# PATHWISE ITÔ CALCULUS

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Abstract. The classical definition of stochastic integral is based on the probabilistic method and is restricted on the semimartingale integration. The Itô's formula obtained is then a stochastic form and hence its application to the finance will lead to some 'model risk'. However, in his seminal paper *Calcul d'Itô sans probabilités*, Hans Follmer provided a pathwise proof of the Itô formula, using the concept of *quadratic variation along a sequence of partitions*. The pathwise integration will enhance the robustness of the hedge strategies and so investigation of this kind of integration is needed.

The theories showed here extend the pathwise integration to more general non-anticipative functionals on the stopped paths and to more irregular paths which may admit a higher order form of quadratic variation. The roughness is recently observed in the implied smile surface for some equities and hence the investigation to the irregular path will lead to a nice application to finance market. We will only talk about the theories in this thesis.

## 1. Introduction to Stochatic Analysis

In this section, I will give a brief introduction to the Itô's formula which is widely used in probability theory.

Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a probability space given a $\sigma-$algebra $\mathcal{F}$, a filtration $\mathcal{F}_t$(which is just a set of $\sigma-$algebras such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s < t$) and a probability measure $\mathbb{P}$. Let's first review some definitions in stochastic process.

**Definition 1.1.** We say a process $H_t$ is a stochastic process if for each time $t$, $H_t$ is a random variable. And we say a process $H_t$ is measurable if the map $H : (\Omega \times \mathbb{F}, \mathcal{F} \otimes Borel(\mathbb{F})) \to (\mathbb{G}, \mathcal{G})$ is measurable, where $\mathbb{F}$ is a subset of $\mathbb{R}$.

We will mainly deal with the case when $\mathbb{F} = \mathbb{R}^+ \cup \{0\}$ and $\mathbb{G} = \mathbb{R}$, which is called a continuous time process.

**Definition 1.2.** We say a stochastic process $H_t$ is adapted if for each time $t$, $H_t$ is $\mathcal{F}_t-$measurable. We say $H_t$ is progressively measurable if for each time $t$, the map $H : (\Omega \times [0, t], \mathcal{F}_t \otimes Borel([0, t])) \to \mathbb{R}$ is measurable.

**Definition 1.3.** We say $T$ is a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ and the $\sigma-$algebra $\mathcal{F}_T$ generated by $T$ consists of all sets $A$ such that $A \cap \{T \leq t\} \in \mathcal{F}_t$.

**Definition 1.4.** We say an adapted stochastic process $M_t$ is a sub(super)-martingale if we have for all $s < t$ such that

$$M_s \leq (\geq)\mathbb{E}[M_t|\mathcal{F}_s]$$

and $M_t$ is a martingale if it is both sub-martingale and super-martingale.

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***Remark***. A theorem of martingale tells us that every martingale has a càdlàg modification, so we will always consider càdlàg martingales afterwards. It is then progressively measurable.

**Definition 1.5.** We say a càdlàg, adapted stochastic process $M_t$ is a local martingale if there exists a sequence of stopping times $T_i$ such that $T_i(\omega)$ is non-decreasing for all $\omega \in \Omega$ and $M_{T_i \wedge t} \mathbb{1}_{T_i > 0}$ is a martingale.

1.1. **Preliminaries on Stochastic Integration.**

**Definition 1.6.** Suppose $A_t$ is a càdlàg process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We say $A_t$ is an increasing process if almost surely all the paths of $A_t$ are non-decreasing. We say $A_t$ is an FV processs if a.s the paths of $A_t$ are of finite variation.

According to Riemann-Stieltjes integration theory, it is possible to obtain the integration of

$$\int_0^t H_t dA_t$$

if the function $H_t$ is pathwise bounded. We can simply view $A_t$ as a random signed measure here. So here we define stochastic integration by pathwise integration. But if the integator is not a pathwise FV process but a local martingale, how should we define the integration with respect to a local martingale? The first question is, can we define the integral with respect to a local martingale just as what we do above?

**Proposition 1.7.** *The sum $\sum_{t_k, t_{k+1} \in \pi_n} H_{t_k}(M_{t_{k+1}} - M_{t_k})$ converges to a limit for all continuous functions $H_t$ when the mesh of $\pi_n$ tends to $0$ only if $M_t$ is of finite variation.*

But we will know from below that a martingale generally is not of finite variation. Hence the approximation method used for defining Riemann-Stieltjes integration fails in local martingale case so it is necessary to use stochastic approach for stochastic integration.

In order to define the stochastic integration, we view integration as a mapping from integrand space to result space. So it is necessary to define this 'mapping' on a dense subset of integration space.

**Definition 1.8.** We say $H$ is a simple predictable process if $H$ has a representation

$$H_t = H_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathbb{1}_{(T_i, T_{i+1}]}(t)$$

where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$ with $|H_i| < \infty$ a.s.,$0 \leq i \leq n$. The collection of simple predictable processes is denoted by $\mathbb{S}$.

