EXISTENCE OF 2-BUBBLE SOLUTION FOR THE HARMONIC MAP HEAT FLOW

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Contents

1. Course of the internship and general mathematical settings

The internship was conducted at the Massachusetts Institute of Technology (MIT) under the guidance of Professor Andrew Lawrie. The schedule was divided between academic courses (including Geometry of Manyfolds, Topics in Differential Equations, and a reading course with Lawrie and his students) and independent research. I was afforded the opportunity to request a meeting with my supervisor at my convenience, despite the necessity of scheduling an appointment due to his demanding schedule and occasional absence from the MIT. One disadvantage of the internship was the lack of mathematical interaction, given that my office was located two floors below the main part of the mathematics department, which made it difficult to engage in impromptu discussions. I must concede that I am somewhat uneasy discussing mathematical subjects with fellow graduate students, given that they were considerably more advanced than me in those areas, which they had specialised in from the outset of their doctoral studies. Furthermore, I encountered a similar challenge with my supervisor. Given my limited progress on my research, I was reluctant to request an appointment, particularly when I had no new findings to present. However, I was fortunate to have a supportive and welcoming graduate student who facilitated my integration into the research group. In part due to this support and the guidance of Andrew Lawrie, I was able to register for a summer university programme on nonlinear partial differential equations at the University of California, Berkeley.

I was assigned the task of solving an open problem at the MIT. The original subject dates back to the 1980s, when Struwe was interested in harmonic mapping (an application with regularity conditions) between two Riemannian manifolds and its evolution. In this context, the mapping is regarded as evolving in order to minimise the energy associated with it. The conventional methodology for addressing this issue entails defining an energy function and postulating that the mapping will evolve in proportion to the rate of decrease in energy as it progresses in the desired direction (the gradient flow). In this case, the greater the decrease in energy, the faster the evolution will occur. The question was whether a certain kind of solution (the 2-bubble solution) could exist in infinite time with additional constraints. Unfortunately, a paper was published by Frank Merle and Kihyum Kim during the internship, which classified all the solutions and proved a stronger result. This paper partly discouraged me from pursuing this line of enquiry further. The next sections will present the result and the sketch of the proof of Kim and Merle, as well as the alternative strategy I was studying during the internship.

2. Introduction

2.1. **General setting and state of the art at the beginning of the internship.**

Consider the Harmonic Map Heat Flow (HMHF) for maps $\mathfrak{u} : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, that is, the gradient flow associated to the Dirichlet energy

$$E(\mathfrak{u}) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \mathfrak{u}|^2 dx,$$

The associated Cauchy problem is given by

$$\begin{cases} \partial_t \mathfrak{u} = \Delta \mathfrak{u} + \mathfrak{u}|\nabla \mathfrak{u}|^2, \\ \mathfrak{u}(0,x) = \mathfrak{u}_0(x) \in \mathbb{S}^2, \end{cases} \quad (t,x) \in [0,T) \times \mathbb{R}^2. \tag{2.1}$$

Consider the *energy-class*

$$\mathcal{E} := H^1(\mathbb{R}^2;\mathbb{S}^2) := \{\mathfrak{u}_0 \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3) \mid |\mathfrak{u}_0(x)|^2 = 1 \text{ for almost every } x \in \mathbb{R}^2\}$$

By the result of Struwe [11], the problem is proved to be well-proved for any initial data in the energy-class, *i.e.* the set of maps with finite energy, and we can associate to every initial data $\mathfrak{u}_0$ a maximal time of existence $T_{\max} \in (0,\infty]$ and a unique solution $\mathfrak{u}(t)$, regular for every $t \in (0, T_{\max})$. A finite maximal time $T_{\max}$ is characterised by the existence of $\varepsilon_0 > 0$ and an at most finite number of points $\{x_\ell\}$ such that

$$\forall R > 0, \quad \limsup_{t \to T_{\max}} E_{B_R(x_\ell)}(\mathfrak{u}(t)) \geq \varepsilon_0,$$

where $E_{\mathcal{D}}(\mathfrak{u}) := \frac{1}{2}\int_{\mathcal{D}} |\nabla \mathfrak{u}|^2 dx$ is a localised energy. **[Struwe85]**. Such x_ℓ are called *bubbling points*. Furthermore, assuming $\mathfrak{u}_0$ to be regular, the energy is continuous and non-increasing as function of $t \in [0, T_{\max})$ and the *energy identity* holds for all $t_1, t_2 \in [0, T_{\max})$ (see **[Struwe85]**)

$$E(\mathfrak{u}(t_2)) + \int_{t_1}^{t_2} \|\mathcal{T}(\mathfrak{u}(t))\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}t = E(\mathfrak{u}(t_1)), \tag{2.2}$$

where $\mathcal{T}(\mathfrak{u}) := \Delta \mathfrak{u} + \mathfrak{u}|\nabla \mathfrak{u}|^2$ is called *tension* of $\mathfrak{u}$. Note that $\mathcal{T}(\mathfrak{u}) = \partial_t \mathfrak{u}$. In particular, $\int_0^{T_{\max}} \|\mathcal{T}(\mathfrak{u}(t))\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}t < \infty$ and there exists $E_{T_{\max}} = \lim_{t \to T_{\max}} E(\mathfrak{u}(t))$.

We further restrict the class of initial data by intersecting space $\mathcal{E}$ with continuous maps $\mathfrak{u}_0$ that tend to a vector $\mathfrak{u}_\infty$ at ∞, *i.e.* satisfies $\lim_{x \to \infty} |\mathfrak{u}_0(x) - \mathfrak{u}_\infty| = 0$. By assigning the point $\mathfrak{u}_\infty$ at the point at infinity, we construct a continuous mapping $\tilde{\mathfrak{u}}_0 : \mathbb{S}^2 \to \mathbb{S}^2$ and we may define the degree of $\mathfrak{u}_0$ to be the one of $\tilde{\mathfrak{u}}_0$. This condition over $\mathfrak{u}$ is preserved by flow, *i.e.* $\lim_{x\to\infty} |\mathfrak{u}(t,x)-\mathfrak{u}_\infty| = 0$ for all t in $[0,T_{\max})$. Under this restriction, $\mathfrak{u}(t)$ is a continuous deformation of $\mathfrak{u}_0$ within its homotopy class, which motivates the introduction by Eells and Sampson [2] of HMHF between Riemmanian manifolds.

