Height theory in Arithmetic geometry

Owen Sabatin

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0.1 Introduction

In this report, we wish to present an introduction to arithmetic geometry and the height theory, as well as discuss some applications to height theory for abelian variety. We first explain the naïve height theory and its application to the Mordell-Weil conjecture. It is straighforward to describe explicitely a height on the projective space $\mathbb{P}^n$ measuring the arithmetic complexity of a point, using the homogenous coordinate. In order to extend this definition to a general projective space X, we first choose an embedding $i : X \to \mathbb{P}^n$ and show that it induce a height theory which depends, up to a bounded term, only on the very ample line bundle $i^*\mathcal{O}(1)$. In fact we can consider a height theory associated to any line bundle L on X. If X is an abelian projective space it is possible to normalize this height using the group structure, this is called the Neron-Tate height, and we can use this height in order to prove the Mordell-Weil conjecture for X.

In the second part we present an Arakelov description of the height. There is a more precise description of this theory, based on global intersection theory and results on Green currents, as explain in [5], but it is usually unnecessary for the applications.

In the last part of this report we apply those tools to the proof of the Bogomolov conjecture. This result states that, on a non torsion subvariety variety X of an abelian variety A, the number of points with small Neron-Tate height (with respect to an ample line bundle) is finite.

They are of course many other applications of height theory in arithmetic. A well-known application is Falting's isogeny theorem and its use for the resolution of Mordell's conjecture. Here Faltings height theory is applied on a moduli-space in order to show that an isogeny class contains only finitely many isomorphism classes of principally polarized abelian variety, but there are a certain number of obstructions to this idea.

This internship was the opportunity to attend the Obseminar "Height bounds and K-stability of Fano manifolds", we studied a recent conjecture on arithmetic Fano manifolds on an explicit bound for their height, we review more specifically the proof of this conjecture for arithmetic toric Fano manifolds, the toric hypothesis allows to reformulate the conjecture in terms of combinatorial data.

I would like to thanks W.Gubler and K.Künnemann for their help and for receiving me at the university of Regensburg. I also thanks D. Biswas for answering some of my questions.

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0.2 Heights and application to the Mordell-Weil conjecture

In this chapter, we fix K a global field and we call M_K the choice of absolute values on K for each the places of K satisfying together the product formula, $\overline{K}$ a choice of algebraic closure of K, and G_K the absolute Galois group of K. We will mainly follow the content of [12].

0.2.1 Height of the projective space and Northcott's theorem

Concretely, the height measures the arithmetic complexity of points. We start by giving the height explicitly on $\mathbb{P}^n$.

Definition 1. For $p = (T_0, ..., T_n) \in \mathbb{P}^n(K)$ a closed point defined by homogeneous coordinates, we set

$$H_K(p) = \Pi_{v \in M_K} \sup_i |T_i|_v$$

and $h(p) = \log(H(p))$ is the logarithmic height on $\mathbb{P}^n(K)$.

This is independent in the choice of homogeneous coordinates by the product formula for the field K.

Example 2. If $K = \mathbb{Q}$, and if $p = (T_0, ..., T_n)$ is written in homogeneous coordinates such that $T_i \in \mathbb{Z}$ have no common factor, then

$$H(p) = \sup_i |T_i|$$

Indeed, for each $v \in M_{\mathbb{Q}}$ finite place, we have $\sup_i |T_i|_v = 1$, otherwise the T_i would have v as common component. Hence only the non-archimidian part of the heights is different from 1.
Take $p = (T_0, ..., T_n)$ a rational point written in homogeneous coordinates, and set $a = T_0\mathcal{O}_K + ... + T_n\mathcal{O}_K$ with $\mathcal{O}_K$ the ring of integer of K and $d = \deg(K/\mathbb{Q})$ then we have

$$H(p) = \frac{1}{N(a)^{1/d}} \Pi_{v \in M_{K,\infty}} \sup_i |x_i|$$

where $N : K^* \to \mathbb{Q}^*$ is the norm map associated to the finite extension $K/\mathbb{Q}$.

Here are some properties of this height:

Proposition 3. *-If L/K is a finite extension, and if $p \in \mathbb{P}^n(K)$, then $H_L(p) = H_K(p)$.*
-Let $\otimes : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ be the Segre embedding defined by $(T_i) \otimes (S_j) = (T_i.S_j)$, then $H(x \otimes y) = H(x) \cdot H(y)$.
-If $\phi : \mathbb{P}^n \to \mathbb{P}^m$ is defined by homogeneous functions of degree d, $\phi_0, ..., \phi_m$. Then $h(\phi(x)) \leq d \cdot h(x) + O(1)$.

This notion is strongly motivated by the following theorem of Northcott

Theorem 4. *If $B > 0$ is a real number, then the set of rational point of height bounded by B, $T := \{p \in \mathbb{P}^n_K | h(p) \leq B\}$ is finite.*

Proof. We limit ourselves to the case where K is a number field, by enlarging the field we can suppose $L/\mathbb{Q}$ to be a Galois extension of Galois group G of cardinal d. Now we have d morphisms $\phi_i : \mathbb{P}^n_K \to \mathbb{P}^n_K$ of degree i that send a point $x = (T_0, ..., T_i)$ to $(s_i(\sigma(T_j)_{\sigma \in G})_{0 \leq j \leq n})$. Those morphisms send a K-rational point to a $\mathbb{Q}$-rational point, now $\phi = (\phi_i) : \mathbb{P}^n_K \to (\mathbb{P}^n_K)^d$ is a finite morphism and if x is of bounded heigths, then $\phi_i(x)$ is of bounded heights as well by the preceding proposition.
Hence we have reduced the problem to $K = \mathbb{Q}$, where it is very explicit: if $x = (T_0, ..., T_n)$ is of bounded height with $T_0, ..., T_n \in \mathbb{Z}$ with no common factor, then $H(x) = \sup_i |T_i|$ is bounded and there are only finitely many possibilities for the values of $T_i \in \mathbb{Z}$. ■

When we have a K-projective variety V and an ample line bundle L on V with a choice of global generators $(s_i)_{0 \leq i \leq n}$, then we get a morphism $\phi : V \to \mathbb{P}^n_K$ associated to it. Then for $x \in V(L)$ a closed point of V we can define the height of x associated to the morphism ϕ by $h_\phi(x) = h(\phi(x))$.
Up to a bounded function, this height only depends on L and not on the choice of global generators, indeed let $(s'_i)_{0 \leq i \leq n}$ be another choice of such generators, we would get a linear K-isomorphism $l : H^0(V, L) \to H^0(V, L)$ sending s_i to s'_i and it would induce a diagram

$$\begin{array}{ccc} V & \xrightarrow{(s_i)} & \mathbb{P}^n_K \\ & \searrow^{(s'_i)} & \downarrow l \\ & & \mathbb{P}^n_K \end{array}$$

Because of the preceding proposition, $h(s'_i(x)) = h(l(s_i(x)))$ is equal to $h(s_i(x))$ up to a bounded function.
This allow us to use the notation h_L for the equivalence class, up to bounded function, of h_ϕ.