We can first define integration on $\mathbb{S}$ with respect to a local martingale $M_t$ by

$$I_M(H)_t = \int_0^t H_s dM_s = H_0 M_0 + \sum_{i=1}^{n} H_i \left( M_{T_{i+1} \wedge t} - M_{T_i \wedge t} \right)$$

**Definition 1.9.** We denote $\mathbb{D}$ the space of adapted processes with càdlàg paths and $\mathbb{L}$ the space of adapted processes with càglàd paths. We also equip $\mathbb{S}, \mathbb{L}, \mathbb{D}$ with topology of uniformly convergence on compact sets in probability, i.e. for each $t > 0$, $\sup_{0 \leq s \leq t} |H_s^n - H_s|$ converges to 0 in probability.

Hence we can view the above integral as $I_M : \mathbb{S} \to \mathbb{D}$. It is easy to prove that $I_M$ is actually a continuous function. Furthermore, a simple predictable process is actually a process in the space $\mathbb{L}$ and we have

**Proposition 1.10.** *The space $\mathbb{S}$ is dense in the space $\mathbb{L}$,*

Then the denseness of $\mathbb{S}$ in $\mathbb{L}$ will give definition of stochastic integration of

$$\begin{aligned} I_M : \mathbb{L} &\to \mathbb{D} \\ H_t &\to \int_0^t H_s dM_s \end{aligned}$$

where $H_t$ is in $\mathbb{L}$ and $M_t$ is a local martingale. And we denote

$$\int_{0+}^t H_t dM_t = \int_0^t H_s dM_s - H_0 M_0$$

Furthermore, the integral given by above limit process is still a local martingale.

***Remark.*** The integrand space can change according to the integrator. Generally for right-continuous martingale we will choose $\mathbb{L}$. But if the martingale is continuous then the integrand could be all progressively measurable processes.

**Definition 1.11.** We say a càdlàg, adapted process $X_t$ is a semimartingale if we have the decomposition

$$X_t = X_0 + M_t + A_t$$

where $M_t$ is a local martingale and $A_t$ is an FV process.

1.2. **Quadratic Variation.** Before introduce the Itô's formula, we need to introduce the quadratic variation. For a stopping time $T$, we denote $X_t^T = X_{T \wedge t}$ for every stochastic process $X$.

**Definition 1.12.** Suppose $X, Y$ are semimartingales. The quadratic variation process of $X$, denoted by $[X, X] = ([X, X]_t)_{t \geq 0}$ is defined by:

$$[X, X] = X^2 - 2 \int X_- dX$$

The quadratic covariation of $X, Y$, also called the bracket process of $X, Y$, is defined by:

$$[X, Y] = XY - \int X_- dY - \int Y_- dX$$

so that the operation $(X, Y) \to [X, Y]$ is bilinear and symmetric.

The quadratic variation can also be defined in another ways

**Proposition 1.13.** *Let $X, Y$ be two local martingales. Then $[X, Y]$ is the unique adapted càdlàg process $A$ with paths of finite variation on compacts satisfying the two properties:*

(1) *$XY - A$ is a local martingale*
(2) *$\Delta A = \Delta X \Delta Y, A_0 = X_0 Y_0$*

and it is actually a limit in the probability sense.

**Proposition 1.14.** *Let $X$ and $Y$ be two semimartingales. Then the bracket process $[X, Y]$ satisfies:*

(1) $[X, Y]_0 = X_0 Y_0, \Delta[X, Y] = \Delta X \Delta Y$

(2) *If $\sigma_n$ is a sequence of random partitions whose mesh tends to 0 a.s and the sup tends to infinity a.s, then*

$$[X,Y] = X_0Y_0 + \sum_i (X^{T^n_{i+1}} - X^{T^n_i})(Y^{T^n_{i+1}} - Y^{T^n_i})$$

*where convergence is uniformly on compacts in probability, and where $\sigma_n$ is the sequence $0 = T^n_0 \le T^n_1 \le \cdots T^n_{k_n}$, with $T^n_i$ stopping times.*

(3) *If $T$ is a stopping time, then $[X^T, Y] = [X, Y^T] = [X^T, Y^T] = [X, Y]^T$.*

***Remark.*** Here we can see that a martingale is generally not of finite variance. Otherwise, consider the term

$$\sum (X^{T^n_{i+1}} - X^{T^n_i})^2 \le \max_i \{X^{T^n_{i+1}} - X^{T^n_i}\} \sum |X^{T^n_{i+1}} - X^{T^n_i}|$$

If $X_t$ is a continuous martingale, then $\max_i\{X^{T^n_{i+1}} - X^{T^n_i}\}$ will tend to 0 and hence $\sum |X^{T^n_{i+1}} - X^{T^n_i}|$ must tends to infinity.

Now since the process $[X,X]$ is nondecreasing with right continuous paths, and since $\Delta[X,X]_t = (\Delta X_t)^2$ for all $t \ge 0$ (with the convention that $X_{0-} = 0$), we can decompose $[X,X]$ path by path into a continuous part and its jump part.

**Definition 1.15.** For a semimartingale $X$, the process $[X,X]^c$ denotes the path by path continuous part of $[X,X]$. We can then write

$$[X,X]_t = [X,X]^c_t + X_0^2 + \sum_{0<s\le t} (\Delta X_s)^2 = [X,X]^c_t + \sum_{0\le s\le t} (\Delta X_s)^2$$

***Remark.*** The concept of quadratic variation can be extended to higher dimension. If we consider a path $X = (X_1, \cdots, X_n) \in \mathbb{D}([0,\infty], \mathbb{R}^n)$, then we say the path $X$ has quadratic variation $[X]_t$ if and only if for any $1 \le i < j \le n$, the quadratic covariation $[X_i, X_j]$ exists.