Harmonic maps ω have a zero tension and are consequently stationary solutions to HMHF 2.1. Harmonic maps in dimension two are conformal (up to change of orientation) and minimize their energy in their homotopy class [4, 3, 9]. The energy of an harmonic map is given by $E(\omega) = 4\pi|\deg(\omega)|$.

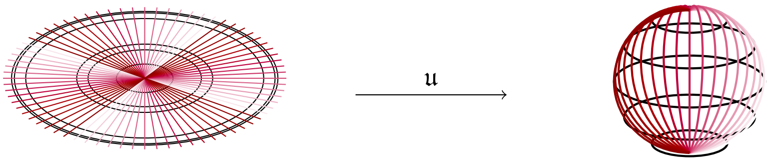
Works of Qing [10], Ding–Tian [12], Wang [1], Qing–Tian [5], and Lin–Wang [**LinWangCVPDE**] provide more refined understanding of the energy identity 2.2, which lead to the description of the solution *along a well-chosen sequence of time* as superposition of a finite number of bubbles (scaling harmonic map) and a time invariant body map. A recent work of Jendrej, Lawrie and Schlag [7] proved that *any sequence of time* can be chosen and so, the continuity of the limit behaviour.

2.2. **The k–equivariant case.** We define *k–equivariant solutions* to be solution to (HMHF) taking the forms of

$$\mathfrak{u}(t,re^{i\theta}) = (\sin v(t,r)\cos k\theta, \sin v(t,r)\sin k\theta, \cos v(t,r)) \in \mathbb{S}^2 \tag{2.3}$$

where $k \in \mathbb{N}$ and r,θ are polar coordinates on $\mathbb{R}^2$.

Remark 2.1. A k-equivariant map is completely described by the latitude on the sphere that is associated to every distance to the origin of $\mathbb{R}^2$. Indeed, all points located at the same distance of the origin are sent on the same latitude on the sphere and longitude is determined from the beginning, uniformely distributed over the sphere.



4-equivariant mapping from $\mathbb{R}^2$ to $\mathbb{S}^2$

Being k–equivariant is a property preserved under the flow of 2.1, which reduces to the scalar equation

$$\text{(HMHF)} \qquad \begin{cases} \partial_t v - \partial_r^2 v - \dfrac{1}{r}\partial_r v = -\dfrac{k^2}{r^2}\dfrac{\sin 2v}{2} & (t,r)\in[0,T)\times\mathbb{R}^+. \\ \qquad v(0,\cdot) = v_0 \end{cases}$$

Indeed, assuming u $k-$equivariant, we have

$$\mathfrak{u}\left(t,re^{i\theta}\right) = \begin{pmatrix} \cos(k\theta)\sin(v(t,r)) \\ \sin(k\theta)\sin(v(t,r)) \\ \cos(v(t,r)) \end{pmatrix}, k\in\mathbb{Z}$$

But 2.1, projected on the 3rd coordinate, gives :

$$\begin{aligned} -\sin v\left[\partial_t v - (\partial_r v)^2\frac{\cos v}{\sin v} - \partial_r^2 v - \frac{1}{r}\partial_r v\right] &= \partial_t \cos v - \left(\partial_r^2\cos v + \frac{1}{r}\partial_r\cos v\right) \\ &= \left(\left[(\partial_r\cos v)^2 + (\partial_r\sin v)^2\right] + \left[\frac{1}{r^2}\left(k^2\sin^2 v\right)\right]\right)\cos v \\ &= \left([\partial_r v]^2 + \frac{k^2}{r^2}\sin^2 v\right)\cos v \end{aligned}$$

i.e.

$$\partial_t v - \partial_r^2 v - \frac{1}{r}\partial_r v = -\frac{k^2}{r^2}\sin v\cos v = -\frac{k^2}{r^2}\frac{\sin 2v}{2} \qquad \text{or} \qquad \sin v = 0$$

and 2.1 becomes

$$\partial_t v - \partial_r^2 v - \frac{1}{r}\partial_r v + \frac{k^2}{r^2}\frac{\sin 2v}{2} = 0, \qquad v(0,r) = v_0(r) \tag{2.4}$$

The energy becomes

$$E(v) = 2\pi\int_0^\infty \frac{1}{2}\left((\partial_r v)^2 + \frac{k^2\sin^2 2v}{r^2}\right) r\mathrm{d}r$$

It is well-known that the energy class within k–equivariance is the disjoint union of the connected components $\mathcal{E}_{\ell,m}$ for $\ell, m \in \mathbb{Z}$, where

$$\mathcal{E}_{\ell,m} \coloneqq \{v : (0,\infty) \to \mathbb{R} : E(v) < \infty,\ \lim_{r\to 0} v(r) = \ell\pi,\ \lim_{r\to\infty} v(r) = m\pi\}. \tag{2.5}$$

Recent works of Jendrej–Lawrie [6] and Kim–Merle [8] give the complete description of the limit behaviour and prove the infinite-time existence for $k \geq 3$. As a corollary, the existence of infinite time 2-bubble solution is known for $k \geq 3$.

More precisely, the unique nontrivial k–equivariant harmonic map is given by

$$Q(r) \coloneqq 2\arctan\left(r^k\right).$$

Unique, in the sense that all nontrivial k–equivariant harmonic maps are of the form

$$\iota Q_\lambda(r) \coloneqq \iota Q\left(\lambda^{-1} r\right),$$

with scale $\lambda \in (0,\infty)$ and sign $\iota \in \{\pm 1\}$, up to an addition by $\ell\pi, \ell \in \mathbb{Z}$. Note that $Q_\lambda \in \mathcal{E}_{0,1}$ and $E(Q) = 4k\pi$.

