

Semiclassical analysis and hyperbolic dynamics

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Contents

- 1 Introduction** **3**
- 1.1 Mathematical Introduction 3

- 2 Semiclassical Analysis** **3**
- 2.1 Definition and first properties 4
- 2.2 Wavefront set 6
- 2.3 Propagation estimates 8

- 3 Dynamics** **9**
- 3.1 Analytic continuation of the resolvent 9
- 3.2 Ruelle zeta function 12
- 3.3 Euler characteristic 14

1 Introduction

I passed my stage at the Laboratoire mathématique d'Orsay under the supervision of Colin Guillarmou between February and May. The original objective of the stage was to study the basics of microlocal and semiclassical analysis, that I obtained mostly from [8] and [7], to be able to read the two original articles [5] [6] in which Faure and Sjorstrand introduced the idea of using microlocal analysis in the study of Anosov flows. Since I completed the task before the time was over, I was assigned to read the articles [2] and [3], whose resume is the subject of this memoir. In the final period of the stage I had the chance to attend some lectures at the "Paris-Saclay conference in Analysis and PDE" organised at the end of May.

I cannot thank enough Colin Guillarmou for his patience and the time he offered me, and for the great insight that his discussions brought to me.

1.1 Mathematical Introduction

Hyperbolic flows are interesting for a variety of reasons and have long been studied. They very often exhibit chaotic and ergodic behaviours, and it's now established that a main tool for their understanding are anisotropic Sobolev spaces. In the same way, dynamical zeta functions have become of common use, so much that an entire book [1] is titled after them.

The goal of this memoir is to present synthetically the articles [2] and [3]. These articles pose themselves in a recent thread of application of microlocal and semiclassical analysis to spectral properties of dynamical systems. Many of the results that one can find were already known to some extent by classical dynamical systems techniques, for instance 71. At the same time new results have been reached so the approach has proved its worth.

This piece will begin with a general introduction to semiclassical analysis, which will contain no proofs and is meant as a taste of the ideas. Delving into technicalities would require orders of magnitudes more space and wouldn't add any value to existing texts, mainly one can refer to [2], appendix [E] in [4], and [9].

We will first introduce the classes of symbols and semiclassical quantization, together with the "rules" of symbolic calculus induced by their identification. Afterwards, we will present a few properties regarding ellipticity and continuity on functional spaces. Finally, we will give the statements of more advanced estimates that are fundamental in the proofs of our results.

After this introduction, we shift our focus on the applications to the spectral theory of Anosov flows. First we will prove Theorem 71 about the existence of meromorphic continuation of the resolvent of the vector field associated with the flow: here we will employ most of the heavy machinery introduced in the first part. Then we will apply these results to show the existence of the meromorphic continuation of the Ruelle zeta function for hyperbolic flows. Here as well the language of microlocal analysis will be necessary to treat a series of trace identities. Finally, we prove a more geometric result about the relation of the Ruelle zeta function with the Euler characteristic of negatively curved surfaces.

The technical nature of the subject makes it difficult to summarise. We will therefore trace the argument and focus on the most important ideas, while not delving too deep in technicalities and leaving out unnecessary proofs that shift away from our focus.

2 Semiclassical Analysis

The idea behind microlocal and semiclassical analyses is the possibility of identifying the space of differential operators with an algebra of symbols that allow for simpler computations. The primer of this relationship is given by the Fourier transform on \mathbb{R}^n that puts in duality differential operators and polynomials. Unfortunately useful topologies don't make the class of differential operators a Banach space, so since Hormander the solution has been to shift to the space of symbols, enlarge the class of allowed symbols inside of $C^\infty(\mathbb{R}^n)$ and then work with the corresponding "pseudodifferential operators" through the Fourier duality.

Semiclassical analysis adds the possibility of studying systems with very high frequencies through the implementation of a semiclassical parameter h that in the idealised scenario tends to 0 and allows to ignore high order terms.

2.1 Definition and first properties

We begin with semiclassical symbol classes. Our interest is in compact manifolds so we will define symbols by a global bound, instead of the more general local one.

Definition 1.

$$S_{1,0}^k(T^*M) = \{a \in C^\infty(M) : \sup_{h \in (0, h_0)} \sup_{(x, \xi) \in T_x^*M} \langle \xi \rangle^{|\beta|-k} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| < \infty\} \quad (2)$$

$b \in S^k(T^*M)$ if there exist $b_\ell \in S_{1,0}^{k-\ell}(T^*M)$ homogeneous of order $k - \ell$ independent of h such that:

$$b(x, \xi) \sim \sum_{\ell=0}^{\infty} b_\ell(x, \xi) \quad \text{that is} \quad b - \sum_{j=1}^{N-1} b_j \in S^{k-N} \quad \forall N$$

$a \in S_h^k(T^*M)$ if there exist $a_\ell \in S^{k-\ell}(T^*M)$ such that:

$$a(x, \xi) \sim \sum_{\ell=0}^{\infty} h^\ell a_\ell(x, \xi) \quad \text{that is} \quad a - \sum_{j=0}^{N-1} h^j a_j \in h^N S_{1,0}^{k-N} \quad \forall N \quad (3)$$

Remark 4. As announced, the prototypical examples of such elements are polynomials in ξ, h , that is $a(x, \xi; h) = \sum_i \sum_{|\alpha| \leq N} h^i a_\alpha^i(x) \xi^\alpha$. Also notice how 2 mimics the decay rate of polynomials of degree k , while instead of becoming identically 0 after $k + 1$ derivatives we allow for fast decay at infinity.

Definition 5.

$$S^{-\infty}(T^*M) = \bigcap_{k \in \mathbb{R}} S_{1,0}^k(T^*M)$$

One says $a \in h^\infty S^{-\infty}(T^*M)$ if:

$$\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h) = O(h^N \langle \xi \rangle^{-N}) \quad \forall N$$

The above symbols are relevant as their associated operators are regularizing and almost all calculation we have is done modulo a symbol of this class.

It will be important in the next sections to have a compactification of T^*M : define $\bar{T}^*M \simeq \{(x, \xi) \in T^*M : |\xi| \leq 1\}$ so that we can identify $T^*M \simeq \{(x, \xi) \in T^*M : |\xi| < 1\} \hookrightarrow \bar{T}^*M$.

Proposition 6. Let $a(x, \xi) \in C^\infty(T^*M)$. Then $a \in S^k$ if and only if $\langle \xi \rangle^{-k} a(x, \xi)$ extends to a smooth function on \bar{T}^*M .

In the same way, if $a \in S^k$, then its Hamiltonian vector field $H_p = \sum_i \partial_{\xi_i} p \partial_{x_i} - \partial_{x_i} p \partial_{\xi_i}$ satisfies $\langle \xi \rangle^{1-k} H_p \in \Gamma(\bar{T}^*M)$.

Given the symbols we can define their quantization. In \mathbb{R}^n this works as a straightforward Fourier transform. On a general manifold the process is more involved: one has to take coordinate charts, transport the symbol in \mathbb{R}^n and pull back the resulting operators. The fact that this operation does preserve the quantities we need is by no means obvious and is due to the fact that semiclassical operators are Pseudo-local, in the sense that they don't enlarge the singular set of a distribution, and can be decomposed as sum of operators almost preserving the support (up to arbitrarily small $\varepsilon > 0$ precision) and smoothing operators. We will see only part of this phenomenon in the part about wavefront sets but will not reference it further in this section.

Definition 7. In the euclidean space we introduce the semiclassical quantization:

$$\begin{aligned} Op_h(a)f(x) = a(x, hD_x)f(x) &= (2\pi h)^{-n} \int e^{i/h(x-y)\cdot\xi} a(x, \xi) f(y) dy d\xi &= \\ &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) f(y) dy d\xi & (8) \end{aligned}$$

Remark 9. Notice that $Op_h(\sum_\alpha a_\alpha(x) \xi^\alpha) = \sum_\alpha a_\alpha(x) (hD_x)^\alpha$.

Proposition 10. *Change of variables Let (ϕ, χ) be a cutoff chart on \mathbb{R}^n . Then for each $a \in S_{1,0}^k(T^*\mathbb{R}^n)$ there exists $b \in S_{1,0}^k(T^*\mathbb{R}^n)$ such that $\chi\phi^*Op_h(a)(\phi^{-1})^*\chi = Op_h(b)$. As a corollary by linearity and by 1 the result is also true for symbols in $S^k(\mathbb{R}^n)$ and $S_h^k(\mathbb{R}^n)$.*

This allows to define pseudodifferential operators on manifolds.

