ASYMPTOTIC SIZE OF THE MAXIMAL CLUSTER FOR POISSON POINT PERCOLATION IN A HYPERBOLIC DISK OF GROWING RADIUS

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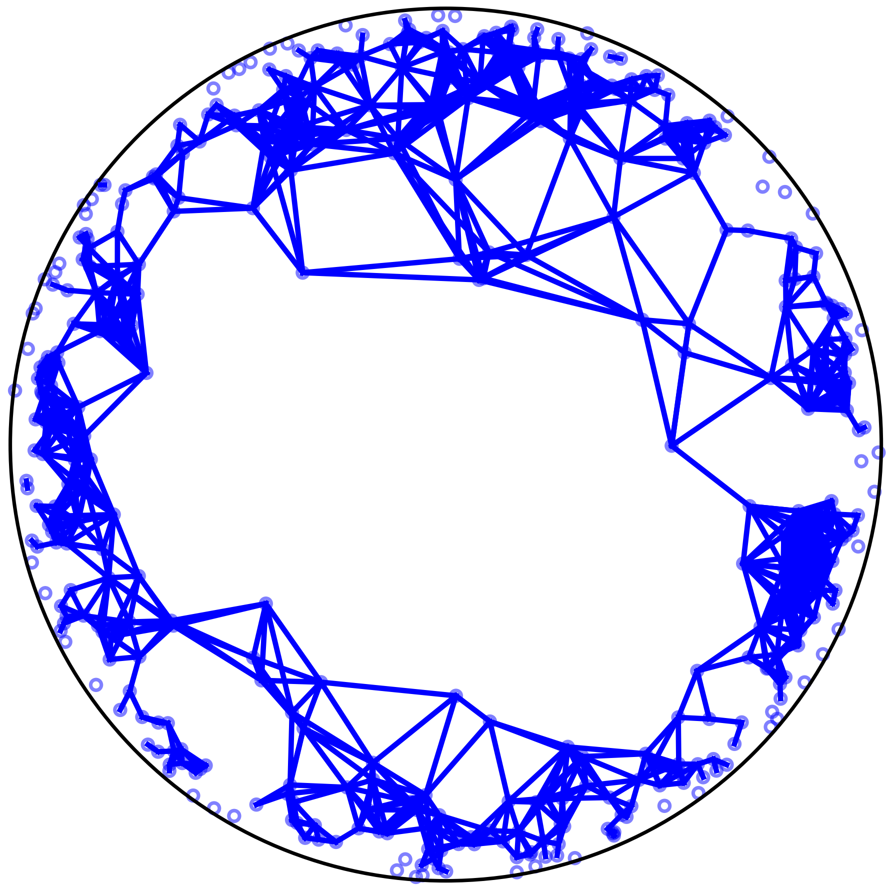

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1. Introduction

Percolation theory is a mathematical field created in 1957 by Broadbent and Hammersley in order to answer the question "Will the center of a porous rock be wet if it is immersed in water ?"

The first model was the following : Consider the grid $G = (V, E)$ where $V = \mathbb{Z}^d$ and E is the set of all the edges connecting two adjacent vertices (i.e. $n = (k_1, \dots, k_d)$ and $m = (l_1, \dots, l_d)$ are connected by an edge if and only if $\sum |k_i - l_i| = 1$). Chose a value of p within $[0, 1]$. Now consider the random subgraph $S = (\mathbb{Z}^d, \widetilde{E})$ of G, where any $e \in E$ is in $\widetilde{E}$ with probability p independently of any other edge. Percolation theory studies the resulting random graph S. A connected component in the graph S is called a cluster. When we chose to keep or not edges, we call the model bond percolation. Another way to generate a subgraph S would be to instead chose to randomly remove vertices, and all edges connected to them. This is called site percolation.

The following theorem is one of the main results on this theory and is due to Burton and Keane, more can be found in [3] :

Theorem 1.1. *There exists a critical probability $p_c = p_c(d)$ such that for any $p < p_c$, there is no infinite open cluster in S a.s., and for any $p > p_c$, there exists exactly one infinite cluster a.s.*

This essentially states that in $\mathbb{Z}^d$, there exist exactly two phases for percolation : if we take few edges, there is almost surely no infinite open cluster as there are not enough connections. But if we take more edges, then suddenly there is a unique infinite open cluster. What happens exactly at the transition p_c is one of the most famous questions on percolation, and any result on this could help understand better the Ising model, which is a very closely related model used in physics.

The random graph S can of course be defined differently. In his articles ([4] and [5]), Lalley considered G to be a lattice within the hyperbolic plane. In this case, and for site percolation, he proved the existence of an additional phase in between the two previous ones where there is coexistence of infinitely many infinite open clusters and infinitely many closed clusters (we decide to label each vertex either "closed" or "open", which result in 2 complementary sets and thus 2 subgraphs where percolation can happen).

Our research during this internship was based on later work from Tykesson, who adapted the result from Lalley's articles to hyperbolic Poisson point percolation, which is a continuous model that we define in the next section. Our results expand on his by considering the restriction of the graph to a disk of growing radius t, and obtaining an asymtotic result on the size of the maximal cluster in this restrained graph.

My internship was under the supervision of Mr. Tobias Muller, professor at the University of Groningen. We had about one meeting every week to talk about my progress and for him to give me guidelines as to what he expected. He gave me recommendations about the articles I should read, and then ideas to prove our results.

In more details, I first had to read a lot of literature on the subject and parallel results (such as Benjamini and Schramm's articles on non-amenable graphs), to get a better grasp on what we were dealing with. Quite a lot of the results were adapting things from existing articles and previous research notes from my supervisor and his colleagues from 8 years ago. They could not finish working on it at the time, so I had to complete it and get all the details right. This was mostly for the cohabitation phase where clusters of the two types exist. After that, we were more in the dark and had to come up with new arguments (using usual methods) and adapt what worked in regular percolation to our model. We had hope that we would be able to publish our results, hence my supervisor asked me to write everything down on a latex document as we made progress. As we proved everything that we wanted, I still had about two weeks to continue working on the article, and although it looks almost finished, some more work has to be done on it. I volunteered to continue working a bit during my holidays, and hopefully my supervisor will be able to finish it and publish it afterwards.

2. Definition of the model

Definition 2.1. *(Poisson point process) A Poisson point process of constant intensity λ is a random set of points ξ within a space E (with an area function area), such that :*

- *for any set $A \subset E$, with area$(A) = c$, the number of points $|\xi \cap A|$ is a random Poisson variable of parameter $\lambda \cdot c$.*
- *for any two subsets $A, B \subset E$, the numbers of points within A and within B are independent from each other.*

Remark. With these properties, note that the union of two Poisson point processes of respective intensity μ and λ is also a Poisson point process with intensity $\mu + \lambda$.

In this paper, we consider a homogeneous Poisson point process $\xi(\lambda) = \xi$ in the hyperbolic plane $\mathbb{H}^2$, with intensity λ. Around each point of ξ, we consider a hyperbolic ball of radius 1, noted $B(x, 1)$, and we denote by $\mathcal{C}$ the set covered by these balls, i.e.

$$\mathcal{C} := \bigcup_{x \in \xi} B(x, 1),$$

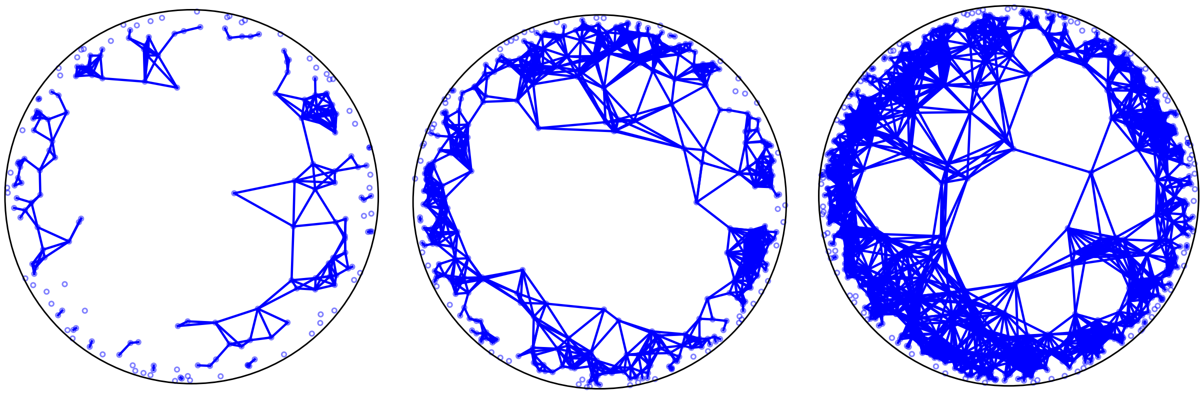
and by $\mathcal{V} := \mathbb{H}^2 \setminus \mathcal{C}$ the vacant set. For two point $x \neq y$ in $\mathbb{H}^2$, we let

$$\left\{x \overset{\mathcal{A}}{\leftrightsquigarrow} y\right\} := \{x \,\text{is connected to}\, y \,\text{by a path laying entirely in the set}\, \mathcal{A}\}$$

and we will refer to the set $\mathcal{C}(x) := \{y \in \mathbb{H}^2, x \overset{\mathcal{C}}{\leftrightsquigarrow} y\}$ as the covered cluster of x. If $x \notin \mathcal{C}$, we then have $\mathcal{C}(x) = \emptyset$.

In this paper, we want to study the restriction of this percolation process to balls of finite radius t around the origin. Towards this end, for any set $\mathcal{A} \subset \mathbb{H}^2$ we define

$$\mathcal{C}_{\mathcal{A}} := \bigcup_{y \in \xi \cap \mathcal{A}} B(y, 1) \qquad \text{and} \qquad \mathcal{C}_{\mathcal{A}}(x) := \{y \in \mathbb{H}^2, x \overset{\mathcal{C}_{\mathcal{A}}}{\leftrightsquigarrow} y\},$$



3 Python simulations of our model in the Poincaré Disk with different intensities. Due to Python limitations, the edge of the disk is not representative of what really happens.

and for $\mathcal{A} = B(o,t)$, we will abbreviate $\mathcal{C}_{B(o,t)}$ to $\mathcal{C}_t$. The objective is to study the size of the maximal cluster of the percolation induced by ξ within $B(o,t)$ for all t. Thus, for any finite cluster $\mathcal{C}_t(x)$ we define its size by $N(\mathcal{C}_t(x)) := |\{\xi \cap \mathcal{C}_t(x)\}|$, and the maximum size of a cluster within $B(o,t)$: $N_t := \max_{x\in B(o,t)} |N(\mathcal{C}_t(x))|$.