Proof. $\partial_r Q = 2k\frac{r^{k-1}}{1+r^{2k}}$,

$\partial_r^2 Q = 2k\frac{(k-1)r^{k-2}\left(1+r^{2k}\right)-2kr^{3k-2}}{(1+r^{2k})^2} = \frac{2kr^{2k}}{1+r^{2k}}\left[k-1-\frac{2kr^{2k}}{1+r^{2k}}\right] = 2k\frac{(k-1)r^{k-2}-(k+1)r^{3k-2}}{(1+r^{2k})^2}$

Using $\sin(2\arctan y) = \frac{2y}{1+y^2}$ and $\tan(2\arctan x) = \frac{2x}{1-x^2}$ (provided $|x| \neq 1$)

$$\begin{aligned}
\frac{k^2}{r^2}\frac{\sin 2Q}{2} &= \frac{k^2}{r^2}\frac{\sin 2\arctan\left(\frac{2r^k}{1-r^{2k}}\right)}{2} \\
&= \frac{k^2}{r^2}\frac{\frac{2r^k}{1-r^{2k}}}{1+\left(\frac{2r^k}{1-r^{2k}}\right)^2} \\
&= \frac{k^2}{r^2}\frac{2r^k\left(1-r^{2k}\right)}{\left(1-r^{2k}\right)^2+4r^{2k}} \\
&= \frac{2k^2r^{k-2}\left(1-r^{2k}\right)}{\left(1+r^{2k}\right)^2}
\end{aligned}$$

And so,

$$\begin{aligned}
-\partial_r^2 Q - \frac{1}{r}\partial_r Q + \frac{k^2}{r^2}\frac{\sin 2Q}{2} &= -\frac{2kr^{2k}}{1+r^{2k}}\left[k-1-\frac{2kr^{2k}}{1+r^{2k}}\right] - 2k\frac{r^{k-2}}{1+r^{2k}} + \frac{2k^2r^{k-2}\left(1-r^{2k}\right)}{\left(1+r^{2k}\right)^2} \\
&= \frac{2kr^{k-2}}{(1+r^{2k})^2}\left[k(1-r^{2k}) - (k-1) + (k+1)r^{2k} - (1+r^{2k})\right] \\
&= \frac{k^2r^{k-2}}{(1+r^{2k})^2}\left[(-k+(k+1)-1)\cdot r^{2k} + (k-(k-1)-1)\right] = 0
\end{aligned}$$

The precise result of [6] and [8] gives that, for all $\ell, p \in \mathbb{Z}$ and $v_0 \in \mathcal{E}_{\ell,p}$, we have the existence of $\{\lambda_j(t)\}_1^{|p-\ell|}$ such that

$$v(t) - \ell\pi - \iota \sum_{j=1}^{|p-\ell|} Q_{\lambda_j(t)} \underset{t\to\infty}{\longrightarrow} 0$$

where $\iota = \operatorname{sgn}(p-l)$ and the precise limit behaviour of $\{\lambda_j(t)\}_1^{|p-\ell|}$ is known, up to a scale factor.

2.3. **Interactions between 2-bubbles.** Let recall that the open question was the existence of a global solution v that decomposes into three parts : $v = Q_{\lambda_1(t)} + Q_{\lambda_2(t)} + g$ with $\|g\|_{\mathcal{E}} \underset{t\to\infty}{\longrightarrow} 0$ but the addition of two bubbles (Q_{λ_1} and Q_{λ_2}) is non-static and an interaction term moves the system.

Let have some exact computation that will be useful for the following reckoning.

$$\begin{aligned}
\partial_r Q_{m,\lambda}(r) &= \frac{1}{\lambda}\partial_r Q\left(\frac{r}{\lambda}\right) && = 2k\frac{\lambda^k r^{k-1}}{\lambda^{2k}+r^{2k}} \\
\partial_r^2 Q_{m,\lambda}(r) &= \frac{1}{\lambda^2}\partial_r^2 Q\left(\frac{r}{\lambda}\right) && = 2k\frac{(k-1)r^{k-2}\lambda^{3k}-(k+1)r^{3k-2}\lambda^k}{(\lambda^{2k}+r^{2k})^2} \\
\frac{\sin 2Q_{m,\lambda}(r)}{2} &= \frac{2r^k\lambda^k\left(\lambda^{2k}-r^{2k}\right)}{\left(\lambda^{2k}+r^{2k}\right)^2} \\
\cos 2Q_{m,\lambda}(r) &= \frac{\lambda^{4k}-6\lambda^{2k}r^{2k}+r^{4k}}{\left(\lambda^{2k}+r^{2k}\right)^2} && = \frac{\left(\lambda^{2k}-r^{2k}\right)^2-4\lambda^{2k}r^{2k}}{\left(\lambda^{2k}+r^{2k}\right)^2} \\
\partial_t Q_{m,\lambda(t)}(r) &= -\frac{\lambda'}{\lambda^2} r\partial_r Q_{m,\lambda}\left(\frac{r}{\lambda}\right)
\end{aligned}$$

Convention. *Until the end,* $g = u - Q_{\lambda_1} - Q_{\lambda_2}$

Notation. $\|g\|_{\mathcal{E}}{}^2 = \int_0^\infty \left[(\partial_r g(r))^2 + \frac{k^2}{r^2}g(r)^2\right] r\mathrm{d}r$

Remark 2.2 (Aim). The open subject is to prove that $\exists u_0$ such that $\lim\limits_{t\to\infty} \|g\|_{\mathcal{E}} = 0$ and $\lim\limits_{t\to\infty} \frac{\lambda_1}{\lambda_2} = 0$

2.4. **Notation.** Let introduce a few sets of notation. Apart from the basic one, additional notations are needed to deal with multi-bubble solutions.

