The Connes-Embedding Problem and some related topics

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Introduction

This thesis is a short view of what I did during my second year internship in University of Copenhagen, Denmark, under the supervision of Magdalena Musat. During this internship, I had the opportunity to attend some Master courses about the field of operator algebras, namely Introduction to Operator Algebras and K-Theory for C*-algebras courses. I was also able to attend a lot of seminars about Groups and Operator Algebras, and some other conferences. On this occasion, I met some famous mathematicians that have a huge importance in the field. That being said, the main part of my internship was about studying the famous Connes Embedding Problem, and this is what I am going to deal with in this thesis.

At the beginning of the 20^{th} century, Hilbert spaces were introduced as a generalization of euclidean vector spaces, for interests both in mathematics and in physics. An interesting object that comes with a Hilbert space is the set of bounded linear operators on a Hilbert space: this is where operator algebras start. Among the different types of operator algebras, one can deal with Banach algebras, C*-algebras,...

In the middle of the 20^{th} century, John von Neumann took interest in a particular kind of operator algebras, the so-called von Neumann algebras, that are subalgebras of $B(H)$ closed under some operations and a specific topology. With von Neumann algebras came a lot of questions.

Among all the people that contribute to von Neumann algebras, one can not forget to deal with Alain Connes, who made a lot of contributions during the second half of the 20^{th} century, solving a lot of open problems, stating other problems or conjectures about von Neumann algebras. In 1976, he stated in one of his famous papers the so-called Connes Embedding Problem, asking whether every von Neumann algebra (with some specific properties) embeds into one particular von Neumann algebra.

In the first part of this thesis, we give some preliminaries that are used along all the thesis. In the second part, we focus more specifically on von Neumann algebras and their fundamental properties. In the third part, we prove that an ultraproduct of von Neumann algebras with traces is still a von Neumann algebra with a trace. In the fourth part, we state the Connes-Embedding Problem in itself and give a reformulation of it. Finally in the fifth part, we give a view of what other things were done during the internship linked to the Connes-Embedding Problem, especially with tensor products on C*-algebras.

I sincerly thank Magdalena Musat, who accepted to take me in an internship and supervised me during all these four months, always giving me good advices and resources to work on this project. I am also grateful to Cyril Houdayer, my mentor at ENS, for making my internship in Copenhagen possible. Finally, I thank Søren Eilers and Mikael Rørdam, the teachers I had in University, and Pieters Spaas and James Hyde, that organized the Group and Operator Algebras seminars each week, and who always found time to talk with me about my internship.

1 Preliminaries

This section gives some preliminaries that are used in the thesis. We first remind some properties of Hilbert spaces and bounded operators on Hilbert spaces. Then we give an introduction to Banach algebras and C*-algebras, and some fundamental theorems that we are using along the following parts.

1.1 The bounded operators on a Hilbert space

Hilbert spaces are a generalization of Euclidean spaces, allowing infinite dimension. In this subsection, we remind the definitions and give some results about linear operators on a Hilbert space.

Definition 1.1.
Let E be a complex vector space. An inner product on E is a map $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathbb{C}$ satisfying the following properties.

(i) It is linear in the first variable: for $x_1, x_2, y \in E$, for $\alpha \in \mathbb{C}$, $\langle \alpha x_1 + x_2, y \rangle = \alpha \langle x_1, y \rangle + \langle x_2, y \rangle$.

(ii) It is conjugate symmetric: for $x, y \in E$, $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

(iii) It is positive definite: for $x \in E$, $\langle x, x \rangle \in \mathbb{R}_+$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

It follows that $\|x\| = \sqrt{\langle x, x \rangle}$ induces a norm on E.

Remark 1.2.
Items (i) and (ii) gives that an inner product is conjugate linear in the second variable: for $x, y_1, y_2 \in E$, $\beta \in \mathbb{C}$, we have $\langle x, \beta y_1 + y_2 \rangle = \bar{\beta} \langle x, y_1 \rangle + \langle x, y_2 \rangle$.

Definition 1.3.
A Hilbert space H is a complex vector space, equipped with an inner product $\langle \cdot, \cdot \rangle$, that is complete when equipped with the norm induced by the inner product.

Definition 1.4.
For $T : H \longmapsto H$ a linear map, let $\|T\| = \sup\limits_{\xi \in H, \|\xi\| \leq 1} \|T\xi\|$. We know that T is continuous if and only if $\|T\| < +\infty$. We then set $B(H) = \{T : H \longrightarrow H \text{ linear } \mid \|T\| < +\infty\}$ the set of bounded linear operators on H. Then it is a complex algebra (the product is the composition) with unit the identity operator, and $\| \cdot \|$ is a norm on $B(H)$.

Proposition 1.5.
$B(H)$ is complete when equipped with the norm above.

Proof. Let $(T_n)_{n \geq 1}$ be a Cauchy sequence in $B(H)$.
For $n, m \geq 1$, $\xi \in H$, $\|T_n \xi - T_m \xi\| \leq \|T_n - T_m\| \|\xi\|$, thus $(T_n \xi)_{n \geq 1}$ is Cauchy in H Hilbert space (in particular complete), hence it converges to a vector that we name $T\xi \in H$.
Using linearity of T_n, it is straightforward to check that T is linear. We now want to show that $T \in B(H)$, i.e it is continuous.
Since $(T_n)_{n \geq 1}$ is Cauchy, there is $n_0 \geq 1$ such that for $n, m \geq n_0$, $\|T_n - T_m\| \leq 1$. In particular

for $\xi \in H$, $\|\xi\| \leq 1$, $\|T_n\xi - T_m\xi\| \leq 1$, so taking the limit on m, $\|T_n\xi - T\xi\| \leq 1$. Thus, $\|T\xi\| \leq 1 + \|T_{n_0}\xi\| \leq 1 + \|T_{n_0}\|$. It follows that T is bounded, as wanted.
Finally, we need to check that $T_n \longrightarrow T$. Let $\varepsilon > 0$, there is $N \geq 1$ such that for $n, m \geq N$, $\|T_n - T_m\| \leq \varepsilon$. Let $\xi \in H$, $\|\xi\| \leq 1$. Then $\|T_n\xi - T_m\xi\| \leq \varepsilon$ for $n, m \geq N$. Taking the limit on m : for $n \geq N$, $\|T_n\xi - T\xi\| \leq \varepsilon$. This is true for any $\xi \in H$ with $\|\xi\| \leq 1$. As a consequence, $\|T_n - T\| \leq \varepsilon$ whenever $n \geq N$, so $(T_n)_{n\geq 1}$ converges to $T \in B(H)$. □

Theorem 1.6 (Riesz representation theorem)**.**
A map $F : H \longrightarrow H$ is a bounded linear operator (on a Hilbert space H) if and only if there is $\eta \in H$ such that for all $\xi \in H$, $F\xi = \langle \xi, \eta \rangle$.