With these notations, we say that there is percolation whenever there exists an infinite (and unbounded) cluster in $\mathcal{C}$. It is proven by Tykesson in [7] that there exists 2 critical intensities $0 < \lambda_c < \lambda_u$ such that :

- for $\lambda \in [0, \lambda_c]$, there exists a unique infinite cluster within the set $\mathcal{V}$, and none in the set $\mathcal{C}$.
- for $\lambda \in]\lambda_c, \lambda_u[$, there exists infinitely many infinite clusters both within the sets $\mathcal{V}$ and $\mathcal{C}$.
- for $\lambda \in]\lambda_u, +\infty[$, there exists a unique infinite cluster within the set $\mathcal{C}$, and none in the set $\mathcal{V}$.

With that in mind, we define the connectivity function τ_λ as :

$$\tau_\lambda(x,y) := \mathbb{P}_\lambda(\mathcal{C}(x) = \mathcal{C}(y)) = \mathbb{P}_\lambda(x \overset{\mathcal{C}}{\longleftrightarrow} y), \qquad \text{for } x, y \in \mathbb{H}^2.$$

The model being invariant under isometry, for $x, y, z, w \in \mathbb{H}^2$ such that $\text{dist}_{H^2}(x,y) = \text{dist}_{H^2}(z,w)$, we have $\tau_\lambda(x,y) = \tau_\lambda(z,w)$. So $\tau_\lambda(x,y)$ is only dependent on $t = \text{dist}_{H^2}(x,y)$, and we will write $\tau_\lambda(t) = \tau_\lambda(x) = \tau_\lambda(o,x)$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{H}^2$ with $\text{dist}_{H^2}(o,x) = t$.

As we consider restrictions of the configuration to $B(o,t)$, we may also define the function $\tau_{\lambda,t}$:

$$\tau_{\lambda,t}(x) := \mathbb{P}_\lambda(\mathcal{C}_t(o) = \mathcal{C}_t(x)), \qquad \text{for } x \in \mathbb{H}^2, \text{dist}_{H^2}(o,x) \leqslant t.$$

Finally, we let $\beta_t(\lambda) := \frac{\log \tau_\lambda(t)}{t}$.

Theorem 2.2. *We have that*

(1) *the limit* $\beta(\lambda) := \lim_{t\longrightarrow\infty} \beta_t(\lambda)$ *exists,*

(2) β *is continuous,*

(3) β *in strictly increasing for* $0 < \lambda < \lambda_u$*,*

(4) $\beta(\lambda_c) = -1$*,*

(5) *and* $\beta(\lambda) = 0$ *for* $\lambda \geqslant \lambda_u$*.*

Theorem 2.3. *For any $\varepsilon > 0$, and for any $\lambda \in]\lambda_c, \lambda_u[$, there exists a.s. a $T = T(\xi) \in \mathbb{R}^+$ such that :*

$$N_t \geqslant e^{(\beta(\lambda)+1-\varepsilon)t} \qquad \text{and} \qquad N_t \leqslant e^{(\beta(\lambda)+1+\varepsilon)t}$$

for every $t \geqslant T$. In consequence, we have

$$\lim_{t \longrightarrow +\infty} \frac{\log N_t}{t} = \beta(\lambda) + 1 \ \text{a.s.}$$

Theorem 2.4. *For any $\lambda < \lambda_c$, there exists a.s. a $T = T(\xi) \in \mathbb{R}^+$ and two constants $0 < c = c(\lambda) < C(\lambda) = C$ such that :*

$$\forall t \geqslant T, \qquad ct \leqslant N_t \leqslant Ct.$$

Theorem 2.5. *For any $\lambda > \lambda_u$, there exists a.s. a $T = T(\xi) \in \mathbb{R}^+$ and two constants $0 < c = c(\lambda) < C(\lambda) = C$ such that :*

$$\forall t \geqslant T, \qquad ce^t \leqslant N_t \leqslant Ce^t.$$

In all this article, for any two points x and y in $\mathbb{H}^2$, we will denote by (x, y) the only geodesic going through both x and y, and $[x, y]$ the part of it that is between the two points, i.e. the geodesic segment connecting x and y. The notation $[x, y)$ then corresponds to the half-geodesic coming from x towards y.

3. Useful tools

This section contains multiple classic definitions and result used in percolation theory. They have been adapted to suit our model, but no proof will be given here.

Definition 3.1. *(Increasing events) If ξ and ξ' are two realisations or our Poisson point process, we write $\xi \subset \xi'$ if any point of ξ is also in ξ'. Most likely, we will see this occur when we consider a realisation ξ of intensity λ and add points to it to have another Poisson point process of intensity $\lambda' > \lambda$.*

An event A is said to be increasing (resp. decreasing) if $\xi \subset \xi'$ implies that $\mathbb{1}_A(\xi) \leqslant \mathbb{1}_A(\xi')$ (resp. $\mathbb{1}_A(\xi) \geqslant \mathbb{1}_A(\xi')$).

For any sets X and Y, the event $X \overset{\mathcal{C}}{\leftrightsquigarrow} Y$ is increasing, and the event $X \overset{\mathcal{V}}{\leftrightsquigarrow} Y$ is decreasing.

Theorem 3.2. *(FKG inequality) If A and B are both increasing or both decreasing events, then*

$$\mathbb{P}_\lambda(A \cap B) \leqslant \mathbb{P}_\lambda(A)\mathbb{P}_\lambda(B).$$

Definition 3.3. *(Disjoint occurrence) Consider two events A and B. We define the event $A \Box B$ called their disjoint occurrence as follows : A configuration ξ is in $A \Box B$ if and only if there exist two disjoint bounded measurable sets $I, J \in \mathbb{H}^2$ so that any configuration ξ' so that $\xi \cap I = \xi' \cap I$ satisfies A, and any configuration ξ'' so that $\xi \cap J = \xi'' \cap J$ satisfies B.*

Theorem 3.4. *(BK inequality) If A and B are both increasing or both decreasing events, then*

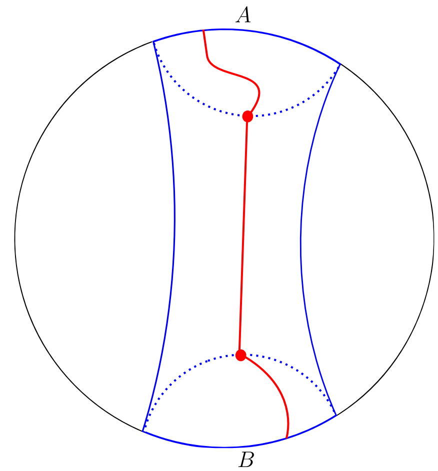
$$\mathbb{P}_\lambda(A \Box B) \leqslant \mathbb{P}_\lambda(A)\mathbb{P}_\lambda(B).$$

Proof of these two results can be found in [6] for the case in $\mathbb{R}^n$. The proof of the FKG inequality is almost identical in our case. The BK inequality can be deduced from the $\mathbb{R}^n$ case by considering the area of $\mathbb{R}^3$ below some curve describing the equivalent intensity when using the Poincaré disk model of the hyperbolic plane.

The next proposition is directly adapted from Lalley's two articles [5] and [4], where he defines sector percolation and uses the definition to prove results that will be of good use to us :

Definition 3.5. *(Sector Percolation) We say that sector percolation occurs when there exist a point $\zeta \in \partial\mathbb{H}^2$ and a path $\gamma \subset \mathcal{C}$ such that $\gamma(t)$ converges to ζ for the euclidean distance on the unit disk model. We define sector percolation within the set $\mathcal{V}$ the same way.*

Sector percolation occurs whenever there is percolation, in particular it happens for the covered cluster whenever $\lambda > \lambda_c$.



Proposition 3.6. *For any $\lambda > \lambda_c$, there exists a constant $C = C(\lambda)$ increasing with λ such that, for any closed half-plane H of boundary arc $\mathcal{A}$ and any point $x \in H$, the probability that there exists a path connecting x to $\mathcal{A}$ while using only balls whose center fall in H is at least C. In a formula :*

$$\mathbb{P}_\lambda \left(x \overset{\mathcal{C}_{H^+}}{\longleftrightarrow} \mathcal{A} \right) \geqslant C.$$

This proposition allows us to prove the following theorem :

Theorem 3.7. *Fix $\lambda > \lambda_c$. For any two disjoint open arcs A and B on the boundary of $\mathbb{H}^2$, consider the strip S bounded by these two arcs and the two geodesics connecting their respective ends. There is a strictly positive probability $p(\lambda)$ for an infinite cluster to join the arcs A and B using only points of $\xi \cap S$, and $p(\lambda)$ is an increasing function in λ.*

A similar version of this theorem holds for vacant paths when sector percolation holds for the vacant cluster (i.e. below λ_u) :

Theorem 3.8. *Fix $\lambda < \lambda_u$. For any two disjoint open arcs A and B on the boundary of $\mathbb{H}^2$, consider the strip S bounded by these two arcs and the two geodesics connecting their respective ends. There is a strictly positive probability $p^{vacant}(\lambda)$ for a vacant path to join the arcs A and B within the strip S, and $p(\lambda)$ is an decreasing function in λ.*

4. Connectivity function

This section contains preliminary results on the connectivity function leading to Theorem 2.2. These next few results are gonna help us prove parts 1. and 2. of Theorem 2.2.

Proposition 4.1. *Let S be a segment of length $l \geqslant 1/2$. Suppose that for every point $x \in S$, $\tau_\lambda(x) \leqslant K$. Then there exists a constant depending only on λ and such that :*

$$\mathbb{P}_\lambda \left(o \overset{\mathcal{C}}{\longleftrightarrow} S \right) \leqslant const \cdot K \cdot (l+1).$$

Proof: On the segment S, consider points x_i at distance $1/2$ of each other. Then conditionally on S being connected to o, consider a point P on the segment that is connected to o. P is at distance at most $1/2$ from one of the x_i, let's say x_j. Consider then the point P' middle of the segment $[P, x_j]$. For any point $y \in B(P', 1/2)$, the ball $B(P', 1)$ contains both P and x_j, thus x_j is also connected to o if there is at least one point of ξ within $B(P', 1/2)$. That happens with probability

$c = 1 - e^{-\lambda A} > 0$, where A is the area of a hyperbolic ball of radius $1/2$. Thus using the FKG inequality (see [6]) :

$$\mathbb{P}_\lambda\left(\exists i, o \overset{\mathcal{C}}{\leftrightsquigarrow} x_i\right) \geqslant \mathbb{P}_\lambda\left(\left\{o \overset{\mathcal{C}}{\leftrightsquigarrow} S\right\} \cap \left\{\text{there is a point of } \xi \text{ within the ball } B\left(P', \frac{1}{2}\right)\right\}\right)$$

$$\overset{\text{FKG}}{\geqslant} c \cdot \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} S\right).$$

Thus leading to

$$\mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} S\right) \leqslant \frac{1}{c} \cdot \mathbb{P}_\lambda\left(\exists i, o \overset{\mathcal{C}}{\leftrightsquigarrow} x_i\right) \leqslant \frac{1}{c} \cdot \sum_i \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x_i\right) \leqslant \frac{1}{c} \cdot (2l+1)K.$$

■

Remark. The same argument can be applied in the case where S is an arc of a circle of length l, and to the function $\tau_{\lambda,t}(t)$.