2.4.1. *Basic notation.*

- For $A \in \mathbb{R}$ and $B \geq 0$, we denote $A \lesssim B$ (or, $A = O(B)$) if $|A| \leq CB$ for some universal constant $C > 0$. If $A, B \geq 0$, we denote $A \simeq B$ if $A \lesssim B$ and $A \gtrsim B$. The dependence on parameters is written in subscripts. Any dependence on k and J (the number of bubbles) will be ignored.
- We use the notation $\mathbb{1}_A$. If A is a statement, it means that it is 1 if A is true and 0 otherwise. If A is a set, then it denotes the indicator function on A; $\mathbb{1}_A(x) = 1$ if $x \in A$ and 0 otherwise.
- δ_{jk} denotes the Kronecker-delta symbol, i.e., $\delta_{jk} = \mathbb{1}_{j=k}$.
- $\chi = \chi(r)$ denotes a smooth radial cutoff function such that $\chi(r) = 1$ if $r \leq 1$ and $\chi(r) = 0$ if $r \geq 2 - \frac{1}{10}$. We denote $\chi_R := \chi(\cdot/R)$ and $\chi_{>R} := 1 - \chi_R$.
- $\langle x \rangle := (1 + |x|^2)^{1/2}$.
- For $k \in \mathbb{N}^*$, denote $|f|_k := \max\{|f|, |rr\partial_r f|, \dots, |r^k r\partial_r^k f|\}$ and $|f|_{-k} := r^{-k}|f|_k$.

2.4.2. *Specific notations.*

- $\langle \cdot, \cdot \rangle$ denotes the inner product given by

$$\langle u, v\rangle = \int_{\mathbb{R}^+} u(r)v(r)r\mathrm{d}r$$

- We recall that $Q(r) := 2\arctan\left(r^k\right)$ and $Q_\lambda(r) := Q(\lambda^{-1}r)$
- Let v be a function, $v_\lambda(r) = v\left(r/\lambda\right)$ and $v_{\underline{\lambda}} = \lambda^{-1}v_\lambda$
- $\Lambda Q_{\underline{\lambda}}(r) := -\,\partial_\lambda Q_{\underline{\lambda}}\left(\frac{r}{\lambda}\right)\Big|_{\lambda=1} = r\partial_r Q_{\underline{\lambda}}(r)$

2.4.3. *Function spaces.* Let $\alpha, \beta, \gamma, \kappa \in \mathbb{R}, T_0 > 0, z : [T_0, \infty) \to \mathbb{R}$ be a continuous function and $v : \mathbb{R}^+ \to \mathbb{R}$ a C^1-function. We define

- $\|z\|_H^2 = \int_0^\infty \left((\partial_r v(r))^2 + \frac{k^2v^2(r)}{r^2}\right) r\mathrm{d}r$
- $\|z\|_{\mathcal{N}_\gamma} = \sup\limits_{t\geq T_0} t^\gamma |z(t)| \qquad (\|z\|_{\mathcal{N}_\gamma(E)} = \|t \mapsto \|z(t)\|_E\|_{\mathcal{N}_\gamma})$
- $\|z\|_{\mathcal{W}_{\alpha,\beta}} = \sup\limits_{\tau\geq t\geq T_0} t^{\beta-\alpha}\left|\int_t^\tau s^\alpha z(s)\mathrm{d}s\right| \qquad (\mathcal{W}_{\{\alpha_1,\alpha_2\},\gamma} = \mathcal{W}_{\alpha_1,\gamma} \cap \mathcal{W}_{\alpha_2,\gamma})$
- $\|z\|_{\mathcal{X}_\gamma} = \|z\|_{\mathcal{N}_\gamma} + \|z'\|_{\mathcal{N}_{\gamma+1}}$
- $\|z\|_{\mathcal{S}_\gamma([T_0,\infty))} = \|z\|_{\mathcal{N}_\gamma([T_0,\infty),H)} + \sup\limits_{t\geq T_0} t^{\gamma-\frac{2}{3}\kappa}\|\tau^{\frac{\kappa}{3}}z(t)r^{-\frac{2}{3}}\|_{L^{\frac{3}{2}}_\tau([t,\infty),L^6_x)}$
- $\|z\|_{\mathcal{Y}_{\gamma+1}([T_0,\infty),E)([T_0,\infty),E)} = \sup\limits_{t\geq T_0} t^\gamma \int_t^\infty \|z(s)\|_E\mathrm{d}s$
- $\|z\|_{\mathcal{Z}_\gamma([T_0,\infty);L^2)([T_0,\infty),L^2)} = \|z\|_{\mathcal{N}_{\beta+1}([T_0,\infty),L^2)} + \|rz\|_{\mathcal{N}_\gamma([T_0,\infty),L^2)} + \|r\partial_t z\|_{\mathcal{N}_{\gamma+1}([T_0,\infty),L^2)}$ (assuming z to be C^1 and with the additional conditions $z \in L^2 \cap H^{-1}$ and $\partial_t z \in H^{-1}$)

3. SKETCH OF MY PERSONAL APPROACH, UNDER LAWRIE'S SUPERVISION

Let's recall the result which I aimed at, proving the existence in infinite time of a 2-bubble solution, which is to prove that

Conjecture 3.1. $\exists v_0 \in \mathcal{E}$, such that the corresponding solution $v(t)$ to (HMHF) admits the decomposition $v(t) - \iota_1 Q_{\lambda_1} - \iota_2 Q_{\lambda_2} \underset{t\to\infty}{\longrightarrow} 0$

Remark 3.1. As the flow remains in its initial connected component $\mathcal{E}_{\ell,m}, v_0$ must be in $\mathcal{E}_{\ell,\ell+2}$ (if $\iota_1\iota_2 = 1$) or $\mathcal{E}_{\ell,\ell}$ (if $\iota_1\iota_2 = -1$) with $\ell \in \mathbb{Z}$.

Remark 3.2. As $E(v(t))$ is non-increasing with time, choosing $v_0 \in \mathcal{E}_{0,2}$ with $E(v_0) < 16\pi$ guarantees, under the assumption that v is global and thanks to a result demonstrated in , that v is a 2-bubble solution (and not a 4-bubble solution for example).

