - p_n t_n)\right) & \text{for } \tau \in [p_n t_n, t_n] \end{cases}$$

Then, by change of variable, we obtain

$$\int_0^{t_n} F(\tilde{\sigma}_n, \tilde{v}_n) = \int_0^{p_n t_n} F(\tilde{\sigma}_n, \tilde{v}_n) + \int_{p_n t_n}^{t_n} F(\tilde{\sigma}_n, \tilde{v}_n) = \frac{p_n t_n}{t}\int_0^t F\left(\sigma, \frac{t}{p_n t_n} v\right) + t_n(1-p_n)\int_0^1 F\left(\sigma_n, \frac{1}{(1-p_n)t_n} v_n\right)$$

Let's compute $\int_0^1 F\left(\sigma_n, \frac{1}{(1-p_n)t_n} v_n\right)$:

$$\begin{aligned}\int_0^1 F\left(\sigma_n, \frac{1}{(1-p_n)t_n} v_n\right) &\leq \frac{\lambda_2}{2}\int_0^1 \sum_{(i,j)\in E} g_{ij}(\sigma_n)\left|\frac{v_n}{(1-p_n)t_n} - \frac{\hbar}{2}\sqrt{w_{ij}}(\log\sigma_{n,i} - \log\sigma_{n,j})\right|^\kappa + \mathcal{G}(\sigma) \\ &\leq 2^{\kappa-1}\lambda_2\int_0^1\left[\sum_{(i,j)\in E} g_{ij}(\sigma_n)\left(\left|\frac{v_n}{(1-p_n)t_n}\right|^\kappa + \left|\frac{\hbar}{2}\sqrt{w_{ij}}|\log\sigma_{n,i} - \log\sigma_{n,j}|\right|^\kappa\right) + \mathcal{G}(\sigma)\right]\end{aligned}$$

$$= 2^{\kappa-1}\lambda_2\left(\left(\frac{\mathcal{W}_\kappa(\mu,\mu_n)}{(1-p_n)t_n}\right)^\kappa + \int_0^1\left[\sum_{(i,j)\in E}\bar{g}_{ij}(\sigma_n)\left(\frac{\hbar}{2}\sqrt{w_{ij}}\right)^\kappa|\sigma_{n,i} - \sigma_{n,j}||\log\sigma_{n,i} - \log\sigma_{n,j}|^{\kappa-1} + \mathcal{G}(\sigma)\right]\right)$$

If we set $p_n = 1 - \mathcal{W}_\kappa(\mu, \mu_n)$, the expression $t_n(1-p_n)\int_0^1 F\left(\sigma_n, \frac{1}{(1-p_n)t_n} v_n\right)$ goes to 0. Finally,

$$\limsup \mathcal{U}(t_n, \mu_n) \leq \limsup \int_0^{t_n} F(\tilde{\sigma}_n, \tilde{v}_n) + \mathcal{U}_0(\tilde{\sigma}) = \int_0^t F(\sigma, v) + \mathcal{U}_0(\sigma) = \mathcal{U}(t, \mu)$$

and $\mathcal{U}$ is upper semicontinuous. □

We want to prove now that $\mathcal{U}$ satisfies the inequality (4.1). To do so, let us consider $(t_0, \rho_0) \in (0,T) \times \mathcal{P}_0(G)$ and $\varphi \in C^1((0,T) \times \mathcal{P}_0(G), l_2)$ such that $u - \varphi$ has a local maximum at (t_0, ρ_0).

Let $\sigma \in \mathcal{W}^{1,\kappa}(0,T;\mathcal{P}(G))$, v Borel such that $\dot{\sigma} + \mathrm{div}_\sigma(v) = 0, \sigma_{t_0} = \rho_0$. Thus, for all $t \leq t_0$,

$$\mathcal{U}(t_0, \rho_0) \leq \mathcal{U}(t, \sigma(t)) + \int_t^{t_0} \left[\mathcal{L}(\sigma, v - \frac{\hbar}{2}\nabla_{\mathcal{W}} S(\sigma)) + \mathcal{G}(\sigma)\right] ds$$

If t is close of t_0, since $u - \varphi$ has a local maximum at (t_0, ρ_0), we have

$$\varphi(t_0, \rho_0) - \varphi(t, \sigma(t)) \leq \int_t^{t_0} \left[\mathcal{L}(\sigma, v - \frac{\hbar}{2}\nabla_{\mathcal{W}} S(\sigma)) + \mathcal{G}(\sigma)\right] ds \tag{4.3}$$

We will attempt to divide this expression by $t - t_0$ and look at the limit when t goes to t_0. We start with some technical lemmas to give a sense to the limit of the left part.

Lemma 4.1
Let $0 \leq t_1 < t_2 \in \mathbb{R}^2$, $\sigma \in \mathcal{W}^{1,\kappa}(t_1, t_2; \mathcal{P}(G))$ and $v \in L^\infty(t_1, t_2; \mathbb{S}^{n\times n})$ such that the contiuity equation $\dot{\sigma} + \mathrm{div}_\sigma(v)$ is verified. Then $\frac{\mathcal{W}_2(\sigma(a), \sigma(b))}{b-a}$ is bounded for all $a, b \in \mathbb{R}^2$ such that $t_1 \leq a < b \leq t_2$.

Proof. Note that $||v||_\sigma \in L^\infty(t_1, t_2; \mathbb{S}^{n\times n})$ since g is continuous on a compact set.

Let $a < b \in [t_1, t_2]^2$, $u(s) := (b-a)s + b$ and $\tilde{v}(s) = (b-a)v(u(s))$. The couple $(\sigma \circ u, \tilde{v})$ is a path connecting $\sigma(a)$ and $\sigma(b)$:

$$\begin{cases} \dot{\overbrace{\sigma \circ u}}(s) = (b-a)\dot{\sigma}(u(s)) \\ \mathrm{div}_{\sigma(u(s))}(\tilde{v}(s)) = (b-a)\mathrm{div}_{\sigma(u(s)}(v(u(s)) \end{cases}$$

$$\mathcal{W}_2^2(\mu(a), \mu(b)) \leq \int_0^1 ||v(s))||^2_{\mu(u(s))} \mathrm{ds} = (b-a)^2 \int_0^1 ||v(u(s))||^2_{\mu(u(s))} \mathrm{ds} = (b-a)\int_a^b ||v(t)||^2_{\mu(t)} \mathrm{dt} \tag{4.4}$$

and so

$$0 \leq \frac{\mathcal{W}_2(\mu(a), \mu(b))}{b-a} \leq \sqrt{\frac{\int_a^b ||v(t)||^2_{\sigma(t)}}{b-a}} \tag{4.5}$$

This last expression is bounded since $||v||_\sigma \in L^\infty(t_1, t_2; \mathbb{S}^{n\times n})$. □

Lemma 4.2
Let $0 < t_1 < t_0$, $\rho \in \mathcal{W}^{1,\kappa}(t_1, t_0; \mathcal{P}(G))$ and v Borel such that $\dot{\rho} + \mathrm{div}_\rho(v) = 0, v(t_0) \in T_{\rho_0}\mathcal{P}(G)$ and $\rho(t_0) = \rho_0$. Let $\varphi \in C^1((0,T) \times \mathcal{P}_0(G), l_2)$. If we assume that $\frac{\mathcal{W}(\rho(t), \rho(t_0))}{t - t_0}$ is bounded when t goes to t_0 then

$$\frac{d}{dt}\varphi(t_0, \rho(t))\Big|_{t=t_0} = (\nabla_{\mathcal{W}}\varphi(\rho_0), v(t_0))_{\rho_0} \tag{4.6}$$

and

$$\frac{d}{dt}\varphi(t, \rho(t))\Big|_{t=t_0} = \partial_t\varphi(t_0, \rho_0) + (\nabla_{\mathcal{W}}\varphi(\rho_0), v(t_0))_{\rho_0} \tag{4.7}$$

