Extending Vendramin's property to finite Coxeter groups

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1 Introduction

I have conducted my intership at the Scuola Galileiana of Padova with the supervision of Professoressa Carnovale. I started by reading a first paper [5] shortly before arriving to Padova. This way I was able to get familiar with the general context of the project: the theory of braided vector spaces, Yetter-Drinfled modules, Hopf and Nichols algebras. I was then handed a problem to tackle, it is the main subject of this memoire. We met every week to discuss about the progression and to look at some new references to finally find a proof. The end of the internship was dedicated to putting the proof onto paper, adding context for the article and making sure that we were using the best possible references.

The general context of the study is the theory of braided vector spaces. A braided vector space V is a vector space with a given braiding function $c : V \otimes V \longrightarrow V \otimes V$ verifying the braid relation: $(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$. We study cases where V is a certain $\mathbb{C}X$ where $X \subset G$ is a stable subset of the conjugation action in the group G. Then the braiding is of the form:

$$c(x \otimes y) = q(x, y).xyx^{-1} \otimes x \text{ for all } x, y \in G$$

with a function $q : G \times G \longrightarrow \mathbb{C}^*$ verifying the cocycle relation (cf. 2.2). We then build the Nichols Algebra of V which we realize as a quotient of the tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ using a certain function constructed from the braiding. This Nichols algebra turns out to be interesting as it has some links to the Fomin-Kirillov algebra in the case where $G = \mathcal{S}_n$. This algebra being itself of big interest as it is linked to the flag variety.

1.1 An example for intuition on the paper

During my internship, I have worked on some constructions based on the study of group of reflections i.e. groups generated by a finite number of reflections (i.e. symmetries realized by a hyperplane). The most common examples are the dihedral groups $\mathcal{D}_n$ and the group of permutations $\mathcal{S}_n$. As an example we can observe the groups $\mathcal{D}_3$ and $\mathcal{D}_4$.

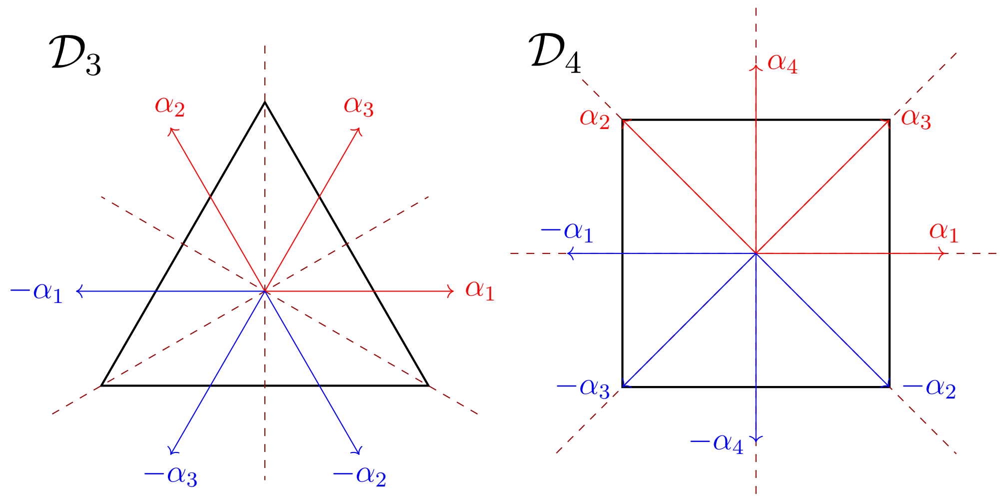

Figure 1: Reflections and roots of the groups $\mathcal{D}_3$ and $\mathcal{D}_4$

We set a reflection group W. We will be studying the set T of all reflections of W (the dotted lines on the drawings) and its associated set of roots Φ. The roots of a reflection are the unit vectors normal to the hyperplane fixed by it ($\pm\alpha_i$ on the drawings). We can define a projection $s : \alpha \in \Phi \longmapsto s_\alpha \in T$ where s_α is the reflection corresponding to the orthogonal hyperplane to α. Yet there are two ways to associate a given reflection $x \in T$ to one of its two roots. We separate the roots Φ in two classes Φ^+ and Φ^-: the positive roots (in red) and the negative roots (in blue). See the following drawings to visualize the reflections and there action on the roots. To simplify the drawings we only write the indexes and not the roots, the sign being given by the color.

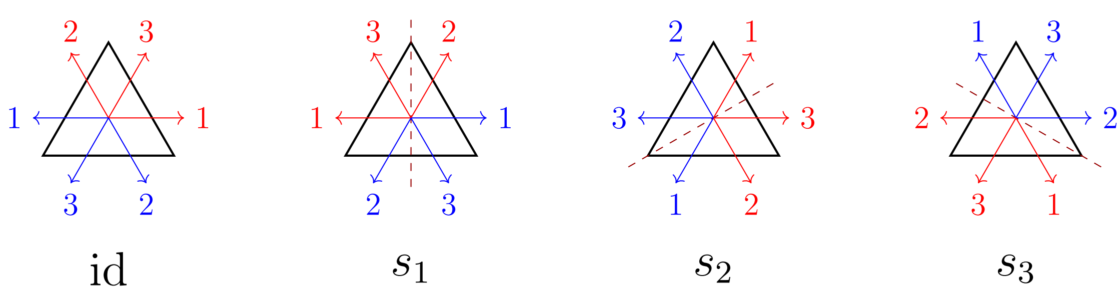

Figure 2: Actions of the reflections s_1, s_2 and s_3 on the root system of $\mathcal{D}_3$

