M1 Internship report: On an equivariant reduction of Yang-Mills

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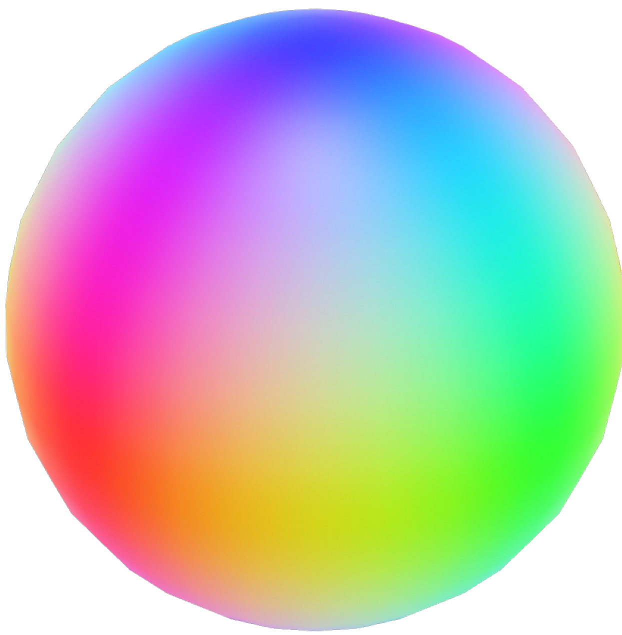

Acknowledgements

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I also want to warmly thank Roland Donninger for giving me the problem I worked on and for supervising me. Each of our meetings instantly cleared out any problems I had. The only regret I had was that we could not have had more of them.

On another note, I want to thank some of my office neighbors Andras Bonk, Frederick Moscatelli and Matthias Ostermann as well as my flatmate Matteo Holzmann for giving me a taste of Austrian culture.

On a musical note, I want to thank mikrokultur, the semmelweissklinik and all the other musicians with whom I had the chance to play during my stay.

Finally, I want to acknowledge the beauty of the Schneeberg mountains for helping me think while hiking my week-ends away.

Contents

1 The Internship itself

Although I expect each internship to be its own unique experience, mine was in a very peculiar setting. The subject I was given by Professor Donninger was at the interface between his specialty (the stability of partial differential equations) and differential geometry, a domain where I was more on my own. The problem was this: an ansatz for the Yang-Mills equations on $SO(n)$ called the "equivariant" ansatz is cited to a lost Romanian paper of 1982. It is never proven nor explained how this ansatz is "equivariant". The aim of this internship was therefore to find a coherent definition of "equivariant" with the ansatz and try to generalize the ansatz.

My internship was divided in three phases:

The first phase was when I looked at the literature on the subject and took a course at the University of Vienna on Principal Fiber Bundles to understand and ask questions. Since I arrived at the beginning of the semester, I was able to follow the course from the top. I also managed to find a copy of the Romanian paper even if I had to go back to Paris to get it.

The second phase was what I expected mathematical research to look like. I first found a coherent algebraic reason for the ansatz to be considered "equivariant". Then I looked for an abstract, and theoretical reason to justify the algebraic simplifications. This allowed me to generalize the ansatz to other Lie groups. During this phase I was surprised at the quantity of computations I had to do. Fortunately I discovered mathematica to help check them near the end of the period.

The third and final phase was one of redacting and writing a research paper. Forunately enough, since I do not like losing myself my own handwritten notes, I had been typing my finding on latex since the beginning. This phase was more of a discovery of all that is needed to publish a paper.

All throughout this internship I had a few obstacles. The first of which was that M1 research internship are uncommon in Austria. At first, I had no office and no access to books at the university library. Fortunately enough I met another student of ENS who had an office and showed me I could ask for one. I still did not get access to books at the university library. Furthermore, my supervisor was already leading a research group, hosting several student courses, director of study at the university of Vienna and hosting a thematic program at the Erwin Schrodinger Institute while having me as an intern. On the one side, this allowed me to follow the thematic program and know about many courses which could be of interest for my problem. On the other side it meant that it could sometimes be difficult to schedule a meeting with my supervisor.

This did not mean I was on my own. I shared my office with four PhD students who were very nice to me. I also attended several talks part of a thematic program on the stability of non-linear waves and general relativity at the Erwin Schrodinger Institute. This part of the internship allowed me to interact with many of the leading mathematicians in the field such as Jérémie Szeftel and Pierre Germain.

In the end, I was able to reduce Yang-Mills equations to non-linear wave equations for the Lie groups $SO^+(p,q)$, $Spin^+(p,q)$ and $SU(n)$ for any base manifolds with the demanded symmetry.

2 A Mathematical Introduction to Yang-Mills theory and a notion of equivariant connections

In this section I selected extracts from the paper I submitted to introduce the necessary context before reducing the Yang-Mills equations in special cases.

2.1 Principal Fiber Bundles

Definition 2.1 (Principal Fiber Bundles)**.** *Let G be a Lie group, M and P manifolds with $\pi : P \longrightarrow M$ smooth. The tuple (P, π, M, G) is called a G-**principal fiber bundle** over M if*

1. *G is a Lie transformation group on P. The action is free and simply transitive on each fibers.*
2. *There exists a bundle atlas $\{(U_i, \phi_i)\}$ consisting of G-equivariant bundle charts. In other words:*
   - *(a) The $\{U_i\}$ cover M.*
   - *(b) $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$ is a diffeomorphism.*
   - *(c) The following diagram commutes:*

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times G \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p_1} & \\ U_i & & \end{array}$$

   - *(d) $\phi_i(p \cdot g) = \phi(p) \cdot g$, where g acts on $U_i \times G$ by $(x, a) \cdot g := (x, a \cdot g)$.*

Now that we have a space where our supplementary information can live, we need the right functions to get it out of our manifold M.