Proof. Indeed, by definition of the Wasserstein differential, we can let $C > 0$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|\rho(t) - \rho|_1 \leq \delta$ then

$$|\varphi(t_0, \rho(t)) - \varphi(t_0, \rho_0) - ((t - t_0)v, \nabla_{\mathcal{W}}\varphi(\rho_0))_{\rho_0}| \leq \epsilon\mathcal{W}(\rho(t), \rho_0) + C|\rho(t) - \rho_0 + \mathrm{div}_{\rho_0}((t - t_0)v|_1$$

Since, $\dot{\rho}(t_0) + \mathrm{div}_{\rho_0}(v) = 0$, we get

$$\limsup_{t\to t_0} \left| \frac{\varphi(t_0, \rho(t)) - \varphi(t_0, \rho_0)}{t - t_0} - (v, \nabla_{\mathcal{W}}\varphi(\rho_0))_{\rho_0} \right| \leq \epsilon\frac{\mathcal{W}(\rho(t), \rho_0)}{t - t_0} + o(1)$$

This is true for all $\epsilon > 0$ so the first part of the lemma is proved, then

$$\begin{aligned}
\varphi(t, \rho(t)) - \varphi(t_0, \rho(t_0)) &= \varphi(t, \rho(t)) - \varphi(t_0, \rho(t)) + \varphi(t_0, \rho(t)) - \varphi(t_0, \rho_0) \\
&= (t - t_0)\partial_t\varphi(t_0, \rho(t)) + (t - t_0)(\nabla_{\mathcal{W}}\varphi(\rho_0), v(t_0))_{\rho_0} + o(t - t_0) \\
&= (t - t_0)(\partial_t\varphi(t_0, \rho_0) + o(1)) + (t - t_0)(\nabla_{\mathcal{W}}\varphi(\rho_0), v(t_0))_{\rho_0} + o(t - t_0) \\
&= (t - t_0)\partial_t\varphi(t_0, \rho_0) + (t - t_0)(\nabla_{\mathcal{W}}\varphi(\rho_0), v(t_0))_{\rho_0} + o(t - t_0)
\end{aligned}$$

and we get the second part of the lemma. □

The hypothesis of this lemma may seem restrictive, mainly because we impose that $v(t_0)$ belongs to the tangent space, we nevertheless show that this condition can be replaced by $v(t_0) \in \mathbb{S}^{n\times n}$. This is the subject of that last lemma:

Lemma 4.3
Let $\sigma \in \mathcal{W}^{1,\kappa}(0, T; \mathcal{P}(G)), v \in L^\infty(0, T; \mathbb{S}^{n\times n})$ such that the continuity equation holds. Then, $w(t) := \pi_{\sigma(t)}(v(t))$ verifies:

1. *w is measurable*
2. *$\dot{\sigma} + \mathrm{div}_\sigma(w(t)) = 0$*
3. *$||w||_{\sigma(t)} \leq ||v||_{\sigma(t)}$*

Proof. First, we see that σ is continuous. Indeed, $\dot{\sigma}_i$ is bounded as a sum of bounded terms:

$$\dot{\sigma}_i = \mathrm{div}_\sigma(v)_i = \sum_{j\in N(i)} g_{ij}(\sigma)\sqrt{w_{ij}}v_{ji}$$

1. $\pi_{\sigma(t)}(v)$ is defined as the minimum of $||v - \cdot||_{\sigma(t)}$ over $T_{\sigma(t)}\mathcal{P}(G)$ and $T_{\sigma(t)}\mathcal{P}(G)$ as the closure of ∇_G in H_σ. Then, by continuity of $w \mapsto ||v - w||_{\sigma(t)}$, we get that

$$\pi_{\sigma(t)}(v) = \min_{\phi\in\mathbb{R}^n} ||v - \nabla_G(\phi)||_{\sigma(t)} \tag{4.8}$$

   For all $\phi \in \mathbb{R}^n$, $t \mapsto ||v - \nabla_G(\phi)||_{\sigma(t)}$ is continuous by continuity of g and σ so $w(t) = \pi_{\sigma(t)}(v)$ is measurable (and also continuous).

2. We have $\dot{\sigma} + \mathrm{div}_\sigma(w) = \dot{\sigma} + \mathrm{div}_\sigma(v) + \mathrm{div}_\sigma(w - v) = 0$.
3. This a direct consequence of the definition of w as minimizer. □

These three lemmas show that if the curve (σ, v) satisfies certain assumptions, then we can compute the desired limit.

Corollary 4.4
Let $\sigma \in \mathcal{W}^{1,\kappa}(0, T; \mathcal{P}(G)), v \in L^\infty(0, T; S^{n\times n})$ such that the continuity equation holds and $w(t) = \pi_{\sigma(t)}v(t)$. Then,

$$\lim_{t\to t_0} \frac{\varphi(t_0, \rho_0) - \varphi(t, \sigma(t))}{t_0 - t} = \partial_t\varphi(t_0, \rho_0) + (\nabla_{\mathcal{W}}\varphi(\rho_0), w)_{\rho_0} = \partial_t\varphi(t_0, \rho_0) + (\nabla_{\mathcal{W}}\varphi(\rho_0), v)_{\rho_0}$$

Therefore, we want to prove that $\forall v_0 \in \mathbb{S}^{n\times n}$, there exists a curve (σ, v) solution of the continuity equation such that $v(t_0) = v_0$.

Proposition. *Let $t_0 > 0$, $v \in L^\infty(0,T;\mathbb{S}^{n\times n})$ continuous and $\rho_0 \in \mathcal{P}_0(G)$. There exists $\delta > 0$ such that the following Cauchy problem has a solution $\mu \in \mathcal{W}^{1,\kappa}(t_0-\delta, t_0; \mathcal{P}(G))$*

$$\begin{cases} \dot{\mu} + \mathrm{div}_\mu(v) = 0 \\ \mu_{t_0} = \rho_0 \end{cases} \tag{4.9}$$

Proof. This a consequence of the Cauchy-Lipschitz theroem. There exists $\epsilon > 0$ such that $1-\epsilon > \rho_{0,i} > \epsilon$ for all $i \in \{1,\dots,n\}$. Let $U =]\epsilon, 1-\epsilon[^n$ and

$$f := \begin{cases} U \times \mathbb{R} \to \mathbb{R}^n \\ (x,t) \mapsto \mathrm{div}_x(v) \end{cases}$$