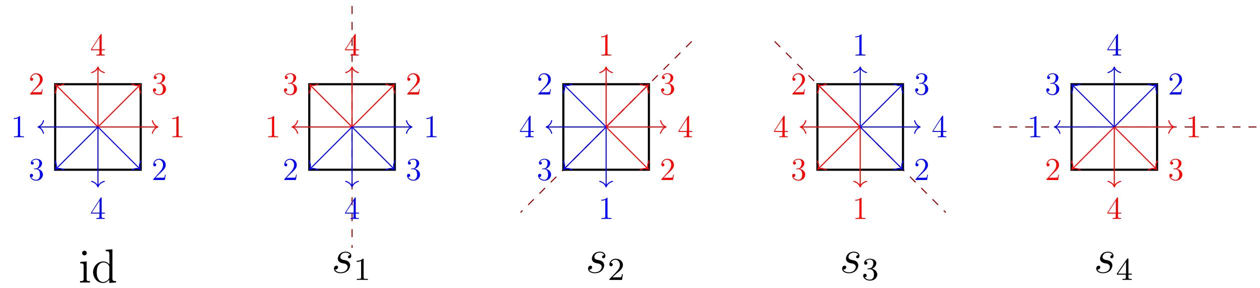

Figure 3: Actions of the reflections s_1, s_2, s_3 and s_4 on the root system of $\mathcal{D}_4$

In this memoire we study racks i.e. couples $(X, \triangleright)$ where X is a set and $\triangleright$ is an operation resembling a conjugation. In a sense we want to compare the most natural rack structure on T given by $x \vdash y = xyx^{-1}$ and the most natural one on Φ given by $\alpha \triangleright \beta = s_\alpha(\beta)$ (of which the action is explicitely shown in Figures 1 and 3) . We will also have to consider the slightly modified rack $\alpha \triangleright_- \beta = -s_\alpha(\beta)$. We notice that $\pm s_\alpha(\beta)$ is a root of the reflection $s_\alpha s_\beta s_\alpha^{-1}$ (Indeed $s_\alpha s_\beta s_\alpha^{-1} s_\alpha(\beta) = -s_\alpha(\beta)$), thus, the difference between the racks $\triangleright$ and $\vdash$ is just a matter of adding signs.

To compare these two racks we need them to be racks of the same set. Thus, we see the rack $\vdash$ of T as a rack on Φ. As said before, $\vdash$ is like $\triangleright$ but without the signs. Thus we can set $\alpha \vdash \beta$ to be the root in $\{\pm s_\alpha(\beta)\}$ of the same sign as β. This corresponds to acting on the numbers without changing the colors, see below for the action induced by $\vdash$.

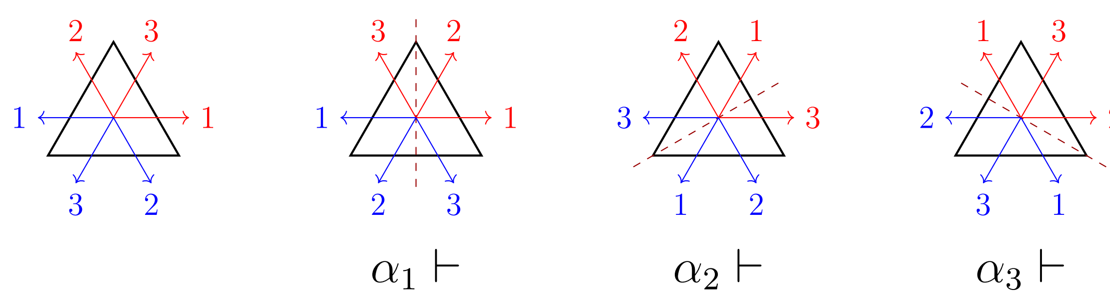

Figure 4: Actions of the roots induced by $\vdash$ on $\mathcal{D}_3$

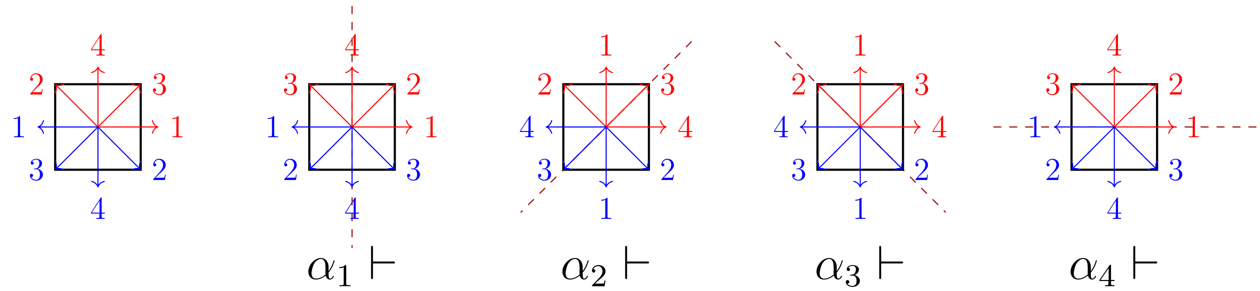

Figure 5: Actions of the roots induced by $\vdash$ on $\mathcal{D}_4$

In this paper we prove that "$\triangleright_-$ and $\vdash$ are twist equivalent"(this doesn't rigourously mean much as we would need to define cocyles). Thus we have a link between the conjugation viewed from the standpoint of symmetries and from the one of roots.

