2-Segal objects

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Introduction

Segal objects were initially introduced by Rezk in [Rez00], and are named after Segal's Γ-spaces. A simplicial set X is a Segal set (or 1-Segal set) when a canonical map

$$X_n \to X_1 \times_{X_0} \times X_1 \times_{X_0} ... \times_{X_0} X_1$$

is a bijection for all $n \in \mathbb{N}$. This definition can be extended to simplicial objects (functors from Δ^{op} to a model category), by replacing the iterated pulllbacks by iterated homotopy pullbacks and by asking for the maps to be weak equivalences instead of bijections. While Segal sets are exactly nerve of small categories, complete Segal spaces, which are Segal objects in Top with a small additional hypothesis, were used by Rezk as a model for $(\infty, 1)$-categories.

Dyckerhoff and Kapranov first introduced 2-Segal objects in [DK19], as a generalization of the Segal condition from Rezk's work. A simplicial set X is said 2-Segal when a family of natural maps

$$X_n \to X_2 \times_{X_1} X_2 \times_{X_1} ... \times_{X_1} X_2$$

are bijections. As for Segal objects, this definition is extended to simplicial objects by replacing the iterated pullbacks by iterated homotopy pullbacks, and by asking for the family of natural maps to be weak equivalences.

Unlike Segal objects, 2-Segal objects encode objects where the composition law might not be unique or unique up to homotopy, but is associative or associative up to homotopy. This intuition is developed in [DK19], where for instance 2-Segal sets are shown to be equivalent to multivalued categories.

Given an exact finitary category, one can associate to it an algebra, called the Hall algebra of the category. The element of the algebra are formal sums of isomorphism classes of objects of the category, and the product law is determined by the short exact sequences of the category. This construction is used in various field of mathematics, like for instance, representation theory, where the finitary exact categories are representation of quiver on a finite field. Following this idea, given a 2-Segal set verifying certain finiteness conditions, one can associate to it an algebra, called the Hall algebra, because of its similarity with the Hall algebra of an exact finitary category. The 2-Segal condition then gives the associativity and the unitality of the algebra. In the first section of this text, we define the notion of Hall algebras for 2-Segal set following [DK19], and we classify the algebras that can be obtained as Hall algebra of a 2-Segal set, by constructing a section to the Hall algebra construction.

There are several motivating cases for the notion of 2-Segal objects, one of them is Waldhausen's S-construction from K-theory. Initially introduced by Waldhausen in [Wal85], the S-construction provides a way to compute the higher K-theory group of a category, as the homotopy group of a certain bisimplical set. Independently, Dyckerhoff and Kapranov in [DK19], and Gálvez-Carrillo, Kock and Tonks in [GKT18], showed that the output of the S-construction on any Waldhausen category is 2-Segal. Adopting the 2-Segal point of view allows to generalize the S-construction to other objects. In [Ber+18], Bergner, Osorno, Ozornova, Rovelli and Scheimbauer showed the input can be broadened to a larger framework than Waldhaussen categories, and that the S-construction induces an equivalence of category between the category of augmented stable double categories and the category of 2-Segal sets. In [Ber+21a], the authors showed the analog results for the model category setting, the S-construction induces a Quillen equivalence between the category of stable double Segal objects and the category of 2-Segal objects. By allowing more general inputs to the S-construction, it becomes possible to define the higher K-theory groups of more general structure, this is for instance discussed in [Ber+21b]. We present this motivation for the study of 2-Segal objects in the second section of this text.

This work was done during an internship at the University of Virginia, under the supervision of Julie Bergner, whom I would like to thank for her availability. Our frequent discussions allowed me to discover a large aspect of her field of work.

My 4 months stay at the University of Virginia was a great opportunity to discover a lot of different mathematics, especially through the weekly geometry seminar and topology seminar of the university's mathematics department, or through my participation to the weekly topology team meeting and to a graduate students reading group on stable homotopy.

1 2-Segal sets and Hall algebras

1.1 Simplicial sets and the Segal condition

In this section, we motivate and introduce the notions of Segal and 2-Segal sets.

We first introduce the notion of simplicial sets. Denote Δ the category whose objects are the finite ordered set $[n] = \{0 < 1 < ... < n\}$ for all $n \in \mathbb{N}$, and whose morphisms are the order-preserving functions. The morphisms of Δ are generated by the *face* maps of the shape

$$\begin{aligned} \delta^i : \quad & [n-1] \to [n] \\ & k \mapsto \begin{cases} k & \text{if } 0 \leq k \leq i, \\ k+1 & \text{if } i+1 \leq k \leq n-1. \end{cases} \end{aligned}$$

And the *degeneracy* maps:

$$\begin{aligned} s^j : \quad & [n] \to [n-1] \\ & k \mapsto \begin{cases} k & \text{if } 0 \leq k \leq j, \\ k-1 & \text{if } j+1 \leq k \leq n. \end{cases} \end{aligned}$$

A *simplicial set* is a functor $\Delta^{op} \to Set$. Denote by $sSet = Fun(\Delta^{op}, Set)$ the category of simplicial set. Given a simplicial set X, we denote by X_n, the set of n-simplices of X, i.e $X_n = X([n])$.

Example 1.1. An important example of simplicial set are the *standard simplices*. The standard simplices are the simplicial sets represented by the objects of Δ, and we denote them by $\Delta[n] = Hom_\Delta([n], \bullet)$, for $n \in \mathbb{N}$. In this context, the Yoneda lemma yields a natural bijection $X_n \cong Hom_{sSet}(\Delta[n], X)$.

Example 1.2. Given a small category C, one can consider the simplicial set $N(C)$, with 0-simplices given by the objects of C, *i.e* $N(C)_0 = Ob(C)$, n-simplices given by $N(C)_n = \{X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} ... \xrightarrow{f_{n-1}} X_n \,|\, f_1, ..., f_{n-1} \in Mor(C)\}$ the set of chains of n composable morphisms, with face maps obtained by composition, and degeneracies by insertion of the identity. This defines the *nerve functor* $N : Cat \to sSet$.

The nerve functor is fully faithful, and the characterization of its image motivates the following definition.