Definition 11. *Let M be a differentiable manifold, consider an atlas and the associated partition of unity (ϕ_j, χ_j) and a further set of cutoff charts $\tilde{\chi}_j$ such that $\chi_j \prec \tilde{\chi}_j$, then for each $a \in S_h^k(T^*M)$ define its semiclassical quantization on the manifold by*

$$Op_h^M(a) = \sum_i \tilde{\chi}_j \phi_j^* Op((\phi_j)_*(a\chi_j)) (\phi_j)_* \tilde{\chi}_j \quad (12)$$

Define then the classes of pseudodifferential operators by $\Psi_h^k(M) = Op_h^M(S_h^k(M))$.

In the general setting (M not compact) these are operators $C_c^\infty(M) \rightarrow C^\infty(M)$. We say that $A \in \Psi_h^k(M)$ is properly supported if its Schwartz kernel has proper support, that is the projections $\pi_1, \pi_2 : \text{supp}(K_A) \rightarrow M$ are proper. In such case, $A : C_c^\infty(M) \rightarrow C_c^\infty(M)$ so we can define its action on distributions by duality: $\langle AT, \phi \rangle = \langle T, A^T \phi \rangle$ is well defined for $T, \phi \in C_c^\infty(M)$ and the right hand side extends continuously to $T \in \mathcal{D}'(M)$ allowing to define the left hand side. In the case of M compact, which is what we're interested in, all $A \in \Psi_h^k$ are proper.

In the general case, it can be shown that any operator $A \in \Psi_h^k(M)$ admits a decomposition $A = A_1 + R$ for $A_1 \in \Psi_h^k(M)$ properly supported and $R \in \Psi^{-\infty}$ smoothing.

Definition 13. *Let $A = A(h) : C_c^\infty(M_2) \rightarrow \mathcal{D}'(M_1)$ be an operator. One writes $A \in h^\infty\Psi^{-\infty}$ or $A = O(h^\infty)_{\Psi^{-\infty}}$ is A is "smoothing", that is each $C^\infty(M_1 \times M_2)$ seminorm of the kernel of A is $O(h^\infty)$. If $a \in h^\infty S^{-\infty}$, then $A = Op(a) \in h^\infty\Psi^{-\infty}$.*

The need to employ change of variables to define the manifold quantization has the problem that it is not intrinsic, and different choices of coordinate charts will hold different results. For this reason we identify a part of the symbol that is actually invariant under such changes and encodes often a good amount of information.

Definition 14. Principal symbol *Suppose $a \in S_{1,0}^k(*M)$ is written in the form 3 We define its principal part to be $a^0 = a_0(x, \xi)$.*

*If $A \in \Psi_h^k(T^*M)$, there exists a unique $\sigma_h(A) \in S^k(\mathbb{R}^n)$ called the Principal Symbol of A such that*

$$a \in S_h^k(T^*M), A = Op_h^M(a) \Rightarrow a^0 = \sigma_h(A) \quad (15)$$

Remark 16. *We have $\sigma_h : \Psi_h^k(M) \rightarrow S^k(T^*M)$ is a linear morphism with kernel $\ker \sigma_h = h\Psi_h^{k-1}(M)$.*

With the principal symbol we can phrase more succintly the algebraic structure of the operations between operators.

Theorem 17. *Suppose $A \in \Psi_h^{k_1}(M)$ and $B \in \Psi_h^{k_2}(M)$. Then:*

- $\sigma_h(A + B) = \sigma_h(A) + \sigma_h(B)$
- $\sigma_h(AB) = \sigma_h(A)\sigma_h(B) \in S_h^{k_1+k_2}(T^*M)$
- $\sigma_h(h^{-1}[A, B]) = -iH_{\sigma_h(A)}\sigma_h(B) = -i \sum_j \frac{\partial \sigma_h(A)}{\partial \xi_j} \frac{\partial \sigma_h(B)}{\partial x_j} - \frac{\partial \sigma_h(B)}{\partial \xi_j} \frac{\partial \sigma_h(A)}{\partial x_j}$
- $\sigma_h(A^*) = \sigma_h(\bar{A})$

Now we can start the study of the functional spaces that work best with semiclassical operators, which turn out to be a semiclassical version of the ordinary Sobolev spaces on manifolds. Theorem 21 is the founding stone of all theorems about continuity between spaces, that are mostly just corollaries.

Definition 18. *The semiclassical Sobolev space H_h^s is defined as the closure of $C^\infty(\mathbb{R}^n)$ in $\mathcal{D}'(\mathbb{R}^n)$ with respect to the norm $\|u\|_{H_h^s(\mathbb{R}^n)} = \|\langle h\xi \rangle^s \hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}$*

Definition 19. *Let M be a manifold, define $H_{h,loc}^s$ as the closure of $C^\infty M$ in $\mathcal{D}'(M)$ for the family of seminorms $\|\phi_*\chi u\|_{H_h^s(\mathbb{R}^n)}$ for all charts (ϕ, χ)*

Remark 20. On a compact manifold the spaces H_h^N and H_h^{-N} are dual, in the sense that the inner product in L^2 has a unique extension to a pairing $\langle u, v \rangle_2$ for $u \in H_h^N$ and $v \in H_h^{-N}$. We have $\langle u, v \rangle \leq \|u\|_{H_h^N} \|v\|_{H_h^{-N}}$.

In the case M isn't compact, the duality is between $H_{s,loc}^N$ and $H_{h,comp}^{-N}$

Theorem 21. Calderon-Vaillancourt Consider $A \in \Psi_h^0$ with compact support, it extends to a continuous operator $L^2(M) \rightarrow L^2(M)$ with

$$\|A\|_{L^2} \leq \sup |\sigma_h(A)| + Ch^{-1/2} \quad (22)$$

Corollary 23. Any operator $A \in \Psi_h^k$ is uniformly bounded in h on compact sets as an operator

$$A : H_{h,comp}^s \longrightarrow H_{h,loc}^{s-k} \quad (24)$$

2.2 Wavefront set

Definition 25. Essential support Let $a \in S_h^k$. We can define its essential support:

$$esssup(a)^c = \{(x_0, \xi_0) \in \bar{T}^*M : \exists W \ni (x_0, \xi_0) \forall \alpha, \beta, N |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha, \beta, N} h^N \langle \xi \rangle^{-N}\} \quad (26)$$

Remark 27. One can represent the *esssup* union of the supports of the parts of a . That is, if $a \sim \sum_j h^j a_j$ like in 3 we have $esssup(a) \sim \bigcup_j supp(a^j)$. We have $esssup(a) = \emptyset$ if and only if $a \in h^\infty \Psi^{-\infty}$. For example:

- $a = he^{-|\xi|^2} \in S^{-\infty}$, but $esssup(a) = T^*M$
- $b = e^{-h|\xi|} \in S^{-\infty}$, and $esssup(b) = \emptyset$

Definition 28. Let $A \in \Psi_h^k$. Taking an atlas with partition of unity (ϕ, χ) , we define:

$$WF_h(a) = \bigcup_{(\phi, \chi)} \tilde{\phi}^{-1}(esssup(a_{\phi, \chi})) \subset \bar{T}^*M \quad (29)$$

where $Op_h(a_{\phi, \chi}) = (\phi^{-1})^* \chi A \chi \phi^*$. In particular, if $a \in S_h^k(T^*M)$, then $WF_h(Op_h^M(a)) = esssup(a)$.

The wavefront set is the set where the action of an operator is not smoothing. By our purposes, it takes the role of the support in the sense that everything happening outside of it is negligible. For all intents and purposes, semiclassical operators act as if they were supported in their wavefront sets.

Definition 30. We say that $A \in \Psi_h^k(M)$ is microlocally supported is $WF_h(A) \subset T^*M$ is compact, and we write $A \in \Psi_h^{comp}$.

We remark that $\Psi_h^{comp} \subset \Psi_h^k$ for all k . In particular $A \in \Psi_h^{comp} : H_h^{-N} \rightarrow H_h^N$ is uniformly bounded in h for all N , so $A : \mathcal{D}' \rightarrow C^\infty$

Definition 31. We say that A and B are microlocally equivalent in U , and we write $A = B + O(h^\infty)_{\Psi^{-\infty}}$, if:

$$WF_h(A - B) \cap U = \emptyset \quad (32)$$

Proposition 33. Given an open covering $(U_j)_{j=1}^n$ of $V \subset M$ compact, there exist $(X_j)_j \subset \Psi_h^0$ such that $WF_h(X_j) \subset U_j$ and on some neighbourhood V we have:

$$\sum_j X_j = I + O(h^\infty)_{\Psi^{-\infty}} \quad (34)$$

Here we introduce the concept of ellipticity. It corresponds to local invertibility inside the class of semiclassical operators.

Definition 35. Let $A \in \Psi_h^k$. We define $ell_h(A) = \{(x, \xi) \in \bar{T}^*M \mid \langle \xi \rangle^{-k} \sigma_h(A) \neq 0\}$.

Theorem 36. Elliptic Parametrix Let $A \in \Psi_h^k$ and E be pseudodifferential, such that $WF_h(A) \subset ell_h(E)$. Then there exists B pseudodifferential such that $A = BE + O(h^\infty)_{\Psi^{-\infty}}$.