Definition 2.2 (Sections)**.** *We call **section** a smooth function $s : M \longrightarrow P$ such that $\pi \circ s = Id_M$.*

We can now switch points of views and define principal bundles from sections:

Theorem 2.3 (Defining a Principal Bundle from sections)**.** *Let G be a Lie group, $\pi : P \longrightarrow M$ a smooth function, then (P, π, M, G) is a principal fiber bundle if and only if*

1. *G is a Lie transformation group that acts simply and freely on the fibers.*
2. *There exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M and local sections $s : U_i \longrightarrow P$ for all $i \in I$.*

Remark. *The bundle atlas functions are defined by the relation*

$$\begin{aligned} \phi_i^{-1} = \psi_{s_i} : U_i \times G &\longrightarrow \pi^{-1}(U_i) \\ (x, g) &\longmapsto s_i(x) \cdot g \end{aligned}$$

Let us look at an important example:

Example 2.1 (The frame bundle of a manifold)**.** *This example is taken from [6].*

Let M be an n-dimensional manifold then we can set

$$GL_x(M) := \{\nu_x = (\nu_1, ..., \nu_n) | \nu_x \text{ is a basis of } T_xM\}$$

Then we define:

$$GL(M) := \amalg_{x \in M} GL_x(M)$$

and

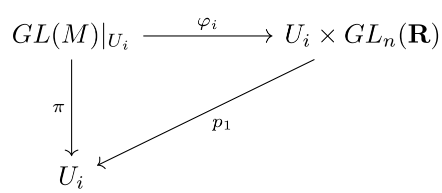
$$\begin{aligned} \pi : GL(M) &\longrightarrow M \\ \nu_x &\longmapsto x \end{aligned}$$

Then $GL_n(\mathbf{R})$ acts on the right with $\nu_x \cdot A = (\sum_{i=1}^n \nu_i A_{i,1}, ..., \sum_{i=1}^n \nu_i A_{i,n})$ which is just the left multiplication.

We then need to get a manifold structure. To do so, take $\{U_i, \phi_i = (x_1, ..., x_n)\}$ a bundle chart of M. For each $\nu_x \in GL(M)|_{U_i}$ there exists a unique $A(x) \in GL_n(\mathbf{R})$ such that $\nu_x = (e_1, ..., e_n) \cdot A(x)$. We then define

$$\begin{aligned} \varphi_i : GL(M)|_{U_i} &\longrightarrow U_i \times GL_n(\mathbf{R}) \\ \nu_x &\longmapsto (x, A(x)) \end{aligned}$$

φ_i is then bijective and we have the following commuting diagram:



We can then verify smoothness to see that we have constructed a $(GL(M), \pi, M, GL_n(\mathbf{R}))$ principal fiber bundle.

Since we will interest ourselves towards Yang-Mills we can look at the frames we are interested in which are the ones invariant by our metric and direct.

Example 2.2 (Metric induced frame bundle)**.** *Given (M, g) a semi-Riemannian manifold of signature (p, q) then set*

$$SO_x(M) = \{\nu_x \in GL_x(M) | g_x(\nu_i, \nu_j) = I_{p,q}, \text{ and } det(\nu_x) = 1\}$$

We analoguously still have a principal fiber bundle: $(SO(M), \pi|_{SO(M)}, M, SO(p, q))$.

2.2 Principal Connection and Connection 1-form

We can now introduce the notion of a connection on a Principal bundle. We want to be able to transport knowledge about one point on the bundle to another.

2.2.1 A geometric distribution

Definition 2.4 (Geometric distribution)**.** *Given a manifold N, a **geometric distribution of dimension** r is a map $\mathfrak{E} : x \mapsto E_x \subset T_x N$ such that for all $x \in N$ there exists a neighborhood U and smooth vector fields on U, $X_1, ..., X_r$ such that $\forall y \in U$, $E_y = span(X_1(y), ..., X_r(y))$.*

This is a sort of generalization of a vector field to the r-th dimension.

We now place ourselves on a principal bundle (P, π, M, G), with the right action defined as above.

Definition 2.5 (Vertical Tangent spaces)**.** *The most basic example of a geometric distribution on our bundle is given by taking its fibers $P_x = \pi^{-1}(x)$.*

We can now go even further, since they are regular submanifolds of P. For any point $u \in P_x$ we can define $Tv_u P := T_u(P_x)$. This is called the vertical tangent space in u.

Proposition 2.6 (on Vertical Tangent Spaces)**.** *We have the following properties:*

1. *$Tv_u = ker(T_u \pi)$.*
2. *Tv_u is linearly isomorphic to $\mathfrak{g}$ with,*

$$\Phi_u : X \mapsto \tilde{X}(u) = \frac{d}{dt}|_{t=0}(u \cdot (exp(tX)))$$

3. *For all $X \in \mathfrak{g}$ we have the following relation with the flow of $\tilde{X}$,*

$$Fl_t^{\tilde{X}}(u) = u \cdot exp(tX) = R_{exp(tX)}(u)$$

Definition 2.7 (Horizontal Tangent space)**.** *Any subspace $Th_u P \subset T_u P$ that is complimentary to $Tv_u P$ is called an **Horizontal Tangent space of** u.*

2.2.2 Principal Connection and Parallel transport

Definition 2.8 (Principal Connection)**.** *A **Connection** on a principal fiber bundle is a geometric distribution of horizontal spaces, that is right invariant. In other words it is a geometric distribution $Th : P \ni u \mapsto Th_u P \subset T_u P$ such that for all $g \in G$ we have*

$$T_u R_g(Th_u P) = Th_{u \cdot g} P$$

This does not look close to the connections we are used to in physics but it is the right notion in this context since it allows us to define parallel transport. But first we need to lift our vector fields in a way that agrees with our connection.

Definition 2.9 (Horizontal Lift of a vector field)**.** *Let $X \in \mathfrak{X}(M)$. A vector field X^* is called an **Horizontal Lift of** X if for all $p \in P$ we have:*

1. *$X^*(p) \in Th_p P$, and*
2. *$T_p \pi(X^*(p)) = X(\pi(p))$*

Theorem 2.10 (Existence and unicity of horizontal lifts)**.** *For any $X \in \mathfrak{X}(M)$ there exists a unique horizontal lift $X^* \in \mathfrak{X}(P)$. Moreover X^* is right invariant.*

Definition 2.11 (Horizontal Lift of a path)**.** *A path* $\gamma^* : I \longrightarrow P$ *is called* ***the horizontal lift*** *of a path* $\gamma : I \longrightarrow M$ *if*

1. $\pi(\gamma^*(t)) = \gamma(t) \, \forall t \in I$ *and*

2. $\frac{d}{dt}\gamma^*$ *is horizontal for all* $t \in I$

Theorem 2.12 (Existence and unicity of horizontal lifts)**.** *Let* $\gamma : I \longrightarrow M$ *be a path in* M*,* $t_0 \in I$ *and* $u \in P_{\gamma(t_0)}$*. Then there exists a unique horizontal lift* γ_u^* *of* γ *such that* $\gamma_u^*(t_0) = u$*.*

We can now define a unique parallel transport associated to our connection.