Theorem 5. *(Weil) There is a unique map $L \mapsto h_L$ from the Picard group $Pic(V)$ to the set of functions $h : V(\overline{K}) \to \mathbb{R}$ up to bounded function with the following properties:*

- *When $L = L' \otimes L''$, then $h_L = h_{L'} + h_{L''}$*
- *When L is ample and $\phi : V \to \mathbb{P}^n_K$ is the associated morphism (with a choice of global generator), then $h_L = h_\phi$*
- *When $L = \mathcal{O}_V$, then $h_L = 0$*

Proof. We first prove unicity. There are some very ample line bundle M on V since V is projective, let M be some choice of such line bundle. Then for every line bundles L, there exists n big enough so that $R := L \otimes M^{\otimes n}$ is very ample. Hence we see by definition that $h_L = h_R - n.h_M$, hence the map is unique.
To prove the existence we need to see the following: if L, L', L'', L''' are four

very ample line bundles with $L \otimes L'^{-1} \simeq L'' \otimes L'''^{-1}$, then we have $h_{L\otimes L'^{-1}} = h_{L''\otimes L'''^{-1}} + O(1)$. But we have the isomorphism of very ample line bundles $L \otimes L''' \simeq L'' \otimes L'$, hence $h_L + h_{L'''} = h_{L''} + h_{L'} + O(1)$ and we get the desire estimate. ■

Combining both theorem 4 and theorem 5 we get the following: If V is a projective K variety and L is any ample line bundle, then the set of rational point of height bounded by a constant $B \in \mathbb{R}$, $\{x \in V(K) | h_L(x) \leq B\}$, is finite.

Later we will see another description of heights in term of Arakelov intersection theory. Before going to the next section we need to see one more fix point lemma

Lemma 6. *If $\phi : S \to S$ is an endomorphism of set and $h : S \to \mathbb{R}$ a function such that $h \circ \phi = \lambda h + O(1)$ with $\lambda > 1$, then there exists a unique function $\tilde{h} : S \to \mathbb{R}$ such that $h = \tilde{h} + O(1)$ and $\tilde{h} \circ \phi = \lambda \tilde{h}$.*

Proof. We first prove unicity. If such a function $\tilde{h}$ exist, we have

$$\tilde{h}(x) = \lim_{n\to+\infty} \frac{1}{\lambda^n} \tilde{h}(\phi^{\circ n}(x))$$

$$= \lim_{n\to+\infty} \frac{1}{\lambda^n} h(\phi^{\circ n}(x)) + \frac{1}{\lambda^n} O(1)$$

$$= \lim_{n\to+\infty} \frac{1}{\lambda^n} h(\phi^{\circ n}(x))$$

This shows unicity, and it suggests a candidate for $\tilde{h}$. The sequence $\frac{1}{\lambda^n} h(\phi^{\circ n}(x))$ is converging: if we let $b = h \circ \phi - \lambda \cdot h$, it is bounded by hypothesis, then

$$\frac{1}{\lambda^{n+m}} h(\phi^{\circ n+m}(x)) = \frac{1}{\lambda^n} h(\phi^n(x)) + \sum_{0\leq i\leq m} \frac{b(x)}{\lambda^{n+i}}$$

The term $\sum_{0\leq i\leq m} \frac{b(x)}{\lambda^{n+i}}$ is close to zero uniformaly in x, m as n is big, so the sequence is Cauchy hence convergent. If we take $n = 0$ we also see that we indeed have $\tilde{h} = h + O(1)$
It remain to check that $\tilde{h} \circ \phi = \lambda \tilde{h}$. But $\frac{1}{\lambda^n} h(\phi^{\circ n}(\phi(x))) = \lambda \frac{1}{\lambda^{n+1}} h(\phi^{\circ n+1}(x))$ so this is clear. ■

0.2.2 Height on Abelian varieties

We focus on the case where $V = A$ is an Abelian variety over K, i.e. a proper (in fact projective) geometrically irreducible group variety over K.

We first recall the theorem of the cube and the rigidity theorem.

Theorem 7. *(Rigidity)*
If V is proper, $V \times W$ geometrically irreducible and $\alpha : V \times W \to U$ is a morphism such that there exists $v_0 \in V$, $u_0 \in U$, $w_0 \in W$ with $\alpha|_{V\times\{w_0\}} = u_0 = \alpha|_{\{v_0\}\times W}$, then α is the constant function u_0.

Proof. See ([10], Theorem 1.1) for the proof. ■

Corollary 8. *If A, B are Abelian varieties and $\phi : A \to B$ is a morphism of variety sending 0_A to 0_B, then ϕ is a morphism of groupe variety.*

Proof. We can define the morphism $r : A \times A \to B$ to be $\phi \circ m - \phi \circ pr_1 - \phi \circ pr_2$. Then r is equal to 0_A on $A \times \{0_A\}$ and $\{0_A\} \times A$ so that r is the constant function 0_A by the rigidity theorem. That is ϕ is a morphism of group variety. ■

Theorem 9. *(Theorem of the cube)*
Let $(V, v), (W, w), (T, t)$ be three pointed proper varities, call $l_{12} : V \times W \to V \times W \times T$, $l_{13} : V \times T \to V \times W \times T$, $l_{23} : W \times T \to V \times W \times T$ the embedding defined by the choice of points.
When L is a line bundle on $V \times W \times T$ such that $l_{12}^* L$, $l_{13}^* L$, and $l_{23}^* L$ are trivial, then L is a trivial line bundle.

Proof. See [11] for the proof. ■

Corollary 10. *If A is an Abelian variety and L a line bundle on A. Let $s_{123} = m \circ (m \circ \pi_{12}, \pi_3), s_{ij} = m \circ \pi_{ij}, s_i := \pi_i : A \times A \times A \to A$ be seven morphisms define by the group structure on A. Then we have the formula*

$$s_{123}^* L - s_{12}^* L - s_{23}^* L - s_{13}^* L + s_1^* L + s_2^* L + s_3^* L = 0$$

Proof. Set $t = s_{123}^* L - s_{12}^* L - s_{23}^* L - s_{13}^* L + s_1^* L + s_2^* L + s_3^* L$. By the theorem of the cube we only need to check that $t|_{\{0\} \times A \times A}$, $t|_{A \times \{0\} \times A}$ and $t|_{A \times A \times \{0\}}$ are zero. By symmetry it suffices to show the first element is zero, but we have $s_{123}(0, y, z) = s_{23}(0, y, z)$, $s_{12}(0, y, z) = s_2(0, y, z)$, $s_{13}(0, y, z) = s_3(0, y, z)$ and $s_1(0, y, z) = 0$ so this is clear. ■