Definition 1.3. A simplicial set X is said to be *1-Segal* if for all $n \geq 0$ the map

$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} ... \times_{X_0} X_1,$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$ is a bijection.

If X is a 1-Segal set, we can retrieve a composition law using the map $X_1 \times_{X_0} X_1 \simeq X_2 \xrightarrow{\delta_1} X_1$. More precisely, we have the following fact:

Proposition 1.4. *A simplicial set is the essential image of the nerve functor N if and only if it is 1-Segal.*

This notion can be extended to simplicial spaces, as explained in the next part, but we can also generalize this condition to higher dimensions. A triangulation $\mathcal{T}$ of a regular $(n+1)$-gon induces a map

$$f_{\mathcal{T}} : X_n \to X_2 \times_{X_1} X_2 \times_{X_1} ... \times_{X_1} X_2.$$

Indeed, if we cyclically label the vertices of the $(n+1)$-gon from 0 to n, then $\mathcal{T}$ determines $n-1$ subsets of $\{0, ..., n\}$ corresponding to the vertices of the triangles in the triangulation. Then consider subcategory of Δ formed by the subsets of 3 elements corresponding to the vertices of the triangles in $\mathcal{T}$, and the subsets of $\{0, ..., n\}$ consisting of 2 elements corresponding to the edges of the triangles, with the inclusion maps. Taking the image by X of this subcategory yields a diagram in Set whose limit is an iterated pullback of faces along their edges. The map $f_{\mathcal{T}}$ is induced by the inclusions of the triangles and edges of $\mathcal{T}$ in $\{0, ..., n\}$.

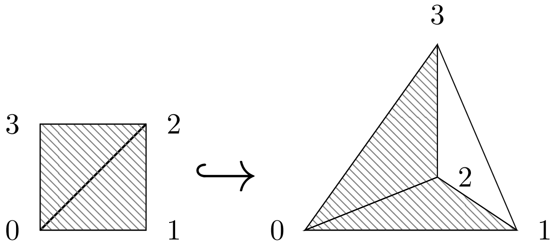
For instance, the triangulation

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 3 \\
1 \arrow[r] & 2 \arrow[u]
\end{tikzcd}$$

induces a map

$$X_3 \to X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,3\}},$$

which comes from the inclusion:



We can now introduce the 2-dimensional generalization of the Segal condition from [DK19].

Definition 1.5. A simplicial set X is a *2-Segal* set if for every $n \geq 3$, for every triangulation $\mathcal{T}$ of the regular $(n+1)$-gon, the induced map $f_{\mathcal{T}} : X_n \to X_2 \times_{X_1} X_2 \times_{X_1} ... \times_{X_1} X_2$ is a bijection.

While 1-Segal sets encode structures with a unique well-defined composition law, 2-Segal sets can be thought of as encoding a structure where composition might not be defined everywhere or might not be unique, but is associative. See [DK19, Part 3.3] for more details.

The 2-Segal condition is equivalent to more convenient conditions. First, we use the following notations from [DK19]. Given S a finite subset of $[n]$, let $X_S = X_{|S|}$ and $X_n \to X_S$ be the map induced by the inclusion of poset full subcategory $S \hookrightarrow [n]$.

Proposition 1.6 ([Ber+21a, Prop 1.17])**.** *If X is a simplicial set, then X is 2-Segal if and only if for every $n \geq 3$, the induced maps*

$$X_n \to X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,...,n\}},$$

and

$$X_n \to X_{\{n-2,n-1,n\}} \times_{X_{\{n-2,n\}}} X_{\{0,...,n-2,n\}},$$

are bijections.

We now introduce some useful definitions for 2-Segal sets.

Definition 1.7. A 2-Segal set X is *reduced* if X_0 is a single point.

Definition 1.8. A 2-Segal set is *unital* if for all $n \geq 2$ and $0 \leq i \leq n-1$, the diagram

$$\begin{tikzcd}
X_{n-1} \arrow[r, "\alpha_i"] \arrow[d, "s_i"] & X_0 \arrow[d, "s_0"] \\
X_n \arrow[r, "\beta_i"] & X_1
\end{tikzcd}$$

is a pullback. With α_i induced by $\{i\} \hookrightarrow [n-1]$ and β_i induced by $\{i, i+1\} \hookrightarrow [n]$.

The unitality condition can be interpreted as the existence of unique neutral elements in the structure encoded by the 2-Segal set, as explained in [DK19].

Finally, we mention the following results.

Proposition 1.9 ([Fel+21])**.** *Every 2-Segal set is unital.*

All of these definitions and results can be extended to 2-Segal objects, which will be discussed in the next part.

1.2 The Hall algebra of a 2-Segal set

Hall algebras appears independently in various fields, such as K-theory or representation theory. In [DK19], the authors define the Hall algebra of finitary 2-Segal sets, and of a class of 2-Segal spaces, providing a general context in which a construction could be called the Hall algebra of a certain structure. In this section, we introduce as in [DK19] the Hall algebra of a 2-Segal set, and we describe the algebras that can be seen as Hall algebras of 2-Segal sets.

Let k be any field of characteristic zero. We say that a simplicial set is *finitary* if the map $(\delta_0, \delta_2) : X_2 \to X_1 \times_{X_0} X_1$ has finite fibers.

Definition 1.10. Let X be a reduced finitary 2-Segal set. The *Hall algebra* $\mathcal{H}(X)$ of X is the k-algebra which has as vector space the free k-vector space spanned by the elements of X_1. If $\mathbb{1}_b$ is the element of the basis of the vector space corresponding to $b \in X_1$, we define the multiplication by

$$\mathbb{1}_b * \mathbb{1}_{b'} = \sum_{b'' \in X_1} |C_{bb'}^{b''}| \mathbb{1}_{b''},$$

with $C_{bb'}^{b''} = \{x \in X_2 \mid \delta_0(x) = b,\ \delta_1(x) = b'',\ \delta_2(x) = b'\}$.

The elements of $C_{bb'}^{b''}$ can be depicted as follows:

$$\begin{tikzcd}
 & * \\
* \arrow[ur, "b''"] \arrow[r, "b'"] & *. \arrow[u, "b"']
\end{tikzcd}$$

We now state the following result, justifying this definition. Intuitively, the associativity is a consequence of the 2-Segal condition and the unit comes from the unitality of the 2-Segal set.