Definition 2.13 (Parallel Transport)**.** *Given a connection* Th*, a path* $\gamma : [a, b] \longrightarrow M$ *and* $u \in P_{\gamma(a)}$ *the* ***parallel transport of*** u ***along*** γ ***with respect to*** Th *is*

$$P_\gamma^{Th}(u) = \gamma_u^*(b)$$

Now that we have seen that we have the right notion of connection, let us encode it in a much simpler way.

2.2.3 Connection 1-form

Definition 2.14 (Connection 1-form)**.** *A* ***connection*** 1***-form*** *on a principal fiber bundle is a* 1*-form* $A \in \Omega^1(P, \mathfrak{g})$ *that satisfies*

1. $R_g^* A = Ad(g^{-1}) \circ A$ *for all* $g \in G$

2. $A(\tilde{X}) = X$ *for all* $X \in \mathfrak{g}$

(As a reminder $(R_g^* A)_u = A_{u \cdot g} \circ T_u R_g$*)*

Now we can show that there is a correspondence.

Theorem 2.15 (Connection and 1-form correspondence)**.** *Connections and connection* 1*-forms on a principal bundle are in bijective correspondence,*

1. *Given a connection* Th *then we can define a connection* 1*-form by*

$$A_u(\tilde{X}(u) \oplus Y_h) := \tilde{X}(u) \, \forall u \in P, \, X \in \mathfrak{g}, \, Y_h \in Th_u P$$

2. *Given a connection* 1*-form* A *we can define a connection* Th *by*

$$Th_u P := ker(A_u) \, \forall u \in P$$

We can finally find an object of the form requested in [3].

Definition 2.16 (The local connection form or Gauge field)**.** *Given a connection form* $A \in \Omega^1(P, \mathfrak{g})$ *and a local section* $s : U \longrightarrow P$ *we define the* ***local connection form or Gauge field***

$$A^s := A_{\circ s} \circ Ts \in \Omega^1(U, \mathfrak{g})$$

To see that this object is useful we'll consider how one can characterize a connection form from local forms.

Theorem 2.17 (Local characterization of the connection form)**.** *Given a covering* $\{(s_i, U_i)\}$ *of* M *with the connection functions* g_{ij} *such that* $s_i(x) = s_j(x) \cdot g_{ij}(x)$ *and* $\mu_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{g})$ *such that* $\mu_{ij}(X) = TL_{g_{ij}^{-1}(x)}(Tg_{ij}(X))$ *we can state:*

If $\{A_i \in \Omega^1(U_i, \mathfrak{g})\}_i$ *is a family of* 1*-forms such that whenever* $U_i \cap U_j \neq$ *we have*

$$A_i = Ad(g_{ij}^{-1}) \circ A_j + \mu_{ij}$$

then there exist a unique connection form $A \in \Omega^1(P, \mathfrak{g})$ *such that* $A^{s_i} = A_i$ *for all* i*.*

Remark. *Notice that since* $g_{ij} = p_2 \circ \psi_{s_j}^{-1} \circ s_i$ *then* g_{ij} *is smooth.*

Example 2.3 (Bijectivity with local covariant derivatives)**.** *Let us come back on example 2.1 with the frame bundle* $GL(M)$*. If we take* $s = (s_1, ..., s_n)$ *a local section of* $GL(M)$ *on* U*, which can be seen as a local frame of coordinate. Then the two following applications are inverse of one another and allow for a global bijectivity between connections on the frame bundle and covariant derivatives or connections on a manifold:*

$$\phi_s : \Omega^1(U, \mathfrak{gl}_n(\mathbf{R})) \longrightarrow \{\textit{Covariant Derivatives on } U\}$$
$$B = \sum_{i,j=1}^{n} \omega_{i,j}\, e_{i,j} \longmapsto \nabla \textit{ such that } \nabla_X s_k := \sum_{i=1}^{n} \omega_{i,k}(X)\, s_i$$

$$\psi_s : \{\textit{Covariant Derivatives on } U\} \longrightarrow \Omega^1(U, \mathfrak{gl}_n(\mathbf{R}))$$
$$\nabla \textit{ such that } \nabla s_i = \sum_{j=1}^{n} \omega_{j,i} \otimes s_j \longmapsto \sum_{i,j=1}^{n} \omega_{i,j}\, e_{i,j}$$

Furthermore, one can show that this bijectivity also holds for connections on $SO(M)$ *and metric-induced Connections. This motivates us to look local* $SO(p,q)$*-connections rather than global* $SO(g)$ *connections.*

Now that we have found the objects we're looking for, let's try to characterize them even more in specific situations.

2.3 Simplifications to the local connection form

To give an abstract justification for the notion of equivariance, we'll now place ourselves on a small enough U_i with a given s_i such that we're on a chart of the form $(U_i, \phi_i = (x_1, ..., x_n))$. To simplify, we'll denote A^{s_i} as $B \in \Omega^1(U_i, \mathfrak{g})$. Further conditions will add on as we progress throughout the section.

2.3.1 General simplifications

Proposition 2.18 (First simplification). *If we write $B_\mu = B(\partial_\mu)$ and we have $f_1, ..., f_r$ a basis of $\mathfrak{g}$ then we can write*

$$\begin{aligned} B &= \sum_{a=1}^{r} \sum_{\mu=1}^{n} B_\mu^a \, f_a \, dx^\mu \\ &= \sum_{\mu=1}^{n} B_\mu \, dx^\mu \\ &= \sum_{a=1}^{r} B^a \, f_a \end{aligned}$$

We will now suppose further structure with a right transformation group action of G, r_g on M. We will take inspiration from [5] to have the same kind of effective simplification.