Lemma 4.2. *The function $\tau_\lambda(t)$ is supermultiplicative, i.e. for any $t_1, t_2 > 0$, we have that*

$$\tau_\lambda(t_1 + t_2) \geqslant \tau_\lambda(t_1)\tau_\lambda(t_2).$$

A similar result can be obtained for the function $\tau_{\lambda,t}(t)$ as well :

$$\tau_{\lambda,t_1+t_2}(t_1 + t_2) \geqslant \tau_{\lambda,t_1}(t_1)\tau_{\lambda,t_2}(t_2).$$

Proof: Let L be a half-line starting at o. Let x and y be points of L at respective distance t_1 and $t_1 + t_2$ from o. Then $\{o \overset{\mathcal{C}}{\leftrightsquigarrow} y\} \subset \{o \overset{\mathcal{C}}{\leftrightsquigarrow} x\} \cap \{x \overset{\mathcal{C}}{\leftrightsquigarrow} y\}$, and thus using the FKG inequality :

$$\tau_\lambda(t_1 + t_2) \geqslant \mathbb{P}_\lambda\left(\{o \overset{\mathcal{C}}{\leftrightsquigarrow} x\}\right)\mathbb{P}_\lambda\left(\{x \overset{\mathcal{C}}{\leftrightsquigarrow} y\}\right) = \tau_\lambda(o, x)\tau_\lambda(x, y) = \tau_\lambda(t_1)\tau_\lambda(t_2).$$

It follows that

$$-\log \tau_\lambda(t_1 + t_2) \leqslant -\log \tau_\lambda(t_1) - \log \tau_\lambda(t_2),$$

and the result on the function $\tau_{\lambda,t}(t)$ is obtained with similar arguments. ■

Remark. The Fekete lemma for subadditive sequences (see [2]) then proves the existence of the limits $\lim_t \frac{\log \tau_\lambda(t)}{t}$ and $\lim_t \frac{\log \tau_\lambda(t)}{t}$, and states that :

$$\beta(\lambda) = \lim_t \frac{\log \tau_\lambda(t)}{t} = \sup \frac{\log \tau_\lambda(t)}{t} \in]-\infty, 0],$$

and

$$\lim_t \frac{\log \tau_{\lambda,t}(t)}{t} = \sup \frac{\log \tau_{\lambda,t}(t)}{t} \in]-\infty, 0].$$

Lemma 4.3. *There exists a constant $c = c(\lambda) > 0$ such that for any $t > 1/2$, $\tau_{\lambda,t}(t) \geqslant c^t$.*

For τ_λ, we have in addition that for any $t \in [0, 1/2]$, $\tau_\lambda(t) \geqslant c$, thus for any $t > 0$, we have $\tau_\lambda(t) \leqslant c^{t+1}$

Proof: Consider a point x at distance t from the origin. On the segment $[o, x]$, consider points x_i at distance $1/2$ from each other (the first one being itself at distance $1/2$ from o). If the last one is at distance less than $1/2$ of x, we move it on the segment away from x to be at distance exactly $1/2$ of it. Then, we have at most $2t$ points. If for each i, there exists a point of ξ in the ball $B(x_i, 1/2)$, then the entire segment is covered. Using the FKG inequality, this happens with probability at least $(1-e^{\lambda A})^{2t}$, where A is the area of a ball of radius 1/2. Moreover, each of these balls is contained within $B(o,t)$ if $t > 1/2$.

For $t < 1/2$, the segment is entirely covered if there exists a point in $B(o, 1/2)$. ■

Proposition 4.4. *The function τ_λ decays exponentially for $\lambda < \lambda_u$, that is, there exists a constant $c = c(\lambda) < 1$ such that for any $t > 0$, we have that*

$$\tau_\lambda(t) \leqslant c^t.$$

Moreover, the constant c can be taken uniformly over any closed interval within $[0, \lambda_u[$, i.e. for any interval $[\lambda_1, \lambda_2]$, there exists a constant $c = c(\lambda_1, \lambda_2) > 0$ such that :

$$\forall \lambda \in [\lambda_1, \lambda_2], \qquad \forall t > 0, \qquad \tau_\lambda(t) \leqslant c^t.$$

Remark. Since $\tau_{\lambda,t}(t) \leqslant \tau_\lambda(t)$, the result is also true for $\tau_{\lambda,t}(t)$.

Proof: Let x be a point at distance t from the origin. Consider the hyperbolic segment $[0, x]$ and the geodesic lines $L_0, L_1, \dots,$ perpendicular to the segment, with L_0 going through o and each L_i at distance 2 from the next one, the minimal distance being achieved along the segment $[0, x]$. Define the event E_i that there exists a vacant bi-infinite path between the geodesics L_i and L_{i+1}. If $|i - j| \geqslant 2$, the events E_i and E_j are thus independent. And, Theorem 3.8 gives us the existence of a constant $C = C(\lambda)$ such that $\mathbb{P}_\lambda(E_i) = C > 0$. The event $\{o \overset{\mathcal{C}}{\leftrightsquigarrow} x\}$ can only happen if none of the events E_i is realised, thus there exists a constant $c < 1$ such that :

$$\tau_\lambda(t) = \mathbb{P}_\lambda(o \overset{\mathcal{C}}{\leftrightsquigarrow} x) \leqslant (1-C)^{\lfloor \frac{t}{4} \rfloor} \leqslant c^t.$$

Theorem 3.8 allows us to take that constant uniformly on any closed interval within $[0, \lambda_u[$, by considering the minimal constant over that interval. ■

Lemma 4.5. *Fix $\lambda > \lambda_c$. Consider a geodesic line L, and write $d := dist_{H^2}(o, L)$ the minimal distance between the origin and L. Suppose that $d \geqslant 1$. Then, there exists a constant $c = c(\lambda)$ such that*

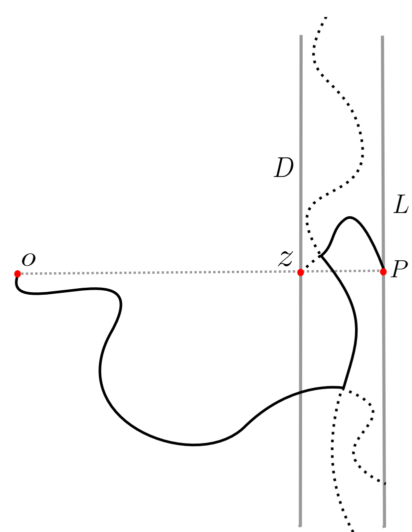
$$\mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} L\right) \leqslant c \cdot \tau_\lambda(d).$$

Proof: Consider the point P of L at distance exactly d from o. Consider the segment $[o, P]$, and place a point z on it at distance $d-1$ of the origin. Let D be the geodesic perpendicular to $[o, x]$ intersecting it at z, and call S the strip delimited by lines D and L. By *Theorem* 3.7, there is a positive probability c that there exists a covered bi-infinite path contained in S, let's call this event A. On the event that A, $\{o \overset{\mathcal{C}}{\leftrightsquigarrow} L\}$ and $\{x \overset{\mathcal{C}}{\leftrightsquigarrow} z\}$ all happen, we have $\{o \overset{\mathcal{C}}{\leftrightsquigarrow} x\}$, thus :

$$c\mathbb{P}_\lambda\left(\{o \overset{\mathcal{C}}{\leftrightsquigarrow} L_1\}\right)\tau_\lambda(1) \overset{\text{FKG}}{\leqslant} \tau_\lambda(t_1),$$

and then

$$\mathbb{P}_\lambda(\{o \overset{\mathcal{C}}{\leftrightsquigarrow} L_1\}) \leqslant \frac{\tau_\lambda(t_1)}{c\tau_\lambda(1)}.$$



■

Lemma 4.6. *For all $\lambda > \lambda_c$, the function τ_λ is ultimately quasi submultiplicative. That is, there exists a constant $C = C(\lambda) < \infty$ such that, for any $t_1, t_2 \geqslant 1$, we have*

$$\tau_\lambda(t_1 + t_2) \leqslant C\tau_\lambda(t_1)\tau_\lambda(t_2).$$

Similarly, we have that

$$\tau_{\lambda,t_1+t_2}(t_1 + t_2) \leqslant C\tau_{\lambda,t_1}(t_1)\tau_{\lambda,t_2}(t_2).$$

Proof: Let us assume that $t_1, t_2 \geqslant 1$. Let L be a half-line starting from o. Denote by x_1, x, x_2 and y the points of L at respective distance $t_1 - 1$, t_1, $t_1 + 1$, t_2 from o. Denote by L_1, L' and L_2 the geodesics perpendicular to L going through the points x_1, x and x_2 respectively, and by H_1 and H_2 the two half-planes delimited by L'. On the event that $\{o \overset{\mathcal{C}}{\leftrightsquigarrow} y\}$, it is necessary that $\{o \overset{\mathcal{C}}{\leftrightsquigarrow} L_1\}$ and $\{y \overset{\mathcal{C}}{\leftrightsquigarrow} L_2\}$, and those events are independent because they rely entirely on the restriction of ξ on the halfplanes H_1 and H_2 respectively. Thus,

$$\tau_\lambda(t_1 + t_2) \leqslant \mathbb{P}_\lambda\left(\{o \overset{\mathcal{C}}{\leftrightsquigarrow} L_1\} \cap \{y \overset{\mathcal{C}}{\leftrightsquigarrow} L_2\}\right) = \mathbb{P}_\lambda\left(\{o \overset{\mathcal{C}}{\leftrightsquigarrow} L_1\}\right)\mathbb{P}_\lambda\left(\{y \overset{\mathcal{C}}{\leftrightsquigarrow} L_2\}\right).$$