Twist equivalence is a less strong version of cohomology. Grossly speaking, the racks are cohomologous if by chosing a good set of positive roots we obtain the same rack (remember that $\vdash$ depends on the definition of Φ^+). To check if $\triangleright_-$ and $\vdash$ are the same all you need to see is if $\alpha \triangleright \beta$ always changes the sign of the root β. That way $\triangleright_-$ sends reds to reds and blues to blues. Observe the following change of Φ^+:

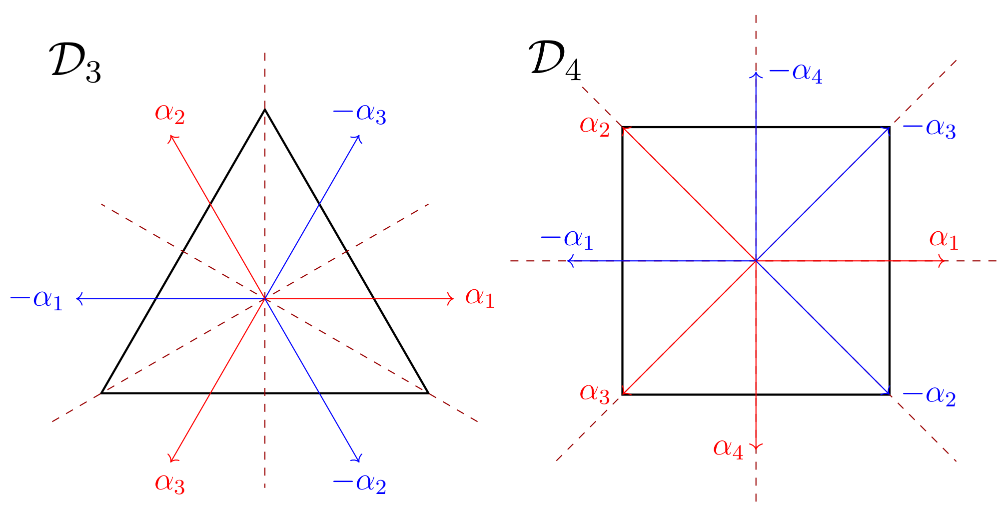

Figure 6: Reflections and roots of the groups $\mathcal{D}_3$ and $\mathcal{D}_4$ with changed set of positive roots Φ^+

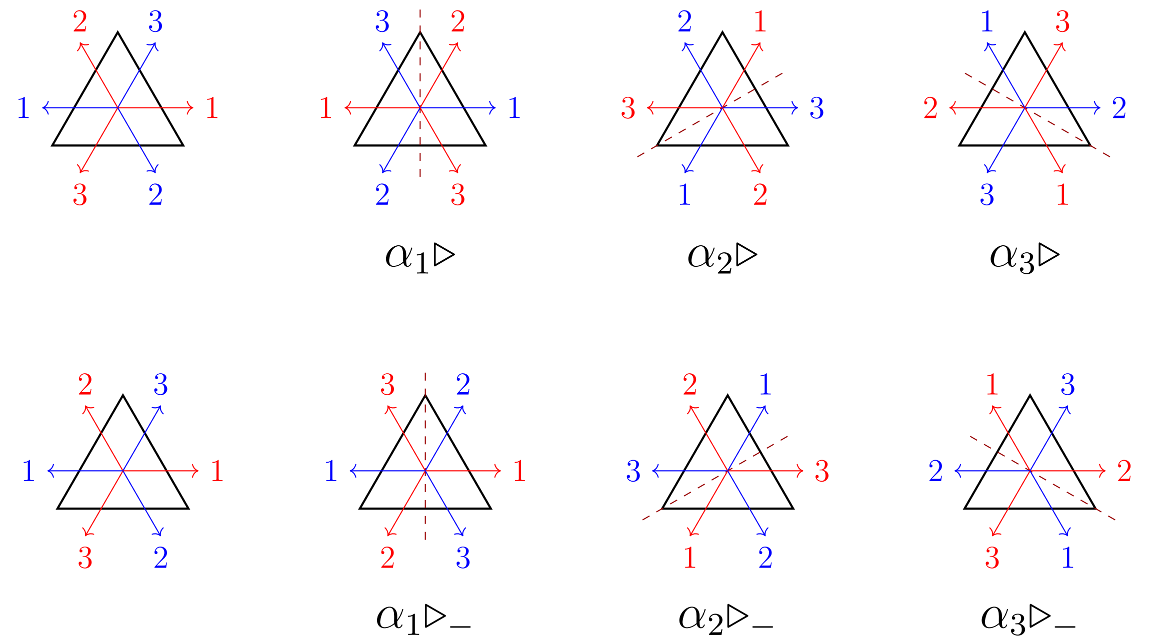

Figure 7: Actions of the roots induced by $\triangleright$ and $\triangleright_-$ with the new set of positive roots on $\mathcal{D}_3$