Definition 2.19 (Adapted Equivariant section). *Given a connection* 1*-form A and H a subgroup of G. We will say that a section $s : U \longrightarrow P$ is an **adapted H-equivariant section** of A if for all $g \in H$*

$$A^{R_g \circ s} = A^{s \circ r_g}$$

We will often shorten this to say that s or A is H-equivariant.

Remark. *This is the case when $R_g \circ s = s \circ r_g$ but this would imply $r_g = Id$ by applying π on both sides.*

Remark. *This represents a connection which preserves the additional structure on the base manifold, similarly to Levi-Civita connections who preserve the metric structure on the base Manifold.*

We then have the following simplification

Proposition 2.20 (Equivariant simplification). *Given an equivariant section bundle, the local* 1*-forms $B = A^s$ satisfy:*

$$Ad(g^{-1})(B(x)) = B(x \cdot g) \circ T_x r_g$$

Proof. If we just apply s^* to the original equation we get for all $x \in M$:

$$\begin{aligned} Ad(g^{-1})(B(x)) &= (R_g^* A)^s(x) = A_{s(x)\cdot g} \circ T_{s(x)} R_g \circ T_x s \\ &= A_{R_g \circ s(x)} \circ T_x(R_g \circ s) = A^{R_g \circ s} = A^{s \circ r_g} \\ &= A_{s(x \cdot g)} \circ T_x(s \circ r_g) = A_{s(x\cdot g)} \circ T_{x\cdot g} s \circ T_x r_g \\ &= A^s(x \cdot g) \circ T_x r_g = B(x \cdot g) \circ T_x r_g \end{aligned}$$

□

2.3.2 Matrix group simplifications

We now place ourselves in the case of $G \subset GL(m, \mathbf{K})$. We then have

Proposition 2.21 (Matrix simplifications). *By linearity we have for all $X \in \mathfrak{g}$*

1. *$TL_g X = g \cdot X$ when L_g is constant*

2. $Ad(g)(X) = g \cdot X \cdot g^{-1}$

This allows us to rewrite the equivariance condition 1 of 2.14 as, for all $g \in G$

$$\begin{aligned}\sum_{\mu=1}^{n} g^{-1} \cdot B_\mu(x) \cdot g \, dx^\mu &= A(s_i(x) \cdot g) \circ T_{s_i(x)\cdot g} R_g \circ T_x s_i \\ &= A(s_i(x) \cdot g) \circ T_x(R_g \circ s_i)\end{aligned}$$

But now we want to get rid of A and come back to B. To do so **we'll now suppose** $m = n$. This means that we can define a local right action on M by

$$\forall g \in G,\, x = (x_1, ..., x_n) \in U_i,\, x \cdot g = g^{-1} \cdot x$$

Where $g^{-1} \cdot x$ is the comparable to the matrix multiplication in coordinates.

We'll now suppose that the section s_i we consider is H-equivariant.

Proposition 2.22 (Equivariant section simplification)**.** *If we suppose the section s_i to be H-equivariant then the condition 1 gives for all $g \in H$:*

$$B(L_g(x)) = Ad(g)(B(x)) \circ (T_x L_g)^{-1}$$

Proof. Using the simplifications in 2.21 and the definition 2.20 we have

$$Ad(g^{-1})(B(x)) = B(x \cdot g) \circ T_x r_g = B(x \cdot g) \circ T_x L_{g^{-1}}$$

We then switch g with g^{-1} and rearrange to get the wanted form. □

Corollary 2.22.1 (Equivariant section simplification with constant left action)**.** *If we suppose the section s_i to be H-equivariant and the group actions to be constant, then condition 1 becomes B is **algebraically H-equivariant**.*

Example 2.4. *To better interpret this, let us place ourselves in the context of an $SO^+(1,2)$ bundle such that locally M looks like time-like coordinate space. We will consider a path γ in M.*

Then notice that with the correspondance seen in previous example we can define a local covariant derivative from B which leads to the same corresponding parallel transport. Furthermore, given a local chart $\phi_U : U \to \mathbf{R}^{2,1}$ we get the following relation if ∇ is g-equivariant:

$$\nabla_{X\cdot g}^{Ad(g)(B)}((\phi_U \circ \gamma(s)) \cdot g) = \nabla_X^B(\phi_U \circ \gamma(s))$$

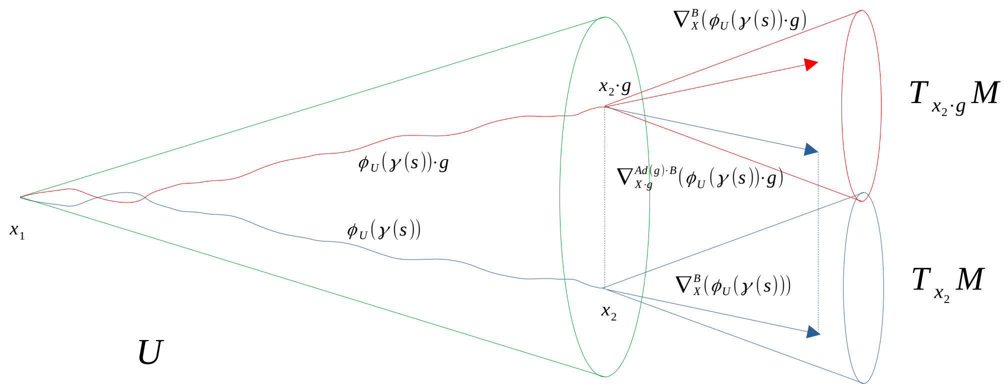
Which can be seen through the following:

To evaluate force, and fields from potentials, which are represented by our connections, we need to look at their induced curvature.

2.3.3 Curvature

To introduce curvature, we need to see that since our connection gives rise to parallel transport, it also allows for a derivative which is the one we will use on the connection form itself to get the curvature. We recommend looking at [6] and [10] for all the details. We will simplify in our context to go faster.