The main idea of the research was to decompose the problem into three pieces :

1. Establish formal ODEs to determine the evolution rate of λ_1 and λ_2. Let's say that we obtain $\lambda_1(t) \sim at^{-\alpha}$ and $\lambda_2(t) \sim bt^{-\beta}$
2. Find a new system to reduce the problem to two ODEs
3. Prove that there is a solution to those ODEs that satisfies a basic condition to prove existence of a global solution to (HMHF)

3.1. **Plausible dynamics for λ_1 and λ_2.**

Notation. Let's note :

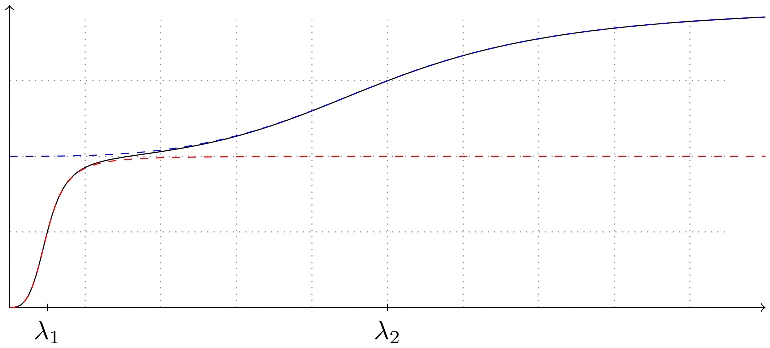
$$\begin{cases} r_i &= \varepsilon_i \left(\frac{r}{\lambda_i}\right)^k, \quad i = 1,2 \\ Q_i &= Q_{\lambda_i} = 2\arctan\left(r_i\right) \\ f(x) &= \sin(2x) \\ a &= r_1 + r_2 \\ b &= 1 - r_1 r_2 \\ x &= a/b \\ f_I &= f\left(Q_1 + Q_2\right) - f\left(Q_1\right) - f\left(Q_2\right) \end{cases}$$

- $Q_1 + Q_2 \equiv 2\arctan\left(\frac{r_1+r_2}{1-r_1r_2}\right) \equiv 2\arctan(x)[\pi]$
- $f\left(Q_1+Q_2\right) = 4\frac{x(1-x^2)}{(1+x^2)^2} = 4\frac{ab(b^2-a^2)}{(a^2+b^2)^2}$
- $a^2 + b^2 = (1+r_1^2)(1+r_2^2)$
- $f(Q_i) = 4\frac{r_i(1-r_i^2)}{(1+r_i^2)^2}$
-

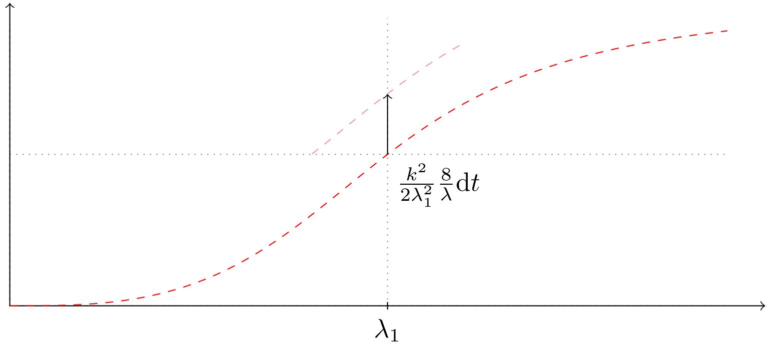
$$\begin{array}{rclcccccccccccc} ab(a^2-b^2) & = & r_1 & +r_2 & -r_1^3 & -6r_1^2r_2 & -6r_1r_2^2 & -r_2^3 & +r_1^4r_2 & +6r_1^3r_2^2 & +6r_1^2r_2^3 & +r_1r_2^4 & -r_1^4r_2^3 & -r_1^3r_2^4 \\ (1+r_2^2)^2r_1(1-r_1^2) & = & r_1 & & -r_1^3 & & +2r_1r_2^2 & & & -2r_1^3r_2^2 & & +r_1r_2^4 & & -r_1^3r_2^4 \\ (1+r_1^2)^2r_2(1-r_2^2) & = & & r_2 & & +2r_1^2r_2 & & -r_2^3 & +r_1^4r_2 & & -2r_1^2r_2^3 & & -r_1^4r_2^3 & \end{array}$$

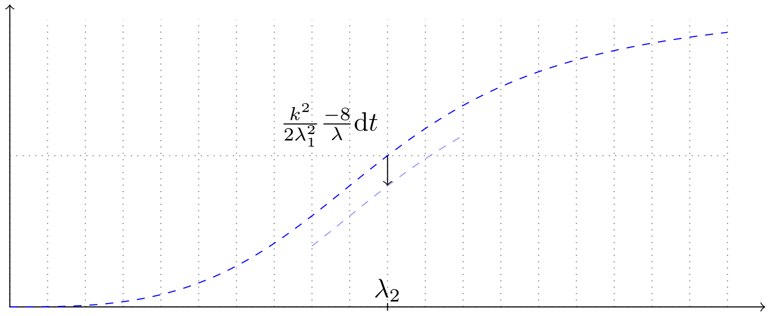
$$f_I = -\frac{2^5 r_1 r_2 (r_1+r_2)(1-r_1r_2)}{(1+r_1^2)^2(1+r_2^2)^2}$$

- $-f_I(\lambda_1) = \operatorname{sgn}\left(r_2\right)\frac{8\frac{1}{\lambda}\left(1-\frac{1}{\lambda^2}\right)}{\left(1+\frac{1}{\lambda^2}\right)^2} \sim \operatorname{sgn}\left(r_2\right)\frac{8}{\lambda}$
- $-f_I(\lambda_2) = \frac{8\lambda\left(1-\lambda^2\right)}{(1+\lambda^2)^2} \sim -\frac{8}{\lambda}$
- For $k = 3, \lambda_2 = 10\lambda_1$