Calling for the previous lemma, we obtain :

$$\tau_\lambda(t_1 + t_2) \leqslant C\tau_\lambda(t_1)\tau_\lambda(t_2).$$

■

Proposition 4.7.

$$\lim_{t\longrightarrow\infty} \frac{\log \tau_{\lambda,t}(t)}{t} = \lim_{t\longrightarrow\infty} \frac{\log \tau_\lambda(t)}{t} = \beta(\lambda).$$

Proof: Fix some $\varepsilon > 0$. For s large enough, by definition of $\beta(\lambda)$, we have

$$\tau_\lambda(s) = e^{\beta_t(\lambda)s} \geqslant e^{(\beta(\lambda)-\varepsilon)s}.$$

Let us fix some s so that the previous inequality is true. Then there exists a constant $m = m(s)$ such that

$$\tau_{\lambda,m}(s) \geqslant e^{(\beta(\lambda)-2\varepsilon)s}.$$

Consider then a half-line L starting at o, and consider the point x on L at distance t from o, and the points y_k on L at distance $s \cdot k$ from o (with $y_0 = 0$). On the event that every y_k is connected to the next within the ball $B(y_k, m)$ for $k \leqslant \lfloor t/s - m\rfloor - 1 = K - 1$, and that y_K is connected to x within the ball $B(y_K, \mathrm{dist}_{H^2}(y_K, x))$, then the event $o \overset{\mathcal{C}_t}{\leftrightsquigarrow} x$ is realised. Thus using the FKG inequality, we have :

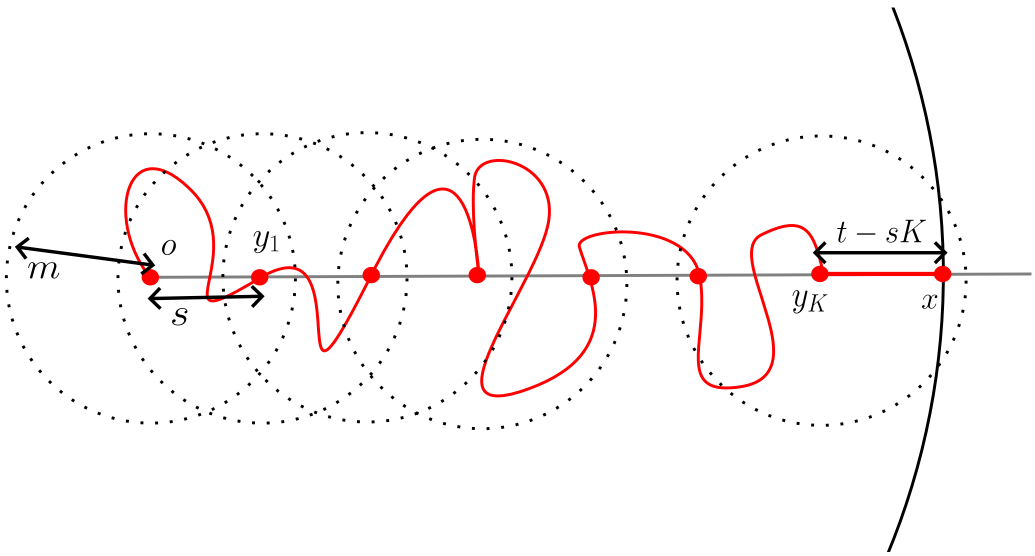
$$\begin{aligned}\tau_{\lambda,t}(t) &\geqslant \mathbb{P}_\lambda\left(\bigcap_{k\leqslant K-1}\left\{y_k \overset{\mathcal{C}_{B(y_k,m)}}{\leftrightsquigarrow} y_k + 1\right\} \cap \left\{y_K \overset{\mathcal{C}_{B(y_K,\mathrm{dist}_{H^2}(y_K,x))}}{\leftrightsquigarrow} x\right\}\right)\\ &\underset{\text{FKG}}{\geqslant} \mathbb{P}_\lambda\left(o \overset{\mathcal{C}_{B(o,m)}}{\leftrightsquigarrow} y_1\right)^K \inf_{r\leqslant m} \tau_{\lambda,r}(r).\end{aligned}$$

Using Proposition 4.4, we know that $\inf_{r\leqslant m} \tau_{\lambda,r}(r)$ is a constant $C = C(\lambda, m)$. Thus we have for t large enough :

$$\tau_{\lambda,t}(t) \geqslant C\tau_{\lambda,m}(s)^{\lfloor \frac{t}{s} - m\rfloor} \geqslant \left(e^{(\beta(\lambda)-2\varepsilon)s}\right)^{\frac{t}{s}-m-1} \geqslant e^{(\beta(\lambda)-3\varepsilon)t}.$$

Therefore, for any $\varepsilon > 0$, we have

$$\lim_{t\longrightarrow\infty} \frac{\log \tau_{\lambda,t}(t)}{t} \geqslant \beta(\lambda) - 3\varepsilon.$$



And clearly, as $\tau_{\lambda,t}(t) \leqslant \tau_\lambda(t)$, we also have

$$\lim_{t \longrightarrow \infty} \frac{\log \tau_{\lambda,t}(t)}{t} \leqslant \beta(\lambda).$$

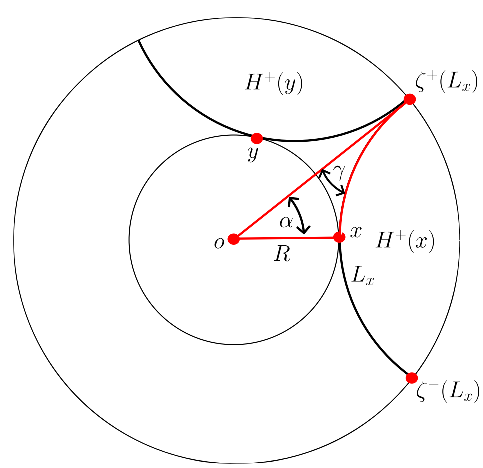
■

Remark. The same argument could be made with all kinds of growing shapes in mind : a square, a cone, a semi-circle... Only in those cases one might have to control the start of the path the same way we had to control its end here, which would only cost a constant that disappears in the exponential.

5. The function β

This section aims to prove Theorem 2.2, which states the existence, continuity and strict increasingness of the function β defined in the introduction.

We start by a few definitions that will be useful for the first two lemmas, and then in the next section. For any $t > 0$, consider the ball $B(o,t)$ and a point $x \in \partial B(o,t)$. Let L_x be the hyperbolic line tangent to the circle $B(o,t)$ and going through x, and $H^+(x)$ the half-plane delimited by L_x that doesn't contain o. L_x and $H^+(x)$ are obviously dependant on t, but we will ommit the parameter t as it is clear at anytime to which t it corresponds.



Proposition 5.1. *For every $x, y \in \delta B(o,t)$, let α_{xy} be the smallest angle between the rays $[ox)$ and $[oy)$. For t large enough, if $\alpha_{xy} \geqslant 4/\cosh(t)$, then $H^+(x) \cap H^+(y) = \emptyset$.*

Proof: Let's denote by $\zeta^+(L_x)$ and $\zeta^-(L_x)$ the limit points of L_x on the circle boundary. Let's consider the triangle consisting of the three points o, x and $\zeta^+(L_x)$, and denote by α, β and γ the corresponding angles at these points. Thus, we have $\gamma = 0$ and $\beta = \pi/2$. Moreover, we have that $\mathrm{dist}_{H^2}(o,x) = t$, so using the second hyperbolic law of cosine, we have :

$$1 = \cos(\gamma) = -\cos(\beta)\cos(\alpha) + \sin(\beta)\sin(\alpha)\cosh(t).$$

Thus

$$\cosh(t) = \frac{1+\cos(\beta)\cos(\alpha)}{\sin(\beta)\sin(\alpha)} = \frac{1}{\sin(\alpha)}.$$

Clearly, $H^+(x) \cap H_y^+ = \emptyset$ if and only if $\alpha_{xy} \geqslant 2\alpha$. Consider then the function $h(\theta) = \theta/2 - \sin(\theta)$. We have $h(o) = 0$ and $h'(\theta) = 1/2 - \cos(\theta) < 0$ for $0 \leqslant \theta \leqslant \pi/3$. Thus, for $\theta \in [0, \pi/6]$, we have $\sin(2\theta) \geqslant \theta$, and $2\theta \geqslant \arcsin(\theta)$. Hence for t large enough :

$$2\alpha = 2\arcsin(1/\cosh(R)) \leqslant 4/\cosh(t).$$

■

Lemma 5.2. *For $t > 0$ and $a, b \in \partial B(o,t)$, let us denote by A_{ab} the event that there exists an infinite vacant path joining $\partial B(o,t)$ to $\partial \mathbb{H}^2$ in between the two rays $[0,a)$ and $[0,b)$. For every interval $[\lambda_1, \lambda_2] \subset]0, \lambda_u[$, there exists a constant $c = c(\lambda_1, \lambda_2) > 0$ and a $\delta > 0$ so that for any $t > 0$ large enough,*

$$\forall a, b \in \partial B(o,t),\ \textit{arclength}(a,b) \geqslant \delta,\ \mathbb{P}_\lambda(A_{ab}) \geqslant c \textit{ and } \textit{dist}_{H^2}(a,b) \geqslant 2.$$

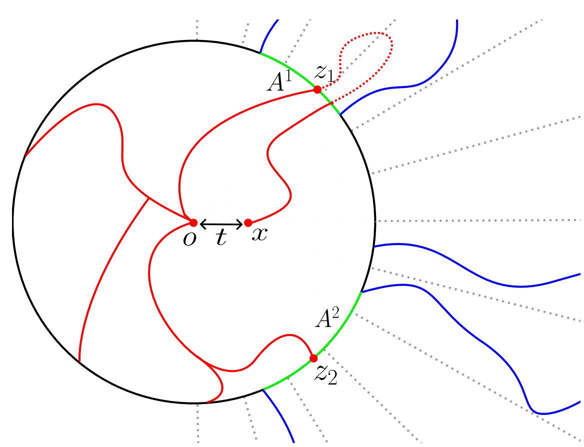
Proof: Consider $a, b \in \partial B(o,t)$ with an arc distance of 8 between each other, and consider the point z on $\partial B(o,t)$ in the middle of a and b. Then, the angle between $\langle oab$ is $8/\sinh(t)$, so the angles $\langle oaz$ and $\langle ozb$ are both $4/\sinh(t)$. Using , we can deduce that the half planes $H^+(a)$, $H^+(z)$ and $H^+(b)$ are all disjoint. Thus, $H^+(z)$ is in between the rays $[o,a)$ and $[o,b)$.