1.3. **Itô's formula.** With the definition of Stochastic Integration, we can then get Ito's formula for semimartingales.

**Theorem 1.16.** *Suppose $f$ is a twice differentiable function on $\mathbb{R}$ and $M_t$ is a semimartingale with above decomposition. We have*

$$\begin{aligned} f(M_t) &= f(M_0) + \int_{0+}^t f'(M_{s-})dM_s + \frac{1}{2}\int_{0+}^t f''(M_{s-})d[M,M]^c_s \\ &+ \sum_{0<s\le t} \{f(M_s) - f(M_{s-}) - f'(M_{s-})\Delta M_s\} \end{aligned}$$

*and we have the Itô isometry here:*

$$\left[\int_{0+}^{\cdot} f'(M_{s-})dM_s\right]_t = \int_{0+}^t |f'(M_{s-})|^2 d[M]_s$$

***Remark.*** This theorem is also called change of variable formula since we can write it in differential form.

$$\begin{aligned} df(M_t) &= f'(M_{s-})dM_s + \frac{1}{2}f''(M_{s-})d[M,M]^c_s \\ &+ \sum_{0<s\le t} \{f(M_s) - f(M_{s-}) - f'(M_{s-})\Delta M_s\}\,\delta_s \end{aligned}$$

## 2. Pathwise Itô Calculus

### 2.1. **Motivations: Pathwise Ito's formula.**

In the paper *Calcul d'Itô sans probabilités*, Hans Follmer provided a pathwise proof of the Itô formula, using the concept of *quadratic variation along a sequence of partitions.*

We suppose $x$ is a real function on $[0,\infty)$, right continuous and has left limit. And we use the notation $x_t = x(t), \Delta x_t = x_t - x_{t-}$.

**Definition 2.1.** Suppose $\{\pi_n\}_{n=1,2\cdots}$ is a sequence of finite subdivision of the real line. We say $x$ has finite quadratic variation along the sequence $\pi_n$ if the following measures

$$\mu_n = \sum_{t_i^n \in \pi_n} \left( x_{t_{i+1}^n} - x_{t_i^n} \right)^2 \delta_{t_i^n}$$

converge weakly to a Radon measure $\mu$ on $[0,\infty)$ who has the decomposition

$$[x,x]_t = [x,x]_t^c + \sum_{s\le t} (\Delta x_t)^2$$

where $[x,x]_t$ is the cumulative distribution function of $\mu$ and $[x,x]^c$ is its continuous part. We will use $Q_\pi([0,T],\mathbb{R})$ to denote the space of all the paths that have quadratic variation along the sequence $\pi_n$. And will also denote the result quadratic variation path as $[x]_\pi$.

***Remark*.** The paper [3] gives equivalent condition for the above convergence. It shows that $x$ has finite quadratic variation along a sequence $\{\pi_n\}$ if and only if càdlàg functions

$$s_n(t) = \sum_{t_i^n \in \pi_n} |x_{t\wedge t_{i+1}^n} - x_{t\wedge t_i^n}|^2$$

converges in $\mathbb{D}([0,\infty),\mathbb{R})$ equipped with Skorohod $J_1$ topology. Here the space $\mathbb{D}([0,\infty))$ is the space for all càdlàg functions on $[0,\infty)$. Furthermore, they are all equivalent to the easy-to-check condition

(1) $q_n(t) = \sum_{t\ge t_i^n\in\pi_n} |x_{t_{i+1}^n} - x_{t_i^n}|^2$ converges to $q(t)$ pointwise
(2) $q(t)$ has decomposition $q(t) = q^c(t) + \sum_{s\le t}(\Delta x(s))^2$. Here $q^c(t)$ is the continuous part of $q(t)$.

Now we can estabilish Ito's formula for this category of real functions.

**Theorem 2.2.** *Suppose $x$ has finite quadratic variation along the sequence $\pi_n$ and $F$ is a $C^2$ function on $\mathbb{R}$. Then we have the Ito's formula*

$$\begin{aligned} F(x_t) &= F(x_0) + \int_0^t F'(x_s)d^\pi x_s + \frac{1}{2}\int_0^t F''(x_{s-})d[x,x]_s \\ &+ \sum_{s\le t}\left[F(x_s) - F(x_{s-}) - F'(x_{s-})\Delta x_s - \frac{1}{2}F''(x_s)\left(\Delta x_s\right)^2\right] \end{aligned}$$

*where we set*

$$\int_0^t F'(x_s)d^\pi x_s = \lim_n \sum_{t_i^n\le t, t_i^n\in\pi_n} F'(x_{t_i^n})(x_{t_{i+1}^n} - xt_i^n)$$

Now we see how to define the integral $\int_0^t F'(x_s)dx_s$ whose integrator is of infinite variation. Furthermore, we can rebuild the stochastic Itô's formula from above version.