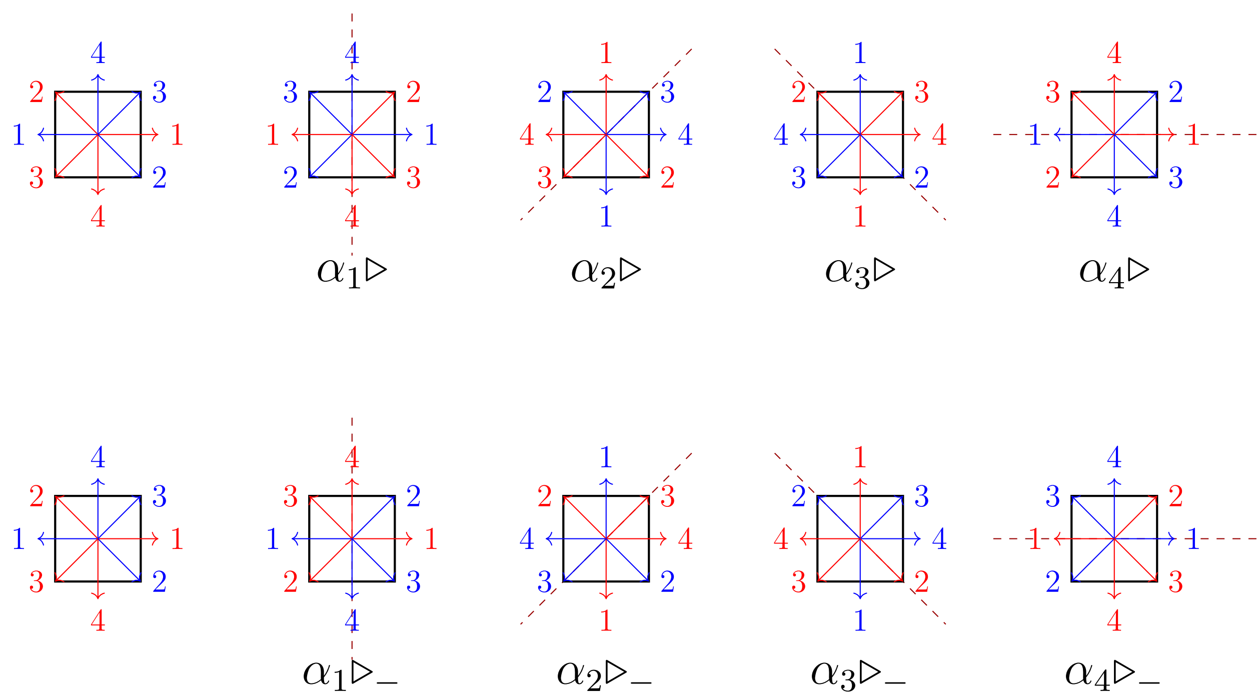

Figure 8: Actions of the roots induced by $\triangleright$ and $\triangleright_-$ with the new set of positive roots on $\mathcal{D}_3$

Here we see that we have cohomology for $\mathcal{D}_3$, yet for $\mathcal{D}_4$ this change of Φ^+ doesn't work as for example $\alpha_1 \triangleright \alpha_4 = \alpha_4$ due to their orthogonality. In the general case we will only have a twist equivalence meaning that we consider a slightly different definition of $\vdash$ in Φ.

This little introduction was to give a bit of intuition to understand what will follow. The mémoire is structured as follows. In Section 2 we will define the general context of the paper and the cocycles used to formally define the problem. In Section 3.1 we define the question and give our strategy to find the proof. Finally in Section 4 we prove the theorem in multiple steps, the last one 4.4 repeats the same arguments given in the previous example.

2 Definitions

2.1 Coxeter groups and root systems

The definitions in this section come from [2, Sections 1.1., 4.2 and 4.4].

Definition 2.1.1. Let $S = \{s_1, ..., s_l\}$ be a set. A matrix $(m_{i,j})_{1\leq i,j\leq l}$ with coefficients in $\{1, 2, ..., \infty\}$ is a Coxeter matrix if it is symmetric and if $m_{i,j} = 1 \iff i = j$.

Then we can define a group W generated by S with relations:

$$(s_i s_j)^{m_{i,j}} = 1 \text{ for all } i, j \in \{1, ..., l\}$$

We call (W, S) a Coxeter system.

Let us consider a Coxeter system $(W, S = \{s_1, ..., s_l\})$ for the rest of the article. We can define the length $l(w)$ of $w \in W$ with respect to S as the minimal number l such that $w = s_{i_1}...s_{i_r}$ with $i_k \in \{1, ..., l\}$, [2, Section 1.4].

We define the geometric representation of (W, S) in the following way:

Definition 2.1.2. Let V be a vector space of dimension $l = |S|$. We chose a basis $\{\alpha_i\}_{1\leq i\leq l}$ of V. We define the bilinear form $(,)$ on V in the following way:

$$(\alpha_i, \alpha_j) = -\cos(\tfrac{\pi}{m_{i,j}}) \text{ for all } i, j \in \{1, ..., l\}.$$

For $\alpha \in V - \{0\}$, we can define its reflection as $s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$ for $v \in V$.

Proposition 2.1.3. *The group W injects itself in $GL(V)$ with the map $s_i \mapsto s_{\alpha_i}$. We assimilate W to its geometrical representation and write $s_i = s_{\alpha_i}$.*

Remark 1.

- We note that $det(w) = (-1)^{l(w)}$ for $w \in W$. Thus any decomposition of the form $w = s_{i_1}...s_{i_r}$ with $i_k \in \{1, ..., l\}$ is such that r has the same parity as $l(w)$. This way if $s \in S$ we have $l(sw) = l(w) \pm 1$.
- We will be using the following geometrical fact: if $f \in O(V)$ and $\alpha \in V - \{0\}$ then $fs_\alpha f^{-1} = s_{f(\alpha)}$.