Proposition 1.11 ([DK19, Part 3.4])**.** *Taking* $\mathbf{1} = s_0(*)$ *as a unit makes* $\mathcal{H}(X)$ *into an associative* k*-algebra.*

Proof. For the associativity, consider a square

$$\begin{tikzcd}
* \arrow[r, "x"] \arrow[d, "a"] & * \\
* \arrow[r, "b"] & * \arrow[u, "c"']
\end{tikzcd}$$

with $a, b, c, d \in X_1$. Since X is 2-Segal, there is a bijection

$$\coprod_{s \in X_1} C_{ba}^{s} \times C_{cs}^{x} \cong \coprod_{t \in X_1} C_{ta}^{x} \times C_{bc}^{t}.$$

Indeed, consider the two triangulations of the square

$$\mathcal{T} : \begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 3 \\
1 \arrow[r] & 2 \arrow[u]
\end{tikzcd} \quad \text{and} \quad \mathcal{T}' : \begin{tikzcd}
0 \arrow[r] \arrow[d] & 3 \\
1 \arrow[r] \arrow[ur] & 2 \arrow[u]
\end{tikzcd}$$

Since X is 2-Segal, the triangulation $\mathcal{T}$ induces a bijection $f_{\mathcal{T}} : X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,3\}} \cong X_3$. Consider the set

$$D_{abc}^{x} = \{u \in X_3 \mid (\delta_{\{2,3\}}, \delta_{\{1,2\}}, \delta_{\{0,1\}}, \delta_{\{0,3\}})(u) = (a, b, c, x)\}.$$

Denote by

$$f = (\delta_{\{2,3\}}, \delta_{\{1,2\}}, \delta_{\{0,1\}}, \delta_{\{0,3\}}) : X_3 \to X_1 \times X_1 \times X_1 \times X_1,$$

and

$$\begin{aligned} g: \quad & X_2 \times_{X_1} \times X_2 \to X_1 \times X_1 \times X_1 \times X_1 \\ & (u,v) \mapsto (\delta_2(u),\delta_0(u),\delta_0(v),\delta_1(v)). \end{aligned}$$

We have pullback squares

$$\begin{tikzcd} D^x_{abc} \arrow[rr, hook] \arrow[d] & & X_3 \arrow[d, "f"] \\ * \arrow[r, "{(a,b,c,x)}"] & X_1 \times X_1 \times X_1 \times X_1 \end{tikzcd}$$

and

$$\begin{tikzcd} \coprod_{s\in X_1} C^s_{ab} \times C^x_{sc} \arrow[rr, hook] \arrow[d] & & X_2 \times_{X_1} X_2 \arrow[d, "g"] \\ * \arrow[r, "{(a,b,c,x)}"] & X_1 \times X_1 \times X_1 \times X_1, \end{tikzcd}$$

and a commutative triangle

$$\begin{tikzcd} X_3 \arrow[rr, equal, "f_{\mathcal{T}}"] \arrow[dr, "f"'] & & X_2 \times_{X_1} \times X_2 \arrow[dl, "g"] \\ & X_1 \times X_1 \times X_1 \times X_1, & \end{tikzcd}$$

thus the two pullback squares are isomorphic and we have a bijection $D^x_{abc} \cong \coprod_{s\in X_1} C^s_{ab} \times C^x_{sc}$. Similarly, $\mathcal{T}'$ induces a bijection between D^x_{abc} and $\coprod_{t\in X_1} C^x_{at} \times C^t_{cb}$. Thus the algebra is associative.

For unitality, since X is unital by Proposition 1.9, if we take the Definition 1.8 with $n = 2$ and $i = 1$ the square

$$\begin{tikzcd} X_1 \arrow[r, "\delta_1"] \arrow[d, "s_1"] & * \arrow[d, "s_0"] \\ X_2 \arrow[r, "\delta_0"] & X_1 \end{tikzcd}$$

is a pullback. Let $a \in X_1$, we want to show that $\mathbf{1} * \mathbb{1}_a = \mathbb{1}_a$. Let $c \in X_1$ and $u \in X_2$ such that $(\delta_0, \delta_1, \delta_2)(u) = (s_0(*), c, a)$. Since $\delta_0(c) = s_0(*)$, the 2-simplex u is an element of the pullback in the above square. Hence there exist $x \in X_1$ such that $u = s_1(x)$. Using the simplicial identities, we deduce that $c = \delta_1(u) = \delta_1(s_1(x)) = \delta_0(s_1(x)) = a$, and therefore $u = s_1(a)$. So u is unique and $\delta_1(u) = a$. We deduce that $\mathbf{1} * \mathbb{1}_a = \mathbb{1}_a$.

Similarly, using the pullback square of Definition 1.8 with $n = 2$ and $i = 0$, we show that $\mathbb{1}_a * \mathbf{1} = \mathbb{1}_a$. □

We are now interested in finding all the k-algebras that can be expressed as the Hall algebra of a 2-Segal set. For that we are going to, given a certain algebra, construct a 2-Segal set which has this algebra as its Hall algebra.

Given a set J, denote $\mathbb{Z}^{(J)}$ the $\mathbb{Z}$-module consisting of sequences of integers indexed by J that are null everywhere except on a finite number of indices.

Theorem 1.12. *A k-algebra A is the Hall algebra of a reduced finitary 2-Segal set if and only if it is of the shape $\mathbb{Z}^{(J)} \otimes_{\mathbb{Z}} k$, where J is a set with a distinguished element $\mathbf{1}$ and $\mathbb{Z}^{(J)}$ is equipped with a ring structure compatible with the sum structure on $\mathbb{Z}^{(J)}$ such that $\mathbb{Z}_+^{(J)}$ is stable by multiplication, and $\mathbb{1}_{\mathbf{1}}$ is the neutral element for the multiplication.*

Proof. Given a reduced finitary 2-Segal set X, take $J = X_1$, $\mathbf{1} = s_0(*)$ and the ring structure on $\mathbb{Z}^{(X_1)}$ described in Definition 1.10.