**Proposition 2.3.** *Suppose $X_t$ is a semi-martingale with canonical decompostion $X_t = M_t + A_t$, where $M_t$ is a local martingale and $A_t$ is an FV process. Then there is a sequence of partitions $\tau_n = \{0 = t_0^n < t_1^n < \cdots < \infty\}$ such that for each time $t$, we have*

$$[X,X]_t = \lim_{n\to\infty} \sum_{t_i^n \in \pi_n} (X_{t_{i+1}^n \wedge t} - X_{t_i^n \wedge t})^2$$

*in probability and almost surely. Specially, we then have the weak convergence*

$$\mu_n = \sum_{t_i^n \in \tau_n} \left(X_{t_{i+1}^n} - X_{t_i^n}\right)^2 \delta_{t_i^n} \to \mu$$

*where the monotonic function associated to $\mu$ is exactly $[X,X]_t$ almost surely.*

***Remark.*** According to the properties of quadratic variation of semimartingale, we see almost surely, the decomposition

$$[X,X]_t = [X,X]_t^c + \sum_{s\le t} (\Delta X_s)^2$$

holds. Hence almost surely, $X$ has the quadratic variation along the sequence $\tau_n$. Hence the pathwise Itô's formula applys here to obtain the stochastic Itô's formula. This shows that we can obtain classical Itô's formula from pathwise Itô's formula.

***Remark.*** The space $Q_\pi([0,T],\mathbb{R})$ is not a vector space and for the path in high dimension, we will need to add additional conditions in the definiton of quadratic variation along $\pi_n$. We say an $n-$dimensional path $x = (x_1, x_2, \cdots, x_n)$ has quadratic variation along the sequence $\pi_n$ if $x_i, x_j, x_i + x_j \in Q_\pi([0,T],\mathbb{R})$ for all $1 \le i < j \le n$. Alternatively, if we consider the matrix function

$$q_n(t) = \sum_{t \ge t_i^n \in \tau_n} (x_{t_{i+1}^n} - x_{t_i^n})(x_{t_{i+1}^n} - x_{t_i^n})^T$$

where we view $x$ as a column vector. Then $x$ has quadratic variation along the sequence $\tau_n$ if and only if $q_n(t)$ converges in $\mathbb{D}([0,T],\mathbb{R}^{n\times n})$ equipped with Skorohod $J_1$ topology. And we denote this space by $Q_\pi([0,T],\mathbb{R}^n)$. For the paths in $Q_\pi([0,T],\mathbb{R}^n)$, we still have the Itô's formula and the proof is similar.

The remarks show that classical Itô's formula can be easily derived from pathwise Itô's formula and this gives motivation to investigate more on the pathwise integration theory and its relation with stochastic analysis. I hereby show some works in this field.

2.2. **Invariance of quadratic variation.** If we look at the definition of the integral $\int_0^t F'(x_s)dx_s$, we see it is given only when we provide a sequence of partitions. Hence a natural question arise here: ***what happens to the result integral if we change the sequence of partitions?*** If we look at the above Itô's formula, we see that actually $\int_0^t F'(x_s)dx_s$ is determined by the process $[x,x]_t$. Hence our question becomes: ***For what families of sequences of partitions, the quadratic variation is invariant?***

This problem is partly solved by Rama Cont and Purba DAS, in their paper *Quadratic variation and quadratic roughness.* The key of this paper is to define a quadratic roughness properties. Before introducing this property, we give some definitions.

We will say that two sequences $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ are asymptotically comparable, denoted $a_n \asymp b_n$, if $|a_n| = O(|b_n|)$ and $|b_n| = O(|a_n|)$ as $n \to \infty$.

**Definition 2.4.** Let $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(\pi_n)}^n = T\}$ be a sequence of partitions of $[0,T]$, where $N(\pi_n)$ is the number of points in $\pi_n$. Define

$$\underline{\pi_n} = \inf_{i=0,\cdots,N(\pi_n)-1} |t_{i+1}^n - t_i^n|, \qquad |\pi_n| = \sup_{i=0,\cdots,N(\pi_n)-1} |t_{i+1}^n - t_i^n|$$

Then we say that $(\pi_n)_{n\geq 1}$ is balanced if

$$\exists c > 0, \forall n \geq 1, \frac{|\pi_n|}{\underline{\pi_n}} \leq c$$

We will denote by $\mathbb{B}([0,T])$ the set of all balanced partition sequences of $[0,T]$

This is a very natural condition to investigate the problem and it contains the dyadic partitions. For two balanced partition sequences $\tau = \{\tau_n\}_{n\geq 1}$ and $\sigma = \{\sigma_n\}_{n\geq 1}$, we say they are comparable if

$$0 < \liminf_{n\to\infty} \frac{|\sigma_n|}{|\tau_n|} \leq \limsup_{n\to\infty} \frac{|\sigma_n|}{|\tau_n|} < \infty$$

Next definition is about coarsening a partition sequence by deleting some points.

**Definition 2.5.** Let $\pi_n = (0 = t_0^n < t_1^n < \cdots < t_{N(\pi_n)}^n = T)$ be a balanced sequence of partitions of $[0,T]$ with vanishing mesh $|\pi_n| \to 0$ and $0 < \beta < 1$. A $\beta-$coarsening of $\pi$ is a sequence of subpartitions of $\pi_n$:

$$A_n = \{0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \cdots < t_{p(n,N(A_n))}^n = T\}$$

such that $\{A_n\}_{n\geq 1}$ is a balanced partition sequences of $[0,T]$ and $|A_n| \asymp |\pi_n|^\beta$.