Figure 1: g-equivariant $SO^+(1,2)$ connection (in color)



Definition 2.23 (Exterior derivative induced by a connection 1-form)**.** *Our connection* 1*-form induces an* ***exterior derivative*** *defined by:*

$$d_B\omega = d\omega + \frac{1}{2}[B \wedge \omega]$$

Where the $dx^\mu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$ *component of* $[B \wedge \omega]$ *is given by the Lie algebra product acting on the components* $[B_\mu, \omega_{\mu_1,...,\mu_k}]$.

Proposition 2.24 (Curvature 2-form)**.** *The curvature* 2*-form through a section bundle covering of* M *or field strength,* $F \in \Omega^2(M, \mathfrak{g})$ *is related to its principal connection form by*

$$\begin{aligned} F_{\alpha,\beta} &= d_B B \\ &= \frac{\partial B_\beta}{\partial x^\alpha} - \frac{\partial B_\alpha}{\partial x^\beta} + [B_\alpha, B_\beta] \end{aligned}$$

Where $[.,.]$ *is the Lie algebra bracket.*

2.4 The right left action

We want to define an appropriate local action from a subgroup of $SO^+(p,q)$ to M. First, we need to know where we want to land. This section takes a lot from chapter three of [8]. Let us fix $O \in M$ and suppose M to be connected.

Definition 2.25 (Group of isometries)**.** *Let us call (M, g)'s **group of isometries** the following:*

$$SO^+(g) := \{\textit{diffeomorphisms } \varphi : M \to M \\ \varphi(O) = O,\ \varphi_* g = g,\ \varphi \textit{ preserves time orientation}\}$$

Intuitively we want our group action to land here. To do so we turn ourselves toward killing vectors. Let us reintroduce killing vectors in a way that is more adapted to our context.

Since we are looking for a Lie Group, there must exists smooth paths φ_ϵ landing in $SO^+(g)$. Furthermore, such smooth paths are charaterized by a fixed point (here O) and their derivative:

$$\frac{d}{d\epsilon}\varphi_\epsilon(x) = \xi(\varphi_\epsilon(x))$$

We suppose their derivative to only depend on the value of the function because that is how it works with Lie Groups. We therefore interest ourselves in the flows of vector fields ξ such that they land in $SO^+(g)$.

Let us begin with a few non trivial properties on such flows.

Proposition 2.26 (Flows)**.** *Given ξ a vector field of M and $x = (x_1, ..., x_n)$ a local atlas we have:*

1. $Fl^\xi_{\alpha\,\epsilon} = Fl^{\alpha\,\xi}_\epsilon$
2. $d_p Fl^\xi_\epsilon = Fl_\epsilon^{d_{Fl^\xi_\epsilon(p)}\xi}$
3. *In local coordinates:* $Jac_x Fl^\xi_\epsilon = Fl_\epsilon^{Jac_{Fl^\xi_\epsilon(x)}\xi}$

Proof. Firstly, notice that given $p \in M$:

$$\frac{d}{d\epsilon}Fl^\xi_{\alpha\,\epsilon}(p) = \alpha\,\xi(Fl^\xi_{\alpha\,\epsilon}(p))$$

Secondly, thanks to Cauchy, let us compute:

$$\begin{aligned}\frac{d}{d\epsilon}d_p Fl^\xi_\epsilon &= d_p(\frac{d}{d\epsilon}Fl^\xi_\epsilon) \\ &= d_p(\xi(Fl^\xi_\epsilon)) = (d_{Fl^\xi_\epsilon(p)}\xi) \circ (d_p Fl^\xi_\epsilon)\end{aligned}$$

Which is exactly what we wanted to show. □

With this, we can see how such vectors that conserve the metric look like:

Proposition 2.27 (Killing Vectors)**.** *We call a vector field ξ of M a non translating **killing vector** if its flow lands in $SO^+(g)$. In local coordinates, this is equivalent to $\xi(O) = 0$ and*

$$(Jac_x\xi)^T \cdot g_x + g_x \cdot (Jac_x\xi) + d_x g(\xi(x)) = 0$$

Or as physicist would write it:

$$\nabla_\rho\xi_\nu + \nabla_\nu\xi_\rho = 0$$

We will note $\mathfrak{k}_O$ the set of such vector fields.

Proof. Firstly, since $\xi(O) = 0$, by definition, $\frac{d}{d\epsilon}Fl^{\xi}_{\epsilon}(O) = 0$ and therefore $Fl^{\xi}_{\epsilon}(O) = O$.

Furthermore, since $d_pFl^{\xi}_0 = Id \in SO^+(g)$ and that the time preservation condition derives from the smoothness in ϵ of the flow. We just have to check that $(Fl^{\xi}_{\epsilon})_*g = g$. To do so we only need to check what happens near $\epsilon = 0$ at all points because:

$$(Fl^{\xi}_{\epsilon+\eta})_*g = (Fl^{\xi}_{\epsilon})_*(Fl^{\xi}_{\eta})_*g$$

Thanks to basic properties of flows. We then see that we need:

$$\frac{d}{d\epsilon}((Fl^{\xi}_{\epsilon})^*g) = 0$$

Which is equivalent at $\epsilon = 0$ in local coordinates to :

$$(Jac_x\xi)^T \cdot g_x + g_x \cdot (Jac_x\xi) + d_xg(\xi(x)) = 0$$

□

In the following we will need proposition 62 of chapter 3 from [8].

Proposition 2.28. *If M is connected and ϕ and ψ are in $SO^+(g)$ such that:*

$$d_O\phi = d_O\psi$$

then $\phi = \psi$.

Theorem 2.29 (The Special Orthornormal Orthochronous Group of g)**.** *If we have fixed tetrads σ such that $g = \sigma^T \cdot I_{p,q} \cdot \sigma$. Then there exists a Lie subgroup K_O of $SO^+(p,q)$ which we will call* ***special orthornormal orthochronous group of*** *g such that:*

$$\begin{aligned}\psi_O : \mathfrak{k}_O &\longrightarrow K\\ \xi &\longmapsto \sigma(O) \cdot Fl_1^{-Jac_O\xi} \cdot \sigma(O)^{-1} = \sigma(O) \cdot exp(-Jac_O\xi) \cdot \sigma(O)^{-1}\end{aligned}$$

defines a surjective exponential for K_O.