Since g is C^∞ on $(0,1)^2$, we can bound ∇g on $[\epsilon, 1-\epsilon]^2$ by a real $M > 0$. Let x, y two vectors of U, we have

$$|\mathrm{div}_x(v) - \mathrm{div}_y(v)|_1 \leq \sum_{i,j} |\sqrt{w_{ij}} v_{ij}||g_{ij}(x) - g_{ij}(y)|_1 \leq |V|_\infty |\sqrt{w}|_\infty \sum_{ij} M(|x_i - y_i| + |x_j - y_j|) \leq C|x-y|_1$$

where C is a constant depending only of (v, G, M). By continuity of g and v, f is continuous. As a consequence, there exists $\delta > 0$ such that the Cauchy problem has a solution on $U \times [t_0 - \delta, t_0]$. Denote μ this solution. We have

$$\frac{d}{dt}\sum_{i=1}^n \mu_i = \sum_{i\neq j} g_{ij}(\sigma)\sqrt{w_{ij}} v_{ji} = \sum_{i<j} g_{ij}(\sigma)\sqrt{w_{ij}} v_{ji} + \sum_{j<i} g_{ij}(\sigma)\sqrt{w_{ij}} v_{ji} = \sum_{i<j} g_{ij}(\sigma)\sqrt{w_{ij}}(v_{ji} - v_{ji}) = 0$$

and so $\sum_i \mu_i$ is constant equal to 1, then $\mu \in \mathcal{P}(G)$. Since $\mu \in U$, $v \in L^\infty(t_0-\delta, t_0)$ and the identity $\dot{\mu} + \mathrm{div}_\mu(v) = 0$, we deduce that $\dot{\mu} \in L^\kappa(t_0 - \delta, t_0; R^n)$ so $\mu \in \mathcal{W}^{1,\kappa}(t_0-\delta, t_0; \mathcal{P}(G))$. ☐

We now have all the necessary tools to carry out the desired calculation. Let $v(t) := v_0 \in L^\infty(0, t_0; \mathbb{S}^{n\times n})$ constant. By the proposition, there exists $t_1 < t_0$ and $(\mu, \delta) \in \mathcal{W}^{1,\kappa}(t_1, t_0; \mathcal{P}(G)) \times \mathbb{R}$ such that $\dot{\mu} + \mathrm{div}_\mu(v) = 0$, $\mu(t_0) = \rho_0$. By the lemma 4.1, $\frac{\mathcal{W}(\rho(t), \rho(t_0))}{t - t_0}$ is bounded when t goes to t_0. Thus, by the lemma (4.2

$$\lim_{t\to t_0} \frac{\varphi(t_0, \rho_0) - \varphi(t, \sigma(t))}{t_0 - t} = \partial_t \varphi(t_0, \rho_0) + (\nabla_\mathcal{W} \varphi(\rho_0), v(t_0))_{\rho_0}$$

We recall that we study this inequality

$$\frac{\varphi(t_0, \rho_0) - \varphi(t, \sigma(t))}{t_0 - t} \leq \frac{\int_t^{t_0} \left[\mathcal{L}(\sigma, v - \frac{\hbar}{2}\nabla_\mathcal{W} S(\sigma)) + \mathcal{G}(\sigma)\right] ds}{t_0 - t}$$

Once again, we use the change of variable between m and v. Let m_v as defined in (3.4). Here, v is continuous and so m_v is. Then, by continuity of $t \mapsto \bar{\mathcal{L}}(\sigma(t), m_v(t)) < +\infty$,

$$\partial_t \varphi(t_0, \rho_0) + (\nabla_\mathcal{W} \varphi(\rho_0), v(t_0))_{\rho_0} \leq \bar{\mathcal{L}}(\rho_0, m_v) + \mathcal{G}(\rho_0)$$

We have

$$(\nabla_\mathcal{W} \varphi(\rho_0), v(t_0))_{\rho_0} = (m_v(t_0), \nabla_\mathcal{W} \varphi(\rho_0)) + \frac{\hbar}{2}(\nabla_\mathcal{W} \mathcal{S}(\rho_0), \nabla_\mathcal{W} \varphi(\rho_0))_{\rho_0}$$

so that

$$\partial_t \varphi(t_0, \rho_0) + (m_v(t_0), \nabla_\mathcal{W} \varphi(\rho_0)) - \bar{\mathcal{L}}(\rho_0, m_v(t_0)) - \mathcal{G}(\rho_0) \leq \mathcal{O}_{\rho_0}(\nabla_\mathcal{W} \varphi(\rho_0)) \tag{4.10}$$

This equation is true for all $v \in \mathbb{S}^{n\times n}$ and so

$$\partial_t \varphi(t_0, \rho_0) + \mathcal{H}(\rho_0, \nabla_\mathcal{W} \varphi(t_0, \rho_0)) - \mathcal{G}(\rho_0) \leq \mathcal{O}_{\rho_0}(\nabla_\mathcal{W} \varphi(\rho_0))$$

$\mathcal{U}$ is a subsolution to the equation (1.1).

4.2 Viscosity supersolution

Proposition. *$\mathcal{U}$ is lower semicontinuous.*

Proof. Let $(\mu, t) \in \mathcal{P}_0(G) \times \mathbb{R}^+$ and $(\mu_n, t_n) \to (\mu, t)$. Let (σ_n, m_n) that realizes the minimum for (μ_n, t_n). Let $\tilde{\sigma}_n, \tilde{m}_n$ by

$$\begin{cases} \tilde{\sigma}_n(\tau) = \sigma_n\left(\frac{t_n \tau}{t}\right) \\ \tilde{m}_n(\tau) = m_n(\tau) \end{cases}$$

such that $\dot{\tilde{\sigma}}_n + \nabla_G \cdot \tilde{m} + \frac{\hbar}{2}\mathrm{div}_{\tilde{\sigma}}(\nabla_G \log \tilde{\sigma}) = 0$. We have,

$$\mathcal{U}(\mu_n, t_n) = \int_0^{t_n} \bar{\mathcal{L}}\left(\sigma_n, m_n\right) + \mathcal{U}_0(\sigma) = \frac{t_n}{t}\int_0^t \bar{\mathcal{L}}\left(\tilde{\sigma}_n, \tilde{m}_n\right) + \mathcal{U}_0(\tilde{\sigma}_n)$$

If we proceed in the same manner as in the proof of theorem (3.1), we can show that a subsequence of $(\tilde{\sigma_n}, m_n)$ can be extracted that converges weakly in $\mathcal{W}^{1,\kappa} \times L^\kappa$ to (σ, m). Note that we should pick some real $t* > \max(\max_{n\in\mathbb{N}} t_n, t)$ and consider the space $\mathcal{W}^{1,\kappa}(0, t*; \mathcal{P}(G)) \times L^\kappa(0, t*; \mathbb{S}^{n\times n})$ even if it means extending the functions as 0 outside their interval of definition. Finally, since $\bar{\mathcal{L}}$ is lower semi-continuous,

$$\liminf \mathcal{U}(\mu_n, t_n) = \liminf \int_0^t \mathcal{L}(\tilde{\sigma}_n, \tilde{m}_n) + \mathcal{U}_0(\tilde{\sigma}_n) \geq \mathcal{U}(\mu, t)$$

□

This shows that $\mathcal{U}(\mu, t)$ is lower semi-continuous.