**Definition 2.6.** Let $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(\pi_n)}^n = T\}$ be a balanced sequence of partitions of $[0,T]$ with $|\pi_n| \to 0$ and $0 < \beta < 1$. We say that $x \in Q_\pi([0,T],\mathbb{R}^d)$ has the quadratic roughness property along $\pi$ with coarsening index $\beta$ on $[0,T]$ if for any $\beta-$coarsening $A_n = \{0 = t_{p(n,0)}^n < t_{p(n,0)^n} < \cdots < t_{p(n,N(A^n))}^n = T\}$ of $\pi$ we have

$$\sum_{j=1}^{N(A_n)} \sum_{p(n,j-1)\leq i\neq i' < p(n,j)} (x_{t_{i+1}^n\wedge t} - x_{t_i^n\wedge t})^t (x_{t_{i'+1}^n\wedge t} - x_{t_{i'}^n\wedge t}) \to 0$$

as $n$ tends to infinity. And we denote by $R_\pi^\beta([0,T],\mathbb{R}^d)$ the set of paths satisfying this property.

Here is the main result of the paper

**Theorem 2.7.** *Let $\pi = \{\pi_n\}_{n\geq 1}$ be a balanced sequence of partitions of $[0,T]$ with $|\pi_n| \to 0$ and*

$$\limsup_n \frac{|\pi_n|}{|\pi_{n+1}|} < \infty$$

*Suppose $x \in C^\alpha([0,T],\mathbb{R}^d) \cap R_\pi^\beta([0,T],\mathbb{R}^d)$ for some $0 < \beta \leq \alpha$. Then for any balanced sequence of partitions $\tau = \{\tau_n\}_{n\geq 1}$, we have $x \in Q_\tau([0,T],\mathbb{R}^d)$ and $[x]_\tau = [x]_\pi$*

***Remark.*** This theorem can be used to give a generic definition of quadratic variation.

We denote the set $\mathcal{P}_\gamma([0,T])$ to be the set of balanced partition sequences $\{\pi_n\}_{n\geq 1}$ satisfying the assumption of above theorem and that $|\pi_n| = o(|\log n|^{-(\epsilon+\gamma)})$ for some $\epsilon > 0$:

$$\mathcal{P}_\gamma([0,T]) = \{\pi \in \mathbb{B}([0,T]), \quad \limsup_{n\to\infty} \frac{|\pi_n|}{|\pi_{n+1}|} < \infty, \quad \exists \epsilon > 0, (\log n)^{\gamma+\epsilon}|\pi_n| \to 0\}$$

and for each $\pi \in \mathcal{P}_\gamma([0,T])$, we denote $\nu(\pi) = \sup\{\nu > \gamma, |\pi_n| = o(|\log n|^{-\nu})\}$. we define furthermore that

$$\mathcal{Q}([0,T],\mathbb{R}^d) = C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) \cap \left( \bigcup_{\pi\in\mathcal{P}_4([0,T])} R_\pi^{\frac{4+\nu(\pi)}{4\nu(\pi)}}([0,T],\mathbb{R}^d) \right)$$

Now it can be proved that all the Brownian sample path is in this set. And we can actually define the map

$$[] : \mathcal{Q}([0,T],\mathbb{R}^d) \to C^0([0,T], S_d^+)$$

such that

$$\forall \tau \in \mathbb{B}([0,T]), \quad \forall x \in Q_\tau([0,T],\mathbb{R}^d) \cap \mathcal{Q}([0,T],\mathbb{R}^d), \quad \forall t \in [0,T], \quad [x]_\tau(t) = [x](t)$$

and we can then call $[x]$ the quadratic variation of $x$. Furthermore, the value $\int_0^t F'(x_s)$ will be determined by this quadratic variation.

***Remark.*** The above remark shows a way to define quadratic variation of some set of paths including Brownian sample paths. But actually not all semimartingale sample paths are in the set $\mathcal{Q}([0,T],\mathbb{R}^d)$. However, from change of time theory in stochastic analysis, any local martingale can be written as a time change of the Brownian motion:

$$M_t = B_{[M]_t}$$

hence when $[M]_t$ has some good continuity, for example Holder continuity, we can show the sample paths of $M_t$ is actually in some holder space. Hence we can extend the set $\mathcal{Q}$ to give a partition-independent quadratic variation definition of $M_t$. Now if we denote the above set by $\mathcal{Q}_{\frac{1}{2}}$, then by defining similarly $\mathcal{Q}_\alpha$ for $\alpha < \frac{1}{2}$ by above procedure. Note the partitions in the definition of the set $\mathcal{Q}_\alpha$ will be different. And we can define the partition-independent quadratic variation for the paths in $\cup_\alpha \mathcal{Q}_\alpha$. And this set will include a lot of semimartingale sample paths.

2.3. **Pathwise integration for non-anticipative functionals.** This method of pathwise integration can be extended for the non-anticipative functionals. Here we begin with the definition of non-anticipative functionals. And we will use different notations for non-anticipative functionals case.