Let $A = \mathbb{Z}^{(J)} \otimes_{\mathbb{Z}} k$ as in the statement of the theorem. We are going to construct a reduced finitary 2-Segal set X such that $\mathcal{H}(X) = A$. We then have $\mathcal{H}(X) = \mathbb{Z}^{(J)} \otimes_{\mathbb{Z}} k$ as required.

Now, for the other implication, notice that for $a_1, ..., a_n \in \mathbb{Z}_+^{(J)}$, we can write uniquely $a_n * ... * a_1 = \sum_{j\in J} \alpha_j \mathbb{1}_j$ with $\alpha_j \in \mathbb{Z}_+$ for all $j \in J$. We then denote $E(a_1, ..., a_n) = \{(c, k) \in J \times \mathbb{N} \,|\, 1 \leq k \leq \alpha_j\}$. We can think of an element (x, k) of $E(a, b)$ as the k-th occurrence of $\mathbb{1}_x$ in the writing of $a * b$ as a sum of $\mathbb{1}_j$ with $j \in J$.

Choose a linear order on J. From now on, we identify $\mathbb{1}_j$ with j. Take $X_0 = *$, $X_1 = J$, and

$$X_n = \{(a_1, ..., a_n, x, k) \,|\, a_1, ..., a_n \in J^n, (x, k) \in E(a_1, ..., a_n)\}.$$

We now define the face maps. For $n = 1$, we take $\delta_0 : X_1 \to X_0$ constant equal to $*$. For $n = 2$, $0 \leq i \leq 2$, we take $\delta_i : X_2 \to X_1$ defined by

$$\delta_i((a_0, a_1, a_2, k)) = \begin{cases} a_1 & i = 0 \\ a_2 & i = 1 \\ a_0 & i = 2 \end{cases},$$

for $(a_0, a_1, a_2, k) \in X_1$.

Let $n \leq 2$, let $0 < i < n$, let $(a_1, ..., a_n) \in J^n$. Notice that

$$a_n * ... * a_1 = \sum_{(y,l)\in E(a_i, a_{i+1})} a_n * ... * a_{i+2} * y * a_{i-1} * ... * a_1, \tag{1}$$

So, using the order on J, we can see the above sum as an ordered partition of the elements of $E(a_1, ..., a_n)$. More precisely, to an element $(x, k) \in E(a_1, ..., a_n)$, using the order on J, we can associate an element $(y, l) \in E(a_i, a_{i+1})$ corresponding to the term of the sum (1) in which the k-th occurrence of x appears, and an integer k' such that $(x, k') \in E(a_1, ..., a_{i-1}, y, a_{i+2}, ..., a_n)$ and the k'-th occurrence of x in $a_n * ... * a_{i+2} * y * a_{i-1} * ... * a_1$ corresponds to the k-th occurrence of x in $a_1 * ... * a_n$. We just defined a bijection

$$f : E(a_1, ..., a_n) \xrightarrow{\simeq} \bigsqcup_{(y,l)\in E(a_i, a_{i+1})} E(a_1, ..., a_{i-1}, y, a_{i+2}, ..., a_n). \tag{2}$$

Let $(a_1, ..., a_n, x, k) \in X_n$, take f as in (2) and write $f((x, k)) = ((y, l), (x, k'))$ with $(y, l) \in E(a_i, a_{i+1})$ and $(x, k') \in E(y, l)$. We define the face map

$$\delta_i(a_1, ..., a_n, x, k) = (a_1, ..., a_{i-1}, y, a_{i+2}, ..., a_n, x, k').$$

For $i = 0$, write

$$a_n * ... * a_1 = \sum_{(y,l)\in E(a_2,...,a_n)} y * a_1,$$

and as above, the k-th occurrence of x appears in one of the $y * a_1$, with index (y,l). Take $\delta_0(a_1,...,a_n,x,k) = (a_2,...,a_n,y,l)$. For $i = n$, write

$$a_n * ... * a_1 = \sum_{(y,l)\in E(a_1,...,a_{n-1})} y * a_n,$$

The k-th occurrence of x corresponds to $(y,l) \in E(a_1,...,a_{n-1})$. Take $\delta_n(a_1,...,a_n,x,k) = (a_1,...,a_{n-1},y,l)$.

We now define the degeneracy maps. Take $s_0(*) = \mathbf{1}$. For $x \in X_1$, take $s_0(x) = (\mathbf{1},x,x,1)$ and $s_1(x) = (x,\mathbf{1},x,1)$. Then for $n \geq 2$, for $0 \leq i \leq n$, the map s_i is obtained by inserting $\mathbf{1}$ in the i-th position, that is, for $(a_1,...,a_n,x,k) \in X_n$, take $s_i(a_1,...,a_n,x,k) = (a_1,...,a_{i-1},\mathbf{1},a_i,...,a_n,x,k)$.

We obtain a simplicial set X. We now show that X is 2-Segal.

Let $n \geq 3$. We want to use Proposition 1.6. We will show that

$$X_n \longrightarrow X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,...,n\}}$$

is a bijection. The other case is similar.

Let $a \in X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,...,n\}}$. Notice that a can be uniquely written as

$$a = ((a_1,a_2,y,l),(y,a_3,...,a_n,z,m)),$$

with $a_1,...,a_n \in X_1$, and with $(y,l) \in E(a_1,a_2), (z,m) \in E(y,a_3,...,a_n)$.

Hence we have the following bijection:

$$X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,...,n\}} \simeq \coprod_{(a_1,...,a_n)\in J^n} \coprod_{(y,l)\in E(a_1,a_2)} E(y,a_3,...,a_n).$$

Furthermore, the decomposition

$$a_n * ... * a_1 = \sum_{(y,l)\in E(a_1,a_2)} a_n * ... * a_3 * y \tag{3}$$

and the order on J induce a bijection

$$E(a_1,...,a_n) \xrightarrow{\simeq} \coprod_{(y,l)\in E(a_1,a_2)} E(y,a_3,...,a_n). \tag{4}$$

But by definition

$$X_n \simeq \coprod_{(a_1,...,a_n)\in J^n} E(a_1,...,a_n),$$

so the map from (4) induces a bijection

$$X_n \longrightarrow X_{\{0,1,2\}} \times_{X_{\{0,2\}}} X_{\{0,2,...,n\}}.$$

We claim that this bijection is exactly the map induce by the face maps. Indeed, the decomposition (3) is exactly the decomposition we used to define δ_1. Hence X is 2-Segal. Now, notice that $\mathcal{H}(X) = A$, which concludes the construction.