Definition 2.1.4. The root system of (W, S) is $\Phi := \{(w(\alpha_i)\}_{i \in \{1,...,l\},\ w \in W}$. The simple system or set of simple roots of Φ is $\Delta = \{\alpha_i\}_{i \in \{1,...,l\}}$.

Proposition 2.1.5. *([2, (4.24)]) The root system Φ of (W, S) can be written as the disjoint union $\Phi = \Phi^+ \bigsqcup \Phi^-$ where Φ^+ (resp. Φ^-) is the set of positive (resp. negative) roots of Φ consisting of positive (resp. negative) linear combinations of Δ.*

Proposition 2.1.6. *([2, Lemma 4.4.3]) If $\alpha \in \Delta$, then $s_\alpha(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}$*

Definition 2.1.7. The Coxeter diagram of (W, S) is the undirected, labelled graph $\Gamma(W)$ defined as follows. Its vertices are the elements of S. Two vertices s_i, $s_j \in S$ are connected by a labelled edge if and only if $m_{i,j} \geqslant 3$ and if this is the case, the label is $m_{i,j}$.

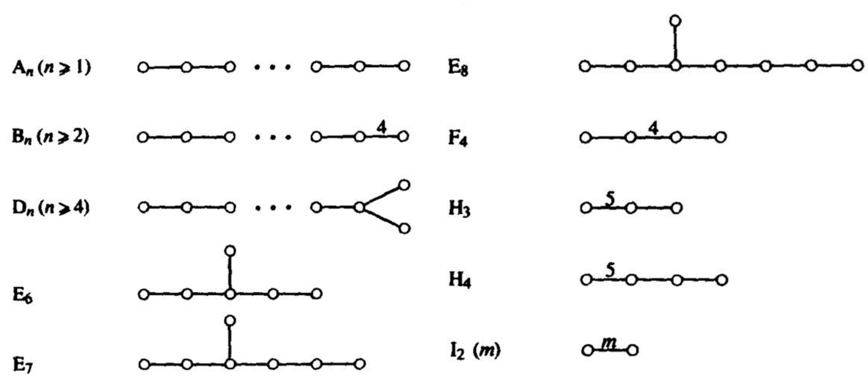

Figure 9: All irreducible finite Coxeter group diagrams [6, Section 2.4]

2.2 Cocycles and twist equivalence

Definition 2.2.1. A rack is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a function, such that the map $x \mapsto i \triangleright x$ is bijective for all $i \in X$, and $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$.

We will study racks of the form $X \subset G$, where G is a group, X is stable by conjugation and $x \triangleright y = xyx^{-1}$ for $x, y \in X$. We will need the following definitions and results:

- Let A be an abelian group. Let X be a rack. A map $q : X \times X \longrightarrow A$ is a **2-cocycle** (here we will say cocycle to simplify) if and only if:

  $$q(x, y \triangleright z)q(y, z) = q(x \triangleright y, x \triangleright z)q(x, z)$$

  for all $x, y, z \in X$. We call $Z^2_R(X, A)$ the set of rack cocycles.

- Let $q, q' \in Z^2_R(X, A)$ by two 2-cocycles. We say that they are **cohomologous** if there exists $\gamma : X \longrightarrow A$ such that:

  $$q(x, y) = \gamma(x \triangleright y)^{-1} q'(x, y)\gamma(y)$$

  for all $x, y \in X$.

- Let A be an abelian group. A **group cocycle** of the group G into A is a map $q : G \times G \longrightarrow A$ such that the following relation is true:

  $$q(x, y)q(xy, z) = q(x, yz)q(y, z) \text{ for all } x, y, z \in G$$

  They form the set $Z^2(G, A)$.

- ([9, Lemma 3.6]) Let A,B be two abelian groups and let $\psi : A \longrightarrow B$ be a group homomorphism. For $\phi \in Z^2(G, A)$ define $\phi_\psi = \psi \circ \phi$. Then $\phi_\psi \in Z^2(G, B)$.

- Let G be a group, let $X \subset G$ be stable by conjugation by itself and let $q, q' \in Z^2_R(X, A)$ be two 2-cocycles on X. We say that q and q' are **twist equivalent** if there exists $\phi \in Z^2(G, A)$ such that:

  $$q(x, y) = \phi(x, y)\phi(x \triangleright y, x)^{-1} q'(x, y)$$

The following proposition gives an insight on what are group cocycles.

Proposition 2.2.2. *([9]) Let G be a group. Let E be a central extension of G of canonical projection $p : E \longrightarrow G$ i.e. the kernel A of p is in the center of E. Let $\rho : G \longrightarrow E$ is a section of p. We define the function $\phi_\rho : G \times G \longrightarrow A$ such that $\rho(xy) = \phi_\rho(x,y)\rho(x)\rho(y)$. Then it is a group 2-cocycle in $Z^2(G, A)$. Additionally, all group cocycles can be obtained that way.*

Proposition 2.2.3. *Let G be a group and let X be a subset of G stable by conjugation. Let E be a central extension of G with kernel A. We denote by $\vdash$ the conjugation of E on itself. Let $q_1, q_2 \in Z^2_R(X, A)$. Then for q_1 and q_2 to be twist equivalent it suffices that there exists a section $\rho : G \longrightarrow E$ such that*

$$\rho(x) \vdash \rho(y) = q_1(x,y)q_2(x,y)^{-1}.\rho(x \triangleright y) \text{ for all } x, y \in X. \tag{2.1}$$

Additionally they are cohomologous E if is a trivial extension.