Remark 1.7.
For $T \in B(H)$, it follows from Riesz representation theorem that there is a unique $T^* \in B(H)$ satisfying, for all $\xi, \eta \in H$, $\langle \xi, T^*\eta \rangle = \langle T\xi, \eta \rangle$. We call it the adjoint of T.

Proposition 1.8.
Let $T \in B(H)$. Then $\|T^*T\| = \|T\|^2$.

Proof. Let $\xi \in B(H)$ with $\|\xi\| \leq 1$. We have:

$$\|T\xi\|^2 = \langle T\xi, T\xi \rangle = \langle T^*T\xi, \xi \rangle \underset{\text{Cauchy-Schwarz}}{\leq} \|T^*T\xi\|\|\xi\| \leq \|T^*T\|.$$

Taking the supremum on ξ, $\|T\|^2 \leq \|T^*T\|$.
Conversly, for $\xi \in H$: we know that $\|T^*T\xi\| = \sup\limits_{\eta \in H, \|\eta\| \leq 1} |\langle T^*T\xi, \eta \rangle|$. But for $\eta \in H$ with $\|\eta\| \leq 1$, we have $|\langle T^*T\xi, \eta \rangle| = |\langle T\xi, T\eta \rangle| \underset{\text{Cauchy-Schwarz}}{\leq} \|T\xi\|\|T\eta\| \leq \|T\|^2$. It follows that $\|T^*T\xi\| \leq \|T\|^2$, and finally $\|T^*T\| \leq \|T\|^2$.
Therefore, $\|T^*T\| = \|T\|^2$. □

1.2 Banach algebras and C*-algebras

Taking inspiration from the structure of $B(H)$, we define some particular operator algebras, namely Banach algebras and C*-algebras, and give some properties on their general structures.

Definition 1.9.
A (unital) Banach algebra $(A, +, \times, \cdot, 1, \|\cdot\|)$ is defined in the following way:

(i) $(A, +, \times, \cdot, 1)$ is a (unital) complex algebra,

(ii) $(A, +, \cdot, \|\cdot\|)$ is a complete complex vector space,

(iii) for all $x, y \in A$, $\|x \times y\| \leq \|x\| \times \|y\|$,

(iv) $\|1\| = 1$.

Definition 1.10.
A (unital) C*-algebra is a Banach algebra A equipped with a mapping $* : a \in A \longmapsto a^* \in A$ satisfying the following properties:

(i) for all $a \in A$, $(a^*)^* = a$,

(ii) for all $a, b \in A$, for all $\alpha \in \mathbb{C}$, $(a + \alpha b)^* = a^* + \bar{\alpha} b^*$,

(iii) for all $a, b \in A$, $(ab)^* = b^* a^*$,

(iv) for all $a \in A$, $\|a^* a\| = \|a\|^2$.

Then for $a \in A$, a^* is called the adjoint of a.

Definition 1.11.
Let A, B be (unital) C*-algebras, a (unital) *-homomorphism $\varphi : A \longrightarrow B$ is a linear multiplicative map, satisfying for all $a \in A$ $\varphi(a^*) = \varphi(a)^*$, and $\varphi(1_A) = 1_B$.

Remark 1.12.
One could define non-unital Banach algebras, C*-algebras, *-homomorphisms, removing all the axioms about 1, but in this thesis we will always work with unital objects and we won't precise anymore that the objects are unital.

Definition 1.13.
Let A be a C*-algebra. A sub-C*-algebra $B \subseteq A$ is a subset containing the unit, closed under the algebraic operations, closed under taking adjoints and closed with respect to the norm.
If it is not necessarly closed with respect to the norm, we just say that it is a sub-*-algebra.

Proposition 1.14.
Let A, B be C*-algebras and $\varphi : A \longrightarrow B$ be a *-homomorphism. Then $\varphi(A)$ is a sub-C*-algebra of B.

Example 1.15.
The following examples are all C*-algebras.

1. Let $A = \mathbb{C}$ equipped with usual operations, the absolute value and the adjoint $z^* = \bar{z}$, then it is a C*-algebra.

2. Let K be a compact Hausdorff space, $A = \mathcal{C}(K)$ equipped with pointwise operations, norm $\|\cdot\|_\infty$ and with the following adjoint operation: $f^*(x) = \overline{f(x)}$ for any $x \in K$. Then it is a C*-algebra. Actually, it turns out that every commutative C*-algebra A is isomorphic to $\mathcal{C}(K_A)$ for some compact Hausdorff space K_A.

3. Let H be a Hilbert space, then $B(H)$ is a C*-algebra with the usual operations and adjoint.

4. In particular for $n \geq 1$, $\mathcal{M}_n(\mathbb{C}) = B(\mathbb{C}^n)$ is a C*-algebra.

Definition 1.16.
Let A be a C*-algebra and let $a \in A$.

(i) We say that a is normal if $aa^* = a^* a$.

(ii) We say that a is self-adjoint if $a^* = a$.

(iii) We say that a is positive if there is $b \in A$ such that $a = b^* b$.

(iv) We say that a is a projection if $a = a^* = a^2$.

(v) We say that a is a unitary if $a^* a = aa^* = 1$.

(vi) We say that a is an isometry if $a^*a = 1$.

(vii) We say that a is a partial isometry if a^*a and aa^* are projections.

Remark 1.17.
A projection is always positive, because $p = p^*p$. Note that a positive element is always self-adjoint and a self-adjoint is always normal, as well as a unitary. However, isometries and partial isometries can be not normal.

Definition 1.18.
We define a (partial) order on the set of self-adjoint elements A_{sa}, given by $a \leq b$ if $b - a$ is positive, for $a, b \in A_{\text{sa}}$.

Definition 1.19.
Let A be a C*-algebra and $\varphi : A \longrightarrow \mathbb{C}$ be a continuous linear map.

(i) We say that φ is positive if for all $a \in A$, if a is positive then $\varphi(a) \geq 0$.

(ii) We say that φ is a state if φ is positive and $\varphi(1) = 1$. It is equivalent to say that $\|\varphi\| = \varphi(1) = 1$.

(iii) We say that φ is tracial if for all $a, b \in A$, $\varphi(ab) = \varphi(ba)$.

(iv) We say that φ is faithful if for all $a \in A$, $\varphi(a^*a) = 0 \implies a = 0$.

We write $S(A) = \{\varphi : A \longrightarrow \mathbb{C} \mid \varphi \text{ is a state}\}$.

Lemma 1.20.
Let A be a C*-algebra and $a \in A$. Then $a = 0$ if and only if for all $\varphi \in S(A)$, $\varphi(a) = 0$.

Theorem 1.21 (Gelfand-Neumark-Segal (GNS) construction)**.**
Let A be a C*-algebra and let φ be a state on A. Then there is a triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$ such that:

(i) H_φ is a Hilbert space, $\xi_\varphi \in H_\varphi$ with $\|\xi_\varphi\| = 1$ and $\pi_\varphi : A \longrightarrow B(H_\varphi)$ is a *-homomorphism,

(ii) for all $a \in A$, $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$,

(iii) ξ_φ is a cyclic vector in H_φ, namely $\pi_\varphi(A)\xi_\varphi$ is dense in H_φ.