We want to prove now that $\mathcal{U}$ satisfies the inequality (4.2). Let $(t_0, \rho_0) \in (0, T) \times \mathcal{P}_0(G)$ and $\varphi \in C^1((0, T)\times\mathcal{P}_0(G), l_2)$ such that $u - \varphi$ has a local minimum at (t_0, ρ_0). There exists $(\sigma, v) \in \mathcal{W}^{1,\kappa}(0, T; \mathcal{P}(G)) \times L^2(0, T; S^{n\times n})$ such that

$$\mathcal{U}(t_0, \rho) = \mathcal{U}(t, \sigma(t)) + \int_t^{t_0} \left[\mathcal{L}(\sigma, v - \frac{\hbar}{2}\nabla_{\mathcal{W}} S(\rho_0)) + \mathcal{G}(\sigma)\right] ds$$

If t is close of t_0, we have

$$\varphi(t_0, \rho_0) - \varphi(t, \sigma(t)) \geq \int_t^{t_0} \left[\mathcal{L}(\sigma, v - \frac{\hbar}{2}\nabla_{\mathcal{W}} S(\rho_0)) + \mathcal{G}(\sigma)\right] ds$$

We can proceed now as in (4.8) and get $w(t)$ measurable such that $w(t) = \pi_{\sigma(t)} v(t)$. By the corollary (4.4), we have

$$\lim_{t\to t_0} \frac{\varphi(t_0, \rho_0) - \varphi(t, \sigma(t))}{t_0 - t} = \partial_t \varphi(t_0, \rho_0) + (\nabla_{\mathcal{W}}\varphi(\rho_0), v)_{\rho_0}$$

We use the change of variable $(v \to m_v)$ and by continuity of σ, m_v, we have

$$\partial_t \varphi(t_0, \rho_0) + (\nabla_{\mathcal{W}}\varphi(\rho_0), v)_{\rho_0} \geq \bar{\mathcal{L}}(\rho_0, m_v(t_0))) + \mathcal{G}(\rho_0)$$

and finally, we obtain the desired formula. $\mathcal{U}$ is a supersolution of (1.1).

$$\partial_t \varphi(t_0, \rho_0) + (m_v(t_0), \nabla_{\mathcal{W}}\varphi(\rho_0)) - \bar{\mathcal{L}}(\rho_0, m_v(t_0)) - \mathcal{G}(\rho_0) \leq \mathcal{O}_{\rho_0}(\nabla_{\mathcal{W}}\varphi(\rho_0)) \tag{4.11}$$

5 Properties of $\mathcal{U}$

5.1 Euler-Lagrange equation

In this subsection, we will take a closer look at the regularity of $\mathcal{U}$ by finding the Euler-Lagrange equation associated with the problem. We will see that we can find one directly around the interior points of the simplex $\mathcal{P}(G)$.

Definition ($\frac{\delta \mathrm{d}}{\delta\sigma}(\sigma)$). *For $\sigma \in \mathcal{P}(G)$, we define the Fréchet derivate of* d *as an element of $\mathbb{R}^n \times \mathbb{S}^{n\times n}$ by $\frac{\delta \mathrm{d}}{\delta\sigma}(\sigma) := \left(\frac{\delta \mathrm{d}_{ij}}{\delta\sigma}(\sigma)\right)_{(i,j)\in E}$ and we write $\frac{\delta \mathrm{d}_{ij}}{\delta\sigma_k}(\sigma)$ the coordinates of $\frac{\delta \mathrm{d}_{ij}}{\delta\sigma}(\sigma)$. It will be useful in the following to remark that*

$$\frac{\delta \mathrm{d}_{ij}}{\delta\sigma_k}(\sigma) = \begin{cases} \partial_1 \bar{g}(\sigma_i,\sigma_j)(\sigma_i - \sigma_j) + \bar{g}(\sigma_i) & \text{if } k = i \\ \partial_2 \bar{g}(\sigma_i,\sigma_j)(\sigma_i - \sigma_j) - \bar{g}(\sigma_i) & \text{if } k = j \\ 0 & \text{if } k \notin \{i,j\} \end{cases}$$

We also denote $\mathrm{A}(\sigma) : \mathbb{S}^{n\times n} \times \mathbb{R}^n$ the adjoint application of $\frac{\delta \mathrm{d}}{\delta\sigma}$ such that

$$< \mathrm{A}m|\phi > \quad = \left\langle m | \frac{\delta \mathrm{d}}{\delta\sigma}(\sigma)\phi \right\rangle, \ \forall m \in \mathbb{S}^{n\times n}, \ \forall \phi \in \mathbb{R}^n$$

Example. *If $g(x,y) = \frac{x-y}{\log x - \log y}$, we have $\bar{g} = 1$ and $\mathrm{d}_{ij}(\sigma) = \sigma_i - \sigma_j$. Then,*

$$\frac{\delta \mathrm{d}_{ij}}{\delta\sigma_k}(\sigma) = \frac{1}{2} A^k_{ij}(\sigma) = \delta_{k,i} - \delta_{k,j}$$

Proposition (Euler-Lagrange equation). *Let $(\mu, T) \in \mathcal{P}_0(G) \times \mathbb{R}^+$ and $(\sigma, m) \in \mathcal{C}(T,\mu)$ that minimizes $\mathcal{U}(t,\mu)$ as in the theorem (3.1). Let p the momentum by $p := \nabla_m \bar{\mathcal{L}}(\sigma, m)$. We have an Euler-Lagrange equation inside the domain $\mathcal{P}_0(G)$:*

$$\forall t_0 \in (0,T) \text{ such that } \sigma_t \in \mathcal{P}_0(G), \tag{5.1}$$

Proof. By continuity of σ, we set $\delta > 0$ and $r > 0$ such that $(\sigma_t)_i > r$, for all $i \in V$ and for all $t \in [t_0 - \delta, t_0 + \delta]$. Let $b \in C^1_c(t_0 - \delta, t_0 + \delta; \mathbb{S}^{n\times n})$ and $\phi = -\nabla_G \cdot b$. For ϵ small enough, we set $\sigma^\epsilon := \sigma + \epsilon\phi \in \mathcal{P}(G)$. We also set $m^\epsilon = m + \epsilon\dot{b} + \frac{\hbar}{2}(\mathrm{d}(\sigma) - \mathrm{d}(\sigma^\epsilon))$ such that $\dot{\sigma}^\epsilon + \nabla_G \cdot m^\epsilon + \frac{\hbar}{2}\nabla_G \cdot \mathrm{d}(\sigma^\epsilon) = 0$.

According to the dynamic programming principle, we have

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \int_{t_0-\delta}^{t_0+\delta} \bar{\mathcal{L}}\left(\sigma + \epsilon\phi, m + \epsilon\dot{b} + \frac{\hbar}{2}(\mathrm{d}(\sigma) - \mathrm{d}(\sigma^\epsilon))\right) + \mathcal{G}(\sigma^\epsilon) &= 0 \\ \Leftrightarrow \int_{t_0-\delta}^{t_0+\delta} -\left\langle \frac{\delta\bar{\mathcal{L}}}{\delta\sigma}(\sigma,m), \nabla_G \cdot b \right\rangle + \left\langle p, \dot{b} - \frac{\hbar}{2}\frac{\delta \mathrm{d}}{\delta\sigma}(\sigma)\nabla_G \cdot b \right\rangle - \left\langle \frac{\delta\mathcal{G}}{\delta\sigma}, \nabla_G \cdot b \right\rangle &= 0 \\ \Leftrightarrow \int_{t_0-\delta}^{t_0+\delta} < \nabla_{\mathcal{W}}\bar{\mathcal{L}}(\sigma,m), b > -\frac{\hbar}{2} < \nabla_G\left(\mathrm{A}(\sigma)p\right), b > + < p, \dot{b} > + < \nabla_{\mathcal{W}}\mathcal{G}, b > &= 0 \end{aligned}$$