Proof. We show that $q_1(x,y)q_2(x,y)^{-1} = \phi_\rho(x,y)^{-1}\phi_\rho(x \triangleright y, x)$, in fact

$$\phi_\rho(x,y)^{-1}\phi_\rho(x \triangleright y, x) = \rho(x \triangleright y)^{-1}(\rho(x)\rho(y)\rho(x^{-1}))$$

as 2.2.2 gives $\phi_\rho(x,y)\rho(x)\rho(y) = \rho(xy) = \rho((x \triangleright y)x) = \phi_\rho(x \triangleright y, x)\rho(x \triangleright y)\rho(x)$. Additionally, if E is trivial then by defining $\gamma = \pi_A \circ \rho$, where π_A is the projection on A, we obtain the cohomology. □

3 Generalizing Vendramin's property to reflection groups

3.1 Defining the problem

Let Φ be the root system and Δ be the simple system associated to (W, S). Denote by $T := \{s_\alpha \text{ for } \alpha \in \Phi\}$ its set of reflections. By Remark 1 the rack T is W-stable and $T = \{ws_\alpha w^{-1} \text{ for } \alpha \in \Delta \text{ and } w \in W\}$.

We will be looking at two specific cocycles on T, these definitions come from [1]. Define $q^-, q^+ : W \times T \mapsto \mathbb{C}^*$ as follows. Let $w \in W$ and $y \in T$ and let α_y denote the positive root associated with the reflection $y \in T$. Then:

$$q^+(w,y) = \begin{cases} 1 \text{ if } w(\alpha_y) \in \Phi^+ \\ -1 \text{ if } w(\alpha_y) \in \Phi^- \end{cases} \tag{3.1}$$

$$q^-(w,y) = \det(w) \tag{3.2}$$

We prove that the restrictions to T of q^+ and q^- are rack cocycles. We will do as in [7, 5.] by proving that:

$$q^\pm(w_1w_2, x) = q^\pm(w_1, w_2 \triangleright x)q^\pm(w_2, x) \text{ for all } w_1, w_2 \in W \text{ and } x \in T. \tag{3.3}$$

This implies the cocycle relation if you notice that $w_1w_2 = (w_1 \triangleright w_2)w_1$.
The equation (3.3) is clear for q^- as $\det(w_1w_2) = \det(w_1).\det(w_2)$.

Let us prove (3.3) for q^+. Notice that $w.\alpha = w(\alpha)$ defines a W-action on Φ. If we consider the bijection $(\epsilon, x) \in \{\pm 1\} \times T \longmapsto \epsilon\alpha_x \in \Phi$ where α_x is the positive root associated to x, then the action on $\{\pm 1\} \times T$ becomes $w.(\epsilon, x) = (\epsilon q^+(w,x), w \triangleright x)$. Let $w_1, w_2 \in W$ and $x \in T$. Then as $w_1w_2.(1,x) = w_1(w_2.(1,x))$, we have
$(q^+(w_1w_2, x), w_1w_2 \triangleright x) = w_1.(q^+(w_2,x), w_2 \triangleright x) = (q^+(w_1, w_2 \triangleright x)q^+(w_2,x), w_1w_2 \triangleright x)$.

Theorem 3.1.1. *([9, Theorem 3.8]) If $W = S_n$ the permutation group with $S = \{(i, i+1), 1 \le i \le n-1\}$ and $T = \{(i,j), 1 \le i,j \le n\}$ then q^+ and q^- are twist equivalent.*

We prove the following generalization of Theorem 3.1.1 to finite Coxeter Groups.

Theorem 3.1.2. *Let W be a finite Coxeter Group. Let q^+ and q^- be the rack cocycles defined by (3.1) and (3.2) respectively, then q^+ and q^- are twist equivalent.*

In addition, q^+ and q^- are cohomologous if and only if $W = \mathcal{D}_n$ the dihedral group of odd order n.

3.2 The Vendramin section

We adapt the constructions given in [9] so as to fit the framework of all finite Coxeter groups. Let (W, S) be a finite Coxeter system. We define $\widetilde{W}$ to be the group generated by $t_1, ..., t_l$, z with relations

$$z^2 = (t_i z)^2 = 1, \quad (t_i t_j)^{m_{ij}} = z^{m_{ij}+1}, \text{ for all } 1 \leq i, j \leq l. \tag{3.4}$$

By construction, z is central and the assignment $t_i \mapsto s_i$ for $i = 1, \ldots, l$ and $z \mapsto 1$ defines a surjective homomorphism $\pi_W : \widetilde{W} \longrightarrow W$ with kernel $\langle z \rangle \cong (\mathbb{Z}/2\mathbb{Z})$. We denote by $\vdash$ the conjugation action in $\widetilde{W}$.

We define the cocycles $q_z^+, q_z^- : T \times T \longrightarrow \langle z \rangle$ by replacing the -1 by z. Then by defining the group isomorphism $\psi : \langle z \rangle \longrightarrow \{\pm 1\}$ we have $q^{\pm} = \psi \circ q_z^{\pm}$. It then suffices to show the twist equivalence of q_z^+ and q_z^-. To acheive this goal, we use Proposition 2.2.3 which leads to the following lemma.