Remark 1.22.
Combining lemma 1.18 with the GNS construction, one can show that for any C*-algebra A, there is an injective *-homomorphism $\pi : A \longrightarrow B(H)$ for some Hilbert space H (in fact we take $H = \bigoplus_{\varphi \in S(A)} H_\varphi, \pi = \bigoplus_{\varphi \in S(A)} \pi_\varphi$). In particular, any C*-algebra is isomorphic to a sub-C*-algebra of $B(H)$ for some H.

Definition 1.23.
Let A be a C*-algebra. A *-representation is a *-homomorphism $\pi : A \longrightarrow B(H)$ for some Hilbert space H. The representation is called faithful if π is injective.

2 On von Neumann algebras

A central object in operator algebras are von Neumann algebras, which are sub-C*-algebras of $B(H)$ that are closed in a specific topology. In this section, we present what von Neumann algebras are and what basic facts are useful in this thesis.

2.1 Topologies on $B(H)$

There is a natural topology on $B(H)$ induced by the norm $\|\cdot\|$. But this is a very strong topology, and we would like to have weaker ones, inducing more freedom. That is why we introduce the strong-operator and weak-operator topologies in this subsection.

Definition 2.1.
Let H be a Hilbert space. We define the following topologies on $B(H)$:

(i) the strong-operator topology (SOT) is the locally convex topology generated by the semi-norms $\|T\|_\xi = \|T\xi\|$ for $\xi \in H$,

(ii) the weak-operator topology (WOT) is the locally convex topology generated by the semi-norms $\|T\|_{\xi,\eta} = |\langle T\xi, \eta\rangle|$ for $\xi, \eta \in H$.

Remark 2.2.
For a net (T_α) in $B(H)$ and $T \in B(H)$, it follows that $T_\alpha \overset{\text{SOT}}{\longrightarrow} T$ if and only if $T_\alpha\xi \longrightarrow T\xi$ for all $\xi \in H$. Besides, $T_\alpha \overset{\text{WOT}}{\longrightarrow} T$ if and only if $\langle T_\alpha\xi, \eta\rangle \longrightarrow \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$.
In particular: $T_\alpha \overset{\|\cdot\|}{\longrightarrow} T \implies T_\alpha \overset{\text{SOT}}{\longrightarrow} T \implies T_\alpha \overset{\text{WOT}}{\longrightarrow} T$ (the last implication follows from Cauchy-Schwarz inequality).

Proposition 2.3.
Let $S \subseteq B(H)$ be a convex subset. Then the SOT-closure and the WOT-closure of S are equal.

Definition 2.4.
A von Neumann algebra is a sub-C*-algebra $\mathcal{M} \subseteq B(H)$ that is closed in the SOT-topology (or equivalently, in the WOT-topology).

Definition 2.5.
A linear functional $\varphi : \mathcal{M} \longrightarrow \mathbb{C}$ is normal if, for any increasing net (x_α) of self-adjoint elements that converges to some $x \in \mathcal{M}$ in the SOT-topology, we have that $\varphi(x_\alpha) \longrightarrow \varphi(x)$.

Definition 2.6.
Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra. The commutant of $\mathcal{M}$ is defined as the set $\mathcal{M}' = \{S \in B(H) \mid \forall T \in \mathcal{M}, ST = TS\}$. It turns out that $\mathcal{M}' \subset B(H)$ is a von Neumann algebra. The set $\mathcal{M} \cap \mathcal{M}'$ is called the center of the von Neumann algebra
A von Neumann algebra is called a factor if it has trivial center, i.e $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$.

2.2 Equivalence of projections and decomposition into types

In von Neumann algebras, projections are central objects, and understanding the behaviour of its projections tells a lot about the structure of a von Neumann algebra. It is even a way to classify von Neumann algebras, using the decomposition into types.

Definition 2.7.
Let $\mathcal{M}$ be a von Neumann algebra and let $p, q \in \mathcal{M}$ be two projections. We say that p, q are Murray-von-Neumann equivalent, and we write $p \sim q$, if there exists $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$.

Example 2.8.
If $\mathcal{M} = B(H)$, then $p \sim q$ if and only if $\dim(p(H)) = \dim(q(H))$, and then v maps $p(H)$ onto $q(H)$ isometrically, and is 0 on $p(H)^\perp$.

Definition 2.9.
Let $p, q \in \mathcal{M}$ be projections in a von Neumann algebra. We say that $p \leq q$ if $q - p$ is positive, equivalently if $pq = qp = p$, i.e if $p(H) \subseteq q(H)$.

Definition 2.10.
Let $\mathcal{M}$ be a von Neumann algebra, and let $p \in A$ be a projection:

- p is minimal if $p\mathcal{M}p = \mathbb{C}p$, or equivalently if for every projection $q \in \mathcal{M}$, $q \leq p \implies q = 0$ or $q = p$,
- p is abelian if $p\mathcal{M}p$ is abelian,
- p is finite if for every $q \in \mathcal{M}$, $p \sim q \leq p \implies q = p$,
- p is infinite if it is not finite.

Definition 2.11.
A von Neumann algebra $\mathcal{M}$ is:

- type I if every non-zero projection majorizes a non-zero abelian projection, or equivalently majorizes a non-zero minimal projection,
- type II if it has no non-zero abelian (or equivalently minimal) projection, but every non-zero projection majorizes a non-zero finite projection,
- type III if it has no finite projection,
- finite if $I \in \mathcal{M}$ is finite,
- infinite if it is not finite,
- type II_1 if it is type II and finite,
- type II_∞ if it is type II and infinite.

Theorem 2.12 (Decomposition into types)**.**
Every von Neumann algebra $\mathcal{M}$ can be written in a unique way as a direct sum (where $\mathcal{M}_X$ is of type X) $\mathcal{M} = \mathcal{M}_I \oplus \mathcal{M}_{II_1} \oplus \mathcal{M}_{II_\infty} \oplus \mathcal{M}_{III}$.

3 Ultraproducts of von Neumann algebras

Now that we have define what von Neumann algebras are, we want to find what constructions we can do on a family of von Neumann algebras. It turns out that if the von Neumann algebras are tracial, then there is a natural way to define ultraproducts, and these ultraproducts are still tracial von Neumann algebras. It is a non-trivial fact, and this is the goal of this section.

3.1 Traces and products

In this subsection, we give some definitions about traces and products of C*-algebras.

Definition 3.1.
A von Neumann algebra $\mathcal{M}$ is tracial if it is equipped with a faithful normal tracial state $\tau : \mathcal{M} \longrightarrow \mathbb{C}$.

Theorem 3.2.
Let $\mathcal{M}$ be a finite factor. Then $\mathcal{M}$ has one and only one tracial state, which is besides faithful and normal.