In view of the existence of Elliptic Parametrix one can retrieve several regularity results. Firstly, the definition 18 of Sobolev spaces can be reformulated by saying that $H_h^s = A^{-1}(L^2)$ for some $A \in \Psi_h^s(M)$ globally elliptic. Using 21 and 36 one can show that this definition is independent of the chosen operator A , and that it coincides with the one given in 18.

At the same time, again by 36 the kernel of any elliptic operator lies in all Sobolev spaces and is therefore in C^∞ . A further analysis can be conducted to prove that in the compact setting elliptic operators are Fredholm.

Theorem 37. Elliptic estimate Let $A, B \in \Psi_h^0$ be with compact support and $E \in \Psi_h^k$ with $WF_h(A) \subset ell_h(E) \cap ell_h(B)$. Fixed N, s then there exist $C > 0, \chi \in C_c^\infty$ such that for each $u \in \mathcal{D}'$ such that $BEu \in H_h^{s-k}$ we have:

$$\|Au\|_{H_h^s} \leq C\|BEu\|_{H_h^{s-k}} + O(h^\infty)\|\chi u\|_{H_h^{-N}} \quad (38)$$

Theorem 39. Microlocal Garding Inequality Let $A \in \Psi_h^{2k}, B, B_1 \in \Psi_h^0$ with compact support, such that:

- $\langle \xi \rangle^{-2k} Re\sigma_h(A) \geq 0$ in a neighbourhood of $\bar{T}^*M \setminus ell_h(B)$
- $WF_h(A) \subset ell_h(B_1)$

Then there exist $\chi \in C_c^\infty$ and $C > 0$ such that $\forall u \in H_h^k$ we have

$$Re\langle Au, u \rangle \geq -C\|Bu\|_{H_h^k}^2 - Ch\|B_1u\|_{H_h^{k-1/2}}^2 - O(h^\infty)\|\chi u\|_{H_h^{-N}}^2 \quad (40)$$

We want to introduce semiclassical parameters to generate families of distributions.

Definition 41. • A family of distributions $u(h) \subset \mathcal{D}'(M)$ is said to be tempered if $\forall \chi \in C_c^\infty$ there exists N such that $\sup_h \|\chi u(h)\|_{H_h^{-N}} \leq Ch^{-N}$

- A family of operators $B(h) : C_c^\infty(M) \rightarrow \mathcal{D}'(M)$ Is tempered if the family of kernels $\mathcal{K}_{B(h)}$ is tempered.

Definition 42. Semiclassical wavefront sets

- Let $u(h) \subset \mathcal{D}'$ be an h -tempered family. One says that $(x_0, \xi_0) \notin WF_h(u)$ if there exists $U \ni (x_0, \xi_0)$ such that for each A pseudodifferential such that $WF_h(A) \subset U$, then $Au = O(h^\infty)_{C^\infty}$.
- Let $B(h) : C_c^\infty(M) \rightarrow \mathcal{D}'(M)$ an h -tempered family of operators. One defines

$$WF_h'(B) = \{(x, \xi, y, \eta) \in \bar{T}^*M^2 : (x, \xi, y, -\eta) \in WF_h(K_{B(h)})\} \quad (43)$$

where $K_{B(h)}$ is the schwartz kernel of $B(h)$ intended as an operator $\mathcal{D}'(T^*M) \rightarrow (T^*M)$

The microlocal (non-semiclassical) wavefront set is similar but is actually a subset of $\partial\bar{T}^*M$, and measures the directions in which the distributions is not C^∞ . We don't give any precise definition, as we do not have the possibility to deal with all the required theory. We only define \mathcal{D}'_Γ to be the set of distributions whose wavefront set is contained in Γ . The wavefront set of distributions is key if one wants to understand how distributions act differently from usual functions. It can be proven that under certain wavefront conditions the operations of multiplication and pullback admit continuous extensions to distributions. This will be important in section 3.2 but again unfortunately developing the theory would require a lot more effort. Here we mention the notions needed to understand at least part of 3.1.

Remark 44. • $WF_h(u) = \emptyset$ s.si $u \in O(h^\infty)_{C^\infty}$

- $WF_h'(A) = \emptyset$ s.si $A = O(h^\infty)_{\Psi^{-\infty}}$

The wavefront set condition can be characterized explicitly in terms of decay in a certain direction. This is explicited in the following proposition, but the definition remains the most useful and operational way to think about it.

Proposition 45. Let $u(h) \subset \mathcal{D}'(\mathbb{R}^n)$ tempered. Then $(x_0, \xi_0) \notin WF_h(u)$ if and only if there exist $U \times V$ conical neighbourhood of (x_0, ξ_0) and $\chi \in C_c^\infty(U)$ such that $\chi(x_0) \neq 0$ and

$$|\hat{\chi}u(\xi)| \leq C_N h^N \langle \xi \rangle^{-N} \quad \forall \xi \in V \quad (46)$$

Clearly the analysis of the wavefront set is closely tied to the analysis of regularity, and it is often the case that by showing that a wavefront set is small enough we get that a distribution is actually smooth.

Lemma 47. *Let $u(h) \in \mathcal{D}'$ be a tempered family of distributions. If for each $A \in \Psi_h^{comp}$, we have $\|Au\|_{L^2} = O(h^\infty)$, then $WF_h(u) \cap T^*M = \emptyset$.*

Finally we give some rules about the composition of wavefront sets.

Proposition 48. *Let $u = u(h) \in \mathcal{D}'$ be tempered, and $A \in \Psi_h^k$. Then:*

- $WF_h(Au) \subset WF_h(u) \cap WF_h(A)$
- $WF_h(u) \subset WF_h(Au) \cap (\bar{T}^*M \setminus ell_h(A))$

Proposition 49. *The following composition rules hold*

- Let $B(h)$ be a semiclassical operator on M , $u(h) \in \mathcal{D}'(M)$ be h -tempered, and $Q \in \Psi_h^{comp}(M)$. Then

$$WF_h(BQu) \cap T^*M_1 \subset \{(x, \xi) \in T^*M : \exists(y, \eta) \in WF_h(u) \cap WF_h(Q) : (x, \xi, y, \eta) \in WF'_h(B)\} \quad (50)$$

- Let $B_1(h), B_2(h)$ be semiclassical operators and $Q \in \Psi_h^{comp}$. Then

$$WF'_h(B_1QB_2) \cap T^*(M \times M) \subset \{(x, \xi, z, \zeta) : \exists(y, \eta) \in WF_h(Q) : (x, \xi, y, \eta) \in WF'_h(B_1), (y, \eta, z, \zeta) \in WF'_h(B_2)\} \quad (51)$$

2.3 Propagation estimates

These are very specific results about propagation of singularities along flows. They will prove fundamental in the second section.

Theorem 52. Propagation of singularities *Let $\mathbf{P} \in \Psi_h^k$, let $\sigma_h(P) = p - iq$ and φ_t the hamiltonian flow of p , that is $\varphi_t = \exp(t(\xi)^{1-k}H_p)$.*

Suppose $A, B, B_1 \in \Psi_h^0$ with compact support, such that:

- $\langle \xi \rangle^{-k}q \geq 0$ on $WF_h(B_1)$
- $\forall(x, \xi) \in WF_h(A) \exists T > 0$ such that $\varphi_{-T}(x, \xi) \in ell_h(B)$ and $\varphi_t(x, \xi) \in ell_h(B_1) \forall t \in [-T, 0]$. Then for each s, N there exist $C > 0, \chi \in C_c^\infty$ such that $\forall u \in H_h^s$ such that $\mathbf{P}u \in H_h^{s-k+1}$ we have:

$$\|A\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1u\|_{H_h^{s-k+1}} + O(h^\infty)\|\chi u\|_{H_h^N} \quad (53)$$

Theorem 54. Source estimates *Consider $P \in \Psi_h^1(\mathcal{M}, (\mathcal{E}))$ such that $\sigma_h(P) = (p - iq)I$, with $p \in S^1(\mathbb{R})$ independent of h , $q \in S_h^0$ and $q \geq 0$. Assume that p is homogeneous of degree 1 for $|\xi|$ large enough. Suppose $L \subset T^*\mathcal{M}$ is a radial source for the hamiltonian flow induced by H_p . There exists $m_0 > 0$ such that*

- For each $B \in \Psi_h^0$ elliptic on $\kappa(L)$ there exists $A \in \Psi_h^0$ elliptic on $\kappa(L)$ such that $u(h) \in \mathcal{D}'$ then for each $m \geq m_0$, if $Au \in H_h^{m_0}$ then

$$\|Au\|_{H_h^m} \leq Ch^{-1}\|B_1Pu\|_{H_h^m} + O(h^\infty) \quad (55)$$

- If $u(h) \in \mathcal{D}'(M, \mathcal{E})$ is h -tempered and $B_1 \in \Psi_h^0(M)$ is elliptic on $\kappa(L)$, then

$$B_1u_0^m, \quad WF_h(Pu) \cap \kappa(L) = \emptyset \Rightarrow WF_h(u) \cap \kappa(L) = \emptyset \quad (56)$$