Proposition 11. *Let A be an Abelian variety. Then there is a unique map $L \mapsto \tilde{h}_L$ from $Pic(A)$ to $A(\overline{K}) \to \mathbb{R}$ with the following properties.*

- (1) *For every L, $h_L = \tilde{h}_L + O(1)$*
- (2) *For $\phi : A \to B$ a morphism of Abelian variety and L a line bundle on B, we have $\tilde{h}_{\phi^* L} = \tilde{h}_L \circ \phi$.*
- (3) *the map $\tilde{h}$ is a morphism of groups from $Pic(A)$ to $\mathbb{R}^{A(\overline{K})}$*

$\overline{h}_L$ is called the Neron-Tate height

Proof. We first assume that L is symmetric: $[-1]^* L \simeq L$, we then have $[2]^* L = L^4$.
We show unicity and then deduce existence. For that we use the multiplication endomorphism $[2] : A \to A$.
By lemma 6 apply to $h = h_L$, $\lambda = 4$, and $\phi = [2]$, there is a unique function $\tilde{h}$ with $\tilde{h}_L(2 \cdot x) = 4\tilde{h}_L(x) = \tilde{h}_{[2]^* L}(x)$ and $\tilde{h}_L = h_L + O(1)$. So $\tilde{h}_L$ is unique and we have our candidate.
For the existence, property (1) is clear.
For (2) if $\phi : A \to B$ is a morphism of Abelian variety, then $\tilde{h}_L \circ \phi = h_L \circ \phi + O(1) = h_{\phi^* L} + O(1)$ because of the definition of naive height is compatible with pullback by definition. Also we have $\tilde{h}_L \circ \phi \circ [2] = \tilde{h}_L \circ [2] \circ \phi = 4\tilde{h}_L \circ \phi$. Hence by unicity of the fix function we indeed get $\tilde{h}_{\phi^* L} = \tilde{h}_L \circ \phi$.
For (3) it is similar, $\tilde{h}_{L+L'} = h_L + h_{L'} + O(1)$ by definition of the height and

property (1), and from both side composing with [2] is the same as multiplying by 4, hence by unicity we get $\tilde{h}_{L+L'} = \tilde{h}_L + \tilde{h}_{L'}$.
Now if L is antisymmetric, we have $[2]^*L = 2L$ and we can apply the same idea. In the general case, $L^2 = (L \otimes [-1]^*L) \otimes (L \otimes [-1]^*L^{-1})$ can be written as the sum of a symmetric line bundle with an antisymmetric one, so we are done. ■

Proposition 12. *For A an Abelian variety and L a line bundle on A, $h_L : A \to \mathbb{R}$ is a quadratic form over $\mathbb{Z}$. That is the function*

$$A \times A \to \mathbb{R} : (x, y) \mapsto \tilde{h}_L(x + y) - \tilde{h}_L(x) - \tilde{h}_L(y)$$

is bilinear.

Proof. This is equivalent to check that the function

$$A \times A \times A \to \mathbb{R} : (x, y, z) \mapsto \tilde{h}_L(x + y + z) - \tilde{h}_L(x + y) - \tilde{h}_L(y + z) - \tilde{h}_L(x + z) + \tilde{h}_L(x) + \tilde{h}_L(y) + \tilde{h}_L(z)$$

is identicaly zero. But by the properties of $\tilde{h}_L$ we can rewrite this function as

$$\begin{aligned}(x, y, z) \mapsto \tilde{h}_{s_{123}^*L}(x, y, z) - \tilde{h}_{s_{12}^*L}(x, y, z) - \tilde{h}_{s_{23}^*L}(x, y, z) - \tilde{h}_{s_{13}^*L}(x, y, z) \\ + \tilde{h}_{s_1^*L}(x, y, z) + \tilde{h}_{s_2^*L}(x, y, z) + \tilde{h}_{s_3^*L}(x, y, z) \\ = h_{s_{123}^*L - s_{12}^*L - s_{23}^*L - s_{13}^*L + s_1^*L + s_2^*L + s_3^*L}(x, y, z)\end{aligned}$$

which is zero by theorem 10. ■

Corollary 13. *For A an Abelian variety, the set of rational torsion points of A is finite.*

Proof. Let L be a symmetric ample line bundle on A, by Northcott's theorem it suffices to show that the height of torsion points are bounded. First we show that $\tilde{h}_L(0_A) = 0$. In fact $\tilde{h}_L(0_A) = \tilde{h}_L([2]0_A) = 4\tilde{h}_L(0_A)$ so that $3\tilde{h}_L(0_A) = 0$. Now if we take $n > 0$ with $nx = 0_A$, then $n^2\tilde{h}_L(x) = \tilde{h}_L(n.x) = 0$: the set of torsion point is of bounded height. ■

Remark 14. Note that there is a converse to the fact that x is a torsion point implies $\tilde{h}_L(x) = 0$. If $x \in A(L)$ is such that $\tilde{h}_L(x) = 0$ then x is torsion. Indeed for every $n \in \mathbb{Z}$ we have $\tilde{h}_L(n.x) = 0$ and by Northcott theorem we have that $\{nx | n \in \mathbb{Z}\}$ is finite. By the pigeon hole principle we get $n, m \in \mathbb{N}$ with $n > m$ such that $nx = mx$ so that $(n - m)x = 0$ and x is torsion.

We need a lemma before going further

Lemma 15. *Let V be a finite dimensional $\mathbb{Q}$-vector space, $\Gamma \subset V$ a subgroup generating V as a vector space and $h : V \to R$ a quadratic form such that for every $B \in \mathbb{R}$, $\Gamma \cap \{x \in V | h(x) \leq B\}$ is finite, then h is positive definite and Γ is a lattice.*

Proof. h is positive definite, otherwise there is $v \in V \setminus \{0\}$ such that $h(v) \leq 0$, by clearing denominator we can choose $v \in \Gamma$. Then $h(n.v) = n^2h(v) \leq 0$ so that the set $\{x \in V | h(x) \leq 0\}$ is infinite contradicting the hypothesis.
Tensoring by $\mathbb{R}$, we saw that if Γ were not a lattice, it would be non discrete in $V_{\mathbb{R}}$. So there is a converging to zero sequence $r_i \in \Gamma \setminus \{0\}$ and $h(r_i)$ converges to $h(0) = 0$, and hence for every $\varepsilon > 0$, $\Gamma \cap \{x \in V | h(x) \leq \varepsilon\}$ is infinite leading to a contradiction. ■

We finally deduce the following structure theorem

Theorem 16. *For A an Abelian variety over a global field K, the group $A(K)$ is isomorphic to $T \oplus \mathbb{Z}^I$ with I a (countable) set and T a finite Abelian group*