Lemma 3.2.1. *Let $W, S, T, \widetilde{W}$ be as above. If there exists a section $\rho : W \mapsto \widetilde{W}$ such that:*

$$\rho(s) \vdash \rho(y) = \begin{cases} \rho(s \triangleright y) z & \text{if } \; s \neq y \\ \rho(s \triangleright y) & \text{if } \; s = y \end{cases} \qquad \forall y \in T, \forall s \in S \tag{3.5}$$

then q^+ and q^- are twist equivalent.

Proof. We want ρ to satisfy (2.1) for $q_1 = q_z^+$ and $q_2 = q_z^-$. We actually show by induction that for all $w \in W$:

$$\rho(w) \vdash \rho(y) = q_z^+(w, y) q_z^-(w, y)^{-1} . \rho(w \triangleright y) \tag{3.6}$$

so the result follows from Proposition 2.2.3. We do not detail further here.

4 Proof of Theorem 3.1.2

4.1 Reflection Conjugacy Graph

Let $W, \widetilde{W}, S, T$ be defined as previously. We build a section $\rho : W \longrightarrow \widetilde{W}$ verifying (2). To build this section we will use a version of the conjugacy graph [4, Section 3.2] of the Coxeter system (W, S).

Definition 4.1.1. The *reflection conjugacy graph* $\tilde{\Gamma}(W)$ of (W, S) is the directed graph defined as follows. Its vertices are the elements of T. Given x, $y \in T$ and $s \in S$ we have a labelled edge $x \xrightarrow{s} y$ if $y = sxs$ and $l(x) < l(y)$.

If $x \xrightarrow{s_i} y$ we will write $x \xrightarrow{i} y$ to simplify notation. For $x \in T$, we write $x \xrightarrow{i}$ (resp. $\xrightarrow{i} x$) if there exists $y \in T$ such that $x \xrightarrow{i} y$ (resp. $y \xrightarrow{i} x$).

Remark 2. Let $x \in T$ and $s \in S$. By Proposition 1.1.9 we have $l(sxs) = l(x)$ or $l(sxs) = l(x) \pm 2$, see [4, Section 3.2].

Definition 4.1.2. Let $\rho_0 : W \longrightarrow TW$ be a section.

We define a section recursively $\rho : W \longrightarrow TW$ by:

$$\rho(x) = \begin{cases} \rho_0(x) \quad \text{if } x \notin T \\ t_i \text{ if } x = s_i \text{ for } 1 \leq i \leq l \\ t_i \triangleright \rho(y) z \text{ if } y \xrightarrow{i} x \text{ for } 1 \leq i \leq l \end{cases} .$$

Theorem 4.1.3. *Let (W, S) be a Coxeter system and T be its reflections. Let $\rho : W \longrightarrow TW$ be the section defined in Definition 4.1.2. Then:*

1. *ρ is well defined.*

2. *ρ satisfies (2).*

So as to not be too long, we will only give the guiding lines of the proof.

4.2 Good definition of the Section

Definition 4.2.1. Let $x, y \in T$ and $a = s_{j_r}...s_{j_1} \in \Sigma$. We write $x \xrightarrow{a} y$ if there exists $x_0 = x, x_1, x_2...x_r = y \in T$ such that $x_{i-1} \xrightarrow{j_i} x_i$ for $i \in \{1, ..., r\}$.

To show that ρ is well defined we need to show that, for all $x \in T$, two paths $s_i \xrightarrow{a} x$ and $s_j \xrightarrow{b} x$ lead to the same $\rho(x)$. The idea is that we can transform one path into the other with group relations that will preserve the value $\rho(x)$.

4.3 Proof that the Section of 4.1.2 verifies (2)

We will be using some notions from [3] in the general case of finite Coxeter Groups.

Proposition 4.3.1. *Let $x \in T$ and $\alpha \in \Delta$. Then*

(i)' $l(x) = l(s_\alpha x s_\alpha) \iff x = s_\alpha x s_\alpha \iff (\alpha_x, \alpha) = 0$ *or* $\alpha_x = \alpha$.

(ii)' $l(x) = l(s_\alpha x s_\alpha) + 2 \iff x \xrightarrow{s_\alpha} \iff (\alpha_x, \alpha) < 0$.

(iii)' $l(x) = l(s_\alpha x s_\alpha) - 2 \iff \xrightarrow{s_\alpha} x \iff (\alpha_x, \alpha) > 0$ *and* $\alpha_x \neq \alpha$.

Proposition 4.3.2. *Let $\alpha_i, \alpha_j \in \Delta$ and $x, x' \in T$ such that $s_j(\alpha_{x'}) = \alpha_x$. Then:*

- $(\alpha_x, \alpha_i) = (\alpha_{x'}, \alpha_i) - 2(\alpha_{x'}, \alpha_j)(\alpha_j, \alpha_i)$.
- $(\alpha_x, \alpha_j) = -(\alpha_{x'}, \alpha_j)$.

Definition 4.3.3. Let $\gamma \in \mathbb{R}$. We define the following sequence:

- $n_0(\gamma) = 1$
- $n_1(\gamma) = 2\gamma$
- $n_p(\gamma) = 2\gamma n_{p-1}(\gamma) - n_{p-2}(\gamma)$

This is the sequence of the Chebyshev polynomials of the second kind in γ.