Definition 3.3.
Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra.
Then $(x, y) \in \mathcal{M}^2 \longmapsto (x \mid y) = \tau(y^*x)$ is a inner product. The trace norm $\|\cdot\|_\tau$ on $\mathcal{M}$ is the norm induced by the inner product: for $x \in \mathcal{M}$, $\|x\|_\tau = \sqrt{(x \mid x)} = \sqrt{\tau(x^*x)}$.

Proposition 3.4.
Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra. Then we have the following results.

(i) For all $x, y \in \mathcal{M}$, $|\tau(y^*x)| = |(x \mid y)| \leq \|x\|_\tau \times \|y\|_\tau$.

(ii) For all $x, y \in \mathcal{M}$, $\|xy\|_\tau \leq \|x\| \times \|y\|_\tau$ and $\|xy\|_\tau \leq \|x\|_\tau \times \|y\|$.

(iii) For all $x \in \mathcal{M}$, $\|x\|_\tau \leq \|x\|$.

Now we fix I an index set, and $(\mathcal{M}_i, \tau_i)_{i\in I}$ a family of tracial von Neumann algebras.

Definition 3.5.
We define the product of the von Neumann algebras, and we note $\prod_{i\in I} \mathcal{M}_i$ (or $\ell^\infty(I; (\mathcal{M}_i)_{i\in I})$), the set:

$$\prod_{i\in I} \mathcal{M}_i = \{(x_i)_{i\in I} \mid \forall i \in I, x_i \in \mathcal{M}_i; \sup_{i\in I} \|x_i\|_i < +\infty\} .$$

It is a C*-algebra, when equipped with pointwise operations and $\|(x_i)_{i\in I}\| = \sup_{i\in I} \|x_i\|_i$.

Question: can we make it tracial? We would like to be able to define a "limit" of $\tau_i(x_i)$.

3.2 Ultrafilters and limits

In this subsection, we introduce what an ultrafilter is and what it means for a family to converge along an ultrafilter.

Definition 3.6.
An (ultra)filter ω on I is a subset $\omega \subseteq \mathcal{P}(I)$ such that:

(i) $I \in \omega, \varnothing \notin \omega$;

(ii) for all $A, B \subseteq I$, if $A \subseteq B$ and $A \in \omega$ then $B \in \omega$;

(iii) for all $A, B \in \omega$, $A \cap B \in \omega$;

(iv) for all $A \subseteq I$, either $A \in \omega$ or $I \setminus A \in \omega$ (ultrafilter).

Remark 3.7.
It turns out that ultrafilters are exactly the maximal filters for the inclusion.

Definition 3.8.
An ultrafilter ω on I is principal if it has a least element $\{i_0\}$.
In that case $\omega = \omega_{i_0} = \{X \subseteq I \mid i_0 \in X\}$.
An ultrafilter is free if it is not principal.

Definition 3.9.
Let ω be an ultrafilter on I and let $(\alpha_i)_{i \in I}, \alpha$ be complex numbers. We say that α_i converges to α with respect to ω, and we write $\lim\limits_{i \to \omega} \alpha_i = \alpha$, if:

$$\forall \varepsilon > 0, \{i \in I \mid |\alpha_i - \alpha| < \varepsilon\} \in \omega \ .$$

We call α the ω-limit of $(\alpha_i)_{i \in I}$.

Proposition 3.10.
Let ω be an ultrafilter on I, $(\alpha_i)_{i \in I}, (\beta_i)_{i \in I}$ be families of complex numbers. The following results hold.

(i) If the ω-limit of $(\alpha_i)_{i \in I}$ exists, then it is unique.

(ii) If $(\alpha_i)_{i \in I}$ is bounded, then it has a (unique) ω-limit, and:

$$\left|\lim_{i \to \omega} \alpha_i\right| \leq \sup_{i \in I} |\alpha_i| \ .$$

(iii) If $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$ converge with respect to ω, then $(\alpha_i + \beta_i)_{i \in I}$ converges and we have $\lim\limits_{i \to \omega} (\alpha_i + \beta_i) = \lim\limits_{i \to \omega} \alpha_i + \lim\limits_{i \to \omega} \beta_i$. Besides, if $(\alpha_i)_{i \in I}$ is bounded, then $(\alpha_i \times \beta_i)_{i \in I}$ converges, and $\lim\limits_{i \to \omega} (\alpha_i \times \beta_i) = \left(\lim\limits_{i \to \omega} \alpha_i\right) \times \left(\lim\limits_{i \to \omega} \beta_i\right)$.

(iv) If $(\alpha_i)_{i \in I}, (\beta_i)_{i \in I}$ are convergent families in $\mathbb{R}$ such that for all $i \in I$, $m \leq \alpha_i \leq \beta_i \leq M$, then $m \leq \lim\limits_{i \to \omega} \alpha_i \leq \lim\limits_{i \to \omega} \beta_i \leq M$.

(v) If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is continuous and if $(\alpha_i)_{i \in I}$ converges, then $(f(\alpha_i))_{i \in I}$ converges and $\lim\limits_{i \to \omega} f(\alpha_i) = f\left(\lim\limits_{i \to \omega} \alpha_i\right)$.

3.3 The tracial ultraproduct

Using ultrafilters and limits along an ultrafilter, we have a way to find some ideals in the product, and then taking the quotient we have a way to build some ultraproducts, in which we are able to define faithful traces.

Proposition 3.11.
Let ω be an ultrafilter on I. Then $\tau_\omega((x_i)_{i\in I}) = \lim\limits_{i\to\omega} \tau_i(x_i)$ is a well-defined tracial state on $\prod\limits_{i\in I} \mathcal{M}_i$.

Remark 3.12.
It is not faithful in general!

Proposition 3.13.
Let $\mathcal{I}_\omega = \{(x_i)_{i\in I} \mid \lim\limits_{i\to\omega} \|x_i\|_{\tau_i} = 0\}$. Then $\mathcal{I}_\omega$ is a closed two-sided ideal in $\prod\limits_{i\in I} \mathcal{M}_i$.

Definition 3.14.
We define the tracial ultraproduct with respect to an ultrafilter ω on I to be the set:

$$\prod_{i\to\omega} \mathcal{M}_i = \left(\prod_{i\in I} \mathcal{M}_i\right) \Big/ \mathcal{I}_\omega \ .$$

It is a C*-algebra, when equipped with *ad hoc* operations. If all $\mathcal{M}_i$ are equal to a same von Neumann algebra $\mathcal{M}$, then we call it the tracial ultrapower.

Remark 3.15.
One can check that if $\omega = \omega_{i_0}$ is principal, then $\prod\limits_{i\to\omega_{i_0}} \mathcal{M}_i \simeq \mathcal{M}_{i_0}$ so the case that is really interesting is the one where ω is free.

Proposition 3.16.
The application $\tau([(x_i)_{i\in I}]) = \tau_\omega((x_i)_{i\in I})$ is a well-defined faithful tracial state on $\prod\limits_{i\to\omega} \mathcal{M}_i$.