Let $X$ be the canonical process on $\Omega = \mathbb{D}([0,T],\mathbb{R}^d)$ and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $X$. We can see the process $Y$ adapted to $\mathbb{F}$ can be represented as a family of functionals $Y(t,\cdot) : \Omega \to \mathbb{R}$ with the property that $Y(t,\cdot)$ only depends on the path stopped at $t$,

$$Y(t,\omega) = Y(t,\omega(\cdot \wedge t))$$

so one can represent $Y$ as

$$Y(t,\omega) = F(t,\omega) \quad \text{for some functional} \quad F : [0,T] \times \mathbb{D}([0,T],\mathbb{R}^d) \to \mathbb{R}$$

where $F(t,\cdot)$ only needs to be defined on the set of paths stopped at $t$. Hence we can view adapted processes as functionals on the space of *stopped paths*: a stopped path is an equivalent class in $[0,T]\times\mathbb{D}([0,T],\mathbb{R}^d)$ for the following equivalent relation:

$$(t,\omega)\sim(t',\omega')\longleftrightarrow(t=t',\omega_t=\omega'_t)$$

where $\omega_t=\omega(t\wedge\cdot)$. We denote the space of stopped paths by $\Lambda_T^d$ and we endow it with a metric

$$d_\infty\left((t,\omega),(t',\omega')\right)=\sup_{u\in[0,T]}|\omega(u\wedge t)-\omega'(u\wedge t')|+|t-t'|$$

Then $(\Lambda_T^d,d_\infty)$ is a complete metric space. And we want to understand the integration and change of variable formula for functionals on $\Lambda_T^d$ which we call non-anticipative functionals.

**Definition 2.8.** A non-anticipative functional on $\mathbb{D}([0,T],\mathbb{R}^d)$ is a measurable map $F:(\Lambda_\infty^d,d_\infty)\to\mathbb{R}$ on the space of stopped paths.

To estabilish the change of variable formula, we will need several continuity and boundness assumptions on the functional $F$. Let's introduce them here:

**Definition 2.9.** A non-anticipative functional $F$ is called:

- continuous at fixed time if for any $t\in[0,T]$, $F(t,\cdot):\Omega\to\mathbb{R}^d$ is continuous under supremum norm.
- left continuous if $\forall(t,\omega)\in\Lambda_T^d,\forall\epsilon>0,\exists\eta>0$ such that $\forall(t',\omega')\in\Lambda_T^d$ if $t'<t$ and $d_\infty\left((t,\omega),(t',\omega')\right)\le\eta$. Then $|F(t,\omega)-F(t',\omega')|\le\epsilon$. And we denote $\mathbb{C}_l^{0,0}(\Lambda_T^d)$ to be the space of left continuous functionals.
- boundness-preserving if for any compact set $K\subset\mathbb{R}^d$ and $t_0<T$, there exists $C_{K,t_0}$ such that

  $$\forall t\le t_0,\forall\omega\in\mathbb{D}([0,T],\mathbb{R}^d),\quad\omega([0,T])\subset K\Rightarrow|F(t,\omega)|\le C_{K,t_0}$$

  We denote by $\mathbb{B}(\Lambda_T^d)$ the set of boundness-preserving functionals.

In order to estabilish Ito's formula for non-anticipative functional, we need notions of differentiability.

**Definition 2.10.** A non-anticipative functional $F$ is said to be

- Horizontally differentiable at $(t,\omega)\in\Lambda_T^d$ if

  $$\mathcal{D}F(t,\omega)=\lim_{h\to0+}\frac{F(t+h,\omega_t)-F(t,\omega_t)}{h}$$

  exists. If $\mathcal{D}F(t,\omega)$ exists for all $(t,\omega)\in\Lambda_T^d$, then $\mathcal{D}F$ is called the horizontal derivative of $F$.
- Vertically differentiable at $(t,\omega)$ if the map:

  $$\begin{aligned}g_{(t,\omega)}:\mathbb{R}^d&\to\mathbb{R}\\ h&\to F(t,\omega+h\mathbb{1}_{[t,T]})\end{aligned}$$

  is differentiable at 0. The gradient at 0 is called vertial derivative of $F$ at $(t,\omega)$:

  $$\nabla_\omega F(t,\omega)=\nabla g_{(t,\omega)}(0)\in\mathbb{R}^d$$

  If $F$ is vertically differentiable at all $(t,\omega)\in\Lambda_T^d$, $\nabla_\omega F:\Lambda_T^d\to\mathbb{R}^d$ defines a non-anticipative functional called vertical derivative of $F$.