Proof. Let us do this step by step:

1. First, let us build an injective Lie algebra morphism:

$$\begin{aligned}\varphi_O : \mathfrak{k}_O &\longrightarrow \mathfrak{so}(p,q)\\ \xi &\longmapsto -\sigma(O) \cdot Jac_O\xi \cdot \sigma(O)^{-1}\end{aligned}$$

We can verify it lands in $\mathfrak{so}(p,q)$ by using the killing vector relation in O which gives us:

$$\begin{aligned}&(Jac_O\xi)^T \cdot g_O + g_O \cdot (Jac_O\xi) + d_Og(\xi(O)) = 0\\ \Leftrightarrow&(Jac_O\xi)^T \cdot g_O + g_O \cdot (Jac_O\xi) = 0\end{aligned}$$

Since $\xi(O) = 0$. We then use $g_O = \sigma(O)^T \cdot I_{p,q} \cdot \sigma(O)$ to get the wanted result.

This is also injective due to since it fixes the derivatives of ξ at O.

We just have to check that it commutes with the Lie algebra product: Notice that one can write:

$$[\xi_1, \xi_2]_p = d_p\xi_2(\xi_1(p)) - d_p\xi_1(\xi_2(p))$$

and therefore:

$$d_p[\xi_1, \xi_2] = d_p\xi_2 \circ d_p\xi_1 - d_p\xi_1 \circ d_p\xi_2 + d_p^2\xi_2(\xi_1(p), .) - d_p^2\xi_1(\xi_2(p), .)$$

Meaning that since $\xi_1(O) = \xi_2(O) = 0$:

$$\begin{aligned} Jac_O([\xi_1, \xi_2]) &= Jac_O\xi_2 \cdot Jac_O\xi_1 - Jac_O\xi_1 \cdot Jac_O\xi_2 \\ &= -[Jac_O\xi_1, \cdot Jac_O\xi_2] \end{aligned}$$

and therefore :

$$\varphi_O([\xi_1, \xi_2]) = [\varphi_O(\xi_1), \varphi_O(\xi_2)]$$

.

2. Secondly, we can notice that $\psi_O = exp \circ \varphi_O$ and since, as shown in [1], the exponential is bijective from $\mathfrak{so}(p, q)$ to $SO^+(p, q)$ we have proven the theorem.

□

Remark. *The minus sign in the application in necessary for multiplication order to coincide.*

K represents the symmetries that are present in the metric g. We can now define the action of K_O on (M, g).

Proposition 2.30. *The special orthonormal orthochronous group of g locally acts on the right of M with:*

$$\begin{aligned} R : M \times K_O &\longrightarrow M \\ (x, \Lambda) &\longmapsto Fl_1^{\psi_O^{-1}(\Lambda)}(x) \end{aligned}$$

Remark. *If we are placed in Minkowski space-time, this coincides with $L_{\Lambda^{-1}}$.*

Proof. All we have to check is that $Fl_1^{\psi_O^{-1}(\Lambda_1)} \circ Fl_1^{\psi_O^{-1}(\Lambda_2)} = Fl_1^{\psi_O^{-1}(\Lambda_2 \cdot \Lambda_1)}$. To do so we will use proposition 62 of chapter 3 from [8] which says that if two local isometries on a connected manifold have the same values in at a point and in their derivatives then they are equal. This is the case and we therefore have a well defined right action. □

Remark (Another point of view)**.** *Using proposition 62 of [8], we can also see the right action R_Λ as the only isometry such that $Jac_O R_\Lambda = \sigma(O)^{-1} \cdot \Lambda^{-1} \cdot \sigma(O)$. We prefer to pass through killing vectors as it gives us an explicit Lie algebra of our connected Lie group K_O.*

With the action in mind we can now consider G-equivariant connections where G is a subgroup of K_O.

Theorem 2.31 (The general semi-Riemannian G-equivariant connection)**.** *Given a connected Riemannian manifold (M, g) and (P, π, G) a principal fiber bundle where G is a Lie subgroup of K_O.*

With the right action we have defined, a local description B of a principal connection that has a G-equivariant section bundle atlas such that $x(O) = 0$ satisfies for all $x \in U$ and $\Lambda \in G$:

$$B(L_\Lambda(x)) = Ad(\Lambda)(B(x)) \circ (T_x L_\Lambda)^{-1}$$

And if we define $\tilde{\Lambda}$ by $T_x L_\Lambda = \sigma(L_\Lambda(x))^{-1} \cdot \tilde{\Lambda}(x) \cdot \sigma(x)$ where $\tilde{\Lambda}(0) = \Lambda$. We then have the following for all x:

1. $\tilde{\Lambda}(x) \in SO^+(p, q)$.
2. $(\Lambda_1 \cdot \Lambda_2)^\sim(x) = \tilde{\Lambda}_1(L_{\Lambda_2}(x)) \cdot \tilde{\Lambda}_2(x)$

And we can take the more appropriate object:

$$\omega_\mu(x) = \sum_{\nu=1}^{n} [\sigma^{-1}(x)]_{\nu,\mu} \, B_\nu(x)$$

which satisfies

$$\omega_\mu(L_\Lambda(x)) = \sum_{\nu=1}^{n} [\tilde{\Lambda}(x)^{-1}]_{\nu,\mu} \, Ad(\Lambda)(\omega(x))_\nu$$

Remark. *In the special cases where $\tilde{\Lambda}$ is constant, we come back to an algebraically K_O-equivariant 1-form.*

With that in mind we can apply our symmetries to reduce Yang-Mills equations.

2.5 Yang-Mills

Yang-Mills equations are a special case of Euler-Lagrange equations, let us introduce the necessary lagrangian formalism before going any further.

2.5.1 Lagrangian Formalism

Firstly we need to define what a Lagrangian is in our context. This mixes the calculus of variation and differential geometry.