Theorem 57. Sink estimates *Consider $P \in \Psi_h^1(\mathcal{M}, (\mathcal{E}))$ such that $\sigma_h(P) = (p - iq)I$, with $p \in S^1(\mathbb{R})$ independent of h , $q \in S_h^0$ and $q \geq 0$. Assume that p is homogeneous of degree 1 for $|\xi|$ large enough. Suppose $L \subset T^*\mathcal{M}$ is a radial sink for the hamiltonian flow induced by H_p . There exists $m_0 > 0$ such that for each $B_1 \in \Psi_h^0$ elliptic on $\kappa(L)$, there exist $A \in \Psi_h^0$ elliptic on $\kappa(L)$ and $B \in \Psi_h^0$ with $WF_h(B) \subset ell(B_1) \setminus \kappa(L)$ such that for each $u(h) \in \mathcal{D}'(\mathcal{M}, \mathcal{E})$ and for $m \leq -m_0$*

$$\|Au\|_{H_h^m} \leq C\|Bu\|_{H_h^m} + Ch^{-1}\|B_1Pu\|_{H_h^m} + O(h^\infty) \quad (58)$$

3 Dynamics

3.1 Analytic continuation of the resolvent

We can now start applying these techniques to the study of flows. Section 3.1 will be the most analytic and here we will use the greatest amount of semiclassical analysis: here we will study the operator given by the vector field associated to the flow, in particular we are interested in its spectral properties and in the wavefront properties of the resolvent.

Definition 59. Anosov flow A flow $\{\phi_t\}_{t \in \mathbb{R}}$ generated by a vector field X on a Riemannian manifold \mathcal{M} is said to be Anosov if there exists a continuous global splitting $T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u$ and constants $C, \lambda > 0$ such that:

$$|(\phi_t)_*v| = |d\phi_t v| \leq ce^{-\lambda t}|v| \quad \forall v \in E_s, \forall t > 0 \quad (60)$$

$$|(\phi_t)_*v| = |d\phi_{-t} v| \leq ce^{\lambda t}|v| \quad \forall v \in E_u, \forall t < 0 \quad (61)$$

Since our instruments live in the cotangent bundle, we consider the dual decomposition. Define E_0^*, E_s^*, E_u^* by $E_0^* = (E_s \oplus E_u)^\perp$, $E_s^* = (\mathbb{R}X \oplus E_s)^\perp$ and $E_u^* = (\mathbb{R}X \oplus E_u)^\perp$. Then notice that:

$$|(\phi_t)^*\xi| = |(d\phi_t)^{-T}\xi| \leq Ce^{-\lambda t}|\xi| \quad \forall \xi \in E_s^*, \forall t > 0 \quad (62)$$

$$|(\phi_t)^*\xi| = |(d\phi_{-t})^{-T}\xi| \leq Ce^{\lambda t}|\xi| \quad \forall \xi \in E_u^*, \forall t < 0 \quad (63)$$

Associated to the flow come the hamiltonian flow on the cotangent bundle:

$$e^{tH_p}(x, \xi) = (\phi_t(x), (d\phi_t(x))^{-T}\xi) \quad (64)$$

generated by the vector field H_p .

To study properties of the flow, we use Anisotropic semiclassical Sobolev spaces, tailored ad hoc for this problem. The Sobolev space we saw in 2.2 are the preimages of $L^2(M)$ under some elliptic operators. We can extend the definition to elliptic operators whose order is variable, that is their symbol will have different decay rates depending on the direction we look at. This will generate a space of function whose regularity is different depending on the direction. Using this we can the construct a space with better regularity properties with respect to differentiation by X that will work well for our questions. The construction relies on a dynamical weight function.

Proposition 65. *There exists $m \in C^\infty T^*\mathcal{M}$ such that $m = 1$ in a neighbourhood of E_s^* , $m = -1$ on a neighbourhood of E_u^* , and $H_p m \leq 0$ on all $T^*\mathcal{M}$.*

Proof. Consider V, V' open and $T > 0$ such that $E_s^* \subset V \subset V'$ and $e^{TH_p}V \subset V'$, $e^{TH_p}(V' \setminus V) \cap V' = \emptyset$. Take $\psi \in C^\infty(T^*\mathcal{M})$ such that $\psi = 1$ on V , $\psi = 0$ in $T^*\mathcal{M} \setminus V'$, and $0 \leq \psi \leq 1$. Then define $m_+(x, \xi) = \frac{1}{T} \int_0^T e_*^{tH_p} \psi(x, \xi) dt$. Define in the same way m_- supported around E_u^* and $m = m_+ - m_-$ satisfies our demands. \square

We can now define a pseudodifferential operator $G \in \Psi^{0+}$ (where $\Psi^{0+} = \bigcap_{\epsilon > 0} \Psi^\epsilon$) satisfying

$$\sigma_G(x, \xi) = m(x, \xi) \log |\xi| \quad (66)$$

Then its exponentials are all elliptic operators and satisfy $e^{sG} \in \Psi^s$. Mimicking 2.2, we define the anisotropic sobolev spaces

$$\mathcal{H}_{sG} = e^{-sG}(L^2(M)) \quad \|u\|_{\mathcal{H}_{sG}} = \|e^{sG}u\|_{L^2} \quad (67)$$

We also need semiclassical adaptation of these. We therefore consider a semiclassical operator $G(h) \in \Psi^{0+}$ such that $\sigma_h(G(h))(x, \xi) = (1 - \chi(x, \xi))\sigma(G)$, and in the same manner define

$$\mathcal{H}_{sG(h)} = e^{-sG(h)}(L^2(M)) \quad \|u\|_{\mathcal{H}_{sG(h)}} = \|e^{sG(h)}u\|_{L^2} \quad (68)$$

These spaces are actually the same: since $\sigma(G(h) - G)$ is bounded we have $\mathcal{H}_{sG} = \mathcal{H}_{sG(h)}$ and the equivalence of norms is expressed through constants dependent on h .

Now that the functional setting is prepared we can advance to the analysis of the flow.

We consider now the vector field X as acting on forms through the Lie derivative. Define:

$$P : \bigoplus \Gamma(\Omega^k) \rightarrow \bigoplus \Gamma(\Omega^k) \quad Pu = -i\mathcal{L}_X u \quad (69)$$

We remark that the principal symbol of P is diagonal:

$$\sigma_P(x, \xi) = \langle \xi, X(x) \rangle = p(x, \xi) \quad (70)$$

As before, we also need a semiclassical version, which is easily given by hP , satisfying $\sigma_h(hP) = p$.

Our main objective is to study spectral properties of P . We can resume the results in the following theorem, though the construction leading to it is as important as the final result.

Theorem 71. Faure-Sjorstrand *Let $C \in \mathbb{R}$ be fixed. For all $s \geq 0$ large enough (depending on C), the unbounded operator $P : \mathcal{D}_s \subset \mathcal{H}_{sG} \rightarrow \mathcal{H}_{sG}$ has discrete spectrum in $\text{Im}(z) > C$. The resolvent $R(\lambda) : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ admits the following Laurent expansion around elements of the spectrum in $\{\text{Im}(\lambda) > C\}$.*

$$R(\lambda) = R_H(\lambda) - \sum_{j=1}^{N(\lambda_0)} \frac{(P - \lambda_0)^{j-1} \Pi_{\lambda_0}}{(\lambda - \lambda_0)^j} \quad (72)$$

where $\Pi_{\lambda_0} : \mathcal{H}_{sG} \rightarrow \mathcal{H}_{sG}$ is the spectral projector. Finally the wavefront sets of the operators involved satisfy $WF'(R_H(\lambda)) \subset \Delta(T^*\mathcal{M}) \cup E_u^* \times E_s^* \cup \Omega_+$ and $WF'(\Pi) \subset E_u^* \times E_s^*$, where $\Omega_+ = \{(e^{tH_p}(x, \xi), x, \xi) : p(x, \xi) = 0, t \geq 0\}$.