Proof. Let L be an ample line bundle.
Choose a basis $(v_i)_{i\in\mathbb{N}}$ of $A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ and set $V_n = \langle v_i \rangle_{i \leq n}$, this is possible since K is countable and hence $A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is too. Then $A(K)/Tors(A(K)) \cap V_n$ generate V_n (by clearing denominator in v_i) and $\tilde{h}_L : V_n \to \mathbb{R}$ is a quadratic positive definite form with $A(K)/Tors(A(K)) \cap V_n$ a lattice in it by the preceding lemma and Northcott's theorem. The image of the morphism $A(K) \to V_{n+1}/V_n$ is a finitely generated subgroup, so it is isomorphic to $\mathbb{Z}$ and we can find a $\mathbb{Z}$-basis w_i of $A(K)/Tors(A(K))$ with $w_i \in V_i$ because $A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the filtred union of the V_i . We deduce that $A(K)/Tors(A(K))$ is free, hence the theorem. ■

0.2.3 Mordell-Weil theorem

The goal of this section is to prove the Mordell-Weil Theorem:

Theorem 17. *Let A be an Abelian variety over K, then $A(K)$ is a finitely generated Abelian group*

Because of the structure theorem 16, we only need to prove a weak form of this theorem

Theorem 18. *Let A be an Abelian variety over K, $n > 0$ an integer, then $A(K)/nA(K)$ is finite*

Indeed if we apply this theorem to $n = 2$ for example, then we see that $A(K)/nA(K) = T/2T \oplus (F_2)^I$ is finite, so that I is a finite set.

To prove this theorem, we first need some results

Theorem 19. *(Hermite) Let K be a number field, S a finite set of place of K and d an integer. Then the set of extensions L/K unramified outside S with bounded degree d is finite.*

Proof. Ramification is compatible with composition. So, because $K/\mathbb{Q}$ is unramified everywhere except at a finite number of places, we reduce to $K = \mathbb{Q}$. Then, beeing unramified outside $S \subset Spec\mathbb{Z}$ for an extension $L/\mathbb{Q}$ means that the discriminant δ_L is not divided by prime not in S. A computation of Hensel shows that

$$v_p(\delta_L) \leq d - 1 + d \log(d)/log(p)$$

for every $p \in S$.
It remains to check that the set of number field with bounded degree and bounded discriminant is finite.
For such a field L, recall that the ring of integer $\mathcal{O}_L$ is a lattice of $\mathbb{R}^{n_1+n_2}$ for n_1 the number of real places and $2n_2$ the number of complex places. By Minkowsky's theorem, there is a constant C bounding in function of the discriminant, and $x \in \mathcal{O}_L$ such that its image $(x = x_0, x_1, ..., x_{n_1+n_2})$ is small enough: $|x_i| \leq C$ for every i. We can choose x to generate the field L and by the inequality on it's conjugate, its minimal polynomial has coefficients bounded by the discriminant of L: they are finitely many choices for such a x. ■

Lemma 20. *(Chevalley) If $\pi : V \to V'$ is a finite étale morphism of variety over K, then there is a finite extension L/K such that for every $x \in V'(K)$, there is $y \in V(L)$ mapping by π to x.*

Proof. By Hermite's theorem 19, it suffices to show that there is $d \in \mathbb{N}$ and S a finite set of place such that for every $x \in V'(K)$, there is an extension L/K of degree less than d and unramified outside S with $y \in V(L)$ mapping by π to x. By quasi-compactness, we can find a finite set of place S such that the constructions of V and V' are defined over $\mathcal{O}_{K,S}$, so we can find models $\mathcal{V}, \mathcal{V}'/Spec\mathcal{O}_K$ of V and V' respectively together with a finite étale morphism $\phi : \mathcal{V} \to \mathcal{V}'$ extending the morphism $\pi : V \to V'$.
A point $x \in V'(K)$ is a morphism $Spec(K) \to V'$, by enlarging S we can assume that this morphism is extended by a morphism $l : Spec(\mathcal{O}_{K,S}) \to mathcalV'$. By base changing ϕ along l, we get a finite étale morphism $\mathcal{V} \times_{\mathcal{V}'} Spec(\mathcal{O}_{\mathcal{K},\mathcal{S}}) = Spec(A) \to Spec(\mathcal{O}_{\mathcal{K},\mathcal{S}})$.
Looking at the generic fiber and taking any factor of $A_{\eta_{\mathrm{Spec}(\mathcal{O}_{K,S})}}/K$, we get an extension L/K unramified outside S of degree bounded by the one of ϕ and the corresponding point of $Spec(A)$ induces a point y on $V(L)$ mapping to x by π. ■

We can now prove the Weak Mordell-Weil theorem

Proof. By enlarging K if necessary, we may assume that every n-torsion point is rational.
We take the finite étale morphism $[n] : A \to A$. For $x \in A(K)/nA(K)$, we choose $\tilde{x} \in A(K)$ representing x. Then, there is $y \in A(L)$ with $n \cdot y = \tilde{x}$, L/K being an extension independent of x by lemma 20. For $\sigma \in G_K$ we have $y - \sigma(y) \in A_n(K)$, hence we also have an application

$$A(K)/nA(K) \to Hom(Gal(L/K), A_n(K)) \colon x \mapsto (p_x : \sigma \mapsto y - \sigma y)$$

It is well defined since if $y' \in A(L)$ satisfies $n(y - y') \in nA(K)$, then there is $r \in A(K)$ with $y - y' - r \in A_n$ so $0 = y - y' - r - \sigma(y - y' - r) = y - \sigma y - (y' - \sigma y')$. This is a morphism of groups because for $\sigma, \tau \in Gal(L/K)$, $p_x \colon (\sigma\tau) = y - \sigma\tau y = y - \sigma y + \sigma(y - \tau y) = p_x(\sigma) + p_x(\tau)$.
It is injective because otherwise, if $y - \sigma(y) = 0$ for every σ, then $y \in A(K)$ and $x = ny \in nA(K)$. Hence $A(K)/nA(K)$ is finite. ■

0.3 Arakelov Geometry

In this chapter we study the Arithmetic intersection theory and some tools used in Arakelov Geometry.