And we consider following classes of functionals

**Definition 2.11.** We define $\mathbb{C}_b^{1,k}(\Lambda_T^d)$ as the set of non-anticipative functional $F$ which are

1. horizontally differentiable with $\mathcal{D}F$ continuous at fixed time
2. $k$ times vertically differentiable with $\nabla_\omega^j F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ for $j = 1, \cdots, k$
3. $\mathcal{D}F, \nabla_\omega F, \cdots, \nabla_\omega^k F \in \mathbb{B}(\Lambda_T^d)$

Now we fix a sequence of partition $\pi$ and suppose $\omega \in Q_\pi([0,T], \mathbb{R}^d)$ and the partition exhausts the jump times in the sense that

$$\sup_{t\in[0,T]-\pi_n} |\omega(t) - \omega(t-)| \to 0 \tag{2.1}$$

as $n \to \infty$. Then the piecewise-constant approximations

$$\omega^n(t) = \sum_{[t_i^n, t_{i+1}^n]\in\pi_n} \omega(t_{i+1}^n -)\mathbb{1}_{[t_i^n, t_{i+1}^n)}(t) + \omega(T)\mathbb{1}_{\{T\}}(t) \tag{2.2}$$

converge uniformly to $\omega$: $\sup_{t\in[0,T]} ||\omega^n(t) - \omega(t)|| \to 0$ as $n \to \infty$. We furthermore define

$$\omega_{t_i^n}^{n,\Delta\omega(t_i^n)} = \omega_{t_i^n}^n + \Delta\omega(t_i^n)\mathbb{1}_{[t_i^n,T]}, \quad \text{then } \omega_{t_i^n -}^{n,\Delta\omega(t_i^n)}(t_i^n) = \omega(t_i^n)$$

Then we can state the pathwise integration and the ito's formula for the non-anticipative functionals.

**Theorem 2.12.** *Let $\omega \in Q_\pi([0,T], \mathbb{R}^d)$ verify (2.1). Then for any $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$, the limit*

$$\int_0^T \nabla_\omega F(t, \omega_{t-}) d^\pi\omega = \lim_{n\to\infty} \sum_{k=1}^{N(\pi_n)} \nabla_\omega F\left(t_i^n, \omega_{t_i^n -}^{n,\Delta\omega(t_i^n)}\right)\left(\omega(t_{i+1}^n) - \omega(t_i^n)\right)$$

*exists and*

$$\begin{aligned} F(T, \omega_T) &= F(0,\omega_0) + \int_0^T \mathcal{D}F(t,\omega_t)dt + \int_0^T \nabla_\omega F(t,\omega_{t-})d^\pi\omega \\ &+ \frac{1}{2}\int_0^T tr\left({}^t\nabla_\omega^2 F(t,\omega_t)d[\omega]_\pi^c\right) \\ &+ \sum_{0<t\le T} \left(F(t,\omega_t) - F(t,\omega_{t-}) - \nabla_\omega F(t,\omega_{t-})\Delta\omega(t)\right) \end{aligned}$$

For a semimartingale $X$, we can find a sequence of partitions $\{\pi_n\}$ such that $X \in Q_\pi([0,T], \mathbb{R}^d)$ a.s. Hence, we can apply the pathwise ito's formula for each state and obtain the Ito's formula for non-anticipative functional.

**Theorem 2.13.** *Let $X$ be an $\mathbb{R}^d-$valued semimartingale and denote, for $t > 0$, $X_{t-}(u) = X(u)\mathbb{1}_{[0,t)}(u) + X(t-)\mathbb{1}_{[t,T]}(u)$. For any $F \in \mathbb{C}_b^{1,2}$, and $t \in [0,T)$,*

$$\begin{aligned} F(t, X_t) &= F(0,X_0) + \int_0^t \mathcal{D}F(s,X_s)ds + \int_0^t \nabla_\omega F(s,X_{s-})dX_s \\ &+ \frac{1}{2}\int_0^t tr\left({}^t\nabla_\omega^2 F(s,X_s)d[X]_s^c\right) \\ &+ \sum_{0<s\le t} \left(F(s,X_s) - F(s,X_{s-}) - \nabla_\omega F(s,X_{s-})\Delta\omega(s)\right) \end{aligned}$$

Hence from what we show above, we can define the integral for

$$\int_0^t Y_s dX_s$$

if $Y_t$ can be expressed as $\nabla_\omega F(s, X_{s-})$ for some non-anticipative functional $F$. In particular, if $F(t,X) = f(t, X(t))$ for some function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, then $\nabla_\omega F(s, X_{s-}) = \nabla f(t, X(t-))$ and we obtain the pathwise integration for $C^{1,2}([0,T] \times \mathbb{R}^d)$. Furthermore, it is showed in the paper [5] that we can have the Ito 'isometry' for pathwise integration. We first define the oscillation of a path $\omega \in \mathbb{D}([0,T], \mathbb{R}^d)$ along a sequence of partitions $\{\pi_n\}$ as

$$osc(\omega, \pi_n) = \max_{[t_j, t_{j+1}] \in \pi_n} \max_{r,s \in [t_j, t_{j+1}]} |\omega_r - \omega_s|$$

Here is the results:

**Theorem 2.14.** *Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of partitions of $[0,T]$, and $\omega \in Q_\pi([0,T], \mathbb{R}^d) \cap C^\nu([0,T], \mathbb{R}^d)$ for $\nu > \frac{\sqrt{3}-1}{2}$ such that $osc(\omega, \pi_n) \to 0$. Now let $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$ with $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$ and satisfies that*

$$\exists K > 0, \forall \omega, \omega' \in \mathbb{D}([0,T], \mathbb{R}^d), \forall t \in [0,T], |F(t,\omega) - F(t,\omega')| \leq K||\omega - \omega'||_\infty$$

*and $\max_{i \in \{0,1,\cdots,N(\pi_n)\}} |F(t_{i+1}^n, \omega) - F(t_i^n, \omega)| \to 0$ as $n \to \infty$. Then we have the isometry:*

$$\begin{aligned} [F(\cdot, \omega.)]_\pi(T) &= \left[\int_0^{\cdot} \nabla_\omega F(s, \omega_{s-}) d^\pi \omega\right]_\pi (T) \\ &= \int_0^T \left| < \nabla_\omega F(s, \omega_{s-})^t \nabla_\omega F(s, \omega_{s-}), d[\omega]_\pi > \right| \end{aligned}$$

2.4. **Pathwise integration for more irregular paths.** The above two works show that we can re-estabilish theorems in Stochastic Analysis using a pathwise approach. But we can also use pathwise approch to extend the theorems in Stochastic Analysis.