An obstacle to our study is the neighbourhood to the zero section, where the dynamics is complicated and the symbol of our operator is not invertible. The solution is to add a potential to the operator: consider $iQ_\delta \in \Psi^{0+}$ such that:

$$\sigma_h(Q_\delta) \geq 0 \quad \sigma_h(Q_\delta) > 0 \text{ on } \{|\xi| < \delta/2\} \quad WF'_h(Q_\delta) \subset \{|\xi| < \delta\} \quad (73)$$

Now define $P_\delta(z) = iP - iQ_\delta - z$, and through the isomorphism $e^{sG} : \mathcal{H}_{sG(h)} \rightarrow L^2$ it is equivalent to the action on L^2 of

$$P_{\delta,s} = e^{sG} P_\delta e^{-sG} = P_\delta + s[G(h), hP] + \mathcal{O}(h^2)_{\Psi_h^{-1+}} \quad (74)$$

The bulk of the work is in the following proposition:

Proposition 75. *Given constants $C_0, \varepsilon > 0$, for all $s > 0$ large enough the operator $P_\delta(z) : \mathcal{D}_{sG(h)} \rightarrow \mathcal{H}_{sG(h)}$ is invertible for $-C_0h \leq \text{Im}(z) \leq 1$ and $|\text{Re}(z)| \leq h^\varepsilon$. Its inverse R_δ satisfies $\|R_\delta(z)\|_{\mathcal{H}_{sG} \rightarrow \mathcal{H}_{sG}} \leq Ch^{-1}$ and $WF'_h(R_\delta(z)) \subset \Delta(T^*\mathcal{M}) \cup \Omega_+$.*

Proof. First of all define the projection $\kappa : T^*\mathcal{M} \setminus 0 \rightarrow \partial T^*\mathcal{M}$. We're mostly interested in $\kappa(E_{s,u}^*) = E_{s,u}^* \setminus \bar{E}_{s,u}^*$, that is the points at infinity of $E_{s,u}^*$. They are respectively source and sink for the hamiltonian flow H_p .

We want to establish the estimate

$$\|u\|_{\mathcal{H}_{sG(h)}} \leq Ch^{-1} \|P_\delta(z)\|_{\mathcal{H}_{sG(h)}} \quad (76)$$

for $u \in \mathcal{D}_{sG(h)}$. By 33 we consider a semiclassical partition of unity to reduce the problem to proving 76 for Au , where $A \in \Psi_h^0$ has wavefront supported in some set S in the five following possibilities: 1) $S \subset (\{p = 0\} \cup \{|\xi| > \delta/2\})^c$; 2) S is a small neighbourhood of $k(E_s^*)$; 3) $S \subset (\{p = 0\} \setminus \bar{E}_u^*)$; 4) S is a small neighbourhood of $(x_0, \xi_0) \in E_u^*$; 5) S is a small neighbourhood of $\kappa(E_u^*)$.

Assume also $\|u\|_{\mathcal{H}_{sG(h)}}$.

Case 1: $S \subset (\{p = 0\} \cup \{|\xi| > \delta/2\})^c$. Here $|\sigma_h(P_{\delta,s})| = |p + \sigma_h(Q_\delta)| \geq c > 0$ by compactness so $P_{\delta,s}$ is elliptic on S . By conjugating with $e^{sG(h)}$ and using 37 we obtain

$$\|Au\|_{\mathcal{H}_{sG(h)}} \leq \|BP_\delta(z)\|_{\mathcal{H}_{sG(h)}} + \mathcal{O}(h^\infty) \quad (77)$$

for some $B \in \Psi_h^0$ with $WF'_h(B) \subset S$.

Case 2: S is a small neighbourhood of $k(E_s^*)$. Here $G(h) \sim 1$ and $\mathcal{H}_{sG(h)}$ is microlocally equivalent to H_h^s . Also, since $\text{Im}(z) \geq -C_0h$, we have $\text{Im}(\sigma_h(P_\delta)) \leq 0$, so we can apply ?? to get that

$$\|Au\|_{\mathcal{H}_{sG(h)}} \approx \|Au\|_{H_h^s} \leq \|B_1 P_\delta(z)\|_{H_h^s} + \mathcal{O}(h^\infty) \approx \|B_1 P_\delta(z)\|_{\mathcal{H}_{sG(h)}} + \mathcal{O}(h^\infty) \quad (78)$$

Case 3: $S \subset (\{p = 0\} \setminus \bar{E}_u^*)$.

Since we're far from \bar{E}_u^* we have that $e^{tH_p}(S) \rightarrow \kappa E_s^*$ for $t \rightarrow \infty$, and since here $\sigma_h(P) \leq 0$, we can apply 52 and conjugate with $e^{sG(h)}$

$$\|Au\|_{\mathcal{H}_{sG(h)}} \leq C \|Bu\|_{\mathcal{H}_{sG(h)}} + Ch^{-1} \|B_2 P_\delta(z)\|_{\mathcal{H}_+} \mathcal{O}(h^\infty) \leq Ch^{-1} (\|B_1 P_\delta(z)\|_{\mathcal{H}_{sG(h)}} + \|B_2 P_\delta(z)\|_{\mathcal{H}_{sG(h)}}) \quad (79)$$

Where B is localised in a neighbourhood of $\kappa(E_s^*)$ and can therefore be estimated by **Case 2**.

Case 4: S is a small neighbourhood of $(x_0, \xi_0) \in E_u^*$.

Like in the previous case for T large enough $e^{tH_p}S \subset \{|\xi| < \delta/2\}$, so like before by propagation of singularities we get

$$\|Au\|_{\mathcal{H}_{sG}(h)} \leq C\|Bu\|_{\mathcal{H}_{sG}(h)} + Ch^{-1}\|B_2P_\delta(z)\|_{\mathcal{H}_+}\mathcal{O}(h^\infty) \leq Ch^{-1}(\|B_1P_\delta(z)\|_{\mathcal{H}_{sG}(h)} + \|B_2P_\delta(z)\|_{\mathcal{H}_{sG}(h)}) \quad (80)$$

Case 5: S is a small neighbourhood of $\kappa(E_u^*)$.

Locally we have $\mathcal{H}_{sG}(h) \approx H_h^{-s}$, and since $\kappa(E_u^*)$ is a sink we can use 57 and with the previous cases we get

$$\|Au\|_{\mathcal{H}_{sG}(h)} \leq Ch^{-1}\|BP_\delta(z)\|_{\mathcal{H}_{sG}(h)} + \mathcal{O}(h^\infty) \quad (81)$$

The combination of all these cases gives finally 76 At the same time, E_s^* and E_v are sink and sources for the inverse time flow e^{-tH_p} . Hence we get the conjugate estimate for $-P_\delta(z)^*$, that is

$$\|v\|_{\mathcal{H}_{-sG}(h)} \leq \|P_{\delta(z)^*v}\|_{\mathcal{H}_{-sG}(h)} + \mathcal{O}(h^\infty) \quad (82)$$

Since both $P_\delta(z)$ and $P_\delta(z)^*$ are closed and injective, we get that $P_\delta(z)$ is invertible.

To obtain the wavefront condition one uses the following lemma

Lemma 83. *Consider $B : C_c^\infty(\mathcal{M}) \rightarrow \mathcal{D}'(\mathcal{M})$ h -tempered. Then $(y, \eta, x, \xi) \notin WF_h'(B)$ if and only if there exist neighbourhoods $U \ni (x, \xi)$ and $V \ni (y, \eta)$ such that*

$$WF_h(f) \subset U \Rightarrow WF_h(Bf) \cap V = \emptyset \quad \forall f(h) \in C_c^\infty(\mathcal{M}) \quad (84)$$

□

Now we can complete the proof of the theorem. In the following we suppose λ is in some compact subset of $\{Im(\lambda) > -C_0\}$. One can choose h small enough that $z = h\lambda$ satisfies $-C_0h \leq Im(z) \leq 1$ and $|Re(\lambda)| \leq h^{1/2}$.

Proposition 85. *Fix $C_0 > 0$. Then for $s > 0$ large enough, $P - \lambda : \mathcal{D}_{sG} \rightarrow \mathcal{H}_{sG}$ is Fredholm of index 0 in the region $\{Im(\lambda) > -C_0\}$.*

Proof. We have that Q_δ is compact as an operator $\mathcal{D}_{sG} \rightarrow \mathcal{H}_{sG}$, and since $\mathcal{D}_{sG}, \mathcal{H}_{sG}$ are isomorphic to $\mathcal{D}_{sG}(h), \mathcal{H}_{sG}(h)$, we have that P is a compact perturbation of the invertible operator P_δ . □

Proposition 86. *For $s > 0$ fixed, there exists C_1 such that for $Im(\lambda) > C_1$, the operator P_λ is invertible with inverse $(P_\lambda)^{-1} = i \int_0^\infty e^{i\lambda t} \phi_{-t}^* dt$ with the integral converging in the operator norm $H^s \rightarrow H^s$ and $H^{-s} \rightarrow H^{-s}$.*

Proof. Given that $\|\phi_t^*\|_{H^{\pm s} \rightarrow H^{\pm s}} e^{C_1 t}$, for $u \in \mathcal{H}_{sG}$ we have

$$u = \int_0^\infty \partial_t(e^{i\lambda t} \phi_t^* u) dt = i \int_0^\infty e^{i\lambda t} \phi_t^* (P - \lambda) u dt \quad (87)$$

The integral converges in operator norm and thus we have the thesis. □

Proof. (Of theorem 71)

Fix C like in the hypothesis. By 85 for some $s > 0$ large enough the Fredholm property is satisfied. By 86 the operator P_λ is invertible for some large λ , so by the Analytic Fredholm theorem Theorem $P - \lambda$ is invertible on $\{Im(\lambda) > C\}$ except on a discrete subset, around which we have an expansion of the form 72. We only need to treat the wavefront.