Definition 2.32 (Lagrangian)**.** *A **Lagrangian** on an n-Manifold M equipped with a principal bundle (P, π, G) is an n-form L depending on the connection (or connection 1-form) and its curvature (or curvature matrix) composed with a fixed set of local sections. ($L = L(B)$)*

Now we'll associate an action to said Lagrangian

Definition 2.33 (Action)**.** *Given a Lagrangian L, its **action** is defined as*

$$S = \int_M L$$

Where we choose the volume form accordingly.

This allows us to define the Lagrangian problem.

Problem 2.34 (Lagrangian optimisation or Principle of Least Action). *What connection gives us minimal action ?*

Definition 2.35 (Euler Lagrange Equations). *The differential equations that arise while solving the Lagrangian problem are called the* ***Euler-Lagrange equations****.*

As is often the case with variational calculus, we will need something to act as an integration by part. If we fix a metric on our manifold then we can look at the following operator.

2.5.2 The Hodge Operator

First we'll introduce a duality between k-forms spaces as seen in [4],[10] and [6].

Definition 2.36 (Hodge Operator). *If we take $\omega \in \Omega^n(M)$ and $\langle .|. \rangle$ an inner product on $\Omega^k(M)$ then we can define for $\alpha \in \Omega^k(M)$*

$$*\alpha \in \Omega^{n-k}(M)$$

Such that for all $\beta \in \Omega^k(M)$ we have $\beta \wedge (*\alpha) = \langle \alpha | \beta \rangle \, \omega$

We see that we therefore need an inner product $\langle .|. \rangle$ on $\Omega(M)$.

Proposition 2.37. *An inner product $\langle .|. \rangle$ on $\Omega^1(M)$ induces one on $\Omega^k(M)$ for all k.*

Proof. We can just pick the inner product defined such that

$$\langle dx_I | dx_J \rangle = \sum_{\sigma \in \Sigma_k} \varepsilon(\sigma) \langle dx_{i_1} | dx_{j_{\sigma(1)}} \rangle ... \langle dx_{i_k} | dx_{j_{\sigma(k)}} \rangle$$

□

Proposition 2.38 (On Pseudo-Riemanninan Manifolds). *Given a metric $g = g_{\mu,\nu}$ on M, if we write its inverse $g^{\mu,\nu}$ then it induces an inner product on $\Omega^1(M)$ by:*

$$\langle dx_i | dx_j \rangle = g^{i,j}$$

Proposition 2.39 (Explicit formula for the Hodge Operator). *If we are given a metric g and take $\omega = \sqrt{|g|} \, dx^1 \wedge ... \wedge dx^n$ (the associated volume form) then the Hodge operator given by the inner product induced by g is given by:*

$$*(dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_k}) = \frac{\sqrt{|g|}}{(n-k)!} \sum_{\nu_1,...,\nu_n=1}^{n} \epsilon_{\nu_1,...,\nu_n} \, g^{\nu_1,\mu_1} ... g^{\nu_k,\mu_k} \, dx^{\nu_{k+1}} \wedge ... \wedge dx^{\nu_n}$$

Proof. Simply see that given $dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_k} \in \Omega(M, \mathfrak{g})$ we get:

$$\begin{aligned} dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_k} \wedge *(dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_k}) &= \frac{\sqrt{|g|}}{(n-k)!} \sum_{\nu_{k+1},...,\nu_n=1}^{n} \epsilon_{\alpha_1,...\alpha_k,\nu_{k+1},\nu_n} \, g^{\alpha_1,\mu_1} ... g^{\alpha_k,\mu_k} \\ &\quad dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_k} \wedge dx^{\nu_{k+1}} \wedge ... \wedge dx^{\nu_n} \\ &= \sqrt{|g|} \, g^{\alpha_1,\mu_1} ... g^{\alpha_k,\mu_k} \, dx^1 \wedge ... \wedge dx^n \end{aligned}$$

□

We now can see how Lie algebra morphisms commute with the Hodge and differential:

Proposition 2.40. *If $\lambda : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a smooth Lie algebra morphism then, taking B a local description of a connection of a G principle bundle we have the following:*

- $* \circ \lambda = \lambda \circ *$
- $\lambda \circ d_B = d_{\lambda \circ B} \circ \lambda$

Proof. The first claim of the proposition is immediate as the Hodge operator does not affect the lie algebra part of 1-forms and λ only affects the lie algebra part.

To prove the second claim, take $\omega \in \Omega(M, \mathfrak{g})$, we have:

$$\begin{aligned} \lambda \circ d_B \omega &= \lambda \circ (d(\omega) + \frac{1}{2}[B \wedge \omega]) \\ &= d(\lambda \circ \omega) + \frac{1}{2}[(\lambda \circ B) \wedge (\lambda \circ \omega)] \\ &= d_{\lambda \circ B}(\lambda \circ \omega) \end{aligned}$$

□

We are now equipped to look at Yang-Mills equations.

2.5.3 Yang Mills Equations

Definition 2.41 (Yang Mills Lagrangian and Action)**.** *Let (P, π, M, G) be a principal bundle of rank k over an n-dimensional manifold M. Let B be a connection 1-form for P with respect to a section bundle covering. Let F be the corresponding curvature 2-form. Let $\langle .|. \rangle$ be an $Ad(.)$ invariant scalar product on $\Omega^1(M, \mathfrak{g})$. Then* ***the Yang Mills Lagrangian*** *is*

$$L(B) = \frac{1}{4}\langle F|F \rangle$$

Its associated ***Yang Mills Action*** *is*

$$S(B) = \int_M L(B)$$

A connection that is a critical point of the associated Lagrangian problem is called a **Yang Mills connection** and the corresponding Euler-Lagrange equations are called the **Yang Mills equations**.

Example 2.5. *In the case of Matrix groups, one can take the $Ad(.)$ invariant product on $\mathfrak{g}$ to be:*

$$\langle A, B \rangle = Trace(A\,B)$$

This then induces an $Ad(.)$ invariant product on $\Omega^k(M, \mathfrak{g}) = \mathfrak{g} \otimes \Omega^k(M)$ given a metric g with $\langle ., . \rangle \otimes g$.