Before introducing the theory we must discuss the intuition of the theory. We want to develop an intersection theory on the space $\operatorname{Spec}\mathbb{Z}$ and more generally on the flat projective spaces above $\operatorname{Spec}\mathbb{Z}$. In fact, we will restrict to the regular case. In the classical setting, namely for algebraic varieties, we can define intersection number in the special case of proper variety: this is because we can define a *pushforward* along proper map for Chow groups, and then we can define the notion of degree on cycles. The problem here is that $\operatorname{Spec}\mathbb{Z}$ being an affine curve, there is no hope we can naïvely provide intersection number on this scheme or on a projective scheme above it. Thus, we must be seeking for a compactification $\widehat{\operatorname{Spec}\mathbb{Z}}$. In order to do that, we consider a point ∞ that corresponds to a fix archimedian absolute value $|.|$ on $\mathbb{Z}$. Then, the good definition of degree for divisor on this scheme is exactly the product formula for the global field $\mathbb{Q}$. More generally, for $\mathcal{X}$ a proper regular flat scheme over $\operatorname{Spec}\mathbb{Z}$, we will look at the compact complex manifold $\mathcal{X}(\mathbb{C})$.

0.3.1 Global Height

In this section we study the global case. We thus fix $\mathcal{X}$ a regular flat projective over $Spec\mathcal{O}_K$.

Definition 21. A metrized line bundle $\overline{L}$ is the data of a line bundle L over $\mathcal{X}$ and a Hermitian metric on the line bundle $L_{\mathbb{C},\sigma}$ of $\mathcal{X}$ for each archimedian places σ of K.

The Arakelov description of the height is an analogue of the degree of a cycle along a line bundle in complex geometry define by intersection theory. This motivate the following proposition analogue to the intersection number in the classical case

Proposition 22. *Let $\overline{L}_1, \cdots, \overline{L}_d$ be metrized line bundles, there is a unique linear map $(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_1)|\cdot) : Z_d(\mathcal{X}) \to \mathbb{R}$ such that*

- *When Z is an integral subscheme of $\mathcal{X}$ and s is a rational section of $\overline{L_d}^m|_Z$, then*

$$m(\hat{c}_1(\overline{L_1}) \cdots \hat{c}_1(\overline{L_d})|Z) = (\hat{c}_1(\overline{L_1}) \cdots \hat{c}_1(\overline{L_{d-1}})|div(s)) - \int_{Z(\mathbb{C})} log(|s|) c_1(L_{1,\mathbb{C}}) \cdots_1 (L_{d,\mathbb{C}})$$

- *When Z is an integral subscheme of dimension 0, then it is a closed point of $\mathcal{X}$ and $\hat{deg}(Z) = log(\#\kappa(Z))$.*

Moreover it is multilinear and symmetric in the variables $\overline{L}_i$.

as in the classical case, we have a projection formula

Proposition 23. *Let $f : \mathcal{X}' \to \mathcal{X}$ be a proper map between regular flat projective scheme over* $\operatorname{Spec}\mathcal{O}_K$*, let $Z \in Z_d(\mathcal{X}')$ be an integral subscheme of $\mathcal{X}'$. Then*

- *When $dim(f(Z)) < d$ we have*

$$(\hat{c}_1(f^*\overline{L}_1), \cdots, \hat{c}_1(f^*\overline{L}_d)|Z) = 0$$

- *When $dim(f(Z)) = d$ we have*

$$(\hat{c}_1(f^*\overline{L}_1), \cdots, \hat{c}_1(f^*\overline{L}_d)|Z) = (\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_d)|f_*(Z))$$

Here $f_*(Z)$ denote $deg(Z/f(Z)).f(Z)$

Proof. We can prove this formula by induction on d.
If $d = 0$ we have to show that $log(\#\kappa(Z)) = log(\#\kappa(f(Z)))$, this is clear since we have a canonical isomorphism of field $\kappa(f(Z)) \to \kappa(Z)$.
In the global case, suppose $d > 0$, and choose s a section of $\overline{L}_d^m$ for some $m > 0$. This induces a section f^*s on $f^*\overline{L}_d^m$. We have

$$m(\hat{c}_1(f^*\overline{L}_1), \cdots, \hat{c}_1(f^*\overline{L}_d)|Z) =$$

$$m(\hat{c}_1(f^*\overline{L}_1), \cdots, \hat{c}_1(f^*\overline{L}_{d-1})|div(f^*s|_Z)) - \int_{Z(\mathbb{C})} log(|f^*s|)c_1(f^*L_{1,\mathbb{C}}) \cdots_1 (f^*L_{d,\mathbb{C}})$$

On the other hand we know that

$$c_1(f^*L_{1,\mathbb{C}}) \cdots_1 (f^*L_{d,\mathbb{C}}) = f^*c_1(L_{1,\mathbb{C}}) \cdots_1 (L_{d,\mathbb{C}})$$

thus

$$\int_{Z(\mathbb{C})} log(|f^*s|)c_1(f^*L_{1,\mathbb{C}}) \cdots_1 (f^*L_{d,\mathbb{C}}) = deg(Z/f(Z)). \int_{f(Z)(\mathbb{C})} log(|s|)c_1(L_{1,\mathbb{C}}) \cdots_1 (L_{d,\mathbb{C}})$$

if $Z \to f(Z)$ is generically finite, the integral is 0 otherwise.
On the other hand

$$f_*div(f^s|_Z) = deg(Z/f(Z)).div(s|_Z)$$

by induction, either $dim(f(Z)) < d$ so that $dim(div(s|_{f(Z)})) < d - 1$ and $(\hat{c}_1(f^*\overline{L}_1), \cdots, \hat{c}_1(f^*\overline{L}_{d-1})|div(f^*s|_Z) = 0$, or $(\hat{c}_1(f^*\overline{L}_1), \cdots, \hat{c}_1(f^*\overline{L}_{d-1})|div(f^*s|_Z) = deg(Z/f(Z)).(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|div(s|_Z)$.
All put together and using the recursion formula

$$m(\hat{c}_1(\overline{L_1}) \cdots \hat{c}_1(\overline{L_d})|Z) = (\hat{c}_1(\overline{L_1}) \cdots \hat{c}_1(\overline{L_{d-1}})|div(s|_Z)) - \int_{Z(\mathbb{C})} log(|s|)c_1(L_{1,\mathbb{C}}) \cdots_1 (L_{d,\mathbb{C}})$$

we deduce the projection formula. ■

A convenient observation is that in the projective case (in fact the important hypothesis here is properness), the arithmetic intersections number change only by a a factor controlled by the archimedian degree if we change the model and the metrics on the line bundles.