Question: is $\prod\limits_{i\to\omega} \mathcal{M}_i$ a von Neumann algebra?

3.4 A caracterisation of von Neumann algebras

We are going to show that the ultraproduct is in fact a C*-algebra. For this, we prove a theorem caracterising von Neumann algebras by completeness of the unit ball for the trace norm.

Theorem 3.17.
Let $\mathcal{M} \subseteq B(H)$ (where H is a Hilbert space) be a C*-algebra equipped with a faithful tracial state τ. The following are equivalent:

(i) $\mathcal{M}$ is a von Neumann algebra and τ is normal;

(ii) $(\mathcal{M})_1$ (the unit ball of $\mathcal{M}$) is complete with respect to the trace norm $\|\cdot\|_\tau$.

Remark 3.18.
Let $(\pi_\tau, H_\tau, \xi_\tau)$ be the GNS representation of $(\mathcal{M}, \tau)$. Then $\mathcal{M}$ is $*$-isometrically isomorphic to $\pi_\tau(\mathcal{M}) \subseteq B(H_\tau)$.
As a consequence $\mathcal{M}$ is a von Neumann algebra if and only if $\pi_\tau(\mathcal{M})$ is a von Neumann algebra. In fact, if $\Phi : \mathcal{N} \longrightarrow B(K)$ is a $*$-monomorphism (where K is a Hilbert space), then $\Phi(\mathcal{N})$ is WOT-closed: one can adapt the proof of corollary 7.1.16 in [KR97] to show that $\Phi : (\mathcal{N})_1 \longrightarrow (B(K))_1$ is WOT-continuous, so $(\Phi(\mathcal{N}))_1$ is WOT-compact hence WOT-closed, and by corollary 19.6 in [Zhu93], $\Phi(\mathcal{N})$ is a von Neumann algebra.

Lemma 3.19.
The application $\tilde{\tau} : y \mapsto \langle y\xi_\tau, \xi_\tau\rangle$ is a faithful tracial state on $\pi_\tau(A)$.
Furthermore $(A, \|\cdot\|_\tau) \longrightarrow (\pi_\tau(A), \|\cdot\|_{\tilde{\tau}})$ is an isometric isomorphism.

Corollary 3.20.
(ii) is equivalent to (ii'): $(\pi_\tau(\mathcal{M}))_1$ is complete with respect to $\|\cdot\|_{\tilde{\tau}}$.

Lemma 3.21.
Assume that $\mathcal{M}$ is a von Neumann algebra. Then τ is normal if and only if $\tilde{\tau}$ is normal.

Corollary 3.22.
(i) is equivalent to (i'): $\pi_\tau(\mathcal{M})$ is a von Neumann algebra and $\tilde{\tau}$ is normal.

Proof. Now we can prove the theorem, showing that (i') and (ii') are equivalent.

$((i') \implies (ii'))$. Assume that $\pi_\tau(\mathcal{M})$ is a von Neumann algebra and that $\tilde{\tau}$ is normal. We want to show that $(\pi_\tau(\mathcal{M}))_1$ is $\|\cdot\|_{\tilde{\tau}}$-complete.
Let $(T_n)_{n\geq 1}$ a Cauchy sequence in $(\pi_\tau(\mathcal{M}))_1$. We build a limit pointwise and show this is the expected limit.
Let $S \in \pi_\tau(\mathcal{M})$. Then for all $n, m \geq 1$:

$$\begin{aligned}
\|T_n S\xi_\tau - T_m S\xi_\tau\|^2 &= \|(T_n - T_m)S\xi_\tau\|^2 \\
&= \langle (T_n - T_m)S\xi_\tau, (T_n - T_m)S\xi_\tau\rangle \\
&= \langle ((T_n - T_m)S)^*(T_n - T_m)S\xi_\tau, \xi_\tau\rangle \\
&= \|(T_n - T_m)S\|_{\tilde{\tau}}^2 \\
&\leq \|T_n - T_m\|_{\tilde{\tau}}^2 \times \|S\|^2 \ .
\end{aligned}$$

Thus $(T_n S\xi_\tau)_{n\geq 1}$ is a Cauchy sequence in H_τ so it converges to some $\eta_S \in H_\tau$.
Now we define $\widetilde{T} : S\xi_\tau \in \pi_\tau(\mathcal{M})\xi_\tau \longmapsto \eta_S \in H$. It is well-defined: if $S_1\xi_\tau = S_2\xi_\tau$ then $(S_1 - S_2)\xi_\tau = 0$, so $\|S_1 - S_2\|_{\tilde{\tau}} = 0$: $S_1 = S_2$. It is straightforward to check this is a linear map. Besides for $n \geq 1, S \in \pi_\tau(\mathcal{M})$, $\|T_n S\xi_\tau\| \leq \underbrace{\|T_n\|}_{\leq 1} \times \|S\xi_\tau\| \leq \|S\xi_\tau\|$, so $\|\widetilde{T}\| \leq 1$. Since $\pi_\tau(\mathcal{M})\xi_\tau$ is dense in H_τ, it extends to some $T \in B(H_\tau)$, with $\|T\| \leq 1$. We want to show that $T \in \pi_\tau(\mathcal{M})$ and that $T_n \underset{\|\cdot\|_{\tilde{\tau}}}{\longrightarrow} T$.

First we are going to show that $T_n \overset{\text{SOT}}{\longrightarrow} T$, and since $(\pi_\tau(\mathcal{M}))$ is SOT-closed (because it is a von Neumann algebra), that will be enough to get the result.
Let $h \in H_\tau$. We want to show that $T_n h \longrightarrow Th$.
Let $\varepsilon > 0$. Since $\pi_\tau(\mathcal{M})\xi_\tau$ is dense, there exists $S \in \pi_\tau(\mathcal{M})$ such that $\|S\xi_\tau - h\| \leq \varepsilon$. Since

$T_nS\xi_\tau \longrightarrow TS\xi_\tau$, there exists $N \geq 1$ such that for all $n \geq N$, $\|T_nS\xi_\tau - TS\xi_\tau\| \leq \varepsilon$. Then for $n \geq N$, we have:

$$\begin{aligned}\|T_nh - Th\| &\leq \|T_nh - T_nS\xi_\tau\| + \underbrace{\|T_nS\xi_\tau - TS\xi_\tau\|}_{\leq\varepsilon} + \|TS\xi_\tau - Th\| \\ &\leq \underbrace{\|T_n\|}_{\leq 1} \times \underbrace{\|h - S\xi_\tau\|}_{\leq\varepsilon} + \varepsilon + \underbrace{\|T\|}_{\leq 1} \times \underbrace{\|S\xi_\tau - h\|}_{\leq\varepsilon} \leq 3\varepsilon \ .\end{aligned}$$

So $T_nh \longrightarrow Th$ for all $h \in H$: $T_n \xrightarrow{\text{SOT}} T$. Thus $T \in \pi_\tau(\mathcal{M})$.
Now, note that:

$$\begin{aligned}\|T_n - T\|_{\tilde{\tau}} &= \langle (T_n - T)^*(T_n - T)\xi_\tau, \xi_\tau\rangle \\ &= \|(T_n - T)\xi_\tau\|^2 \\ &= \|T_n\xi_\tau - T\xi_\tau\|^2 \longrightarrow 0 \text{ by SOT-convergence.}\end{aligned}$$

So $T_n \longrightarrow T$ for the trace norm: $(\pi_\tau(\mathcal{M}))_1$ is complete.
Note that we didn't use the fact that $\tilde{\tau}$ is normal.