Using Proposition 3.6, we then deduce that there is a positive probability at least $c = c(\lambda_1, \lambda_2)$ that z is connected to $\partial \mathbb{H}^2$ by a vacant path remaining in $H^+(z)$ for any $\lambda \in [\lambda_1, \lambda_2]$. On that event, A_{ab} is realised. ■

Proposition 5.3. *For every interval $[\lambda_1, \lambda_2] \subset [0, \lambda_u[$, there exists a constant $C = C(\lambda_1, \lambda_2)$ and a $T = T(\lambda_1, \lambda_2)$ so that for any $\lambda \in [\lambda_1, \lambda_2]$, for any $\forall t \geqslant T$ and any $x \in \mathbb{H}^2$ such that $\textit{dist}_{H^2}(o,x) = t$,*

$$\frac{\tau_\lambda(t)}{2} \leqslant \mathbb{P}_\lambda(o \overset{\mathcal{C}_{t^2}}{\leftrightsquigarrow} x) = \tau_{\lambda,t^2}(t).$$

Proof: Using the same notations as in the previous lemma, consider t large enough so that the lemma holds for t^2. On the boundary of $B(o, t^2-1)$, consider $\left\lfloor \frac{2\pi\sinh(t^2-1)}{\delta} \right\rfloor$ points denoted $a_1, \ldots, a_n$ regularly spaced (the length of arc between two consecutive points is then constant equal to $\delta' \approx \delta$, with $\delta' \geqslant \delta$), and denote by s_i the area between $\partial B(o,t)$, $\partial \mathbb{H}^2$, $[o, a_i)$ and $[o, a_{i+1})$ (if $i = n$, we consider the rays $[o, a_n)$ and $[o, a_1)$ instead). Consider a point x at distance t of the origin. We define the event A that there exists a sequence of t^3 consecutive points a_i so that the events $A_{a_i, a_{i+1}}$ are realised (if $i = n$, we consider the event A_{a_n, a_1} instead).



Because $\mathrm{dist}_{H^2}(a_i, a_{i+1}) \geqslant 2$, the distance between the strips s_i and s_j for $|i-j| \geqslant 2$ is at least 2, hence the events $A_{a_i,a_{i+1}}$ and $A_{a_j,a_{j+1}}$ are independent. Hence for t large enough :

$$\mathbb{P}_\lambda(A) \leqslant \sum_i \mathbb{P}_\lambda\left(\bigcap_{j=i}^{i+t^3-1} A_{a_j,a_{j+1}}\right) \leqslant \left\lfloor \frac{2\pi \sinh(t^2-1)}{\delta'} \right\rfloor (1-c)^{\left\lfloor \frac{t^3}{2} \right\rfloor} \leqslant const \cdot e^{t^2-1} e^{\ln(1-c)\left\lfloor \frac{t^3}{2} \right\rfloor}.$$

We want to show that the probability of the event $B := \{o$ is connected to x but any path connecting the two points has to use points of ξ outside of $B(o,t^2)\}$ is exponentially small. On the event A^C, there exist no arc on the ball $B(o,t^2-1)$ of length more than $(\delta'+1)t^3$ without an infinite vacant path coming out of it. We define the arcs A_i to be the disjoint open arcs of $\partial B(o,t^2-1)$ bounded by two starting points of such vacant infinite paths.

We start by exploring the cluster $\mathcal{C}_{t^2}(o)$: starting from o, we look within $B(o,2) \cap B(o,t^2)$ for points of ξ. If some point x is found this way, we add the area $B(x,1)$ to our cluster and search for new points within $B(x,2) \cap B(o,t^2)$. Once there is no more point to add, we have explored an area E covering the entire cluster $\mathcal{C}_{t^2}(o) \cap B(o,t^2)$ plus its neighbouring $B(o,t^2)$ which define its boundary. In a formula, we have :

$$E = \left(\bigcup_{y \in \mathcal{C}_{t^2}(o)} B(y,2)\right) \cap B(o,t^2).$$

Moreover, the configuration ξ on $\mathbb{H}^2 \setminus E$ is independent from the part of the configuration that we explored.

Condition on the event that paths from o to x exist, but that any of these paths exits $B(o,t^2)$. Then the cluster $C_{t^2}(o)$ has to have intersection points with $\partial B(o,t^2-1)$, and does not contain x. We can define the set O to be the component of x within $B(o,t^2) \setminus E$. Because the path from x to o has to leave the ball $B(o,t^2)$, it means that there exists a path from x to $\partial B(o,t^2-1)$ within the set O. Therefore, the set $O \cap \partial B(o,t^2-1)$ is an arc of ends z_1 and z_2. We define A^1 and A^2 to be the arcs A_i and A_j such that $z_1 \in A^1$, and $z_2 \in A^2$. On the event A, because vacant paths block the connection of the paths outside of $B(o,t^2-1)$, the point x has to be connected within O to one of the arcs A^1 or A^2. We thus define the events :

$$F := o \overset{\mathcal{C}_{B(o,t^2)}}{\leftrightsquigarrow} \partial B(o,t^2-1), \qquad \text{and} \qquad G := x \overset{\mathcal{C}_O}{\leftrightsquigarrow} A^1 \cup A^2.$$

As it is, G is not independent from F because O is correlated to the event F. However, we can replace the configuration ξ by another Poisson point process outside of O. Indeed, consider a configuration ξ' independent from ξ on $B(o,t^2)$, with the same parameter λ. The event G is included in the event G' that x is connected to $A^1 \cup A^2$ with the configuration $(\xi \cap O) \cup (\xi' \cap (B(o,t^2) \setminus O)$, that has the same law as ξ, and G' is independent from F. Because the arcs A^1 and A^2 are at most of length $(c'+1)t^3$, we have $\mathbb{P}_\lambda(G') \leqslant const \cdot (c'+1)t^3 \tau_\lambda(t^2-t)$ using Proposition 4.1. Hence for t large enough :

$$\mathbb{P}_\lambda(B) \leqslant \mathbb{P}_\lambda(A) + \mathbb{P}_\lambda(F \cap G) \leqslant \mathbb{P}_\lambda(A) + \mathbb{P}_\lambda(G') \leqslant const \cdot e^{t^2} e^{\ln(1-c)\left\lfloor \frac{t^3}{2} \right\rfloor} + const \cdot (\delta'+1)t^3 \tau_\lambda(t^2)$$

Using Proposition 4.4, we may conclude that both terms are decreasing exponentially in t^2, whereas $\tau_\lambda(t)$ is decreasing exponentially in t, proving the result. ■

All these results aim at proving the folliwing continuity result, which we will in turn use to prove the continuity of β :

Lemma 5.4. *The function $\tau_{\lambda,t}(s)$ is continuous in (λ, t, s) for $t > s$.*

Proof: Fix two triplets (λ, t, s) and (λ', t', s'), with $t > s$ and $t' > s'$. We can assume that $s' > s$. Fix a ray r going from o and fix two points x and x' on r at distance t and t' from o. Consider the event A that in $\xi(\lambda')$, there exists a covered path from o to x within $B(o, s)$ but not from o to x' in $B(o, s')$, or the opposite. A may happen only if at least one of the two following condition happens :

- there exist a point of ξ in the area $B(x, 1)\Delta B(x', 1)$,
- there exist a point of ξ in the area $B(o, s') \setminus B(o, s)$.

Write $f(|t' - t|) = area(B(x, 1)\Delta B(x', 1))$. We have :

$$|\tau_{\lambda',t^2} - \tau_{\lambda',t'^2}| \leqslant \mathbb{P}_\lambda(A) \leqslant 1 - e^{-f(|t'-t|)\lambda'} + 1 - e^{2\pi(\cosh(s')-\cosh(s))\lambda'}.$$

To compare $\tau_{\lambda,t}(s)$ to $\tau_{\lambda',t}(s)$, we will use a coupling argument. Assume that $\lambda' - \lambda = \delta > 0$. Consider a realisation of $\xi(\lambda)$, and an independent Poisson point process ζ of intensity δ on $\mathbb{H}^2$. Then the process $\xi'(\lambda') = \xi \cup \zeta$ is a Poisson point process of intensity λ', it thus has the same distribution as $\xi(\lambda')$. Hence, $\tau_{\lambda',t}(s) - \tau_{\lambda,t}(s)$ is the probability that o is connected to x within $B(o, s)$ in $\xi'(\lambda')$ but not in $\xi(\lambda)$. It may happen only if $\zeta \cap B(o, s)$ contains at least one point. Thus :

$$|\tau_{\lambda',t}(s) - \tau_{\lambda,t}(s)| \leqslant 1 - e^{2\pi(\cosh(s)-1)\delta}.$$

Hence, we have :

$$\begin{aligned}\tau_{\lambda',t'}(s') - \tau_{\lambda,t}(s) &\leqslant |\tau_{\lambda',t'}(s') - \tau_{\lambda',t}(s)| + |\tau_{\lambda',t}(s) - \tau_{\lambda,t}(s)| \\ &\leqslant 3 - e^{2\pi(\cosh(s)-1)|\lambda'-\lambda|} - e^{-f(|t'-t|)\lambda'} - e^{2\pi(\cosh(s')-\cosh(s))\lambda'},\end{aligned}$$

which goes to 0 when $||(\lambda, t, s) - (\lambda', t', s')||_1 \longrightarrow 0$. ■

Proposition 5.5. *The function $\beta(\lambda)$ is continuous for $\lambda > \lambda_c$.*

Proof: For all $\lambda > \lambda_c$, and $s \geqslant 3$, Lemma 4.2 and Lemma 4.6 give us

$$\tau_\lambda(s)^2 \leqslant \tau_\lambda(2s) \leqslant C(\lambda)\tau_\lambda(s)^2.$$

Thus by induction, we can show that for all $k \geqslant 0$, we have

$$\tau_\lambda(s)^{2^k} \leqslant \tau_\lambda(2^k s) \leqslant C(\lambda)^{1+2+\ldots+2^{k-1}}\tau_\lambda(s)^{2^k}.$$

By taking the log of this inequality and dividing by $2^k s$, we then get

$$\frac{\log \tau_\lambda(s)}{s} \leqslant \frac{\log \tau_\lambda(2^k s)}{2^k s} \leqslant \frac{\log \tau_\lambda(s)}{s} + \frac{\log C(\lambda)}{s}.$$

Fix some $\lambda^* > \lambda_c$ and $\delta > 0$ such that $\lambda^* - \delta > \lambda_c$. By Theorem 3.7 there exists a constant C such that the previous inequality holds with said constant, for all $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$, and then using Proposition 5.3 :

$$\frac{\log \tau_{\lambda,s^2}(s)}{s} - \frac{\log(2)}{s} \leqslant \frac{\log \tau_\lambda(2^k s)}{2^k s} \leqslant \frac{\log \tau_{\lambda,s^2}(s)}{s} + \frac{C + \log(2)}{s}.$$