Proposition 24. *Let $\mathcal{X}, \mathcal{X}'$ be to regular flat projective scheme over* $\operatorname{Spec} \mathcal{O}_K$ *being models of a variety X over* $\operatorname{Spec} K$*, let $\overline{L}_1, \cdots, \overline{L}_d$ (resp. $\overline{L}'_1, \cdots, \overline{L}'_d$) be metrized line bundles on $\mathcal{X}$ (resp. on $\mathcal{X}'$) such that $L_{i,\mathbb{Q}} \simeq L'_{i,\mathbb{Q}}$ for each $1 \leq i \leq d$.*
Then there exist $C > 0$ such that

$$|(\hat{c}_1(\overline{L}'_1), \cdots, \hat{c}_1(\overline{L}'_d)|\mathcal{Z}) - (\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_d)|\mathcal{Z}')| \leq C.(c_1(\overline{L}_1), \cdots, c_1(\overline{L}_d)|Z)$$

For every integral subscheme $\mathcal{Z}$ (resp $\mathcal{Z}'$) of $\mathcal{X}$ (resp $\mathcal{X}'$) with $Z := \mathcal{Z}_\mathbb{Q}, \mathcal{Z}'_\mathbb{Q}$

Proof. We can find a model $\mathcal{X}''$ dominating both models $\mathcal{X}$ and $\mathcal{X}'$, hence by the projection formula we can assume that $\mathcal{X} = \mathcal{X}'$ and $\mathcal{Z} = \mathcal{Z}'$.
We can prove the result in the special case where $\overline{L}_i = \overline{L}_i'$ for $i < d$, so that in the general case we replace the line bundles one by one using this special case.
In this case we observe that

$$|(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1}), \hat{c}_1(\overline{L}_d')|\mathcal{Z}) - (\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1}), \hat{c}_1(\overline{L}_d)|\mathcal{Z})| =$$

$$|(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1}), \hat{c}_1(\overline{L}_d' \otimes \overline{L}_d^{-1})|\mathcal{Z})|$$

We know that $(\overline{L}_d' \otimes \overline{L}_d)|_{\mathbb{Q}} = \mathcal{O}_K$, so we have the canonical meromorphic section $s = 1$ for the line bundle $\overline{L}_d' \otimes \overline{L}_d$.
The recursion formula applied to this section gives

$$|(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1}), \hat{c}_1(\overline{L}_d' \otimes \overline{L}_d^{-1})|\mathcal{Z})| =$$

$$|(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|div(s|_{\mathcal{Z}})) - \int_{Z(\mathbb{C})} log(|1|)c_1(\overline{L_1}) \cdots c_1(\overline{L}_d)|$$

but the variety $X(\mathbb{C})$ is projective hence compact, so that all metrics on the trivial line bundle are equivalent, thus we have

$$|\int_{Z(\mathbb{C})} log(|1|)c_1(\overline{L_1}) \cdots c_1(\overline{L}_d)| \leq C'.\int_{Z(\mathbb{C})} c_1(\overline{L_1}) \cdots c_1(\overline{L}_d) = C'.(c_1(\overline{L}_1), \cdots, c_1(\overline{L}_{d-1})|Z)$$

for some constant $C' > 0$, this controls the second term.
For the first term, choose $n \in \mathbb{N}$ such that $n.s$ and $n.s^{-1}$ are global section of $\overline{L}_d' \otimes \overline{L}_d^{-1}$ and define effective divisors.
Thus we have $|(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|div(s|_{\mathcal{Z}}))| \leq |(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|div(n|_{\mathcal{Z}}))|$.
Decomposing n into a product of prime numbers, it suffices to bound $|(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|\mathcal{Z}|_{\mathbb{F}_p}))|$ for a fixed prime number p. Now $\mathcal{Z}$ is a flat scheme over Spec $\mathbb{Z}$, hence the degree at the generic fiber and at the special fiber must be the same, we deduce the equality $(\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|\mathcal{Z}|_{\mathbb{F}_p})) = (\hat{c}_1(\overline{L}_1), \cdots, \hat{c}_1(\overline{L}_{d-1})|Z))|$ and this immediately proves the inequality result. ■

Definition 25. Let $(\mathcal{X}, \mathcal{L})$ be a regular flat projective model of (X, L). The height associated to this model is defined by

$$h_{\mathcal{L}}(Z) := \frac{(\hat{c}_1(L)^{d+1}|\mathcal{Z})}{[K : \mathbb{Q}](d+1)(c_1(L)^d|Z)}$$

for $Z \in Z^d(X)$ and $\mathcal{Z}$ is its closure in $\mathcal{X}$.

0.3.2 Hilbert-Samuel arithmetic theorem

As in classical algebraic-geometry, we can link arithmetic intersection invariant with Euler characteristic invariant. The strongest form of this fact lies in the arithmetic Grothendieck-Riemann-Roch theorem. However in practice, we usually restrict to the Hilbert-Samuel corollary. Hence we should first describe the context.

Let $\overline{L}$ being an ample metrized line bundle and $\mathcal{Y}$ being the closure in $\mathcal{X}$ of a subclosed variety $Y \subset X$ of dimension d. We set $V_n = H^0(Y, L^n|_Y) \otimes_K \mathbb{R}$, $\Gamma_n = H^0(\mathcal{Y}, L^n|_{\mathcal{Y}})$. The metric on L induces a sup norm and a L^2-norm on the $\mathbb{R}$-vector space V_n, and Γ_n is a lattice of this vector space, we also set B_n to be the unit ball for the sup norm.

Theorem 26. *we have the asymptotic behaviour for n big enough,*

$$-\log(Vol(V_n/\Gamma_n)) + log(Vol(B_n)) = n^{d+1}\frac{(\hat{c}_1(\overline{L})^{d+1}|Y)}{(d+1)!} + o(n^{d+1})$$

As explained before, this result was historically proved as a corollary of the arithmetic Riemann-Roch theorem and a result on analytic torsion by Bismut-Vasserot, see [4] for the details. Fortunately there is a more direct proof by A.Abbes and T.Bouches, inspired by the classical proof for Hilbert-Samuel theorem using the short exact sequence

$$0 \to H^0(Y, L^n) \to H^0(Y, L^{n+1}) \to \bigoplus_{Z \subset Y} H^0(Z, L^{n+1}|_Z)^{\oplus ord_Z(div(s))} \to 0$$

the injective map on the left is the multiplication induce by some global section s of L, in the arithmetic case we have to compare the default for additivity at the level of metric, this is done by some analytic results.

We can use this theorem and Minkowski's theorem to show the existence of a "small" global section:

Corollary 27. *Suppose* $(\hat{c}_1(\overline{L})^{d+1}|Y) > 0$. *If* n *is big enough, then there is a global section* $s \in \Gamma_n$ *such that* $|s|_{sup} \leq 1$

Proof. The preceding theorem gives

$$-log(Vol(V_n/\Gamma_n)) + log(Vol(B_n)) = n^{d+1}\frac{(\hat{c}_1(\overline{L})^{d+1}|Y)}{(d+1)!} + o(n^{d+1}) \geq nlog(2)$$

For n big enough, thus by Minkowski's theorem there is a non-zero section s in the lattice Γ_n which is itself contained in B_n. That concludes the proof. ■

0.4 The Bogomolov Conjecture

We fix a number field K and we call M_K a set of representatives for each absolute value on K for all the places of K satisfying the product formula, as we did in the first chapter.