($(ii') \implies (i')$) Assume that $(\pi_\tau(\mathcal{M}))_1$ is $\|\cdot\|_{\tilde{\tau}}$-complete. First we are going to show that $\pi_\tau(\mathcal{M})$ is a von Neumann algebra.
By a corollary of the Kaplansky density theorem (see [Zhu93] corollary 19.6), $\pi_\tau(\mathcal{M})$ is a von Neumann algebra if and only if $(\pi_\tau(\mathcal{M}))_1$ is SOT-closed. Let $T \in \overline{(\pi_\tau(\mathcal{M}))_1}^{\text{SOT}}$, then there is a net $(T_\alpha)_{\alpha\in\Lambda}$ in $(\pi_\tau(\mathcal{M}))_1$ such that $T_\alpha \xrightarrow{\text{SOT}} T$. Then $\|T_\alpha\xi_\tau - T\xi\tau\| \longrightarrow 0$. But for $\alpha, \beta \in \Lambda$, $\|T_\alpha - T_\beta\|_{\tilde{\tau}} = \|T_\alpha\xi_\tau - T_\beta\xi_\tau\|$, thus (T_α) is a Cauchy net for $\|\cdot\|_{\tilde{\tau}}$ in $(\pi_\tau(\mathcal{M}))_1$. By assumption, it converges to some $T' \in (\pi_\tau(\mathcal{M}))_1$. We aim at showing $T' = T$.
For $S \in \pi_\tau(\mathcal{M})$, we have:

$$\|T_\alpha S\xi_\tau - T'S\xi_\tau\| = \|(T\alpha - T')S\|_{\tilde{\tau}} \leq \|S\| \times \|T_\alpha - T'\|_{\tilde{\tau}} \underset{\alpha}{\longrightarrow} 0 \ .$$

So $T_\alpha S\xi_\tau \underset{\alpha}{\longrightarrow} T'S\xi_\tau$, then using the fact that $\|T_\alpha\|, \|T'\| \leq 1$ we deduce that $T_\alpha h \longrightarrow T'h$ for all $h \in H$ (by the same proof as above, for $(i') \implies (ii')$). So $T_\alpha \xrightarrow{\text{SOT}} T'$, thus $T = T' \in \pi_\tau(\mathcal{M})$.
Now we want to show that $\tilde{\tau}$ is normal. In fact, it is SOT-continuous: if $T_\alpha \xrightarrow{\text{SOT}} T$, then:

$$|\tilde{\tau}(T_\alpha) - \tilde{\tau}(T)| = |\tilde{\tau}(T_\alpha - T)| \leq \|T_\alpha - T\|_{\tilde{\tau}} = \|T_\alpha\xi_\tau - T\xi_\tau\| \underset{\alpha}{\longrightarrow} 0 \ .$$

This concludes the proof of the theorem. □

3.5 The tracial ultraproduct is a von Neumann algebra

Using the theorem of the previous subsection, we can now show that the ultraproduct is in fact a von Neumann algebra, and we give some further results about properties that are preserved by ultraproducts.

Theorem 3.23.
The C*-algebra $\prod_{i\to\omega} \mathcal{M}_i$ equipped with the faithful tracial state τ is a von Neumann algebra, and τ is normal.

Proof. By theorem 3.17, it is enough to show that $\left(\prod_{i\to\omega}\mathcal{M}_i\right)_1$ is $\|\cdot\|_\tau$-complete.

Let $([x^{(n)}])_{n\geq 1}$ be a Cauchy sequence in $\left(\prod_{i\to\omega}\mathcal{M}_i\right)_1$ for the trace norm. Since the canonical projection $\prod_{i\in I}\mathcal{M}_i \longrightarrow \prod_{i\to\omega}\mathcal{M}_i$ is a surjective $*$-homomorphism, we can pick $x^{(n)} \in \prod_{i\in I}\mathcal{M}_i$ so that $\|x^{(n)}\| = \|[x^{(n)}]\| \leq 1$.
Besides, since a Cauchy sequence is convergent if and only if it has a subsequence that converges, we can assume taking a subsequence that $\|[x^{(n+1)}] - [x^{(n)}]\|_\tau \leq \frac{1}{2^{n+1}}$.
Let for $p \geq 1$, $E_p = \left\{ i \in I \mid \|x_i^{(p+1)} - x_i^{(p)}\|_{\tau_i} \leq \frac{1}{2^p}\right\}$.
Since $\frac{1}{2^{p+1}} \geq \|x^{(p+1)} - x^{(p)}\|_{\tau_\omega} = \lim_{i\to\omega} \|x_i^{p+1)} - x_i^{(p)}\|_{\tau_i}$, it follows that $E_p \in \omega$.
Let for $k \geq 0$, $F_k = \bigcap_{1\leq p\leq k} E_p$ (so $F_0 = I$), then $F_k \in \omega$ by finite intersection. Finally, let $F = \bigcap_{k\geq 0} F_k = \bigcap_{p\geq 1} E_p$. Note that we have $F \subseteq \ldots \subseteq F_{k+1} \subseteq F_k \subseteq \ldots \subseteq F_1 \subseteq F_0$. Then we can write $I = F \sqcup \bigsqcup_{k\geq 0} (F_k \setminus F_{k+1})$. Intuitively, $i \in F_k \setminus F_{k+1}$ if we can control the gap between $x_i^{(n)}$ and $x_i^{(n+1)}$ until the step k, and $i \in F$ means that we can control the gap at every step. We now want to build a limit to $([x^{(n)}])_{n\geq 1}$.
For $i \in I$, if $i \in I \setminus F$, then there exists $n_i \geq 1$ such that $i \in F_{n_i} \setminus F_{n_i+1}$. Let $x_i = x_i^{(n_i)}$.
If $i \in F$, then for all $n \geq 1$, $\|x_i^{(n+1)} - x_i^{(n)}\|_{\tau_i} \leq \frac{1}{2^n}$.
As a consequence, for $m \geq n$, $\|x_i^{(m)} - x_i^{(n)}\|_{\tau_i} \leq \sum_{k=n}^{m-1} \frac{1}{2^k} \leq \frac{1}{2^{n-1}}$. Thus the sequence $(x_i^{(n)})_{n\geq 1}$ is Cauchy in $(\mathcal{M}_i)_1$ for the trace norm, and the latter is complete by theorem 3.17. So $(x_i^{(n)})_{n\geq 1}$ converges to some $x_i \in (\mathcal{M}_i)_1$ for the trace norm.
Finally, we define $x = (x_i)_{i\in I}$. We want to check that $[x^{(n)}]$ converges to $[x]$ for $\|\cdot\|_\tau$.
Note that for $i \in I$, $\|x_i\|_i \leq 1$, so $x \in \left(\prod_{i\in I}\mathcal{M}_i\right)_1$. Thus $\|[x]\| \leq \|x\| \leq 1$: $[x] \in \left(\prod_{i\to\omega}\mathcal{M}_i\right)_1$.
Let $n \geq 1$ and let $i \in F_n$. If $i \in F$, then by the previous calculation, $\|x_i - x_i^{(n)}\|_{\tau_i} \leq \frac{1}{2^{n-1}}$. Otherwise, there exists $n_i \geq n$ such that $x_i = x_i^{(n_i)}$. If $n_i = n$, then $\|x_i - x_i^{(n_i)}\|_{\tau_i} = 0 \leq \frac{1}{2^{n-1}}$. If $n_i > n$:

$$\|x_i - x_i^{(n)}\|_{\tau_i} \leq \sum_{p=n}^{n_i-1} \underbrace{\|x_i^{(p+1)} - x_i^{(p)}\|_{\tau_i}}_{\leq \frac{1}{2^p}} \leq \sum_{p=n}^{n_i-1} \frac{1}{2^p} \leq \frac{1}{2^{n-1}} .$$

Thus, for $i \in F_n$, $\|x_i^{(n)} - x_i\|_{\tau_i} \leq \frac{1}{2^{n-1}}$, so $\{i \in I \mid \|x_i^{(n)} - x_i\|_{\tau_i} \leq 2^{n-1}\} \supseteq F_n \in \omega$. It follows that $\|[x^{(n)}] - [x]\|_\tau = \|x^{(n)} - x\|_{\tau_\omega} \leq 2^{-n+1} \longrightarrow 0$, so $[x^{(n)}] \longrightarrow [x]$ for the trace norm, as wanted.
Using theorem 3.17, we conclude that $\prod_{i\to\omega}\mathcal{M}_i$ is a von Neumann algebra, and that τ is normal:

it is a tracial von Neumann algebra. $\square$

Corollary 3.24.
If $(\mathcal{M}_i)_{i\in I}$ are finite factors and ω is an ultrafilter on I, then $\prod_{i\to\omega}\mathcal{M}_i$ is a von Neumann algebra.

Proof. In fact a finite factor has one and only one tracial state, which is besides faithful and normal (theorem 3.2). $\square$

Theorem 3.25.
Let $(\mathcal{M}_i, \tau_i)_{i\in I}$ be tracial von Neumann algebras, ω an ultrafilter on I.
Assume that $F = \{i \in I \mid \mathcal{M}_i \text{ is a factor}\} \in \omega$. Then $\prod_{i\to\omega}\mathcal{M}_i$ is a factor. Besides, it is a II_1 factor if and only if for all $k \geq 1$, $\{i \in I \mid \dim \mathcal{M}_i \geq k^2\} \in \omega$.

Corollary 3.26.
The following results hold.

(i) An ultraproduct of finite factors is a finite factor.

(ii) An ultraproduct of II_1 factors is a II_1 factor.

4 The Connes-Embedding Problem

Now that we have a lot of prerequisites, we are able to state the Connes-Embedding problem. We first give the construction of an important von Neumann algebra, namely the hyperfinite II_1 factor, then we state the problem. Finally we give a reformulation of this problem using a Kirchberg theorem about approximation of unitaries in trace.

4.1 The hyperfinite II_1 factor

The hyperfinite II_1 is one of the famous examples of von Neumann algebras that is not $B(H)$ for some H. We give here its definition and basic properties.

Definition 4.1.
A von Neumann algebra $\mathcal{M}$ is hyperfinite if there is an increasing sequence $(\mathcal{M}_n)_{n\geq 1}$ of finite-dimensional sub-*-algebras of $\mathcal{M}$ such that $\bigcup\limits_{n\geq 1}\mathcal{M}_n$ is SOT-dense in $\mathcal{M}$.

Definition 4.2.
Let $H = \ell^2(\mathbb{N})$ (infinite dimensional separable Hilbert space). For $n \geq 1$, let $A_n = \mathcal{M}_{2^n}(\mathbb{C})$. We define an embedding $\varphi_n : A_n \longrightarrow B(H)$ by $\varphi_n(x) = \begin{pmatrix} x & 0 & 0 & \dots \\ 0 & x & 0 & \dots \\ 0 & 0 & x & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ (identifying $B(H)$ with matrixes with countably many entries).
We define $\mathcal{R} = \overline{\bigcup\limits_{n\geq 1}\varphi_n(A_n)}^{\mathrm{SOT}} \subseteq B(H)$: it is a von Neumann algebra.

Proposition 4.3.
The von Neumann algebra $\mathcal{R}$ is a hyperfinite II_1 factor.

Proposition 4.4.
For $n \geq 1$, $\tau_n = \dfrac{1}{2^n}\mathrm{Tr}$ is a faithful normal tracial state on A_n, we then define a trace on $\bigcup\limits_{n\geq 1}\varphi_n(A_n)$ by $\tau(\varphi_n(a)) = \tau_n(a)$. Then τ extends to the unique normal faithful tracial state on $\mathcal{R}$.

Proposition 4.5.
Every hyperfinite II_1 factor is isomorphic to $\mathcal{R}$.
Besides, if $\mathcal{M}$ is a II_1 factor, there is an injective *-homomorphism $\pi : \mathcal{R} \longrightarrow \mathcal{M}$.

4.2 The Connes-Embedding Problem

In 1976, Alain Connes asked whether the ultrapowers of $\mathcal{R}$ contains every II_1 factor as a subalgebra. This is known as the Connes-Embedding problem, and at this time it remains an open problem.

Proposition 4.6.
Using corollary 3.25, we know that for any free ultrafilter ω on $\mathbb{N}$, $\mathcal{R}^\omega$ is a II_1-factor.

Problem 4.7 (Connes-Embedding Problem (CEP))**.**
Let ω be a free ultrafilter on $\mathbb{N}$.
Does every separable finite tracial von Neumann algebra $(\mathcal{M},\tau)$ admit a trace-preserving embedding into $\mathcal{R}^\omega$?

Remark 4.8.
For a von Neumann algebra, "separable" means "with respect to the SOT-topology".
The Connes-Embedding Problem asks whether $\mathcal{R}^\omega$ contains any finite von Neumann algebra as a sub-von Neumann algebra.

4.3 A reformulation of the Connes-Embedding Problem

In 1993, Kirchberg reformulated the Connes-Embedding Problem, approximating unitaries in a von Neumann algebra by unitaries in a matrix algebra, with respect to their traces.