Especially,





- But $\partial_r\left(Q_1+Q_2\right)(\lambda_1) \sim \partial_r\left(Q_1\right) = \frac{k}{\lambda_1}$
  And

$$Q_{1(t+\mathrm{d}t)}(\lambda_1) - Q_{1(t)}(\lambda_1) \quad\sim\quad -\frac{k^2}{2\lambda_1^2} f_I \mathrm{d}t \tag{3.1}$$

$$Q_1(\lambda_1(t+\mathrm{d}t)) - Q_1(\lambda_1(t)) \quad\sim\quad (\lambda_1(t+\mathrm{d}t)-\lambda_1(t))\,\partial_r Q_1(\lambda_1) \tag{3.2}$$

$$Q_{1(t+\mathrm{d}t)}(\lambda_1(t+\mathrm{d}t)) \quad=\quad Q_{1(t)}(\lambda_1(t)) = \frac{\pi}{2} \tag{3.3}$$

We get that $\frac{k^2}{2\lambda_1^2} f_I \mathrm{d}t \sim (\lambda_1(t+\mathrm{d}t)-\lambda_1(t))\,\partial_r Q_1(\lambda_1)$

and so $\lambda_1(t+\mathrm{d}t)-\lambda_1(t) \sim -\dfrac{4\lambda_1}{\lambda k}\dfrac{k^2}{\lambda_1^2}\mathrm{d}t$ *i.e.*

$$\partial_t \lambda_1 = -4k\lambda_1^{k-1}\lambda_2^{-k} \tag{3.4}$$

- Same, $\partial_r\left(Q_1+Q_2\right)(\lambda_2)\sim\partial_r\left(Q_2\right)=\frac{k}{\lambda_2}$
  But

$$Q_{2(t+\mathrm{d}t)}(\lambda_2)-Q_{2(t)}(\lambda_2)\quad\sim\quad-\frac{k^2}{2\lambda_2^2}f_I\mathrm{d}t \tag{3.5}$$

$$Q_2(\lambda_2(t+\mathrm{d}t))-Q_2(\lambda_2(t))\quad\sim\quad(\lambda_2(t+\mathrm{d}t)-\lambda_2(t))\,\partial_rQ_2(\lambda_2) \tag{3.6}$$

$$Q_{2(t+\mathrm{d}t)}(\lambda_2(t+\mathrm{d}t))\quad=\quad Q_{2(t)}(\lambda_2(t))=\frac{\pi}{2} \tag{3.7}$$

We get that $\frac{k^2}{2\lambda_2^2}f_I\mathrm{d}t\sim(\lambda_2(t+\mathrm{d}t)-\lambda_2(t))\,\partial_rQ_2(\lambda_2)$
and so $\lambda_2(t+\mathrm{d}t)-\lambda_2(t)\sim\dfrac{4\lambda_2}{\lambda k}\dfrac{k^2}{\lambda_2^2}\mathrm{d}t$ *i.e.*

$$\partial_t\lambda_2=4k\lambda_1^k\lambda_2^{-k-1} \tag{3.8}$$

- Using 3.4 and 3.8, we get

$$\lambda_1\partial_t\lambda_1=-\lambda_2\partial_t\lambda_2\qquad i.e.\qquad\partial_t\lambda_1^2=-\partial_t\lambda_2^2 \tag{3.9}$$

$$\partial_t\lambda_1\sim-\frac{4k}{C^k}\lambda_1^{k-1}\qquad\text{et}\qquad\lambda_2\sim C \tag{3.10}$$

i.e.

$$\lambda_1\sim C^{\frac{k}{k-2}}\sqrt[k-2]{\frac{k-2}{4kt}}\quad\text{and}\quad\lambda_2\sim C\qquad(\text{if }k\geq3) \tag{3.11}$$

3.2. **Implementation of the Lyapunoch-Schmidt scheme for (HMHF).** Consider an k-equivariance case $k\geq3$. We define

$$\kappa:=\frac{1}{k-2}$$

- For $\alpha\in\mathbb{R}$ fixed, $\mathcal{H}_\alpha$ notes $\left\{\lambda(t)\backslash|t^\alpha\lambda(t)|=O(1)|t^{\alpha-1}\lambda'(t)|=O(1)\right\}$

3.2.1. *Resolution of the projected problem.* Let $(\lambda_1,\lambda_2)\in\mathcal{H}$. We want to solve

$$\begin{cases}E+L[g]+N(g)=c_1Q_{\lambda_1}+c_2Q_{\lambda_2}\\ \langle g,Q_{\lambda_i}\rangle=0,\qquad i=1,2\end{cases}$$

3.2.2. *Contraction mapping principle.* We will use the contraction mapping principle. Let $(\lambda_1,\lambda_2)\in X_{2\kappa}\times X_\kappa$ be a pair of trajectories and $g\in$ espace à définir$([T_0,\infty))$ for T_0 which will be defined later. We define the mapping,

$$(C_1(\lambda_1,\lambda_2,g),C_2(\lambda_1,\lambda_2,g),G(\lambda_1,\lambda_2,g)):=(c_1,c_2,g)$$

as the solution of the linear equation

$$\begin{aligned}\partial_th-\Delta h+f_\ell(h)[h]&=F^I_{\lambda_1,\lambda_2}+N_{\lambda_1,\lambda_2}(g)+({\lambda_1}'+c_1)\Lambda Q_{\underline{\lambda_1}}+({\lambda_2}'+c_2)\Lambda Q_{\underline{\lambda_2}}\\ \left\langle\Lambda Q_{\underline{\lambda_1}},h\right\rangle&=0\\ \left\langle\Lambda Q_{\underline{\lambda_2}},h\right\rangle&=0\end{aligned}$$

The next step of the proof was to prove thanks to the contraction mapping principle was to prove that there exists a solution to the linear equation.

Finally, the last step was to prove that there is (c_1, c_2, g) solution to the linear solution with $c_1 \equiv 0$ and $c_2 \equiv 0$.