This equation holds for all $b \in C^1_c(t_0 - \delta, t_0 + \delta; \mathbb{S}^{n\times n})$, we deduce the following equation

$$\nabla_{\mathcal{W}}\bar{\mathcal{L}}(\sigma, m) - \frac{\hbar}{2}\nabla_G \circ \mathrm{A}(\sigma)(p) + \nabla_{\mathcal{W}}\mathcal{G}(\sigma) = \dot{p} \text{ a.e. in } [t_0 - \delta, t_0 + \delta] \tag{5.2}$$

□

5.2 Duality

Let $(\mu, t) \in \mathcal{P}_0(G) \times \mathbb{R}^+$. Under certain assumptions, it is possible to express an alternative form of $\mathcal{U}$ using duality arguments. We use here the method developed in [4]. In the following, we assume that the function $\bar{g} = 1$ and that the hamiltonian $\mathcal{H}$ is 1-homogeneous with respect to σ. We need to extend the definition of $\mathcal{U}_0$ and $\mathcal{G}$ to $\mathbb{R}^n$ by setting

$$\begin{cases} \mathcal{G}(0) = 0 \\ \mathcal{G}(s\rho) := s\mathcal{G}(\rho) \text{ for any } s > 0, \text{ for any } \rho \in \mathcal{P}(G) \\ \mathcal{U}_0(\rho) := +\infty \text{ for any } \rho \notin \mathcal{P}(G) \end{cases}$$

Let's define a function K by

$$K(\sigma, m, \phi) := \int_0^t \left(\bar{F}(\sigma, m) + < \phi, \dot{\sigma} + \nabla_G \cdot (m + \frac{\hbar}{2}\mathrm{d}(\sigma)) > \right) ds + \mathcal{U}_0(\sigma_0) \tag{5.3}$$

Provided we have enough regularity, if $\sigma_t = \mu$, we get by integration by parts,

$$K(\sigma, m, \phi) = \int_0^t \left(\bar{F}(\sigma, m) - < \dot{\phi}, \sigma > - < \nabla_G \phi, m + \frac{\hbar}{2}\mathrm{d}(\sigma) > \right) ds + (\phi_t, \mu) - (\phi_0, \sigma_0) + \mathcal{U}_0(\sigma_0)$$

The dual formulation turns out to be the minimax interversion for this idendity. To lighten the notations, we denote $\mathcal{A}(\sigma, m) := \int_0^t \bar{F}(\sigma, m) + \mathcal{G}(\sigma)$. In the following, we shall consider

$$K := \begin{cases} L^\kappa(0, t; \mathbb{R}^n) \times L^{\kappa'}(0, t; \mathbb{S}^{n\times n}) \times W^{1,\kappa'}(0, t; \mathbb{R}^n) \to \mathbb{R} \\ (\sigma, m, \phi) \mapsto \mathcal{A}(\sigma, m) + (\phi_t, \mu_t) - (\phi_0, \sigma_0) + \mathcal{U}_0(\sigma_0) - \int_0^t \left(< \nabla_G \phi, m + \frac{\hbar}{2}\mathrm{d}(\sigma) > + < \dot{\phi}, \sigma > \right) ds \end{cases}.$$

The following functional spaces will be useful in our analysis:

- for any $e \geq 1$, let $A^e := L^\kappa(0, t; [0, e]^n) \times L^{\kappa'}(0, t; \mathbb{S}^{n\times n})$ and $A := L^\kappa(0, t; \mathcal{P}(G)) \times L^{\kappa'}(0, t; \mathbb{S}^{n\times n})$
- for any $l > 0$, let $B^l := \{\phi \in W^{1,\kappa'}(0, t; \mathbb{R}^n) \mid |\phi|_{W^{1,\kappa'}} \leq l\}$ and $B := W^{1,\kappa'}(0, t; \mathbb{R}^n)$

The map $\phi \mapsto K(\sigma, m, \phi)$ is linear wich provides

$$\sup_{\phi \in B^l} K(\sigma, m, \phi) = \mathcal{A}(\sigma, m) + \mathcal{U}_0(\sigma_0) + l\epsilon(\sigma, m)$$

and

$$\sup_{\phi \in B} K(\sigma, m, \phi) = \begin{cases} \mathcal{A}(\sigma, m) + \mathcal{U}_0(\sigma_0) & \text{if } (\sigma, m) \in \bar{\mathcal{C}}(t, \mu) \\ +\infty & \text{else} \end{cases}$$

where $\epsilon(\sigma, m) := \sup_{\phi \in B^1} < \phi_t, \mu > - < \phi_0, \sigma_0 > - \int_0^t \left(< \nabla_G \phi, m + \frac{\hbar}{2}\mathrm{d}(\sigma) > + < \dot{\phi}, \sigma > \right)$.

It will also be useful to remark that $\begin{cases} \epsilon(\sigma, m) = 0 & \text{if } (\sigma, m) \in \bar{\mathcal{C}}(t, \mu) \\ \epsilon(\sigma, m) \geq 0 & \text{else} \end{cases}$ and that ϵ is convex and weakly lower semicontinuous as a supremum of such functions.

Proposition. *For any $l > 0$, $e \geq 1$ and $A_* \in \{A, A^e\}$*

$$\inf_{(\sigma,m)\in A_*} \sup_{\phi \in B^l} K(\sigma, m, \phi) = \sup_{\phi \in B^l} \inf_{(\sigma,m)\in A_*} K(\sigma, m, \phi) \tag{5.4}$$

Proof. B^l is a convex linear topological space and compact for the weak topology, A_* is a convex linear topological set. The map $(\sigma, m) \mapsto K(\sigma, m, \phi)$ is lower semicontinuous and the map $\phi \mapsto K(\sigma, m, \phi)$ is upper semi continuous on B^l. The set $\{(\sigma, m) | K(\sigma, m, \phi) \leq \lambda\}$ is convex by convexity of $(\sigma, m) \mapsto K(\sigma, m, \phi)$. If $\mathcal{A}(\sigma, m) = +\infty$, then $\{\phi | K(\sigma, m, \phi) \geq \lambda\} = B^l$ which is a convex set. Else, $K(\sigma, m, \cdot)$ is linear and $\{\phi | K(\sigma, m, \phi) \geq \lambda\}$ is convex. □

Lemma 5.1
For any $l > 0$, there exists $(\sigma^{*,l}, m^{*,l})$ wich minimizes $\mathcal{A} + l\epsilon$ over A^e.

Proof. Admitted, the proof can be found in [4]. □

Proposition. *For any $e \geq 1$ and $A_* \in \{A, A^e\}$,*

$$\inf_{(\sigma,m)\in A_*} \sup_{\phi\in B} K(\sigma, m, \phi) = \sup_{\phi\in B} \inf_{(\sigma,m)\in A_*} K(\sigma, m, \phi) \tag{5.5}$$

Proof. We denote (σ^*, m^*) the minimizer as in the second item of the theorem (3.1). We remark that $\epsilon(\sigma^*, m^*) = 0$ by definition, therefore using the previous proposition, we get

$$\mathcal{A}(\sigma^*, m^*) + \mathcal{U}_0(\sigma_0^*) = \sup_{\phi\in B^l} K(\sigma^*, m^*, \phi) \geq \inf_{(\sigma,m)\in A_*} \sup_{\phi\in B^l} K(\sigma, m, \phi) = \sup_{\phi\in B^l} \inf_{(\sigma,m)\in A_*} K(\sigma, m, \phi)$$