Let A being an Abelian variety and $X \subset A$ being a strict closed subvariety. We say that X is a torsion variety if it is the translation of a sub-abelian variety of A by a torsion point.
We also fix $\mathcal{L}$ any ample line bundle on A and define $X\{\varepsilon\} = \{x \in X(\overline{K}) | \hat{h}_{\mathcal{L}}(x) \leq \varepsilon\}$ where $\hat{h}$ is the Neron-Tate height as define before and $\varepsilon > 0$. The aim of this chapter is to prove the following

Theorem 28. *(Bogomolov conjecture) If X is not a torsion subvariety, then there exists $\varepsilon > 0$ such that $X\{\varepsilon\}$ is not Zariski dense in X.*

Before going through the proof, we must discuss how one can recover the Neron-Tate normalized height on an Abelian variety by using the theory of height defined by metrized line bundle.

Proposition 29. *Let X be a variety over $\mathbb{C}$, $f : X \to X$ be an endormorphism of variety and L a line bundle on X such that we have a fixed isomorphism $i : L^d \to f^*L$ for some $d > 1$. Then there exists a unique Hermitian metric on L such that i becomes an isometry*

Proof. This is similar to the Neron-Tate normalisation defined in Chapter 1: first we see that for any metric $|.|$ on L, we have a new metric $|.|'$ such that i is an isometry for f^*L endowed with the metric induced by $|.|$ on L. And L^d is endowed with the metric induced by $|.|'$ on L: we simply take the metric $|.|_1$ on L^d induced via i for the canonical metric on f^*L just described. Then, set $|.|'$ to be the d-th root of $|.|_1$. The map $|.| \mapsto |.|'$ is contracting with a factor $\frac{1}{d} < 1$, hence by completeness of the space of metric, we have a unique fixed point for this map. But a fixed point is precisely a metric such that i becomes an isometry. ■

Take L to be a symmetric line bundle on A, then recall that we have the relation $L^4 \simeq [2]^*L$. Thus we can use the preceding proposition to define a unique metric on L such that this isomorphism is an isometry. Using the Neron-model $\mathcal{A}$ of A, the new Hermitian line bundle $\overline{L}$ on $\mathcal{A}$ define a height function $h_{\overline{L}}(\cdot)$ on cycles, this height satisfy the relation $h_{\overline{L}}(2.Z) = 4.h_{\overline{L}}(Z)$ so we recover the Neron-Tate height on points by unicity.

0.4.1 Equidistribution

In this section, $\mathcal{X}$ is a regular projective variety over $\mathcal{O}_K$, $X := \mathcal{X}_{\mathbb{Q}}$ is the corresponding variety over K, and $\overline{L}$ is a Hermitian ample line bundle on $\mathcal{X}$. Moreover, we fix an archimedian place σ of K and write $X(\mathbb{C})$ for the corresponding complex manifold define by this place.

We start by the following observation

Lemma 30. *Let (x_n) be a generic sequences of the subclosed subset $Y \subset X$, then we have*

$$\liminf_n h_{\overline{L}}(x_n) \geq h_{\overline{L}}(Y)$$

Proof. For $a \in \mathbb{R}$ we write $\overline{L}(a)$ to be the modification of the Hermitian line bundle $\overline{L}$ where we scale the Hermitian metric at every archimedian places by a factor e^a. In particular we get the formula $h_{\overline{L}(a)}(Z) = h_{\overline{L}}(Z) - a$ for every cycle Z. Hence for $\varepsilon > 0$, setting $a = h_{\overline{L}}(Y) - \varepsilon$ we get $h_{\overline{L}(a)}(Y) = \varepsilon > 0$.
In this case we can apply the Hilbert-Samuel theorem to $\overline{L}(a)$ and we have, for m big enough, the existence of a non trivial section $s \in \overline{L}(a)^m$ with $|s|_{sup} \leq 1$. We set Z to be the support of the divisor $div(s)$ or Y. Of course, for n big enough x_n is in $Y \setminus Z$ by hypothesis, hence by the inductive formula for the height we get that $h_{\overline{L}(a)}(x_n) \geq 0$.
As a consequence $\liminf_n h_{\overline{L}}(x_n) - h_{\overline{L}}(Y) + \varepsilon \geq 0$ and $\liminf_n h_{\overline{L}}(x_n) \geq h_{\overline{L}}(Y) - \varepsilon$ for every $\varepsilon > 0$. From that, we have the inequality $\liminf_n h_{\overline{L}}(x_n) \geq h_{\overline{L}}(Y)$. ∎

Proposition 31. *If (x_n) is a generic sequence in $Y \subset X$ a closed subvariety of dimension d such that $h_{\overline{L}}(x_n) \to h_{\overline{L}}(Y)$, then the sequence of measures $\mu_n = \frac{1}{\#O(x_n)} \sum_{r \in O(x_n)} \sigma_r$ on $X(\mathbb{C})$ converge to the measure*

$$\mu = \frac{c_1(\overline{L})^{\boxtimes d}}{deg_{\overline{L}}(Y)}|_Y$$

Proof. Let f be a smooth function on $Y(\mathbb{C})$, we want to show the following

$$\lim_n \frac{1}{\#O(x_n)} \sum_{r \in O(x_n)} f(r) = \int_{Y(\mathbb{C})} f . \frac{c_1(\overline{L})^{\boxtimes d}}{deg_{\overline{L}}(Y)}|_Y$$

Here $O(x)$ is the corresponding Galois orbit of the point x in $Y(\mathbb{C})$.
We look at the family $\overline{L}(-\lambda . f)$ to be Hermitian line bundles $\overline{L}$ where we scale the Hermitian metric at the archimedian place we have fixed in the first place by a factor $e^{-\lambda . f}$, where $\lambda > 0$. Then, we look at the limit as λ is approaching 0.
The idea is to compute the derivative of $\lambda \mapsto h_{\overline{L}(-\lambda . f)}(Y)$ at 0. More precisely using the description of the height using arithmetic intersection theory we get

$$h_{\overline{L}(-\lambda . f)}(Y) = \frac{1}{(d+1)[K:\mathbb{Q}]deg_{\overline{L}}(Y)} \sum_l \binom{d+1}{l} (\hat{c}_1(\overline{L})^{d+1-l} . \hat{c}_1(\overline{O_X}(-\lambda . f))|Y)$$

The first term of the sum is $h_{\overline{L}}(Y)$, we can use the inductive formula for the intersection number by using the canonical section of $\overline{O}_X$, namely the unit section 1, to describe the second term as $\lambda(d+1) . \int_{Y(\mathbb{C})} f . \frac{c_1(\overline{L})^{\boxtimes d}}{deg_{\overline{L}}(Y)}|_Y$, all the other terms are monic in λ of degree bigger than 2.
Thus we get

$$h_{\overline{L}(-\lambda . f)}(Y) = h_{\overline{L}}(Y) + \lambda . \int_{Y(\mathbb{C})} f . \frac{c_1(\overline{L})^{\boxtimes d}}{deg_{\overline{L}}(Y)}|_Y + P(\lambda)$$