Remark. *If we compare this to electromagnetism, this corresponds to a free system with no current or charge. Furthermore, we still find that curvature is the force field of the system.*

Theorem 2.42. *A connection* 1*-form* B *is solution to problem 2.34 if and only if:*

$$*d_B * F = 0$$

Proof. See [6] and [10] for the full details but the basic idea is this: If we derive at 0 we get $\frac{dF_{B+\epsilon H}}{d\epsilon} = d_B H$. Then $\int \langle d_A\omega|\eta\rangle = \int *\eta \wedge d_A\omega = \pm \int d_A * \eta \wedge \omega \mp \int d_A(*\eta \wedge \omega) = \pm \int \langle *d_A * \eta, \omega\rangle$ if we suppose M to be compact (we can then generalize). Afterward $\frac{dL(B+\epsilon H)}{d\epsilon} = 2\int \langle F_B, d_B H\rangle = 2\int \langle *d_B * F_B, H\rangle$ for all $H \in \Omega^1(M, \mathfrak{g})$. This needs to be 0 since we are looking for connections minimising the Lagrangian, therefore $*d_B * F_B = 0$. □

We will now follow [3] and [6] to simplify the equation.

Proposition 2.43 (First rewriting)**.** *Yang mills in the context of a section bundle covering with a diagonal metric* g *can be rewritten as, for all* $\alpha \in \{1, ..., n\}$

$$\sum_{\beta=1}^{n} g^{\beta,\beta}\ ([B_\beta, F_{\alpha,\beta}] + \frac{\partial F_{\alpha,\beta}}{\partial x^\beta} + F_{\alpha,\beta}(\frac{\partial}{\partial x^\beta}\log(\sqrt{|g|}\, g^{\beta,\beta}\, g^{\alpha,\alpha})) = 0$$

Proof. We can start by using theorem 2.42 to get $*d_B * F = 0$. Using $F = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} F_{\alpha,\beta}\, dx^\alpha \wedge dx^\beta$. We can compute component by component before adding them all up. Without writing the sums on $\nu_3, ..., \nu_n$ explicitly we get:

$$\begin{aligned} d_B * (F_{\alpha,\beta}\, dx^\alpha \wedge dx^\beta) =& \frac{\sqrt{|g|}}{(n-2)!}\, \epsilon_{\alpha,\beta,\nu_3,...,\nu_n}\, g^{\alpha,\alpha}\, g^{\beta,\beta} \\ & [(\frac{\partial F_{\alpha,\beta}}{\partial x^\alpha} + [B_\alpha, F_{\alpha,\beta}])\, dx^\alpha \wedge dx^{\nu_3} ... \wedge dx^{\nu_n} \\ & + (\frac{\partial F_{\alpha,\beta}}{\partial x^\beta} + [B_\beta, F_{\alpha,\beta}])\, dx^\beta \wedge dx^{\nu_3} ... \wedge dx^{\nu_n}] \\ & + \frac{\epsilon_{\alpha,\beta,\nu_3,...,\nu_n}}{(n-2)!}\, F_{\alpha,\beta} \\ & \times [\frac{\partial}{\partial x^\alpha}(\sqrt{|g|}\, g^{\alpha,\alpha}\, g^{\beta,\beta})\,, dx^\alpha \wedge dx^{\nu_3} ... \wedge dx^{\nu_n} \\ & + \frac{\partial}{\partial x^\beta}(\sqrt{|g|}\, g^{\alpha,\alpha}\, g^{\beta,\beta})\,, dx^\beta \wedge dx^{\nu_3} ... \wedge dx^{\nu_n}] \end{aligned}$$

Then we can compute the Hodge operator of each terms:

$$\begin{aligned} & * (\frac{\sqrt{|g|}}{(n-2)!}\, \epsilon_{\alpha,\beta,\nu_3,...,\nu_n}\, g^{\alpha,\alpha}\, g^{\beta,\beta}\, dx^\alpha \wedge dx^{\nu_3} ... \wedge dx^{\nu_n}) \\ & = (-1)^n\, g^{\alpha,\alpha}\, dx^\beta \end{aligned}$$

$$\begin{aligned} & * (\frac{\epsilon_{\alpha,\beta,\nu_3,...,\nu_n}}{(n-2)!}\, \frac{\partial}{\partial x^\alpha}(\sqrt{|g|}\, g^{\alpha,\alpha}\, g^{\beta,\beta})\, dx^\alpha \wedge dx^{\nu_3} ... \wedge dx^{\nu_n}) \\ & = (-1)^n\, g^{\alpha,\alpha}\, \frac{\partial}{\partial x^\alpha}(\log(\sqrt{|g|}\, g^{\alpha,\alpha}\, g^{\beta,\beta})\, dx^\beta \end{aligned}$$

Summing it all up we get the wanted formula for all α. □

Corollary 2.43.1. *If g is constant and diagonal we get that Yang Mills reduces to*

$$\sum_{\beta=1}^{n} g^{\beta,\beta}\ ([B_\beta, F_{\alpha,\beta}] + \frac{\partial F_{\alpha,\beta}}{\partial x^\beta}) = 0$$

We are now ready to see what our group equivariant simplifications bring to the table in the preprint `https://doi.org/10.48550/arXiv.2406.04171`.

3 Further applications

The simplifications brought forward by the equivariant ansatz can be used in all the places the invariant ansatz [7] was also used:

1. Finding solutions to the Einstein Yang-Mills system as in [9].
2. Studying the stability of such solutions to the Yang-Mills equation as in [2].

Since this ansatz is applicable to the strong interaction through $SU(3)$. An equivariant ansatz could be more encouraging towards quark confinement as it preserves the symmetries of an equivariant quark wave function:

$$|\psi(x,t)\rangle = \phi(r,t)\,\frac{x}{r}$$

which is represented on the cover of the report.

A Notations and glossary

$\mathfrak{X}(M)$	The vector space of vector fields over M
$\mathfrak{g}$	The lie algebra associated to the lie group G
Fl_s^Y	The flow of Y evaluated at time s
R_g	$R_g(h) = h \cdot g$
L_g	$L_g(h) = g \cdot h$
$Ad(g)$	$d_e(conj_g)$
$\Omega^k(M, A)$	The space of k-forms with values in A
f^*X	The pullback of X with respect to f
$ker(\phi)$	$\{x \mid \phi(x) = 0\}$

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