For $\varepsilon > 0$, we can take S large enough so that $\frac{C+\log(2)}{S} \leqslant \varepsilon$. For all $s > S$, by taking the limit on k, we thus have :

$$\frac{\log \tau_{\lambda,s^2}(s)}{s} - \varepsilon \leqslant \beta(\lambda) \leqslant \frac{\log \tau_{\lambda,s^2}(s)}{s} + \varepsilon.$$

Hence, we have that $\log(\tau_{\lambda,s^2}(s))/s$ converges uniformly on every segment to $\beta(\lambda)$. The functions $\lambda \mapsto \log(\tau_{\lambda,s^2}(s))/s$ are all continuous in λ by Lemma 5.4, thus their limit β is also continuous. ■

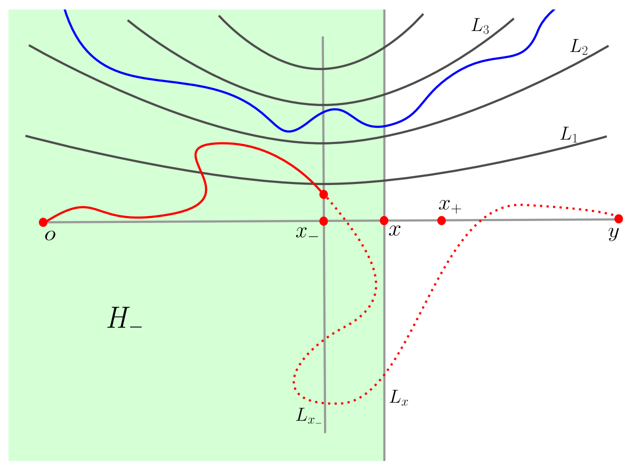
To prove the continuity below λ_c, we will define the function $\sigma_\lambda(x) := \max_{y \geqslant x} \tau_\lambda(y)$. We know thanks to Lemma 4.3 and Proposition 4.4 that $\tau_\lambda(x)$ is bounded above and below by two exponentials, i.e. there exists two constants c and C (again depending on λ, but that can be taken uniformly on any interval below λ_u) such that $c \cdot c^t \leqslant \tau_\lambda(t) \leqslant C^t$. Thus there exists a constant $D > 1$ so that $\sigma_\lambda(x)$ is in fact the maximum over all $y \in [x, Dx]$.

Proposition 5.6. *The function $\beta(\lambda)$ is continuous for $\lambda < \lambda_u$.*

Proof: We prove that the function σ_λ has similar properties to τ_λ below λ_u, thus allowing us to conclude via the same argument as previously.

The hard part is to prove that for every $0 < \lambda^* < \lambda_u$, and some $\delta(\lambda^*) > 0$ so that $\lambda^* + \delta < \lambda_u$ and $\lambda^* - \delta > 0$, there exists a constant C so that for all $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$ and all t_1, t_2 large enough, we have

$$\sigma_\lambda(t_1 + t_2) \leqslant C(t_1 + t_2)^2 \sigma_\lambda(t_1)\sigma_\lambda(t_2).$$



Here we have only represented the lines above L_{x_-}. In blue, the vacant bi-infinite path that cannot be crossed by our covered path. A similar situation would happen below $[o, y]$, and for x_+.

Fix some $t_1, t_2 > 1$, $s_1 \geqslant t_1$ and $s_2 \geqslant t_2$ so that $\sigma_\lambda(t_1 + t_2) = \tau_\lambda(s_1 + s_2)$. Fix some ray L coming from o, and let x_-, x, x_+ and y be the points of L at respective distance $s_1 - 1, s_1, s_1 + 1$ and $s_1 + s_2$ from o. Consider the lines L_{x_-}, L_x and L_{x_+} perpendicular to L going through x_-, x and x_+ respectively, and the lines $L_1, L_2, \ldots$, (and L_{-1}, L_{-2} on the other side of L) parallel to L, each at distance 2 of the previous one, and the minimal distance occurs between the intersection points with L_{x_-}. For each i, there is a probability at least $C = C(\lambda, \delta)$ that there exists a vacant bi-infinite cluster in between L_i and L_{i+1}. Thus, the probability that there exists no vacant path in any of the *const* $\cdot\, (t_1 + t_2)$ first strips is decreasing exponentially. Note that the function $\sigma_\lambda(t_1 + t_2) \geqslant \tau_\lambda(t_1 + t_2)$ has a lower bound $\mathbb{P}_\lambda([0, y] \subset \mathcal{C})$ that is also exponential in $t_1 + t_2$, thus there exists some constant D so that

$$\mathbb{P}_\lambda(\text{no strip between } L \text{ and } L_{D\cdot(t_1+t_2)}) \leqslant \sigma_\lambda(t_1 + t_2)/8.$$

We can obtain the same formulas with the same argument along the line L_{x_+}. Thus the probability that there exists a vacant path between L and $L_{D\cdot(t_1+t_2)}$ for L_{x_-} and for L_{x_+}, and one in between L and $L_{-D\cdot(t_1+t_2)}$ for both L_{x_-} and L_{x_+} is at least $1 - \sigma_\lambda(t_1 + t_2)/2$. This means that the event that in addition to all this, there exists a covered path from o to y has a probability greater than $\sigma_\lambda(t_1 + t_2)/2$. On that event, the covered path has to cross the line L_{x_-} within distance $D \cdot (t_1 + t_2)$ from x_-. Hence there exist a point $P_- \in L_{x_-}$ at distance at most $D \cdot (t_1 + t_2)$ from x_-, and a point $P_+ \in L_{x_+}$ at distance at most $D \cdot (t_1 + t_2)$ from x_+, so that the probability that there exists a path from o to y that crosses L_{x_-} for the first time at P_- and crosses L_{x_+} for the last time at the point P_+ is at least $\frac{\sigma_\lambda(t_1+t_2)}{(const\cdot D\cdot(t_1+t_2)+1))^2}$ (see Proposition 4.1). Denote by $\mathbb{H}_-$ and

$\mathbb{H}_+$ the half-planes delimited by L_x. We have finally :

$$\begin{aligned}\sigma_\lambda(t_1+t_2)=\tau_\lambda(s_1+s_2) &\leqslant (const\cdot D\cdot(t_1+t_2))^2\mathbb{P}_\lambda\left(o\overset{\mathcal{C}_{H_-}}{\leftrightsquigarrow}P_-\cap y\overset{\mathcal{C}_{H_+}}{\leftrightsquigarrow}P_+\right)\\ &\leqslant (const\cdot D\cdot(t_1+t_2))^2\sigma_\lambda(s_1-1)\sigma_\lambda(s_2-1)\\ &\leqslant \frac{(const\cdot D\cdot(t_1+t_2))^2}{\sigma_\lambda(1)^2}\sigma_\lambda(s_1)\sigma_\lambda(s_2).\end{aligned}$$

These 3 properties allow us to conclude following the proof of Proposition 5.6 and replacing instances of τ_λ by σ_λ. ■

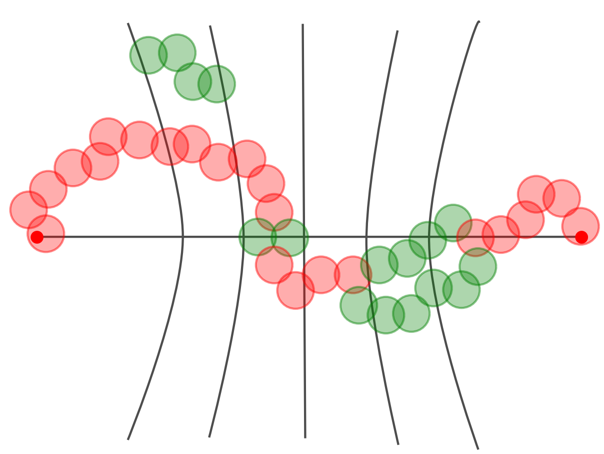
With the two last propositions, we have proven that β is continuous in λ.

Lemma 5.7. *For $\lambda \geqslant \lambda_u$, we have $\beta(\lambda)=0$.*

Proof: For $\lambda \geqslant \lambda_u$, the event that o is in an infinite cluster has a positive probability p_λ. Also, because of the unicity of the infinite cluster and the FKG inequality, the probability that o is connected to any point $x\in\mathbb{H}^2$ is at least $\theta(\lambda)^2=\mathbb{P}_\lambda(\mathcal{C}(o)\text{ is infinite})\mathbb{P}_\lambda(\mathcal{C}(x)\text{ is infinite})>0$. So for all $t>0$, $\tau_\lambda(t)\geqslant\theta(\lambda)^2>0$, and $\log(\tau_\lambda(t))/t\longrightarrow 0$, thus $\beta(\lambda)=0$. ■

Definition 5.8. *(Pivotal points) Let A be an increasing event, and ξ a configuration where A is realised. A pivotal point for the event A in the configuration ξ is a point of ξ with the property that removing it makes it so that A is no longer true.*

Theorem 5.9. *On the interval $[0,\lambda_u[$, the function β is strictly increasing.*



The pivotal points are in green. The first and fourth strip have a crossing BK-independant from the path. The second does not, but still has pivotal points.