For any $l > 0$ we have the following inequalities:

$$\mathcal{A}(\sigma^*, m^*) \geq \mathcal{A}(\sigma^{*,l}, m^{*,l}) + l\epsilon(\sigma^{*,l}, m^{*,l}) \geq \mathcal{A}(\sigma^{*,l}, m^{*,l})$$

Denote $(\sigma^\infty, m^\infty)$ a point of accumulation of $(\sigma^{*,l}, m^{*,l})$. By the minimal property of (σ^*, m^*), the last inequality and the lower semi-continuity of $\mathcal{A}$ and ϵ,

$$\epsilon(\sigma^\infty, m^\infty), \liminf_{l\to+\infty} l\epsilon(\sigma^{*,l}, m^{*,l}) = 0 \text{ and } \mathcal{A}(\sigma^*, m^*) = \mathcal{A}(\sigma^\infty, m^\infty)$$

Finally, by definition of $(\sigma^{*,l}, m^{*,l})$, we get that

$$\mathcal{A}(\sigma^\infty, m^\infty) \leq \liminf_{l} \sup_{B^l} \inf_{(\sigma,m)\in A^e} K(\sigma, m, \phi) \leq \sup_{B} \inf_{(\sigma,m)\in A^e} K(\sigma, m, \phi)$$

But $K(\sigma^\infty, m^\infty, \cdot) = \mathcal{A}(\sigma^\infty, m^\infty)$ so $\mathcal{A}(\sigma^\infty, m^\infty) = \sup_B K(\sigma^\infty, m^\infty, \phi) \geq \inf_{(\sigma,m)\in A_*} \sup_B K(\sigma, m, \phi)$ To summarize what we did,

$$\inf_{(\sigma,m)\in A^e} \sup_{\phi} K(\sigma, m, \phi) \leq \sup_{\phi} \inf_{(\sigma,m)\in A_*} K(\sigma, m, \phi).$$

The reverse inequality is always true and the inequality finds out to be an egality. □

Lemma 5.2

1. *For any $e \geq 1, \inf_{(\sigma,m)\in A^e} K(\sigma, m, \phi) =< \phi_t, \mu > -\mathcal{U}_0^*(\phi_0) - e\int_0^t H_+(\dot{\phi}, \nabla_G\phi)$*
2. $\inf_{(\sigma,m)\in A^\infty} K(\sigma, m, \phi) = \begin{cases} < \phi_t, \mu > -\mathcal{U}_0^*(\phi_0) & \text{if } H_+(\dot{\phi}, \nabla_G\phi) = 0 \text{ a.e. in } (0,t) \\ +\infty & \text{else} \end{cases}$

   where $H(a, b) := \sup_{\sigma\in\mathcal{P}(G)} \left\{\mathcal{H}(\sigma, b) - \mathcal{G}(\sigma) + \frac{\hbar}{2} < b, \mathrm{d}(\sigma) > + < a, \sigma >\right\}$ and $H_+(a, b) := \max(0, H(a, b))$.

Proof. 1. Let $e \geq 1$, $\inf_{(\sigma,m)\in A^e} K(\sigma, m, \phi) =$

$$\inf_{\sigma\in L^\kappa([0,e]^n)} \mathcal{U}_0(\sigma_0) + < \phi_t, \mu > - < \phi_0, \sigma_0 > - \sup_{m\in L^\kappa} \left\{\int_0^t < \dot{\phi}, \sigma > -\bar{F}(\sigma, m) + < \nabla_G\phi, m + \frac{\hbar}{2}\mathrm{d}(\sigma) >\right\} =$$

$$\inf_{\sigma\in L^\kappa([0,e]^n)} \mathcal{U}_0(\sigma_0) + < \phi_t, \mu > - < \phi_0, \sigma_0 > - \int_0^t \mathcal{H}(\sigma, \nabla_G\phi) - \mathcal{G}(\sigma) + < \nabla_G\phi, \frac{\hbar}{2}\mathrm{d}(\sigma) >$$

Since $\mathcal{H}$ and $\mathcal{G}$ are 1-homogeneous with respect to σ, we have

$$\inf_{(\sigma,m)\in A^e} K(\sigma, m, \phi) =< \phi_t, \mu > -\mathcal{U}_0^*(\phi_0) - e\int_0^t H(\dot{\phi}, \nabla_G\phi)$$

2. We let e goes to $+\infty$ to verify that second part. $\square$

Lemma 5.3
Given $\phi \in W^{1,\kappa'}(0,t;\mathbb{R}^n)$, there is $\bar{\phi} \in W^{1,\kappa'}(0,t;\mathbb{R}^n)$ such that $H_+(\dot{\bar{\phi}}, \nabla_G \bar{\phi}) = 0$ and

$$\inf_{(\sigma,m)\in A^1} K(\sigma,m,\phi) = \inf_{(\sigma,m)\in A^1} K(\sigma,m,\bar{\phi})$$

Proof. Let $O := \{H(\dot{\phi}, \nabla_G \phi < 0\}$ measurable and to avoid trivalities, assume $\mathcal{L}^1(O) > 0$. Set

$$\alpha = -\int_0^t \chi_O H(\dot{\phi}(s), \nabla_G \phi(s))ds$$

The function $\sigma \mapsto \mathcal{H}(\sigma, \nabla_G \phi) - \mathcal{G}(\sigma)$ is bounded on $\mathcal{P}(G)$, then

$$|\dot{\alpha}| \leq \sup_{\sigma\in\mathcal{P}(G)} \left| \mathcal{H}(\sigma,b) - \mathcal{G}(\sigma) + \frac{\hbar}{2} < \nabla_G\phi, \mathrm{d}(\sigma) > + < \dot{\phi}, \sigma > \right| \leq |\dot{\phi}| + \hbar|\nabla_G \phi| + C \in L^{\kappa'}(0,t;\mathbb{R})$$

Let $\bar{\phi} := (\phi_i + \alpha)_i$. We have $\nabla_G \bar{\phi} = \nabla_G \phi$ and for any $\sigma \in \mathcal{P}(G)$, $(\bar{\phi}, \sigma) =< \phi, \sigma > + \dot{\alpha}$. Then, $H_+(\dot{\bar{\phi}}, \nabla_G \bar{\phi}) = 0$.

By the lemma 5.2,

$$\inf_{(\sigma,m)\in A^1} K(\sigma,m,\bar{\phi}) =< \phi_t, \mu > -\mathcal{U}_0^*(\phi_0) - \int_0^t \chi_O H(\phi(\dot{s}), \nabla_G \phi(s)) = \inf_{(\sigma,m)\in A^1} K(\sigma,m,\phi)$$ $\square$

Lemma 5.4
Given $\phi \in W^{1,\kappa'}(0,t;\mathbb{R}^n)$ such that $H(\dot{\phi}, \nabla_G \phi) \leq 0$, there is $\bar{\phi} \in W^{1,\kappa'}(0,t;\mathbb{R}^n)$ such that $H(\dot{\bar{\phi}}, \nabla_G \bar{\phi}) = 0$ and

$$(\bar{\phi}_t, \mu) - \mathcal{U}_0^*(\bar{\phi}_0) \geq < \phi_t, \mu > -\mathcal{U}_0^*(\phi_0)$$

Proof. Admitted here. The approach is very similar to the previous lemma $\square$

Proposition.

$$\inf_{(\sigma,m)\in A^\infty} \sup_{\phi\in B} K(\sigma,m,\phi) = \sup_{\phi\in B} \inf_{(\sigma,m)\in A^\infty} K(\sigma,m,\phi)$$