□

However, as shown by the following example, two different choices of basis J of the same algebra might not yield the same simplicial set, since the construction depends on the choice of J.

Example 1.13. Take $A = \mathbb{Q}[X]/(X^2 - 1)$, and take $J_1 = (1, X)$, and $J_2 = (1, 2X)$. We have two decompositions $A = \mathbb{Z}^{(J_1)} \otimes \mathbb{Q} = \mathbb{Z}^{(J_2)} \otimes \mathbb{Q}$. With J_1, Construction 1.2 gives a simplicial set X such that $|X_2| = 8$. But with J_2, since $2X * 2X = 4 = 1 + 1 + 1 + 1$, we obtain a simplicial set Y such that $|Y_2| = 20$.

Even if J is fixed, the following example show that the obtained simplicial set depends on the choice of the order on J.

Example 1.14. Consider the algebra A spanned by the generators $G = \{a, b, c, d, x, y, z\}$ and the relations

$$\begin{gathered} x * y = a + b \\ y * z = c + d \\ a * z = 2 \\ b * z = 1 \\ x * c = 2 \\ x * d = 1 \\ v * w = 0 \text{ for all the other product of generators } v, w. \end{gathered}$$

Take $J = G \cup \{1\}$. Such relations define an associative k-algebra of the shape $\mathbb{Z}^{(J)} \otimes k$, with ${\mathbb{Z}_+}^{(J)}$ stable by multiplication, as required by the hypothesis of 1.2.

Let $\leq_1$ be a linear order on J such that $a \leq_1 b$ and $c \leq_1 d$. Let X be a simplicial set obtained using the Construction 1.2 with such an order. Let $\leq_2$ be a linear order such that $a \leq_2 b$ and $d \leq_2 c$. Let Y be a simplicial set obtained with such an order.

We now show that X and Y are not isomorphic.

Suppose that there exist an isomorphism $f : X \to Y$. First notice that $f_1 : X_1 = J \to Y_1 = J$ is the identity. Indeed, since $1 = s_0(*)$, we must have $f_1(1) = 1$. Next, notice that y is the only generator that can be multiplied on the left and on the right and yield an non-zero result. Such a fact can be noticed in the structure of X and Y, hence $f_1(y) = y$. With the same kind of arguments, we can notice that each generator has a unique role in the structure of X and Y, and we deduce that f_1 is the identity.

Now, we notice that $x * y * z = 3$. Consider the 3-simplex $s = (z, y, x, 1, 3) \in X_3$, knowing that the 1-simplex are unchanged by f, we can write $f(s) = (z, y, x, 1, k) \in Y_1$ with $1 \leq k \leq 3$.

If we write the decomposition of $x * y * z$ with the order $\leq_1$, we have

$$\begin{aligned} (x * y) * z &= a * z + b * z = \underbrace{1 + 1}_{a*z} + \underbrace{1}_{b*z} \\ x * (y * z) &= x * c + x * d = \underbrace{1 + 1}_{x*c} + \underbrace{1}_{x*d} \end{aligned}$$

Hence we can compute $\delta_1(s) = (b, x, 1, 1)$, and $\delta_2(s) = (z, d, 1, 1)$. Notice that $\delta_2\delta_1(s) = b$ and $\delta_0\delta_2(s) = d$. We claim that Y has no such 3-simplex.

Indeed, if we write the ordered decomposition with $\leq_2$, we obtain

$$\begin{aligned} (x * y) * z &= a * z + b * z = \underbrace{1 + 1}_{a*z} + \underbrace{1}_{b*z} \\ x * (y * z) &= x * d + x * c = \underbrace{1}_{x*d} + \underbrace{1 + 1}_{x*d} \end{aligned}$$

There are no index k such that the k-th occurrence of 1 in $x * y * z$ appears at the same time in the decomposition of $x * d$ and $b * z$, hence we cannot have $\delta_2\delta_1(f(s)) = b$ and $\delta_0\delta_2(f(s)) = d$.

Despite those examples, there might some algebraic properties of the algebra that are transferred to the topological properties of the obtained simplicial sets, regardless of the choice of the basis J, or of the order on J. Here are some example where the construction is independent of these choices.

Example 1.15. If G is a group, then if we consider the group algebra $k[G] = \mathbb{Z}^{(G)} \otimes k$, and if we impose $J = G$, then the construction 1.2 produces the simplicial set BG, the classifying space of G.

Example 1.16. More generally, if A is an algebra of the shape $\mathbb{Z}^{(J)} \otimes k$, and if for every $j, j' \in J$, we have $j * j' \in J$, then the simplical set produced by the construction does not depend of the order on J. Such an algebra is of the shape $\mathbb{Z}^{(M)} \otimes k$, with M a monoid, and the obtained simplicial set is isomorphic to BM, the classifying space of the monoid.

Example 1.17. If $A = k$, the only valid choice of J is $J = \{1\}$, hence the produced simplicial set is always a point.

2 2-Segal objects

In this section, we discuss the notion of 2-Segal objects, a generalization of the previous definition to simplicial object in a nice model category. We will motivate this notion by discussing examples of 2-Segal objects that appears in K-theory, which are obtained as the output of the Waldhaussen S-construction.

2.1 Homotopy limits and colimits

In this section we discuss the notion of homotopy limits, which we will need to extend the Segal condition to simplicial objects. For an exhaustive treatment of homotopy limits, see [Hir03, Chapter 18]

First, we mention few classical model structures.

Example 2.1. There is a model structure on *Set* called the *trivial model structure* such that

- The weak equivalences are the isomorphisms,
- Every map is a cofibration and a fibration.

Example 2.2. There is a model structure $sSet_{Qu}$ on *sSet* called the *classical Quillen structure* such that

- The cofibrations are the monomorphisms,
- The weak equivalences are morphisms whose geometric realization is a weak equivalence of topological space,
- The fibrant objects are the Kan complexes.