Proof: Consider $\lambda<\lambda_u$. For any point $x\in\mathbb{H}^2$ at distance $t=2k$, $k\in\mathbb{N}$ from the origin, consider the geodesic segment $[0,x]$ and the geodesic lines $L_0,L_1,\dots$ perpendicular to that segment, with L_0 going through o, each at distance 2 from the next one, the distance being minimal along the segment $[0,x]$. Let's call s_i the strip in between the geodesics L_{i-1} and L_i. For any s_i, the probability that there is a vacant bi-infinite path in that strip is $c>0$. If we define $E_i:=\{L_{i-1}\overset{\mathcal{C}}{\leftrightsquigarrow}L_i\}$, then if $|i-j|\geqslant 2$, the events E_i and E_j are independent. Moreover, $\mathbb{P}_\lambda(E_i)=(1-c)$. If $\{o\overset{\mathcal{C}}{\leftrightsquigarrow}x\}$ is realised, consider the set P_t of pivotal points for the event. For any $\varepsilon>0$, on the event that $|P_t|\leqslant\varepsilon t$, there exists a set of $\lfloor k-\varepsilon t\rfloor$ strips s_i such that there exists another path from L_{i-1} to L_i. By "another path", we mean here a path that can be covered by a set of balls disjoint from the set that covers the path from o to x. Let $\mathcal{E}_k=\{I\subset\{1,\dots,k\},|I|=\lfloor\varepsilon t\rfloor\}$. The BK inequality yields :

$$\begin{aligned}\mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x \cap |P_t| \leqslant \varepsilon t\right) &\leqslant \mathbb{P}_\lambda\left(\bigcup_{I\in\mathcal{E}_k}\{o \overset{\mathcal{C}}{\leftrightsquigarrow} x\} \,\square\, \{\forall j \notin I, E_j\}\right)\\ &\leqslant \sum_{I\in\mathcal{E}_k}\mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\cdot \mathbb{P}_\lambda\left(\forall j\notin I, E_j\right)\\ &\leqslant \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\cdot \sum_{I\in\mathcal{E}_k}\mathbb{P}_\lambda\begin{pmatrix}\forall j\in\{1,\dots,\lfloor(k-\varepsilon t)/2\rfloor\}, E_{2i_j-1}\text{ where}\\ \{1,\dots,k\}\setminus I = \{x_{i_1}<x_{i_2}<\ldots<x_{i_{\lceil k-\varepsilon t\rceil}}\}\end{pmatrix}\\ &\leqslant \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\cdot \sum_{I\in\mathcal{E}_k}(1-c)^{\lfloor(k-\varepsilon t)/2\rfloor}\\ &\leqslant \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\cdot \binom{k}{\lfloor\varepsilon t\rfloor}(1-c)^{\lfloor(k-\varepsilon t)/2\rfloor}.\end{aligned}$$

As $k = t/2$, we can use the Stirling formula to show that

$$\begin{aligned}\binom{k}{\lfloor\varepsilon t\rfloor} &\sim \frac{\sqrt{2\pi k}\cdot k^k\cdot e^{k-\lfloor\varepsilon t\rfloor}e^{\lfloor\varepsilon t\rfloor}}{e^k\sqrt{2\pi\lfloor\varepsilon t\rfloor}\,\lfloor\varepsilon t\rfloor^{\lfloor\varepsilon t\rfloor}\sqrt{2\pi\left(k-\lfloor\varepsilon t\rfloor\right)}\left(k-\lfloor\varepsilon t\rfloor\right)^{k-\lfloor\varepsilon t\rfloor}}\\ &\sim \frac{\sqrt{t}\cdot t^{\frac{t}{2}}}{2^{\frac{t}{2}}\sqrt{2\varepsilon t}\left(t\left(\varepsilon-\frac{\alpha(\varepsilon t)}{t}\right)\right)^{t\left(\varepsilon-\frac{\alpha(\varepsilon t)}{t}\right)}\sqrt{2\pi\left(\frac{t}{2}-\varepsilon t\right)}\left(t\left(\frac{1}{2}-\varepsilon+\frac{\alpha(\varepsilon t)}{t}\right)\right)^{t\left(\frac{1}{2}-\varepsilon+\frac{\alpha(\varepsilon t)}{t}\right)}}\\ &\sim \frac{1}{2\sqrt{\pi\varepsilon\left(\frac{1}{2}-\varepsilon\right)}}\frac{1}{\sqrt{t}\left[\sqrt{2}\left(\varepsilon-\frac{\alpha(\varepsilon t)}{t}\right)^{\left(\varepsilon-\frac{\alpha(\varepsilon t)}{t}\right)}\left(\frac{1}{2}-\varepsilon+\frac{\alpha(\varepsilon t)}{t}\right)^{\left(\frac{1}{2}-\varepsilon+\frac{\alpha(\varepsilon t)}{t}\right)}\right]^t},\end{aligned}$$

where $\alpha(\varepsilon t) = \varepsilon t - \lfloor\varepsilon t\rfloor < 1$. We define $D(\varepsilon) = \sqrt{2}\cdot\varepsilon^\varepsilon(1/2-\varepsilon)^{1/2-\varepsilon}$. $D(\varepsilon)$ goes to 1 as ε goes to 0. And $\varepsilon - \alpha(\varepsilon t)/t \leqslant \varepsilon$, thus we can find an ε small enough so that $1/4-\varepsilon > 1/5$, and so that for any $\delta < \varepsilon$, we have $\mathbb{D}(\delta) > (1-c)^{1/4}$. With such an ε fixed, we have :

$$\begin{aligned}\mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x \cap |P_t| \leqslant \varepsilon t\right) &\leqslant \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\frac{1}{2\sqrt{\pi\varepsilon\left(\frac{1}{2}-\varepsilon\right)}}\frac{(1+o_t(1))}{\sqrt{t}\cdot D\left(\varepsilon-\frac{\alpha(\varepsilon t)}{t}\right)^t}(1-c)^{t\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)-1}\\ &\leqslant \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\cdot\frac{1}{(1-c)2\sqrt{\pi\varepsilon\left(\frac{1}{2}-\varepsilon\right)}}\frac{1}{\sqrt{t}}\left(\frac{(1-c)^{\frac{1}{5}}}{D\left(\varepsilon-\frac{\alpha(\varepsilon t)}{t}\right)}\right)^t(1+o_t(1))\\ &\leqslant \mathbb{P}_\lambda\left(o \overset{\mathcal{C}}{\leftrightsquigarrow} x\right)\cdot const(\varepsilon)\cdot\left(\frac{(1-c)^{\frac{1}{5}}}{(1-c)^{\frac{1}{4}}}\right)^t(1+o_t(1)).\end{aligned}$$

Thus there exists a constant $a < 1$ such that $\mathbb{P}_\lambda(|P_t| \leqslant \varepsilon t \mid o \overset{\mathcal{C}}{\leftrightsquigarrow} x) \leqslant a^t$ for t large enough.

Then, on the event that $o \overset{\mathcal{C}}{\leftrightsquigarrow} x$ with $|P_t| \geqslant \varepsilon t$, we consider $\delta > 0$ and use the following coupling argument between $\xi(\lambda)$ and $\xi(\lambda-\delta)$. Start with a realisation $\xi(\lambda)$ of the Poisson point process with intensity λ. Fix a $\delta > 0$ smaller than λ. Then remove independently each point of $\xi(\lambda)$ with probability δ/λ. In the end, we obtain a new Poisson point process of intensity $\lambda-\delta$ on $\mathbb{H}^2$, thus creating a coupling between $\xi(\lambda)$ and $\xi(\lambda-\delta)$.

At $\lambda - \delta$, for the event $o \overset{\mathcal{C}}{\leftrightsquigarrow} x$ to still hold, every pivotal point needs to remain. Each one of them remains independently with probability $\frac{\lambda-\delta}{\lambda}$. Thus we have

$$\tau_{\lambda-\delta}(t) \leqslant \tau_\lambda(t) \left(\mathbb{P}_\lambda \left(|P_t| \leqslant \varepsilon t \right) + \left(\frac{\lambda - \delta}{\lambda} \right)^{\lfloor \varepsilon t \rfloor} \right).$$

Thus we can find a constant $b < 1$ so that $\tau_{\lambda-\delta}(t) \leqslant \tau_\lambda(t) b^t$ for t large enough, and we obtain :

$$\beta(\lambda - \delta) \leqslant \lim \frac{\log \tau_\lambda(t)}{t} + \log(b) < \beta(\lambda).$$

■

Lemma 5.10. *For all $\lambda > \lambda_c$, we have $\beta(\lambda) > -1$.*

Proof: Fix a $\lambda > \lambda_c$. The mean area of the cluster $\mathcal{C}(o)$ is infinite because its size is infinite with positive probability, and a.s. there exists no finite ball with an infinite number of points. Then

$$\infty = \mathbb{E}[\text{area}(\mathcal{C}(o))] = \int_{t>0} 2\pi \sinh(t) \tau_\lambda(t).$$

Since $\tau_\lambda(t)$ is smaller than $e^{(\beta(\lambda)+\varepsilon)t}$ for any ε and $t \geqslant T(\varepsilon)$, then for any $\varepsilon > 0$, it must be true that

$$\infty = \int_{t>T} 2\pi \sinh(t) \tau_\lambda(t) \leqslant \int_{t>T} 2\pi \sinh(t) e^{(\beta(\lambda)+\varepsilon)t}.$$

And it is necessary that $\beta(\lambda) \geqslant -1$ for the second integral to be divergent for any $\varepsilon > 0$. The strict increasingness of β then gives us $\beta(\lambda) > 1$. ■

It is also true that for $\lambda < \lambda_c$, we have $\beta(\lambda) \leqslant -1$. Proving this last result would end the proof of Theorem 2.2. However this needs an argument developed later.

6. Galton-Watson trees

We first want to define a regular rooted tree starting at o and so that the progenies of two vertices at the same level of the tree are independent.

Let's consider the ball of radius R around the origin, where R will be chosen later to have nice properties. Remember the notations introduced in the last section : we denote by L the horizontal hyperbolic line through the origin, and H^+ the upper half-plane. For a point $x \in \partial B(o, R)$, we let L_x be the hyperbolic line tangent to the circle $B(o, t)$ and going through x, and $H^+(x)$ the half-plane delimited by L_x that doesn't contain o. Finally let γ_x be the hyperbolic isometry sending o to x and H^+ to $H^+(x)$.

We also remind the reader of a lemma we proved in the previous section :

Proposition 6.1. *For every $x, y \in \delta B(o, R)$, let α_{xy} be the smallest angle between the rays $[ox)$ and $[oy)$. For R large enough, if $\alpha_{xy} \geqslant 4/\cosh(R)$, then $H^+(x) \cap H^+(y) = \emptyset$.*

Remark. Note that with an angle $\alpha = 4/\cosh(R)$ between the points x and y on $\partial B(o, R)$, we can use the hyperbolic cosine rule to determine the distance d between x and y :

$$\cosh(d) = \cosh(R)^2 - \cos(\alpha) \sinh(R)^2 \sim \cosh(R)^2 - \sinh(R)^2 + \left(\frac{4}{\cosh(R)} \right)^2 \sinh(R)^2 + o(1) = 17 + o_R(1).$$

Thus $d = \text{arccosh}(17) + o(1) \geqslant 2$ for R large enough.

Choose R large enough for two points x, y of $\partial B(o,R)$ at angle at least $4/\cosh(R)$ to have disjoint half-planes $H^+(x)$ and $H^+(y)$. We define the set

$$\mathcal{U}_R := \{(R, 4k/\cosh(R)), k \in \{1, \dots, \pi\cosh(R)/4 - 1\} \text{ and } 4k/\cosh(R) \in [\pi/4, 3\pi/4]\},$$

where (r,θ) are the hyperbolic polar coordinates of the points. Clearly, $\mathcal{U}_R \subset H^+$.