Proof. Admitted here. The proof is once again very similar to [4]. $\square$

We can now state the main theorem of this part:

Theorem 5.5
Let $(t,\mu) \in \mathbb{R} \times \mathcal{P}_0(G)$, we have the dual formulation

$$\mathcal{U}(t,\mu) = \sup_{\phi\in B}\{< \phi_t, \mu > -\mathcal{U}_0^*(\phi_0) | H(\dot{\phi}, \nabla_G \phi\}) = 0$$

Proof. Let's define

$$I_{\mathcal{C}(\sigma_0,\sigma_1)}(\sigma,m) := \begin{cases} 0 & \text{if } (\sigma,m) \in \bar{\mathcal{C}}(l,\mu) \\ +\infty & \text{else} \end{cases}$$

For any $(\sigma,m) \in A^\infty$, we have

$$\sup_{\phi\in B} K(\sigma,m,\phi) = \mathcal{A}(\sigma,m) + I_{\mathcal{C}(\sigma_0,\sigma_1)}(\sigma,m)$$

so that

$$\inf_{(\sigma,m)\in A^\infty} \sup_{\phi\in B} K(\sigma,m,\phi) = \inf_{(\sigma,m)\in A^\infty} \left\{\mathcal{A}(\sigma,m) + I_{\mathcal{C}(\sigma_0,\sigma_1)}(\sigma,m)\right\}$$

Using $\bar{\mathcal{C}}(t,\mu) \subset A^\infty$ and the last proposition, we get

$$\min_{(\sigma,m)\in\bar{\mathcal{C}}(t,\mu)} \mathcal{A}(\sigma,m) = \inf_{(\sigma,m)\in A^\infty} \sup_{\phi\in B} K(\sigma,m,\phi) = \sup_{\phi\in B} \inf_{(\sigma,m)\in A^\infty} K(\sigma,m,\phi)$$

We conclude using the lemmas 5.4 and 5.2. □

Outlook: With the dual writing of $\mathcal{U}$, we hope to can extend the Euler-Lagrange equation to the boundary of $\mathcal{P}(G)$. If $\bar{g} \neq 1$, the term $\mathrm{d}(\sigma)$ causes the impossibility of using the usual inversion theorems. Nevertheless, the duality could still hold. In all cases, those results are useful for studying the regularity of (σ^*, m^*). Under certain assumptions, we show in that way that they are C^∞.

6 Appendix

In this appendix, we'll take a closer look at the κ-Monge-Kantorovitch metric used in this manuscript. In particular, we'll show that the κ-Monge-Kantorovitch is finer than the canonical norm.

Let us begin by demonstrating the property for the particular case where G has a cardinality of 2.

Lemma 6.1
Let G defined by $V = \{1,2\}$ and $E = \{(1,2),(2,1)\}$. Let $\rho^0, \rho^1 \in \mathcal{P}(G)$, there exists $C, C' > 0$, such that $\mathcal{W}_\kappa^\kappa(\rho^0,\rho^1) \leq C|\rho^0 - \rho^1|^{C'}$.

Proof. Let $(\rho^0,\rho^1) \in \mathcal{P}(G)^2$ such that $\rho_1^0 < \rho_1^1$. Define the function $G : \tau \mapsto \int_0^\tau \frac{\mathrm{d}r}{\sqrt{g(r,1-r)}}$. G is strictly increasing and continuously differentiable, therefore it has an inverse function.

Let $\rho : [0,1] \to \mathcal{P}(G)$ by $\rho_1(t) = G^{-1}(G(\rho_1^0) + (G(\rho_1^1) - G(\rho_1^0))t)$ and $\rho_2(t) = 1 - \rho_1(t)$. Let us also define $m_{12} = -m_{21}$ by $\sqrt{w_{12}}m_{21} = \dot{\rho}_1(t)$ such that the equality $\dot{\rho} = \nabla_G.m$ is satisfied. We have then $\dot{\rho}_1 = (G(\rho_1^1) - G(\rho_1^0))\sqrt{g_{12}(\rho)}$. Thus,

$$\begin{aligned}\mathcal{W}_\kappa^\kappa(\rho^0,\rho^1) \leq \int_0^1 g_{12}(\rho)^{1-\kappa} m_{12}^\kappa &= \int_0^1 g_{12}(\rho)^{1-\frac{\kappa}{2}} \left(\frac{G(\rho_1^1) - G(\rho_1^0)}{\sqrt{w_{12}}}\right)^\kappa \\ &= \frac{(G(\rho_1^1) - G(\rho_1^0))^{\kappa-1}}{w_{12}^{\kappa/2}} \int_{\rho_1^0}^{\rho_1^1} g(r,1-r)^{\frac{1-\kappa}{2}} \mathrm{d}r\end{aligned}$$ □

We conclude by the hypothesis on the function g which make the right integral bounded and the function G lipschitz.

Let's look now to the general case. The first step in our approach is to reduce the problem to the previous case. It is relatively straightforward when ρ^0 and ρ^1 differ in only two vertices. This first lemma contributes to this direction:

Lemma 6.2
Let $(\mu^l)_{l\in\mathbb{N}} \in \mathcal{P}(G)^\mathbb{N}$ and $(\rho^l)_{l\in\mathbb{N}} \in \mathcal{P}(G)^\mathbb{N}$ such that $|\rho^l - \mu^l|_{l_\infty} \to 0$. We assume that for all $l \in \mathbb{N}$, there exists i_1, i_2 such that $w_{i_1,i_2} > 0$ and for all $i \in [|1,\ldots,n|]$, we have

$$(i \neq i_1 \text{ or } i \neq i_2) \Rightarrow \rho_i^l = \mu_i^l$$

Then, we have

$$\mathcal{W}_\kappa^\kappa(\mu^l,\rho^l) \to 0$$

Proof. Let $l \in \mathbb{N}$, i_0, i_1 as in the hypothesis of the lemma and denote G_2 a graph of size 2 defined by $V = \{i_1, i_2\}$. Let $s^l = \rho^l_{i_1} + \rho^l_{i_2}$. ($s$ is independent of l). If $s^l = 0$, that means that $\rho^l = \mu^l$ and we are done. Else, let $\tilde{\mu}^l = (\frac{\mu^l_{i_1}}{s^l}, \frac{\mu_{i_2})}{s^l}$ and $\tilde{\rho}^l = (\frac{\rho^l_{i_1}}{s^l}, \frac{\rho^l_{i_2}}{s^l})$ and $(\tilde{\sigma}^l, \tilde{v}^l)$ a path connecting $\tilde{\mu}$ and $\tilde{\rho}^l$. We aim to construct a path connecting μ^l and ρ^l by considering σ^l defined by

$$\sigma^l_i = \begin{cases} \tilde{\sigma}^l_i & \text{if } i \in \{i_1, i_2\} \\ \mu^l_i = \rho^l_i & \text{else} \end{cases} \text{ and } v^l_{ij}(t) = \begin{cases} s^l \tilde{v}^l_{ij}(t) & \text{if } (i,j) \in \{(i_1, i_2), (i_2, i_1)\} \\ 0 & \text{else} \end{cases}$$

We have $\dot{\rho}^l + \text{div}^l_\rho(v) = 0$. Consequently,

$$\mathcal{W}^\kappa_\kappa(\mu^l, \rho^l) \le \int_0^1 \left(\sum_{(i,j)\in E} g(\sigma^l)|v^l_{ij}|^\kappa \right) = (s^l)^\kappa \int_0^1 \left(\sum_{(i,j)\in E} g(\tilde{\sigma}^l)|\tilde{v}^l_{ij}|^\kappa \right) \le \mathcal{W}^\kappa_\kappa(\tilde{\mu}^l, \tilde{\rho}^l)$$

□

Now, we can apply the particular case where $|G = 2|$ to get the result.