4. Result of Kim and Merle [8]

4.1. **Statement of the result and brief overview.** The paper shows that for all $\ell, p \in \mathbb{Z}$ and $v_0 \in \mathcal{E}_{\ell,p}$, we have the existence of $\{\lambda_j(t)\}_1^{|p-\ell|}$ such that

$$v(t) - \ell\pi - \iota \sum_{j=1}^{|p-\ell|} Q_{\lambda_j(t)} \underset{t\to\infty}{\longrightarrow} 0$$

where $\iota = \operatorname{sgn}(p-l)$ and the precise limit behaviour of $\{\lambda_j(t)\}_1^{|p-\ell|}$ is known, up to a scale factor. Note that the solution is defined in infinite time. The result roughly says that the solution decomposes into the sum of *scale-decoupled bubbles* and a vanishing radiation. More precisely,

Theorem 4.1. *Let $k \geq 3$. Let $\ell, m \in \mathbb{Z}$ and $v_0 \in \mathcal{E}_{\ell,m}$ be an initial data. Then, the corresponding solution $v(t)$ to* (HMHF) *is global and admits the decomposition*

$$v(t) - \ell\pi - \iota \sum_{j=1}^{|p-\ell|} Q_{\lambda^{\text{ex}}_{j,L_\infty}(t)} \underset{t\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{E} \tag{4.1}$$

for some real $L_\infty \in (0,\infty)$ and the rates $\lambda^{\text{ex}}_{j,L_\infty}(t)$ are given by

$$\lambda^{\text{ex}}_{j,L_\infty}(t) := L_\infty^{1+2\alpha_j} \cdot \frac{\beta_j}{t^{\alpha_j}}$$

with

$$\begin{cases} \alpha_j &= \frac{1}{2}\left(\frac{k}{k-2}\right)^{j-1} - \frac{1}{2}, \\ \kappa &= -\dfrac{4\int_0^\infty (\Lambda Q)^3 y^{k-1} dy}{\int_0^\infty (\Lambda Q)^2 y dy} < 0, \\ \beta_1 &= 1 \quad \text{and} \\ \beta_j &= \left(\frac{\alpha_j}{|\kappa|}\right)^{\frac{1}{k-2}} \beta_{j-1}^{\frac{k}{k-2}} \qquad \forall j \geq 2 \end{cases} \tag{4.2}$$

Remark 4.2. In the case of a single bubble, results were obtained on the assumption that the initial condition is already close to a bubble, which is the only non-trivial k-equivariant harmonic map. The primary challenge lies in the fact that two bubbles interact with one another, and it is not possible to assume that the radiation is as small as desired. If one begins with a null radiation, a non-zero radiation will immediately be created, which will vanish as the two bubbles separate. However, this radiation will not be sufficiently small if the two bubbles are not distant enough from one another. The difficulty lies in bounding this remaining radiation. The main contribution of Kim and Merle's paper is to refine the multi-bubble model to include the residual interaction and relies on the following statement to prove global existence.

Proposition 4.3 (Bubble decomposition)**.** *Let $k \in \mathbb{N}$, let $\ell, m \in \mathbb{Z}$, and let $v(t)$ be the solution to* (HMHF) *with initial data $v(0) = v_0 \in \mathcal{E}_{\ell,m}$, defined on its maximal interval of existence $[0, T_+)$.*

(Global solution) *If $T_+ = \infty$, there exist a time $T_0 > 0$, an integer $N \geq 0$, continuous functions $\lambda_1(t), \dots, \lambda_N(t) \in C^0([T_0, \infty))$, signs $\iota_1, \dots, \iota_N \in \{-1, 1\}$, and $g(t) \in \mathcal{E}$ defined by*

$$v(t) = m\pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_j(t)} - \pi) + g(t), \tag{4.3}$$

such that

$$\|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \xrightarrow[t\to\infty]{} 0, \tag{4.4}$$

where above we use the convention that $\lambda_{N+1}(t) = \sqrt{t}$.

(Blow-up solution) *If $T_+ < \infty$, there exist a time $T_0 < T_+$, integers m_∞, m_Δ, a mapping $u^* \in \mathcal{E}_{0,m_\infty}$, an integer $N \geq 1$, continuous functions $\lambda_1(t), \dots, \lambda_N(t) \in C^0([T_0, T_+))$, signs $\iota_1, \dots, \iota_N \in \{-1, 1\}$, and $g(t) \in \mathcal{E}$ defined by*

$$v(t) = m_\Delta \pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_j(t)} - \pi) + v^* + g(t), \tag{4.5}$$

such that

$$\|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \xrightarrow[t\to T_+]{} 0, \tag{4.6}$$

where above we use the convention that $\lambda_{N+1}(t) = \sqrt{T_+ - t}$.

Remark 4.4. The speed rate of λ_{N+1} is fixed by energy conditions, considering that initial energy is finite.

Remark 4.5. Proving $\lambda_N \underset{t\to T_+}{\not\to} 0v$ is now sufficient to get the global existence of a solution.

4.2. **Some additional specific notation.** Let $v, J \in \{1, 2, \dots\}$, and $k \in \mathbb{N}_{\geq 3}$ be given, and let define $u = r^{-D} v$, and $f(v) = \frac{k^2}{2}\{2v - \sin(2v)\}$.

- For functions g and h identified as radial parts of functions on $\mathbb{R}^N$, we denote the integral $\int g = c_N \int_0^\infty g(r) r^{N-1} dr$ and the inner product $\langle g, h \rangle = \int g(r) h(r)$, where c_N is the volume of the unit $(N-1)$-sphere.
- Once $\dot{H}^1$-scalings and the generator Λ are defined by either of the two ways above, we set $\phi_{\underline{\lambda}}(r) := \frac{1}{\lambda^2} \phi_\lambda$ and $\Lambda_{-1}\phi := \Lambda\phi + 2\phi$.
- For $\ell, m \in \mathbb{Z}$, define $\mathcal{H}_{\ell,m} := \{u = r^{-D} v : v \in \mathcal{E}_{\ell,m}\}$, where $\mathcal{E}_{\ell,m}$ was defined previously. Let $\mathcal{H} := \mathcal{H}_{0,0}$. Note that $\mathcal{H}$ is naturally identified with $\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^N)$.
- For $\vec{\iota} = (\iota_1, \dots, \iota_J) \in \{\pm 1\}^J$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_J) \in (0, \infty)^J$, we define

  $$\overline{\lambda}_j := \sqrt{\lambda_j \lambda_{j-1}},\ \mu_j := \lambda_j / \lambda_{j-1},\ \overline{\iota}_j := \iota_j \iota_{j-1} \text{ for } j \in \{2, \dots, J\}.$$