We now discuss the notions of homotopy limits and colimits, which we will use to define the Segal and 2-Segal objects.

Notice that limits or colimits don't behave well with respect to the passage to the homotopy category. For description of certain homotopy limits in Top, see [Dug08]. For instance in Top, as explained in [Dug08], consider the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \\ D^2 & . \end{tikzcd}$$

The pushout of this diagram in Top is S^2. But the diagram is level-wise equivalent to the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

whose pushout is $*$, so the homotopy type of the pushout is not the same for the two diagrams, despite the fact that they are level-wise weakly equivalent. The colimit functor is not homotopy invariant. The homotopy colimit functor can be thought as a correction of this functor to make it homotopy invariant. We can make this definition more precise if we use the notion of derived functor.

Definition 2.3. Let C be a category with weak equivalences, a functor $F : C \to D$ is *left derivable* if there exists a functor $Q : C \to C$ together with a natural weak equivalence $Q \to Id$ such that $F \circ Q$ sends weak equivalences to isomorphisms. We then take $\mathbb{L}F = F \circ Q$ the *left derived functor* of F. It is independent of the choice of Q.

Dually, F is *right derivable* if there exist $R : C \to C$ and a natural weak equivalence $Id \to R$ such that $F \circ R$ sends weak equivalences to isomorphisms. We then take $\mathbb{R}F = F \circ R$ the *right derived functor* of F.

Then, $\mathbb{L}F : C \to D$ is homotopical, so it can be factored as $\mathbb{L}F : Ho(C) \to D$. When D is a category with weak equivalences, we take the $\mathbb{L}F$ as the left derived functor of $C \xrightarrow{F} D \to Ho(D)$. Therefore, $\mathbb{L}F$ can be seen as a functor $Ho(C) \to Ho(D)$.

Let C be a bicomplete category with weak equivalences and let D be a small category. The limit functor $lim_D : C^D \to C$ is the right adjoint to the constant functor $const : C \to C^D$. The functor $const$ sends an object to the constant diagram at this object. The colimit functor $colim_D : C^D \to C$ is the left adjoint of the constant functor.

Now, we say that a weak equivalence on C^D is an object-wise weak equivalence, and we can define the notions of homotopy limits and colimits.

Definition 2.4. The homotopy limit (resp. homotopy colimit) is defined, if it exists, as the right (resp. left) derived functor of the functor $lim_D : C^D \to C$ (resp. $colim_D : C^D \to D$).

We then have the following adjunction:

$$\begin{tikzcd} Ho(C^D) \arrow[r, shift left, "holim_D"] \arrow[r, phantom, "\top"] & Ho(C), \arrow[l, shift left, "const"] \end{tikzcd}$$

And its dual for homotopy colimits. Note that since in general $Ho(C^D) \ncong Ho(C)^D$, the homotopy limit is not the limit in the homotopy category.

A consequence of the definition is that the homotopy limit only depends on the homotopy types of the objects of the diagram.

In some cases, the homotopy limit functor can be seen as a composition of a functor $H : C^D \to C$ with the localization functor $C \to Ho(C)$. The functor H is then also called the homotopy limit. This is for instance the case when the injective model structure exists on C^D, the homotopy limit becomes the limit of a fibrant replacement of the diagram.

To illustrate these definitions, we restrict ourselves to the category of simplicial sets, with the classical Quillen model structure. The homotopy limits and homotopy colimits always exist in $sSet$, and there is an explicit construction for them called the bar construction. However, output of this construction are hard to compute. In the rest of the text, we are mainly interested in homotopy pullbacks in $sSet$, so we now describe them.

A *Kan complex* is a simplicial set X such that every horn of X has a filler, meaning that for every map $\Lambda^i[n] \to X$, there is a lift

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d] & X \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd} .$$

We can give an explicit description of homotopy pullbacks over a Kan complex. If X is a Kan complex, and A and B are simplicial sets, together with the data of morphisms $A \to X$ and $B \to X$, denote by $A \times^h_X B$ the simplicial set

$$A \times_{Map(\{0\},X)} Map(\Delta[1], X) \times_{Map(\{1\},X)} B,$$

where $Map(C, D)$ is the simplicial mapping space, such that $Map(C, D)_n = Hom_{sSet}(C \times \Delta[n], D)$. The simplicial set $A \times^h_X B$ is an explicit description of the homotopy pullback when X is a Kan complex. Since Kan complexes are the fibrant objects of $sSet$, one can use a fibrant replacement functor to compute the homotopy pullback when X is not a Kan complex, however this makes the homotopy limit harder to describe.

2.2 2-Segal objects

We use the same framework as in [Ber+21a]. Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ be a combinatorial closed symmetric monoidal category in which all objects are cofibrant. Possible choices of $\mathcal{C}$ include for instance $sSet$ the category of simplicial sets with the classical Quillen model structure, or Set, with the model structure where the weak equivalence are isomorphisms.

A *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. When $\mathcal{C} = sSet$, we call simplicial object in $\mathcal{C}$ a bisimplicial set, or a simplicial space. We can extend the definitions of Section 1.1 to simplicial objects.

Definition 2.5. A simplicial object X is said *1-Segal* if for all $n \geq 0$ the map

$$X_n \to X_1 \times^h_{X_0} X_1 \times^h_{X_0} ... \times^h_{X_0} X_1,$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$ is a weak equivalence.

While 1-Segal sets are describing structure with an unique well-defined composition law, 1-Segal objects describe a structure where the composition is defined up to homotopy.

As discussed in section 1.1, a triangulation $\mathcal{T}$ of a regular $(n+1)$-gon yields a diagram in $\mathcal{C}$, an we can consider the homotopy limit of this diagram. So, as in 1.1, it induces a map $f_{\mathcal{T}} : X_n \to X_2 \times^h_{X_1} X_2 \times^h_{X_1} ... \times^h_{X_1} X_2$.

Definition 2.6. A simplicial object X is a *2-Segal object* if for every $n \geq 3$, for every triangulation $\mathcal{T}$ of the regular $(n+1)$-gon, the induced map $f_{\mathcal{T}} : X_n \to X_2 \times^h_{X_1} X_2 \times^h_{X_1} \ldots \times^h_{X_1} X_2$ is a weak equivalence.