We would like to use this lemma wisely by proceeding as follows: we consider given ρ^l and ρ^∞, the idea is to construct a sequence connecting ρ^l and ρ^∞, where each term differs from its neighbor in only two vertices. Because of the condition $w_{i_1,i_2} > 0$ in the hypothesis of the lemma, this is not immediate.

Lemma 6.3
Let $\rho^\infty \in \mathcal{P}(G), \rho^l \in \mathcal{P}(G)^\mathbb{N}$ with $\rho^l \neq \rho^\infty$, such that $|\rho^l - \rho^\infty|_{l_\infty} \to 0$ then there exists $\bar{\rho}^l \in \mathcal{P}(G)$ such that

- $B(\bar{\rho}^l, \rho^\infty) := \{i \in V | \bar{\rho}^l_i \neq \rho^\infty_i\} \subsetneq \{i \in V | \rho^l_i \neq \rho^\infty_i\} := B(\rho^l, \rho^\infty)$ *for all* $l \in \mathbb{N}$
- $\mathcal{W}_\kappa(\bar{\rho}^l, \rho^l) \to 0$

Proof. Let $l \in \mathbb{N}$ and pick $(i_0, i_1) \in V^2$ such that $i_0 \neq i_1$, $\rho^l_{i_0} > \rho^\infty_{i_0}$ and $\rho^l_{i_1} < \rho^\infty_{i_1}$. Define $\bar{\rho}^l$ by

$$\bar{\rho}^l_i = \begin{cases} \rho^l_i & \text{if } i \in V \backslash \{i_0, i_1\} \\ \rho^\infty_{i_0} & \text{if } i = i_0 \\ \rho^l_{i_1} + \rho^l_{i_0} - \rho^\infty_{i_0} & \text{if } i = i_1 \end{cases}$$

We verify first that $\bar{\rho}^l \in \mathcal{P}(G)$ and that $B(\bar{\rho}^l, \rho^\infty) \subset B(\rho^l, \rho^\infty) \backslash \{i_0\}$. Since the graph is connected, there exists $j_0, \dots, j_{n_0} \in V$ such that $w_{j_i j_{i+1}} > 0$ for all $i \in [|0, n_0 - 1|]$, $j_0 = i_0, j_{n_0} = i_1$.

Define $\bar{\rho}^{l,0}, \dots, \bar{\rho}^{l,n_0} \in \mathcal{P}(G)$ by

$$\bar{\rho}^{l,0} = \rho^l \text{ and } \bar{\rho}^{l,i}_j = \begin{cases} \rho^l_{j_i} + \rho^l_{i_0} - \rho^\infty_{i_0} & \text{if } j = j_i \\ \rho^l_j & \text{if } j \in V \backslash \{i_0, j_i\} \\ \rho^\infty_{i_0} & \text{if } j = i_0 \end{cases}$$

Remark that $\bar{\rho}^{l,n_0} = \bar{\rho}^l$ and that let's apply the lemma (6.3) to get that

$$\mathcal{W}_\kappa(\bar{\rho}^{l,i}, \bar{\rho}^{l,i+1}) \to 0$$

□

We can conclude the proof of the lemma since $0 \le \mathcal{W}_\kappa(\bar{\rho}^l, \rho^l) \le \sum_0^{n_0-1} \mathcal{W}_\kappa(\bar{\rho}^{l,i}, \bar{\rho}^{l,i+1})$.

Theorem 6.4
Let $\rho^\infty \in \mathcal{P}(G), \rho^l \in \mathcal{P}(G)^\mathbb{N}$ with $\rho^l \neq \rho^\infty$ such that $|\rho^l - \rho^\infty|_{l_\infty} \to 0$. Then, $\mathcal{W}_\kappa(\rho^l, \rho^\infty) \to 0$.

Proof. Let's apply the lemma (6.3) recursively to construct a finite number of sequences $(\bar{\rho}^{l,i})_{i\in\{0,\dots,n_0\}}$ with $n_0 \le |G|$ such that

- $\bar{\rho}^{l,0} = \rho^l$
- $\bar{\rho}^{l,i+1}$ is constructed by taking $\rho^l = \bar{\rho}^{l,i}$ in the lemma
- $\bar{\rho}^{l,n_0} = \rho^\infty$

We have then

$$\forall i \in \{0, \dots, n_0 - 1\}, \mathcal{W}_\kappa(\bar{\rho}^{l,i}, \bar{\rho}^{l,i+1}) \to 0$$

Then, $\mathcal{W}_\kappa(\rho^l, \rho^\infty) \leq \sum_{i=0}^{n_0-1} \mathcal{W}_\kappa^\kappa(\bar{\rho}^{l,i}, \bar{\rho}^{l,i+1}) \to_{l\to+\infty} 0$. We are done. □

References

[1] E. Bayraktar, A. Cecchin, A. Cohen and F. Delarue, Finite state mean field games with Wright–Fisher common noise, J. Math. Pures Appl. 147 (2021), 98–162.

[2] Caines, P. E., Huang, M., and Malhamé, R. P., Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, Commun. Inf. Syst. 6 (2006), no. 3, 221–251

[3] P. Cardialaguet, Notes on mean-field games, (from P.-L. Lions' lectures at Collège de France), 2013, https://www.ceremade.dauphine.fr/cardalia/MFG20130420.pdf.

[4] W. Gangbo, W. Li and C. Mou, Geodesics of minimal length in the set of probability measures on graphs, ESAIM Control Optim. Calc. Var. 25 (2019), Paper No. 78, 36 pp.

[5] W. Gangbo, S. Mayorga and A. Święch, Finite dimensional approximations of Hamilton-Jacobi-Bellman equations in spaces of probability measures, SIAM J. Math. Anal. 53 (2021), no. 2, 1320–1356.

[6] W. Gangbo, T. Nguyen and A. Tudorascu, Hamilton-Jacobi equations in the Wasserstein space, Methods Appl. Anal. 15 (2008), no. 2, 155–183. Control Optim. Calc. Var. 24 (2018), no. 1, 437–461.

[7] Gomes D.A., Mohr J. and Souza R.R. Discrete time, finite state space mean field games. Journal de Mathématiques Pures et Appliquées, 93(3) (2010), 308-328.

[8] O. Hijab, Infinite-dimensional Hamilton-Jacobi equations with large zeroth-order coefficient, J. Funct. Anal. 97 (1991), no. 2, 311–326.

[9] J.-M Lasry, P.-L Lions, Mean field games. Jpn. J. Math. 2, 229–260 (2007). https://doi.org/10.1007/s11537-007-0657-8

[10] J.-M. Lasry and P.-L. Lions, Jeux à champ moyen. I. Le cas stationnaire, C. R. Acad. Sci.Paris, 343 (2006), 619–625

[11] J.-M. Lasry and P.-L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal, C.R. Acad. Sci. Paris, 343 (2006), 679–684

[12] H. Pham and X. Wei, Bellman equation and viscosity solutions for mean-field stochastic control problem, ESAIM