  Set $\overline{\lambda}_1 := 0,\ \overline{\lambda}_{J+1} := \infty,\ \mu_1 := 0,\ \mu_{J+1} := 0$.
- For a function ϕ, we define $\phi_{;j} := \iota_j \phi_{\lambda_j}$ and $\phi_{\underline{;j}} := \frac{1}{\lambda_j^2} \phi_{;j}$.
- We denote $W_{\vec{\iota},\vec{\lambda}} = \sum_{j=1}^J W_{;k} = \sum_{j=1}^J \iota_k W_{\lambda_k}$.
- Multi-bubble linearized operator is given by $H_{\vec{\lambda}} := -\Delta - \sum_{j=1}^J r^{-2} f'(Q_{\lambda_j})$.
- For $j \in \{1, \dots, J\}$, we denote $\iota_{1\dots j}$ and $\lambda_{1\dots j}$ as shorthands for $\iota_1, \dots, \iota_j$ and $\lambda_1, \dots, \lambda_j$, respectively.
- $\mathcal{D} := \sum_{j=2}^J \frac{\mu_j^{2D}}{\lambda_j^2}$.
- For $\alpha \in (0, \infty)$, define $\mathcal{P}_J(\alpha) := \{\vec{\lambda} \in (0, \infty)^J : \max_{2 \leq \ell \leq J} \mu_\ell < \alpha\}$.

4.3. Sketch of the construction of the refined profile.

Proposition 4.6.

- *$\exists \alpha_0 \ll 1, A_0 \gg 1$ as $\forall J \in \mathbb{N}^*, \{\iota_i\}_J, \vec{\lambda} \in \mathcal{P}(\alpha_0)$, there exists a modified profile U such that*

$$U = \sum_{k=1}^{J} \iota_k r^k Q_{\lambda_k} + \tilde{U}$$

where $\tilde{U}$ satisfies some bounds that make it small enough.
- *U solves the equation*

$$\Delta U + r^{-(k+2)} f(\sum Q_{\lambda_i}) = -\sum_{j=2}^{J} \iota_j \iota_{j-1} \frac{\kappa \mu_j^k}{\lambda_j^2} \Lambda(r^k Q_{\lambda_j}) + \Psi, \tag{4.7}$$

where the main term has the size (recall $\mathcal{D} = \sum_{j=2}^{J} \frac{\mu_j^{2k}}{\lambda_j^2}$)

$$\Big\| \sum_{j=2}^{J} \iota_j \iota_{j-1} \frac{\kappa \mu_j^k}{\lambda_j^2} \Lambda(r^k Q_{\lambda_j}) \Big\|_{L^2}^2 \simeq \mathcal{D} \qquad (\text{with } \mu_j = \lambda_j / \lambda_{j-1}) \tag{4.8}$$

and the remainder Ψ satisfies the estimates

$$\|\Psi\|_{L^2} \lesssim \delta(\alpha^*) \sqrt{\mathcal{D}}, \tag{4.9}$$

Remark 4.7. The estimate (4.9), saying that the inhomogeneous error is smaller than the main term (4.8) in the L^2 norm, is crucial in the proof of spacetime estimate.

The idea is to construct by induction a profile $U \sim \sum \iota_i Q_{\lambda_i}$ such that

$$\Delta U_J + f(U_J) \approx -\sum_{j=2}^{J} \frac{c_j \iota_j}{\lambda_j^2} \Lambda(r^k Q_{\lambda_j}) \qquad \text{and} \qquad c_j \approx \iota_j \iota_{j-1} \kappa. \tag{4.10}$$

where c_j is fixed to satisfy some orthogonality conditions.

4.4. Spacetime estimate.

Such construction gives important estimates which let us affirm that

Proposition 4.8 (Spacetime estimate). *Let $u(t)$ be a solution. Then, there exist $t_0 < T$ and a C^1-curve $\{\lambda_1, \dots, \lambda_J\} : [t_0, T) \to \mathcal{P}_J(\alpha_0)$ such that the following holds on the time interval $[t_0, T)$:*

- *(Decomposition) We still have (4.3) with the new choice of parameters $\{\lambda_1, \dots, \lambda_J\}(t)$. The refined decomposition of $u(t)$ as*

$$u(t) = U(\{\iota_1, \dots, \iota_J\}, \{\lambda_1, \dots, \lambda_J\}(t); \cdot) + g(t) =: U(t) + g(t) \tag{4.11}$$

satisfies the orthogonality conditions

$$\langle g(t), \mathcal{Z}_{\lambda_k(t)} \rangle = 0 \qquad \forall k \in \{1, \dots, J\}. \tag{4.12}$$

- *(Spacetime estimate) Recall that $\mathcal{D} = \sum_{j=2}^{J} \mu_j^{2D} / \lambda_j^2$. We have*

$$\int_{t_0}^{T} \Big\{ \mathcal{D} + \|g(t)\|_{\dot{H}^2}^2 \Big\} dt < +\infty. \tag{4.13}$$

Remark 4.9. This L_t^2-control is crucial when modulation equations are integrated later ; it guarantees that nonlinear terms of $g(t)$ do not contribute to the dynamics of $\{\lambda_1(t), \dots, \lambda_J\}$. Moreover, the modified profile U was introduced to achieve this L^2 in time control. Due to its construction, μ_j's also turn out to enjoy some integrability in time.

4.5. **Link with the initial problem.** The open subject that was initially presented is now rendered inconsequential by this new result. Indeed, all initial conditions within $\mathcal{E}_{\ell,\ell+2}$, with ℓ an integer, yield a two-bubble solution with global existence in time.

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