Suppose that λ is not in the spectrum. Rename $R_\delta(z) = (hP - z - iQ_\delta)^{-1}$, and observe that

$$R(\lambda) = h(R_\delta(z) - iR_\delta(z)Q_\delta R_\delta(z)) - R_{\delta(z)Q_\delta R(\lambda)Q_\delta R_\delta(z)} \quad (88)$$

By 75 $WF_h'(R_\delta(z) - iR_\delta(z)Q_\delta R_\delta(z)) \cap T^*(\mathcal{M} \times \mathcal{M}) \subset \Delta(T^*\mathcal{M}) \cup \Omega_+$.

Again by 75 we find $WF_h'(R_\delta(z)Q_\delta R(\lambda)Q_\delta R_\delta(z)) \cap T^*(\mathcal{M} \times \mathcal{M}) \subset \Delta(T^*\mathcal{M}) \cup v_\delta$ where $v_\delta = \{(\rho, \rho') | t, s \geq 0 : e^{tH_p}(\rho) \in WF_h(Q_\delta), e^{-tH_p}(\rho') \in WF_h(Q_\delta)\}$. Hence

$$WF(R_\delta) \cap T^*(\mathcal{M} \times \mathcal{M}) \subset \Delta(T^*\mathcal{M}) \cup \Omega_+ \cup v_\delta \quad (89)$$

But $R(\lambda)$ is independent of h, δ , so

$$WF(R_\delta) \cap T^*(\mathcal{M} \times \mathcal{M}) \subset \Delta(T^*\mathcal{M}) \cup \Omega_+ \cup \bigcap_{\delta > 0} v_\delta = \Delta(T^*\mathcal{M}) \cup \Omega_+ \cup (E_u^* \times E_s^*) \quad (90)$$

The case of λ_0 in the spectrum is similar: one takes 88 with $(\lambda - \lambda_0)^{-N(\lambda_0)}R(\lambda)$ and by taking derivatives one can prove the required condition. \square

3.2 Ruelle zeta function

The Ruelle zeta function ζ_R is built around periodic orbits like the Riemann zeta function is built around primes. The hope is that it can capture interesting properties of the flow, and as we will see in 3.3 this is at least sometimes true. At the same time, the analysis of ζ_R requires much work, for example a priori it is not even well defined. The objective of this section is to introduce it and prove that it can be meromorphically extended to the whole complex plane.

The strategy proceeds by using Guillemin's trace formula to link the value of ζ_R to the integral of the trace, which we'll introduce later, of the transfer operator e^{itP} . Then by approximating the trace with smooth functions and switching the integration order the resolvent shows up and we find a meromorphic function.

Before defining ζ_R , we state two lemmas that we will not prove. Here L is a constant of the flow defined by the property $\|f \circ \phi_t\|_{C^2} \leq Ce^{L|t|}\|f\|_{C^2} \quad \forall f \in C^2$ for some positive C .

Lemma 91. *Let $\{\phi_t\}_{t \in \mathbb{R}}$ be an Anosov flow on a compact manifold \mathcal{M}^n with a smooth measure μ . Fix $t_e > 0$, then there exists $C \in \mathbb{R}$ such that*

$$\mu \otimes dt(\{(x, t) : t_e \leq t \leq T, d(x, \phi_t(x)) \leq \varepsilon\}) \leq Ce^n e^{nLT} \quad \forall T, \varepsilon > 0 \quad (92)$$

Lemma 93. *Under the same hypotheses, let $N(T)$ the the number of closed trajectories of period smaller than T . Then $N(T) \leq Ce^{(2n-1)LT}$.*

Consider the set of periodic orbits γ for the flow. For each one of them denote T_γ and T_γ^\sharp the period and the primitive period, respectively. To each such orbit, one can associate the linearized Poincaré map $\mathcal{P}_\gamma = d\phi_{-T_\gamma}(x)|_{E_s^* \oplus E_u^*}$ for some $x \in \text{spt}(\gamma)$.

We can now define the Ruelle zeta function:

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{iT_\gamma^\sharp \lambda}) \quad (94)$$

Remark that by 93 the function is well defined and holomorphic for $\text{Im}(\lambda)$ large enough.

It can be shown that:

$$\det(I - \mathcal{P}_\gamma) = \sum_{k=0}^{n-1} \text{tr} \wedge^k \mathcal{P}_\gamma \quad |\det(I - \mathcal{P}_\gamma)| = (-1)^{\dim E_s} \det(I - \mathcal{P}_\gamma) \quad (95)$$

Se using the logarithm expansion $\log(1+z) = \sum_{k \geq 0} (-1)^k \frac{z^k}{k}$, we have

$$\zeta_R(\lambda) = \exp\left(-\sum_{\gamma^\sharp} \sum_{m \geq 1} \frac{e^{iT_\gamma^\sharp \lambda m}}{m}\right) = \exp\left(-\sum_{\gamma} e^{iT_\gamma \lambda} T_\gamma^\sharp / T_\gamma\right) = \prod_{k=0}^{n-1} \exp\left(-\sum_{\gamma} \frac{e^{iT_\gamma \lambda} T_\gamma^\sharp \text{tr} \wedge^k \mathcal{P}_\gamma}{|\det(I - \mathcal{P}_\gamma)| T_\gamma}\right)^{(-1)^{k+\dim E_s}} \quad (96)$$

Therefore it is sufficient to study the meromorphic expansion of the functions

$$f_k(\lambda) = -i \sum_{\gamma} \frac{e^{iT_\gamma \lambda} T_\gamma^\sharp \text{tr} \wedge^k \mathcal{P}_\gamma}{|\det(I - \mathcal{P}_\gamma)|} = \frac{\partial}{\partial \lambda} \log \exp\left(-\sum_{\gamma} \frac{e^{iT_\gamma \lambda} T_\gamma^\sharp \text{tr} \wedge^k \mathcal{P}_\gamma}{|\det(I - \mathcal{P}_\gamma)| T_\gamma}\right) \quad (97)$$

The proof we will give relies on the use of Trace identities, that is properties of the trace of various operators. What we're going to use is not the usual functional analytic trace tr , but a generalization to distribution called the Flat trace tr^b realised informally as $\text{tr}^b A = \int_M K_A(x, x) dx$ for $A : \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ with kernel K_A , where the integral is meant as a distribution in \mathcal{D}' . The above definition can be applied

under the hypothesis that K_A can be pulled back under the immersion $\iota : M \ni x \mapsto (x, x) \in M \times M$, which can be realised under the wavefront condition $WF'(K_A) \cap N(T^*M) = \emptyset$ for $N(T^*M) = \{(x, \xi, x, -\xi) \in T^*M \times T^*M\}$ the conormal bundle. We will not go any further in explaining the details, we just remark that the shift by some positive time the reader will see in the integrals is purposefully made to meet such wavefront condition.

As announced we need a way to approximate the trace with smooth functions and we need estimates to switch orders of integration. Here are the respective results.

Let $\psi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ near 0. There exists a family of $F_\varepsilon \in C^\infty(\mathcal{M} \times \mathcal{M})$ such that

$$E_\varepsilon(x, y) = \frac{1}{F_\varepsilon(x, y)} \psi\left(\frac{d(x, y)}{\varepsilon}\right) \quad (98)$$

satisfies $E_\varepsilon(1_{\mathcal{M}}) = 1_{\mathcal{M}}$ and $C^{-1}\varepsilon^n \leq E_\varepsilon \leq C\varepsilon^n$. We have then

Lemma 99. *We have $E_\varepsilon \in \Psi^{-\infty}$ and if B follows the wavefront set condition $WF'(B) \cap \Delta(T^*\mathcal{M}) = \emptyset$, then $\text{tr}^\flat B = \lim_{\varepsilon \rightarrow 0} \text{tr} E_\varepsilon B E_\varepsilon$, where the right hand side is the usual trace in L^2 .*

The proof is quite technical and relies on alternative characterizations of symbol classes.