Thus if $\gamma = \cos(\theta)$, we have the relation:

$$n_p(\gamma) = \sin((p+1)\theta)/\sin(\theta) \quad .$$

Proposition 4.3.4. *Let $m \in \mathbb{N}$ and $\gamma = cos(\frac{\pi}{m})$.*
Then $n_{m-1} = 0$ and $n_p > 0$ for $p \in \{0, ..., m-2\}$.

Proof. This stems from the formula. □

We will prove a central lemma to prove that ρ satisfies (2).

Lemma 4.3.5. *Let x in T and $i \in \{1, ..., l\}$ such that $s_i x s_i = x$ and $x \neq s_i$. Then there exists $j \in \{1, ..., l\}$ and $k_0, k_1, k_2..., k_r \in \{i, j\}$ and $x_0, ..., x_r \in T$ such that*

$$x_r \xrightarrow{k_{r-1}} x_{r-1} \xrightarrow{k_{r-2}} ... \; x_1 \xrightarrow{k_0 = j} x_0 = x$$

where the word $k_0 k_1 k_2 ... k_r$ is an alternation of j and i with $k_0 = j$, and either

1. *$x_r = s_{k_r}$, the integer $m_{i,j}$ is even and the length $r = m_{i,j}/2 - 1$.*
2. *$s_{k_r} x_r s_{k_r} = x_r$ with $s_{k_r} \neq x_r$ and $r = m_{i,j} - 1$.*

Proof. We set Φ a root system for W where all of the vectors have length 1.
Thus $(\alpha_i, \alpha_j) = -\cos(\pi/m_{i,j}) := -\gamma_{i,j}$ for all $j \in \{1, ..., l\}$.

If $x = s_j \in S$ then it is case 1. as then $m_{i,j} = 2$ and so $r = 0$. If not then we can write $x_1 \xrightarrow{j} x$ for some $j \in \{1, ..., l\}$ and $x_1 \in T$. By Propositions 4.3.1 and 4.3.2, We know that $(\alpha_x, \alpha_j) = -(\alpha_{x_1}, \alpha_j) := \delta > 0$ as $x_1 \xrightarrow{j} x$. Moreover, $(\alpha_x, \alpha_i) = 0$ as $s_i x s_i = x$.

We will build a path $x_p \xrightarrow{k_{p-1}} x_{p-1} \xrightarrow{k_{p-2}} ... \; x_1 \xrightarrow{k_0 = j} x_0 = x$ by double induction on p such that $k_0 k_1 k_2 ... k_p$ is an alternation of j and i with $k_0 = j$ and with:

$$(\alpha_{x_q}, \alpha_{k_q}) = \delta . n_q(\gamma_{i,j}) \text{ for all } q \in \{0, ..., p\}$$

This construction ending when one of the two cases of the lemma is reached. Noting that $n_{m_{i,j}-1}(\gamma_{i,j}) = 0$.

Theorem 4.3.6. *Let W be a finite reflection group with Coxeter generators $S = \{s_1, ..., s_l\}$ and set of reflections T, let $\widetilde{W}$ be its the central extension from Definition* (3.4) *and ρ be the section defined in Definition 4.1.2. Then ρ satisfies* (2)*.*

Proof. Let $x \in T$ and $s_i \in S$. Let us compute $\rho(s) \triangleright \rho(x)$.

- Suppose $x = s_i$, then $\rho(s_i) \triangleright \rho(x) = \rho(x)$.

- Suppose $x \stackrel{i}{\longrightarrow} y$ for some $y \in T$, then $\rho(y) = t_i \triangleright \rho(x)z$. Thus $\rho(s_i) \triangleright \rho(x) = z\rho(s_i \triangleright x)$. Conjugating by t_i a second time we get $\rho(s_i) \triangleright \rho(y) = z\rho(s_i \triangleright y)$.

- Suppose $x \neq s_i$ and $s_i x s_i = x$. We want to prove that $\rho(s_i \triangleright x) = z\rho(x)$. We use Lemma 4.3.5. If x is in case 1. a z appears because of the relations in $\widetilde{W}$ 3.4. Moreover the case 2. enables to proceed by induction on the length of x.

□

4.4 When q^+ and q^- are cohomologous

If $W = \mathcal{D}_n$ with an odd n, then q^+ and q^- are cohomologous as in that case $\widetilde{\mathcal{D}_n}$ is a trivial extension using Proposition 2.2.3. We prove that it is in fact the only case where q^+ and q^- are cohomologous.

Lemma 4.4.1. *Let (W, S) be a finite Coxeter group. Suppose there exists $s, s' \in S$ where (ss') is of even order m. Then q^+ and q^- are not cohomologous.*

Proof. If q^+ and q^- are cohomologous we have $\gamma : W \longrightarrow \mathbb{C}^*$ such that:

$$q^-(x, y) = \gamma(x \triangleright y)^{-1} q^+(x, y)\gamma(y) \text{ for all } x, y \in T.$$

Write $\sigma = (s's)^{m/2-1}s'$. We know that $s\sigma s = \sigma$. Then $\gamma(s \triangleright \sigma) = \gamma(\sigma)$, thus $q^-(s, \sigma) = q^+(s, \sigma)$. This is a contradiction as $s(\alpha_\sigma) = \alpha_\sigma \in \Phi^+$. See the introductive diagram 8 to illustrate this proof. □

Thus the only finite Coxeter groups such that q^+ and q^- are cohomologous are the odd Dihedral groups according to the classification [3, Chapitre VI, §4, Théorème 1]. This concludes the proof of Theorem 3.1.2.

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