With $P(\lambda)$ a polynomial divisible by λ^2.
On the other hand the preceding lemma shows the inequality

$$\liminf_n h_{\overline{L}(-\lambda.f)}(x_n) \geq h_{\overline{L}(-\lambda.f)}(Y)$$

and by hypothesis we know that $\liminf_n h_{\overline{L}}(x_n) = h_{\overline{L}}(Y)$. We also know that $h_{\overline{L}(-\lambda.f)}(x_n) = h_{\overline{L}}(x_n)+\lambda\frac{1}{[K:\mathbb{Q}]\#O(x_n)}\sum_{r\in O(x_n)} f(r)$. Hence we get the inequality

$$\liminf_n \lambda.\frac{1}{[K:\mathbb{Q}]\#O(x_n)}\sum_{r\in O(x_n)} f(r) \geq \lambda.\int_{Y(\mathbb{C})} f.\frac{c_1(\overline{L})^{\boxtimes d}}{deg_{\overline{L}}(Y)}|_Y + P(\lambda)$$

dividing both sides by λ and passing to the limit with $\lambda \to 0$ we have

$$\liminf_n \frac{1}{\#O(x_n)}\sum_{r\in O(x_n)} f(r) \geq \int_{Y(\mathbb{C})} f.\frac{c_1(\overline{L})^{\boxtimes d}}{deg_{\overline{L}}(Y)}|_Y)$$

Now the smooth function f being arbitrary, we can apply this inequality for $-f$ instead of f and we get $\liminf_n \frac{1}{\#O(x_n)}\sum_{r\in O(x_n)} f(r) \geq \int_{Y(\mathbb{C})} f.\frac{c_1(\overline{L})}{deg_{\overline{L}}(Y)}|_Y$. We have the convergence result for all smooth function and then for all measurable function by density. ■

0.4.2 Proof of the Bogomolov conjecture

The conjecture is independent of the ample line bundle chosen: if L and L' are two ample line bundles, there is a $n \in \mathbb{N}$ big enough so that $L^n \otimes L'^{-1}$ and $L'^n \otimes L^{-1}$ are ample. The resulting Neron-Tate heights for these two line bundles are thus positive, we deduce that $nh_L \geq h'_L \geq \frac{1}{n}h_L$ and the independance become clear.
If the conjecture is true for an Abelian variety A and B is isogenous to A with $i : B \to A$ a fixed isogeny, then the conjecture is true for B as well. To see this note that since the conjecture is independent of the ample line bundle chosen we can set L any ample line bundle on A and $L' = f^*L$. We have $h'_L(x) = h_L(f(x))$ and if (x_n) is a generic sequence in $X \subset B$ then $f(x_n)$ is a generic sequence in $f(X)$ as well since f is finite surjective.

Let $G(X)$ be the stabilizer of X, the first step is to reduce to the case $G(X) = 1$. For that let $A' = A/G$ and A'' be the reduce connected component of the neutral element of $G(X)$, $X' = X/G$. Then $A' \times A''$ is isogenous to A and X' is such that its stabilizer is zero.

Then the independance of the conjecture under isogeny, and the independance of the ample line bundle choosen allow us to reduce to the case $G(X) = 1$.

Proposition 32. *If $G(X) = 1$, then the map $\alpha^m : X^m \to A^{m-1}, (x_1, \cdots, x_m) \mapsto (x_2 - x_1, \cdots, x_m - x_1)$ is birational on its image for m large enough.*

Proof. First we prove that for m large enough, α^m is generically finite. Let $x = (x_1, \cdots, x_m) \in X^m$ be such that $(X - x_1) \cap \cdots \cap (X - x_m) = \{1\}$. This is possible since $\cap_{x\in X}(X - x) = G(X) = 1$ and the intersection holds already on a finite subset of the index by quasi-compacity. But $(X - x_1) \cap \cdots \cap (X - x_m)$

is precisely $f^{-1}(f(x))$, hence α_m is generically finite. Fix such an m from now on.
Choose $U \subset X^m$ such that α^m is smooth restricted to U, this is possible since in caracteristic 0 every finitely generated extension factorize as a standard transcendental extension followed by a separable extension.
Then comparing dimension, recalling that α_m is generically finite, $\alpha_{m'} : U \times X^{m'-m} \to \alpha_{m'}(X^{m'})$ is étale, because we can choose again m' large enough so that one of the fiber is reduce to a point by the same idea as before, we see that this étale map is generically an isomorphism. ■

Now we can prove the Bogomolov conjecture. Suppose by contradiction that we have a generic sequence (x_n) such that $h_{\overline{L}}(x_n)$ converge to 0, and fix m large enough so that the lemma above holds for $f := \alpha_m$.
Since the number of strict sub varieties in X is at most countable, we can find a generic sequence $l_n := (x_{i_n^1}, \cdots, x_{i_n^m})_n$ in X^m with i_n^j diverging to $+\infty$ as n tend to $+\infty$.
Then $h_{\overline{L}^{\boxtimes m}}(X^m) \leq \liminf h_{\overline{L}^{\boxtimes m}}(l_n) = 0$ by lemma 30. Likewise, if we set $y_n = f(l_n)$, y_n is still a generic sequence in the image of f. We also have $h_{\overline{L}^{\boxtimes m-1}}(y_n) \to 0$ by quadraticity of the Neron-Tate height, hence we have $h_{\overline{L}^{\boxtimes m-1}}(f(X)) = 0$. By the equidistribution theorem, we have that the sequence of measures $\delta(l_n)$ converge to $\frac{c_1(L)^{\boxtimes m}}{deg(X^m)}\delta_{X^m}$ and the sequence $\delta(y_n)$ converge to $\frac{c_1(L)^{\boxtimes m-1}}{deg(f(X^m))}\delta_{f(X^m)}$. Because we have $\delta(l_n) = f^*\delta(y_n)$, we deduce the equality between distributions $\frac{c_1(L)^{\boxtimes m}}{deg(X^m)} = f^* \frac{c_1(L)^{\boxtimes m-1}}{deg(f(X^m))}$. Now f is birational and both sides are defined by differential forms, so the equality holds as differential forms on a dense open subset, hence it holds everywhere. Besides, $c_1(\overline{L}^{\boxtimes m})$ and $f^*c_1(\overline{L}^{\boxtimes m-1})$ are proportional on X^m. But $c_1(L)^{\boxtimes m}$ is strictly positive on the diagonal of X^m and $f^*c_1(L)^{\boxtimes m-1}$ vanishes on the diagonal of X^m, hence we have a contradiction.

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