These sets allows us to define a regular embedded tree starting at o by iteration of the functions γ_{x_k} : we define $\mathcal{Z}_0 := o$, $\mathcal{Z}_1 := \mathcal{U}_R$, and by iteration for $k \geqslant 1$:

$$\mathcal{Z}_k := \{y \in \mathbb{H}^+, y = \gamma_{x^1}(\gamma_{x^2}(\dots(\gamma_{x^k}(o))\dots)), \text{where}\, x^1, \dots, x^k \in \mathcal{U}_R\}.$$

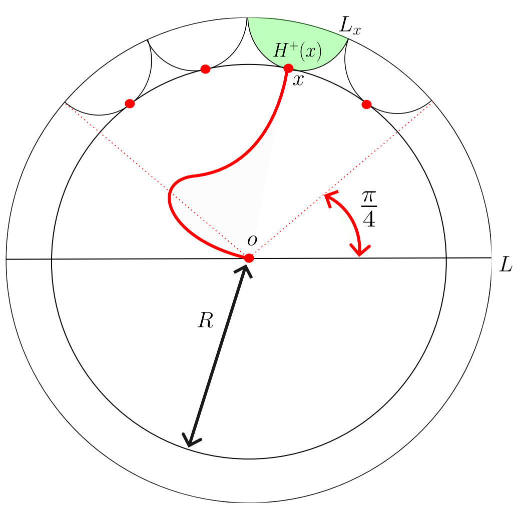
The set $\mathcal{Z} = \bigcup_k \mathcal{Z}_k$ is to be the set of vertices of the tree $\mathcal{T}$. For every $k \geqslant 1$ and $y \in \mathcal{Z}_k$, we have $y = \gamma_{x_1}(\gamma_{x_2}(\dots(\gamma_{x_k}(o))\dots))$, and we define its parent by

$$parent(y) := \gamma_{x_1}(\gamma_{x_2}(\dots(\gamma_{x_{k-1}}(o))\dots)),$$

The edges of $\mathcal{T}$ are the pairs $(parent(y), y)$ for every $y \in \mathcal{Z} \setminus \{o\}$.

Proposition 6.2. *For any $k \geqslant 1$ and $y \in \mathcal{Z}_k$, we have that $H^+(y) \subset H^+(parent(y))$.*

Moreover, for any distinct $y_1, y_2 \in \mathcal{Z}_k$, we have $H^+(y_1) \cap H^+(y_2) = \emptyset$.



Within the tree $\mathcal{T}$, we will construct a Galton-Watson tree. Towards this end, we define for each $y \in \mathcal{Z}$ the set $H_R^+(y) := H^+(y) \cap B(y,R)$. We define by iteration the sets $W_k \subset \mathcal{Z}_k$ the following way : $W_0 := \mathcal{Z}_0 = \{o\}$, and

$$W_{k+1} := \left\{ \begin{array}{c} y \in \mathcal{Z}_{k+1}, parent(y) \in W_k \\ \text{and } parent(y) \overset{H_R^+(y)}{\longleftrightarrow} y \end{array} \right\}.$$

Using Proposition 6.2, we observe that for distinct points $y_1, y_2 \in W := \cup W_k$, the sets $H_R^+(y_1)$ and $H_R^+(y_2)$ are disjoint, thus their respective sets of offsprings are independent. In addition, the sets $H_R^+(parent(y))$ and $H_R^+(y)$ are also disjoint, so the offspring of y is independent from the offspring of $parent(y)$. Due to the invariance of the model to isometries, we know that the size of the set of offsprings are i.i.d. random variables.

We denote by $\mathcal{T}_W$ the Galton-Watson tree thus defined.

Proposition 6.3. *For any $\varepsilon > 0$ for R large enough, we have*

$$\mathbb{E}[|W_1|] \geqslant \cosh(R) e^{(\beta(\lambda)-\varepsilon)R}.$$

Proof: For any point $y \in \mathcal{U}_R$, we can consider a sector $S_y(R)$ of angle $\pi/2$ centered around the ray $[0,y)$, and of length R. With this definition, $S_y \subset H_R^+(o)$, thus

$$\mathbb{P}_\lambda(o \overset{\mathcal{C}_{H_R^+(o)}}{\longleftrightarrow} y) \geqslant \mathbb{P}_\lambda(o \overset{\mathcal{C}_{S_y(R)}}{\longleftrightarrow} y).$$

For any $\varepsilon > 0$, by definition of β and Section 4, we have for R large enough :

$$\mathbb{P}_\lambda(o \overset{\mathcal{C}_{S_y(R)}}{\longleftrightarrow} y) = e^{\frac{\log \tau_{\lambda,R}(R)}{R} R} \geqslant e^{\left(\beta(\lambda) - \frac{\varepsilon}{2}\right)R}.$$

Moreover, $|\mathcal{Z}_1| \sim 1/2 \cdot \pi \cosh(R)/4$, so there exists a constant k such that $|\mathcal{Z}_1| \geqslant k\cosh(R)$ for any R. Thus for R large enough :

$$\mathbb{E}[|W_1|] = \sum_{y\in\mathcal{Z}_1} \mathbb{P}_\lambda(o \overset{\mathcal{C}_{H_R^+(o)}}{\leftrightsquigarrow} y) \geqslant k\cosh(R)e^{\left(\beta(\lambda)-\frac{\varepsilon}{2}\right)R} \geqslant \cosh(R)e^{(\beta(\lambda)-\varepsilon)R}.$$

■

Theorem 6.4. *For any $\varepsilon > 0$, there exists a $T = T(\xi) < \infty$ such that*

$$\forall t \geqslant T, N_t \geqslant e^{(\beta(\lambda)+1-\varepsilon)t}.$$

Proof: Fix some $\varepsilon > 0$. Using Proposition 6.3 and Proposition 4.7, we prove that for R large enough, we have $\mathbb{E}[|W_1|] > 1$, and the Galton-Watson tree $\mathcal{T}_W$ has a positive probability to survive. For every $k \in \mathbb{N}$, any point in W_k has to belong to a ball within $\mathcal{C}_{Rk}(o)$, meaning that $N_{Rk} \geqslant |W_k|$.

It is shown in [1] (pages 230-238) that the limit

$$\lim_{k\longrightarrow\infty} \frac{|W_k|}{\mathbb{E}[|W_1|]^k} = Z$$

exists almost surely (where Z is a non-negative random variable), and that conditioned on the survival of the tree $\mathcal{T}_W$, we have $Z > 0$ a.s. Thus, for almost any configuration ξ so that the tree survives, there exists a threshold $K = K(\xi)$ such that for any $k \geqslant K$, we have :

$$|W_k| \geqslant \left(\frac{1}{2}\mathbb{E}[|W_1|]\right)^k \geqslant e^{-k\log 2}e^{-k\log 2}e^{Rk}e^{(\beta(\lambda)-\varepsilon)Rk} \geqslant e^{(1+\beta(\lambda)-\varepsilon-2\frac{\log 2}{R})Rk}.$$

In that last line, we can increase R again and fix it so that $\frac{\log 2}{R} \leqslant \varepsilon$, hence finally, for any $k \geqslant K(\xi)$:

$$|W_k| \geqslant e^{(1+\beta(\lambda)-3\varepsilon)Rk}.$$

Then, if the tree $\mathcal{T}_W$ survives, we have for every $t \geqslant RK(\xi)$:

$$N_t \geqslant N_{R\lfloor \frac{t}{R}\rfloor} \geqslant |W_{R\lfloor \frac{t}{R}\rfloor}| \geqslant e^{(1+\beta(\lambda)-3\varepsilon)R\lfloor \frac{t}{R}\rfloor} \geqslant e^{(1+\beta(\lambda)-4\varepsilon)t}.$$

Now, although the tree $\mathcal{T}_W$ has a positive probability to survive, it is not certain. We thus define the trees $\mathcal{T}_W(y)$ the same way for every $y \in \mathcal{Z}$. For $y \in \mathcal{Z}_l$, we then define $W_l(y) := \{y\}$ and $W_k(y)$ the same way as before for any $k \geqslant l$. With probability one we can find $l \geqslant 0$ and $y \in \mathcal{Z}_l$ so that the tree $\mathcal{T}_W(y)$ survives. Following the previous arguments, we obtain a.s. a threshold $K(\xi) < \infty$ such that for any $k \geqslant K(\xi)$:

$$|W_k(y)| \geqslant \left(\frac{1}{2}\mathbb{E}[|W_1|]\right)^{k-l} \geqslant e^{(1+\beta(\lambda)-3\varepsilon)R(k-l)} \geqslant e^{(1+\beta(\lambda)-4\varepsilon)Rk},$$

and for t large enough (where "large enough" exists a.s.),

$$N_t \geqslant e^{(1+\beta(\lambda)-5\varepsilon)t}.$$

■

This Galton-Watson tree result also allows us to finish the proof of Theorem 2.2 :

Lemma 6.5. *For $\lambda < \lambda_c$, we have $\beta(\lambda) < -1$.*

Proof: By strict increasingness, we only need to prove that $\beta(\lambda) \leqslant -1$. Suppose that there exists $\lambda < \lambda_c$ with $\beta(\lambda) > -1$. Then using Proposition 6.3, the tree as defined has a positive probability to survive, which in turn means that the cluster $\mathcal{C}(o)$ has a positive probability to be infinite. This is impossible since $\lambda < \lambda_c$. ■

7. A quick note on further results

We have in these few pages only proven Theorem 2.2 and the lower bound in theorem Theorem 2.3. The upper bound of that theorem was proven using a moment argument.

The methods used for the inequalities of Theorem 2.4 and Theorem 2.5 were pretty similar to those presented here. In fact, below λ_c, the lower bound was also obtained by finding trees inside our clusters. The upper inequality though was a lot trickier and a new exploration process had to be defined to make sure to count every single point within a set cluster.

As a final remark, note that some of the results presented here for the covered cluster have an equivalent for the vacant cluster. While it is kind of intuitive that it should be true, the vacant clusters have a different geometry and different rules than the covered clusters. Most notably : a point is vacant if there is an empty ball around it, so instead of "a point covers an area", it becomes "an empty area means an empty point", which proves to be problematic in the proofs.

To be more specific, we have proven within the proof of Theorem 2.5 that the equivalent function β^{vacant} is strictly decreasing in λ by adapting the pivotal point argument in Theorem 5.9. In addition, the Galton-Watson tree construction can be adapted to suit the vacant clusters, and thus the arguments that come from it still hold.

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