The characterization from Proposition 1.6 extends to this context, when asking for weak equivalences instead of isomorphisms.

Proposition 2.7 ([Ber+21a, Prop 1.17])**.** *If X is a simplicial set, then X is 2-Segal if and only if for every $n \geq 3$, the induced maps*

$$X_n \to X_{\{0,1,2\}} \times^h_{X_{\{0,2\}}} X_{\{0,2,\ldots,n\}},$$

and

$$X_n \to X_{\{n-2,n-1,n\}} \times^h_{X_{\{n-2,n\}}} X_{\{0,\ldots,n-2,n\}},$$

are weak equivalences.

In the rest of this section, we will discuss an example of 2-Segal objects, as the output of the Waldhausen S-construction.

2.3 The Waldhausen S-construction

The Waldhausen S-construction provides another way to compute higher K-theory groups. In [DK19], the S-construction can take in input a general type of categories, namely proto-exact categories.

Definition 2.8. [DK19] A *proto-exact category* is the data of a category $\mathcal{E}$ together with two class of morphisms $\mathcal{M}$ and $\mathcal{C}$, whose elements are called *admissible monomorphisms* (represented $\rightarrowtail$) and *admissible epimorphisms* (represented $\twoheadrightarrow$) such that:

1. $\mathcal{E}$ is pointed, *i.e.* it has an object $*$ which is both final and initial. Any morphism $* \to A$ is in $\mathcal{M}$, and any morphism $A \to *$ is in $\mathcal{C}$.

2. The classes $\mathcal{M}$ and $\mathcal{C}$ are closed under compositions and contains all isomorphisms.

3. A commutative square in $\mathcal{E}$

$$\begin{tikzcd} A_2 \arrow[r, tail, "i"] \arrow[d, two heads, "j_2"] & A_1 \arrow[d, two heads, "j_1"] \\ A_2' \arrow[r, tail, "i'"] & A_1' \end{tikzcd}$$

   with $i, i' \in \mathcal{M}$ and $j_1, j_2 \in \mathcal{C}$ is cartesian if and only if it is cocartesian.

4. Any diagram in $\mathcal{E}$

$$\begin{tikzcd} & A_1 \arrow[d, two heads, "j_1"] \\ A_2' \arrow[r, tail, "i'"] & A_1' \end{tikzcd}$$

   with $i' \in \mathcal{M}$ and $j_1 \in \mathcal{C}$ can be completed to a bicartesian square as in 3., with i admissible monomorphism and j_2 admissible epimorphism.

5. Any diagram in $\mathcal{E}$

$$\begin{tikzcd}
A_2 \arrow[r, tail, "i"] \arrow[d, two heads, "j_2"] & A_1 \\
A'_2 &
\end{tikzcd}$$

with $i \in \mathcal{M}$ and $j_2 \in \mathcal{C}$ can be completed to a bicartesian square as in 3., with i' admissible monomorphism and j_1 admissible epimorphism.

Example 2.9. An abelian category is proto-exact, with $\mathcal{M}$ all the monomorphisms and $\mathcal{C}$ all the epimorphisms.

We can now give the version of the Waldhausen S-construction presented in [DK19, Part 2.4]. Let $\mathcal{E}$ be a proto-exact category. We are going to associate a simplicial space to $\mathcal{E}$.

Let $T_n = Fun([1],[n])$ the ordered set formed of ordered pairs (i,j) with $0 \leq i \leq j \leq n$, with $(i,j) \leq (k,l)$ if and only if $i \leq k$ and $j \leq l$. A functor $F : T_n \to \mathcal{E}$ is a commutative diagram :

$$\begin{tikzcd}
F(0,0) \arrow[r] & F(0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & F(0,n-1) \arrow[r] \arrow[d] & F(0,n) \arrow[d] \\
& F(1,1) \arrow[r] & \cdots \arrow[r] & F(1,n-1) \arrow[r] \arrow[d] & F(1,n) \arrow[d] \\
& & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
& & & F(n-1,n-1) \arrow[r] & F(n-1,n) \arrow[d] \\
& & & & F(n,n)
\end{tikzcd}$$

Now denote $\mathcal{W}_n(\mathcal{E})$ the full subcategory of $Fun(T_n, \mathcal{E})$ formed of the diagram F such that:

1. For all $0 \leq i \leq n$, $F(i,i) \cong *$,
2. All horizontal morphisms are in $\mathcal{M}$ and all vertical morphisms are in $\mathcal{C}$,
3. Each square is bicartesian.

Finally, assuming $\mathcal{E}$ is small, take $S_n(\mathcal{E})$ the nerve of the subcategory of $\mathcal{W}_n(\mathcal{E})$ with the same object but only isomorphisms. Since $Fun([1], \bullet)$ is a simplical set, we get a bisimplicial set $S_\bullet(\mathcal{E})$. This bisimplicial set, has the following properties.

Proposition 2.10. *The bisimplicial set $S_\bullet(\mathcal{E})$ is a reduced 2-Segal space.*

Proof. We give a sketch of the proof in a discrete context. We restrict ourselves to the framework of [Ber+18], in which the input of the construction is a stable double category. It can be seen as a proto-exact category, where the pullbacks and pushouts are unique, and not only unique up to isomorphisms as before. Hence the subcategory $\mathcal{W}_n(\mathcal{E})$ contains no isomorphisms that are not identities. So the nerve $N(\mathcal{W}_n)$ is a discrete simplicial set, so the simplicial space $S_\bullet(\mathcal{E})$ is a simplicial set.