Theorem 4.9 (Kirchberg, 1993)**.**
Let $(\mathcal{M},\tau)$ be a tracial separable finite von Neumann algebra. The following are equivalent.

(i) $\mathcal{M}$ admits a trace-preserving embedding into $\mathcal{R}^\omega$.

(ii) For all $\varepsilon>0$, for all $n\geq 1$, for all unitaries $u_1,u_2,\dots,u_n\in\mathcal{M}$, there is $k\geq 1$ and there are unitaries $v_1,\dots,v_n\in\mathcal{M}_k(\mathbb{C})$ such that for all $1\leq i,j\leq n$:
$$|\tau(u_i^*u_j)-\mathrm{tr}_k(v_i^*v_j)|<\varepsilon$$
(where $\mathrm{tr}_k=\dfrac{1}{k}\mathrm{Tr}_{\mathcal{M}_k(\mathbb{C})}$).

Proof. We give here the proof of the direction ($\Longrightarrow$). The other one is technical and requires more prerequisites.
Embedding $\mathcal{M}$ in $\mathcal{R}^\omega$, and since it is trace-preserving, we can assume $u_1,\dots,u_n\in\mathcal{U}(\mathcal{R}^\omega)$. For a unitary u_i, and since $\mathcal{R}^\omega$ is a von Neumann algebra, we can write it $u_i=\exp(2i\pi a_i)$ for $a_i\in\mathcal{R}^\omega_{\mathrm{sa}}$.
Let $\pi:\ell^\infty(\mathcal{R})\longrightarrow\mathcal{R}^\omega$ be the quotient mapping. Then we can find $b_i\in\ell^\infty(\mathcal{R})_{\mathrm{sa}}$ such that $\pi(b_i)=a_i$. Setting $w_i=\exp(2i\pi b_i)\in\ell^\infty(\mathcal{R})$, we have $\pi(w_i)=u_i$.
In particular $w_i=(w_i(k))_{k\geq 1}$ with $w_i(k)$ unitary in $\mathcal{R}$, and by definition of the trace on $\mathcal{R}^\omega$, $\tau(u_i^*u_j)=\tau_\omega(w_i^*w_j)=\lim\limits_{k\to\omega}\tau_{\mathcal{R}}(w_i(k)^*w_j(k))$. Since any finite intersection of elements in ω is non-empty, one can find $k\geq 1$ such that for all $1\leq i,j\leq n$, $|\tau(u_i^*u_j)-\tau_{\mathcal{R}}(w_i(k)^*w_j(k))|<\dfrac{\varepsilon}{2}$.
Now, since $\bigcup\limits_{n\geq 1}\varphi_n(A_n)$ is SOT-dense in $\mathcal{R}$, one can find $p\geq 1$ and $v_j\in\mathcal{U}(\mathcal{M}_{2^p}(\mathbb{C}))$ such that $\|w_j(k)-\varphi_p(v_j)\|_{\tau_{\mathcal{R}}}<\dfrac{\varepsilon}{4}$. Using Cauchy-Schwarz inequality and the definition of $\tau_{\mathcal{R}}$, it is straightforward to check that $|\tau_{\mathcal{R}}(w_i(k)^*w_j(k))-\mathrm{tr}_{2^p}(v_i^*v_j)|<\dfrac{\varepsilon}{2}$. Setting $k=2^p$, we then have the result. □

5 Tensor products and the Connes-Embedding Problem

There are strong links between the Connes-Embedding Problem and tensor products of C*-algebras generated by free groups. In this section, we give a sketch of what are tensor products on C*-algebras and what are the links with the Connes-Embedding Problem.

Definition 5.1.
Let A, B be C*-algebras. We define the algebraic tensor product $A \odot B = \text{Span}\{a \otimes b \mid a, b \in A\}$, when $(a,b) \in A \times B \longmapsto a \otimes b \in A \odot B$ is bilinear. It is a vector space.
Defining $(a \otimes b)(c \otimes d) = ac \otimes bd$ and $(a \otimes b)^* = a^* \otimes b^*$, we make it a *-algebra.

Definition 5.2.
We define the maximal norm $\|\cdot\|_{\max}$ on $A \odot B$ by:

$$\|x\|_{\max} = \sup\{\|\pi(x)\| \mid \pi : A \odot B \longrightarrow B(H) \text{ is a *-representation}\}.$$

We define the maximal tensor product $A \otimes_{\max} B$ as the completion of $A \odot B$ with respect to the maximal norm $\|\cdot\|_{\max}$.

Definition 5.3.
We define the minimal norm $\|\cdot\|_{\min}$ on $A \odot B$ in the following way: let $\pi : A \longrightarrow B(H)$, $\rho : B \longrightarrow B(K)$ be faithful representations. We set:

$$\left\|\sum_{k=1}^{n} a_k \otimes b_k\right\|_{\min} = \left\|\sum_{k=1}^{n} \pi(a_k) \otimes \rho(b_k)\right\|_{B(H \otimes K)}.$$

We define the minimal tensor product $A \otimes_{\min} B$ as the completion of $A \odot B$ with respect to the minimal norm $\|\cdot\|_{\min}$.

Remark 5.4.
It turns out that the minimal norm does not depend on the choice of the faithful representations. Actually, $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ are respectivly the smallest and the biggest norms that make $A \odot B$ a C*-algebra. In general the minimal and maximal tensor products are different : the way to make the tensor product a C*-algebra is not unique.

Definition 5.5.
Let G be a discrete group. We define $C^*(G)$ the C*-algebra generated by G : it is the smallest C*-algebra containing the elements of G, where $g^* = g^{-1}$ (i.e we want the elements of G to be unitaries in $C^*(G)$). More precisely, it is the completion of $\left\{\sum_{g \in G} a_g g \mid \{g \in G \mid a_g \neq 0\} \text{ finite}\right\}$ with respect to the norm $\left\|\sum_{g \in G} a_g g\right\| = \sum_{g \in G} |a_g|$.

Theorem 5.6 (Kirchberg)**.**
Let H be a Hilbert space and $\mathbb{F}$ be a free group. Then:

$$C^*(\mathbb{F}) \otimes_{\min} B(H) = C^*(\mathbb{F}) \otimes_{\max} B(H).$$

Theorem 5.7.
The following are equivalent:

(i) The Connes-Embedding Problem holds.

(ii) $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.

Conclusion

Finally, the Connes-Embedding problem is a very important subject for von Neumann algebras. On the one hand, a positive answer would mean that understanding $\mathcal{R}^\omega$ is enough to understand every separable finite von Neumann algebra. On the other hand, a negative answer would lead to study what possible counterexamples are and what they look like.

Besides, the Connes-Embedding problem is linked to many other problems or conjectures. In the fifth part of this thesis, we mentioned the link with the uniqueness of the C*-tensor product on $C^*(\mathbb{F}_\infty)$ with itself. But one can find links and equivalences with many other statements, it is for instance linked with hyperlinear groups and sofic groups, but one can also find some links with computer science and quantum information theory.

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