Lemma 100. *Suppose $t_0 > 0$. There exists C such that*

$$\|E_\varepsilon \phi_{-T}^* E_\varepsilon\|_{tr} \leq C\varepsilon^{-n-2} e^{CT} \int_T^{T+1} |\text{tr}^\flat E_\varepsilon \phi_{-t}^* E_\varepsilon| dt \leq C e^{CT} \quad \forall \varepsilon > 0 \quad \forall T > t_0 \quad (101)$$

Proof. Firstly we have

$$\|E_\varepsilon \phi_{-T}^* E_\varepsilon\|_{tr} \leq \|E_\varepsilon\|_{tr} \|\phi_{-T}^*\|_{L^2 \rightarrow L^2} \|E_\varepsilon\|_{L^2 \rightarrow L^2} \leq \quad (102)$$

$$= C e^{CT} \|(-\Delta + 1)^{-k}\|_{tr} \|(\Delta + 1)^k E_\varepsilon\|_{L^2 \rightarrow L^2} \leq C_1 e^{CT} \varepsilon^{-2k} \quad (103)$$

And secondly

$$\int_T^{T+1} |\text{tr}(E_\varepsilon \phi_t^* E_\varepsilon)| dt = \int \int_{\mathcal{M} \times \mathcal{M}} E_\varepsilon(x, y) E_\varepsilon(\phi_t(y), x) dx dy dt \leq \quad (104)$$

$$= C\varepsilon^{-2n} \int \int_{\mathcal{M} \times \mathcal{M}} \mathbf{1}_{d(x, y) \leq \varepsilon} \mathbf{1}_{d(x, \phi_t(y)) \leq \varepsilon} dx dy dt \leq C\varepsilon^{-n} \int \int_{\mathcal{M}} \mathbf{1}_{d(y, \phi_t(y)) \leq 2\varepsilon} dy dt \leq C e^{nLT} \quad (105)$$

where we used both the explicit formulation of E_ε and 91. \square

Here comes the crucial point: Guillemin's trace formula allows us to tie together approximations of f_k with the integrated trace of e^{itP} : the integral will resemble that of 86 and will let us

Let's consider Guillemin's Trace formula:

$$\text{tr}^\flat e^{-itP} = \sum_\gamma \frac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|} \quad (106)$$

where the left hand side is well defined under the condition $WF'(e^{itP}) \cap N^*(\mathbb{R}^+ \cup \Delta(T^*\mathcal{M})) = \emptyset$ which can be verified to be true.

Denote by $\mathcal{E}_0^k = \{\omega \in \Gamma(\Omega^k) : \iota_X \omega = 0\}$. Then the restriction of 106 satisfies:

$$\text{tr}^\flat e^{-itP}|_{\mathcal{E}_0^k} = \sum_\gamma \frac{T_\gamma^\sharp \text{tr} \wedge^k \mathcal{P}_\gamma \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|} \quad (107)$$

Consider $0 < t_0 < \inf_\gamma T_\gamma$, take $T \geq t_0$ and $\chi_T \in C_c^\infty(\mathbb{R})$ with $\text{supp} \chi_T \subset [t_0/2, T+1]$ and $\chi_T = 1$ around $[t_0, T]$. $P_k = P|_{\mathcal{E}_0^k}$. By 107

$$-i \sum_\gamma \frac{\chi_T(T_\gamma) e^{iT_\gamma \lambda} T_\gamma^\sharp \text{tr} \wedge^k \mathcal{P}_\gamma}{|\det(I - \mathcal{P}_\gamma)|} = -i \int_{t_0/2}^{T+1} \chi_T(t) e^{it\lambda} \text{tr}^\flat e^{-itP_k} dt = -i \text{tr}^\flat \int_{t_0/2}^{T+1} \chi_T(t) e^{it(\lambda - P_k)} dt \quad (108)$$

Therefore we can write

$$f_k(\lambda) = -i \lim_{T \rightarrow \infty} \text{tr}^\flat \int_{t_0/2}^{T+1} \chi_T(t) e^{it(\lambda - P_k)} dt = \quad (109)$$

$$= -i \lim_{T \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} tr^b \int_{t_0/2}^{T+1} \chi_T(t) E_\varepsilon e^{it(\lambda - P_k)} E_\varepsilon dt = -i \lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} tr^b \int_{t_0/2}^{T+1} \chi_T(t) E_\varepsilon e^{it(\lambda - P_k)} E_\varepsilon dt \quad (110)$$

If $R_k(\lambda)$ is the restriction of the resolvent to \mathcal{E}_0^k , that is the inverse of $(P_k - \lambda)$, we have by 86 that $f_k(\lambda) = -\lim_{\varepsilon \rightarrow 0} tr E_\varepsilon e^{it_0(\lambda - P_k)} R_k(\lambda) E_\varepsilon$. By 71 $e^{it_0(\lambda - P_k)} R_k(\lambda)$ satisfies the hypotheses of 99 and therefore $f_k(\lambda) = -e^{it_0\lambda} tr^b(e^{-it_0 P_k} R_k(\lambda))$ which is meromorphic. Finally, it can be shown that the poles have integer residues, so that from 96 and 97 one can deduce meromorphic continuation of ζ_R .

3.3 Euler characteristic

One can prove that the geodesic flow of a strictly negatively curved Riemannian manifold Σ , which is a flow on $T\Sigma$, is Anosov. These form an important class of Anosov flows with particular regularity and geometric properties. For instance, closed orbits of the flow are now geodesics so naturally we can expect geometric properties of the manifold to emerge from our study. In this section we will show how for a surface the Ruelle zeta function vanishes at the origin with order equal to the Euler characteristic of the surface.

The strategy of proof is straightforward: we tie the order of vanishing of ζ_R to the dimensions of some distribution spaces, whose dimensions we will compute.

Consider Σ a 2-dimensional Riemannian manifold, let ζ_R be the Ruelle zeta function of the geodesic flow on its cotangent bundle $T^*\Sigma$. If Σ is negatively curved, the flow is Anosov, so that ζ_R is well defined and all the previous results apply.

Theorem 111. *Let (Σ, g) be a 2-dimensional surface with negative curvature, let $\chi(\Sigma)$ be its Euler characteristic. Then the function $s \mapsto s^{\chi(\Sigma)} \zeta_R(s)$ is holomorphic and nonvanishing in 0.*

Let $M = S^*\Sigma = \{(x, \xi) \in T^*M : |\xi| = 1\}$ be the unitary cotangent bundle of Σ . The entirety of the work is actually done on Σ . Indeed, if $j : S^*M \hookrightarrow T^*M$ is the canonical immersion, the form $\alpha = j^*(\xi dx)$ is a contact form on M and it generates the (restriction of the) geodesic flow of Σ . Noticeably $\iota_X(\alpha) = 1$, $\iota_X(d\alpha) = 0$, and $u \wedge d\alpha = (\iota_X u) dvol_M$.

Recall the first Betti number $b_1(X) = \dim H^1(X, \mathbb{C})$, satisfying $\chi(X) = 2 - b_1(X)$ for X any differential manifold.

By the following we get $\chi(M) = 2 - \dim H^1(M, \mathbb{C}) = 2 - \dim H^1(M, \mathbb{C}) = \chi(\Sigma)$

Lemma 112. *Let $\pi : M \rightarrow \Sigma$ be the projection map. Then $\pi^* : H^1(\Sigma, \mathbb{C}) \rightarrow H^1(M, \mathbb{C})$ is an isomorphism.*

Let $X = \ker(\alpha)$ be the generator of the geodesic flow. For $k = 0, 1, 2$ define $\Omega_0^k(M) \subset \Omega^k(M)$ the forms annihilated by ι_X .

Rename the operator

$$P_k = P|_{\Omega_0^k} = -i\mathcal{L}_X = -i\iota_X d : \Omega_0^k \rightarrow \Omega_0^k \quad (113)$$

By 71 each P_k admits a meromorphic resolvent R^k with decomposition around poles, in particular around 0.

$$R^k(z) = R^{k, hol}(z) + \sum_{j=0}^N \frac{(-P_k - z)^{j-1} \Pi_k}{z^j} \quad (114)$$

From this decomposition we try to recover the normal and generalised eigenspaces for P_k . Remark that Π_k extends continuously to an operator $\Pi_k : \mathcal{D}'_{E_u^*} \rightarrow \mathcal{D}'_{E^*u}$.

Definition 115. *Define $m_k = \dim(\text{range}(\Pi_k))$.*

For each k define

$$\zeta_k(\lambda) = \exp\left(-\sum_{\gamma} \frac{e^{iT_\gamma \lambda} T_\gamma^\# \det \wedge^k \mathcal{P}_\gamma}{|\det(I - \mathcal{P}_\gamma)| T_\gamma}\right) \quad (116)$$

so that by 96 we get

$$\zeta_S(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda) \zeta_2(\lambda)} \quad (117)$$

and since m_k is the order of vanishing of ζ_k in 0, it is sufficient to prove that $\chi(M) = -m_0 + m_1 - m_2$. We will prove that $m_0 = m_2 = 1$ and $m_1 = b_1(M)$. We try to exploit 114 to find a relation between the m_k and eigenvectors.

Definition 118. *Let*

$$Res^k = \{u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \cap \Omega_0 : P_k u = 0\} = \{u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) : \iota_X u = 0 \ \iota_X(du) = 0\} \quad (119)$$

that is the kernels of the P_k on $\mathcal{D}'_{E_u^*}$.