We can show that it is a 2-Segal set. Any $s \in \mathcal{S}_3(\mathcal{E})$ can be depicted as

$$\begin{tikzcd}
* \arrow[r, tail] & a_{01} \arrow[r, tail] \arrow[d, two heads] & a_{02} \arrow[r, tail] \arrow[d, two heads] & a_{03} \arrow[d, two heads] \\
 & * \arrow[r, tail] & a_{12} \arrow[r, tail] \arrow[d, two heads] & a_{13} \arrow[d, two heads] \\
 & & * \arrow[r, tail] & a_{23} \arrow[d, two heads] \\
 & & & *
\end{tikzcd}.$$

We can represent the face maps, obtained by composition. The face $\delta_1(s)$ can be depicted as:

$$\begin{tikzcd}
* \arrow[r, tail, bend left] & a_{02} \arrow[r, tail] \arrow[d, two heads, bend left] & a_{03} \arrow[d, two heads, bend left] \\
 & * \arrow[r, tail] & a_{23} \arrow[d, two heads] \\
 & & *
\end{tikzcd}$$

where the bend arrow are obtained by composition of the arrows of s. Similarly $\delta_3(s)$ can be depicted as:

$$\begin{tikzcd}
* \arrow[r, tail] & a_{01} \arrow[r, tail] \arrow[d, two heads] & a_{02} \arrow[d, two heads] \\
 & * \arrow[r, tail] & a_{12} \arrow[d, two heads] \\
 & & *
\end{tikzcd}$$

thus the data of $(\delta_1(s), \delta_3(s))$ is enough to uniquely determine s. Indeed, if we color this data in the diagram:

$$\begin{tikzcd}
\color{red}{*} \arrow[r, tail, red] & \color{red}{a_{01}} \arrow[r, tail, red] \arrow[d, two heads, red] & \color{red}{a_{02}} \arrow[r, tail, blue] \arrow[d, two heads, red] & \color{blue}{a_{03}} \arrow[d, two heads] \\
 & \color{red}{*} \arrow[r, tail, red] & \color{red}{a_{12}} \arrow[r, tail] \arrow[d, two heads, red] & a_{13} \arrow[d, two heads] \\
 & & \color{red}{*} \arrow[r, tail] & \color{blue}{a_{23}} \arrow[d, two heads] \\
 & & & *
\end{tikzcd}$$

And the rest of the data is recover using pushouts, by property 5 of Definition 2.8.

We showed that the map $(\delta_1, \delta_3) : \mathcal{S}_3(\mathcal{E}) \to \mathcal{S}_{\{0,2,3\}}(\mathcal{E}) \times_{\{0,2\}} \mathcal{S}_{\{0,1,2\}}(\mathcal{E})$ is a bijection. The other cases are treated similarly. □

We motivate this example using some notion of K-theory. For more details see [Wei13, Part IV.6]. First, we need to define the geometric realization of a bisimplicial set.

Definition 2.11. The geometric realization BX of a bisimplicial set X is obtained by taking a copy of $\Delta^p \times \Delta^q$ for each element of $X_{p,q}$, and by identifying the horizontal and vertical faces with the corresponding $\Delta^p \times \Delta^{q-1}$ or $\Delta^{p-1} \times \Delta^q$, and collapsing its degeneracies.

We can consider the homotopy groups of a bisimplicial set, as the homotopy groups of its geometric realization. If $\mathcal{E}$ is a proto-exact category, we define the K-theory groups of $\mathcal{E}$ as $K_n(\mathcal{E}) = \pi_{n+1}(S_\bullet(\mathcal{E}))$. This definition coincides with the other definitions of the K-theory groups. We precise this in the case of an exact category.

Definition 2.12. An *exact category* is the data of a pair $(\mathcal{C}, E)$, with $\mathcal{C}$ an additive category and E a family of sequences

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0,$$

such that there is an embedding of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{A}$ such that E is the class of all sequences as above of $\mathcal{C}$ that are exact in $\mathcal{A}$. We also require $\mathcal{C}$ to be closed by extension, meaning that if we have a short exact sequence in $\mathcal{C}$ as above, with $A, C \in \mathcal{C}$, then B is isomorphic to an object of $\mathcal{C}$.

And for an exact category $\mathcal{E}$, the group $K_0\mathcal{E}$ is usually defined as the abelian group generated by the generators $[C]$ for each object C of $\mathcal{E}$, and by the relation $[A]+[C]=[B]$ for each short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{E}$.

Notice that $[0]$ is the neutral element, that $[A \oplus B] = [A] + [B]$, and that $[A] = [A']$ if A and A' are isomorphic.

Proposition 2.13 ([Wei13, Prop IV.8.4])**.** *If $\mathcal{E}$ is an exact category, the group $\pi_1(S_\bullet(\mathcal{E}))$ coincides with the group $K_0\mathcal{E}$ defined above.*

Proof. Let X be any bisimplicial set such that $X_0 = *$. We use the classical fact that the fundamental group of a CW-complex only depends on its 2-skeleton.

First notice that $\pi_0(X) = *$. We deduce that $\pi_1(X)$ is the group generated by the generators $X_{1,0}$ (since $X_{0,1} = *$), and two types of relations. The first relations are those coming from the cells of the shape $\Delta^1 \times \Delta^1$ associated to $X_{1,1}$, which identify two elements of $X_{1,0}$ if there exists an element of $X_{1,1}$ connecting them. The other relations comes from the cells Δ^2 corresponding to the elements of $X_{2,2}$. For $x \in X_{2,0}$, notice that x imposes the relation $\delta_1(x) = \delta_2(x)\delta_0(x)$. Furthermore, notice that if two elements of $X_{2,0}$ are connected by an edge of $X_{2,1}$ then there respective faces are in the same connected component of X_1. We deduce that $\pi_1(X)$ is the group generated by the elements of $\pi_0(X_1)$ as generators, and with the relations $\delta_1(x) = \delta_2(x)\delta_0(x)$, for $x \in \pi_0(X_2)$.

We apply this with $X = S_\bullet(\mathcal{E})$. First, we have $S_0(\mathcal{E}) = N(\mathcal{W}_0(\mathcal{E})) = N(*) = *$. Next, $\pi_0(S_1(\mathcal{E}))$ correspond to the isomorphism classes of objects of $\mathcal{E}$. Finally, $\pi_0(S_2(\mathcal{E}))$ correspond to the isomorphism classes of short exact sequences of the shape $s = \{0 \rightarrowtail A \rightarrowtail B \twoheadrightarrow C \twoheadrightarrow 0\}$. For such a sequence we have $(\delta_0, \delta_1, \delta_2)(s) = (C, B, A)$. Hence we have $\pi_1(S_\bullet(\mathcal{E})) = K_0\mathcal{E}$. ☐

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