The generalised eigenspace for 0 (of dimension m_k) is usually larger than the kernel, and the equality is given by an idempotency condition.

Proposition 120. *In general $m_k \geq \dim Res^k$. We have $m_k(0) = \dim Res^k$ if the following is satisfied:*

$$\forall u \in \mathcal{D}'_{E_u^*} \quad P_k^2 u = 0 \Rightarrow P_k u = 0 \quad (121)$$

So finally our objective is to prove

Proposition 122. *The following are true*

- $\dim Res^0 = \dim Res^2 = 1$
- $\dim Res^1 = b_1(M)$
- *The condition 121 is true for $k = 0, 1, 2$*

The most interesting case is the one for $k = 1$, we will therefore assume true the result for $k = 0, 2$. Notice that in this case

$$Res^0 = \{c \in \mathbb{C}\} \quad Res^2 = \{c d\alpha : c \in \mathbb{C}\} \quad (123)$$

We now proceed to prove for $k = 1$:

Lemma 124. *Let $\Gamma \subset T^*M \setminus 0$ be a closed conic set. Assume that $u \in \mathcal{D}'(M, \Omega^k)$, $du \in C^\infty(M, \Omega^{k+1})$. Then there exist $v \in C^\infty(M, \Omega^k)$ and $w \in \mathcal{D}'(M, \Omega^{k-1})$ such that $u = v + dw$.*

We assume this result as well.

Proposition 125. *Let $P \in \Psi^1(M, \mathcal{E})$ be pseudodifferential and self-adjoint on $L^2(M, \mathcal{E})$, with principal symbol $-iX$. Suppose that $u \in \mathcal{D}'(M, \mathcal{E})$ satisfies $Im\langle Pu, u \rangle \geq 0$ and $Pu \in C^\infty(M, \mathcal{E})$, then $u \in C^\infty(M, \mathcal{E})$.*

Proof. It is sufficient to prove that $Au \in C^\infty$ for all $A \in \Psi_h^{comp}$ (see 30). Again we finish if we can prove that for each $A \in \Psi_h^{comp}$ there exists $B \in \Psi_h^{comp}$ such that

$$\|Au\|_{L^2} \leq Ch^{1/2}\|Bu\|_{L^2} + \mathcal{O}(h^\infty) \quad (126)$$

One can construct a function $\chi \in C_c^\infty(T^*M, [0, 1])$ such that

$$supp(1 - \chi)^* M \setminus 0 \quad H_p \chi \leq 0 \text{ near } E_u^* \quad H_p \chi < 0 \text{ on } E_u^* \cap WF_h(A) \quad (127)$$

So by taking a modification of $Op_h^M(\chi)$ we find F such that

$$F \in \Psi_h^{comp}, \quad \sigma_h(F) = \chi \quad WF_h(I - F) \subset T^*M \setminus 0 \quad F^* = F \quad (128)$$

We can also find $A_1 \in \Psi_h^{comp}$ such that $WF_h(A_1) \subset T^*M \setminus 0$ and $WF_h(A_1) \cap E_u^* = \emptyset$ and

$$-\frac{1}{2}H_p \chi - |\sigma_h(A_1)|^2 \geq C^{-1}|\sigma_h(A)|^2 \quad (129)$$

Since $Pu \in C^\infty$, we have $WF_h(Pu) \cap \bar{T}^*M \setminus 0 = \emptyset$, so $(I - F)Pu = \mathcal{O}(h^\infty)_{C^\infty}$, so

$$-Im\langle Pu, u \rangle_{L^2} \leq \mathcal{O}(h^\infty) \quad (130)$$

And by self-adjointness of F, P we find $-Im\langle FPu, u \rangle = \frac{1}{2i}\langle [P, F]u, u \rangle_{L^2}$. Observe that $\frac{1}{2i}[P, F] \in \Psi_h^{comp}$ and $\sigma_h(\frac{1}{2i}[P, F]) = -\frac{1}{2}H_p \chi$.

By ?? applied to the operator $\frac{1}{2i}[P, F] + A_1^* A_1 - C^{-1}A^* A$ we get

$$\|Au\|_{L^2}^2 \leq C\|A_1 u\|_{L^2}^2 + \frac{C}{2i}\langle [P, F]u, u \rangle_{L^2} + Ch\|u\|_{L^2}^2 + \mathcal{O}(h^\infty) \quad (131)$$

Since $WF_h(u)^* M \subset E_u^*$ we have $A_1 u = \mathcal{O}(h^\infty)_{L^2}$ and therefore we are done. \square

Lemma 132. *Suppose $u \in Res^1$. Then there exists $\phi \in \mathcal{D}'_{E_u^*}$ such that $u - d\phi \in \Gamma(\Omega^1)$ and $d(u - d\phi) = 0$. The cohomology class of $u - d\phi$ is independent of the chosen ϕ , and $u \mapsto [u - d\phi_u]$ is an isomorphism $Res^1 \simeq H^1(M)$*

Proof. First, we have that $u \in Res^1 \Rightarrow du = 0$. Indeed we have $du \in Res^2$ so $du = cd\alpha$, also $\iota_X u = 0$ so $u \wedge d\alpha = 0$, hence

$$0 = \int_M u \wedge d\alpha dVol_M = \int_M du \wedge \alpha dVol_M = c vol(M) \quad (133)$$

so $c = 0$ and $du = 0$. By 124 we find $\phi \in \mathcal{D}'_{E_u^*}$ as required. If another $\psi \in \mathcal{D}'_{E_u^*}$ satisfies the thesis, then $d(\phi - \psi) \in C^\infty(M)$ so again by 124 $\phi - \psi \in C^\infty(M)$ and so $[u - d\phi] = [u - d\psi]$.

Suppose now that $[u - d\phi] = 0$, so we may assume $u = d\phi$. Then we have that $X\phi = \iota_X u = 0$ so $\phi \in Res^0$, then $\phi = c \in \mathbb{C}$ and $u = 0$. We now need to prove injectivity and surjectivity. Consider $v \in \Gamma(\Omega^1)$ closed. If we find $\phi \in \mathcal{D}'_{E_u^*}$ such that $X\phi = -\iota_X v$ we are done. We have that

$$\int_M \iota_X v dVol_M = \int_M v \wedge d\alpha = \int_M (dv) \wedge \alpha = 0 \quad (134)$$

So we conclude by the following:

Lemma 135. *Suppose $f \in C^\infty(M)$ and $\int_M f dVol_M = 0$. Then there exists $u \in \mathcal{D}'_{E_u^*}$ such that $Xu = f$.* \square

Now we can finish:

Lemma 136. *Condition 121 is satisfied for $k = 1$, that is*

$$u \in \mathcal{D}'_{E_u^*}(M, \Omega_1), \quad \iota_X u = 0 \quad \iota_X(du) \in Res^1 \Rightarrow \iota_X(du) = 0 \quad (137)$$

Proof. Let $v = \iota_X(du) \in Res^1$. Now $\alpha \wedge du = adVol_M$ for $a \in \mathcal{D}'_{E_u^*}$. Then $\int_M adVol_M = \int_M u \wedge d\alpha = \int_M \iota_X u dVol_M = 0$. Also since $X\alpha = 1$ and $\iota_X(d\alpha) = 0$ are constant, we have $\mathcal{L}_X \alpha = \mathcal{L}_X d\alpha = 0$, and since $v \in Res^1$ we get $dv = 0$. Then $(Xa)dVol_M = \mathcal{L}_X(\alpha \wedge du) = \alpha \wedge dv = 0$. Then $Xa = 0$ so $a = 0$ and so $\alpha \wedge du = 0$. Then $du = \alpha \wedge \iota_X du = \alpha \wedge v$, so by 132 we find

$$w \in \Gamma(M, \Omega^1) \quad \phi \in \mathcal{D}'_{E_u^*}(M) \quad v = w + d\phi \quad dw = 0 \quad (138)$$

Since $\iota_X v = 0$ we get $X\phi = -\iota_X w$ and integration by parts gives

$$0 = Re \int_M du \wedge \bar{w} = Re \int_M \alpha \wedge d\phi \wedge \bar{w} = re \int_M \phi \bar{w} \wedge d\alpha = -Re \langle X\phi, \phi \rangle_{L^2} \quad (139)$$

By 125 we obtain then $\phi \in C^\infty(M)$ and so $v \in \Gamma(\Omega^1)$. Since $(e^{tX})^* v = v$ we have

$$\langle v(x), z \rangle = \langle v(e^{tX}x), de^{tX}(x) \cdot z \rangle \quad (x, z) \in TM, t \in \mathbb{R} \quad (140)$$

But for $t \rightarrow \pm\infty$ the right hand side vanishes for $z \in E_{s,u}(x)$, respectively. Since $\iota_X v = 0$, we get $v = 0$. \square

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