The Stochastic Analysis deals with martingales, who have finite quadratic variation. But what will happen to more irregular paths? If we consider some paths that have infinite quadratic variation but have finite 'pth'variantion along some sequence of partitions, what will be integration theory for these paths? For example, the sample paths for fractional Brownian Motions. There is another work by Rama Cont and Nicolas Perkowski which deals with more irregular continuous paths.

Similar to the definition of quadratic variation, we here give the definition of pth variation along a sequence of partitions.

**Definition 2.15.** Let $p > 0$, a continuous path $x \in C([0,T], \mathbb{R})$ is said to have $p$th variation along a sequence of partition $\{\pi_n\}_{n\geq 1}$ if $osc(S, \pi_n) \to 0$ and the sequence of measures

$$\mu_n := \sum_{t_j^n \in \pi_n} \delta(\cdot - t_j^n) |x_{t_{j+1}^n} - x_{t_j^n}|^p$$

converge weakly to a measure $\mu$ without atoms. In this case, we write $x \in V_p(\pi)$ and $[x]^p(t) := \mu([0,t])$, and we call $[x]^p$ the pth variation of $x$.

***Remark.*** The weak convergence of measures on $[0,T]$ is equivalent to pointwise convergence of cumulative distribution function on the continuous points of terminal measure. Hence above definition is equivalent to $\exists [x]^p \in C([0,T],\mathbb{R})$ such that

$$\sum_{t_j^n \in \pi_n} |x_{t\wedge t_{j+1}^n} - x_{t\wedge t_j^n}|^p \to [x]^p(t)$$

for all $t \in [0,T]$. Moreover, this pointwise convergence is equivalent to uniform convergence on $[0,T]$.

**Theorem 2.16.** *Let $p \in \mathbb{N}$ be even, $\{\pi_n\}$ be a given sequence of partitions of $[0,T]$. Suppose $x \in V_p(\pi)$ and $f \in C^p(\mathbb{R})$. Then we have the change of variable formula*

$$f(x(t)) = f(x(0)) + \int_0^t f'(x_s)dx_s + \frac{1}{p!}\int_0^t f^{(p)}(x_s)d[x]_s^p$$

*where the integral is defined by*

$$\int_0^t f'(x_s)dx_s := \lim_{n\to\infty} \sum_{[t_j^n, t_{j+1}^n]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(x_{t_j^n})}{k!}\left(x_{t\wedge t_{j+1}^n} - x_{t\wedge t_j^n}\right)^k$$

There is also a similar formula for non-anticipative functionals with irregular paths.

**Theorem 2.17.** *Let $p$ be an even integer, let $F \in \mathbb{C}_b^{1,p}(\Lambda_T^1)$, and let $x \in V_p(\pi)$ for a sequence of partition $\{\pi_n\}$ with vanishing mesh size $|\pi_n| \to 0$. Then the functional change of variable formula*

$$F(t,x_t) = F(0,x_0) + \int_0^t \mathcal{D}F(s,x_s)ds + \int_0^t \nabla_\omega F(s,x_s)d^\pi x + \frac{1}{p!}\int_0^t \nabla_\omega^p F(s,x_s)d[x]_s^p$$

*holds, where*

$$\int_0^t \nabla_\omega F(s,x_s)d^\pi x = \lim_{n\to\infty} \sum_{i=0}^{N(\pi_n)-1} \sum_{k=1}^{p-1} \frac{1}{k!}\nabla_\omega F(t_i^n, x_{t_i^n -}^n)\left(x_{t\wedge t_{i+1}^n} - x_{t\wedge t_t^n}\right)^k$$

*and $x^n$ is defined as in (2.2).*

The extension of integration of irregular path to higher dimension is not completely the same as the quadratic case. It is the same that $V_p(\pi)$ is not a vector space and in order to define the quadratic variation of $(x_1, x_2)$, [4] requires $x_1 + x_2, x_1, x_2 \in Q_\pi([0,T],\mathbb{R})$. And by expansion, we are able to define $[x_1, x_2]$ which then will be the condition that $(x_1,x_2) \in Q_\pi([0,T],\mathbb{R}^2)$. However for general even $p$, the expansion will involve a lot of terms and we want to give another convenient definition for $p$th variation for higher dimensional paths. We will consider symmetric tensor space $Sym_p(\mathbb{R}^d)$ here.

**Definition 2.18.** Let $p \in \mathbb{N}$ be even and $x \in C([0,T],\mathbb{R}^d)$. Let $\{\pi_n\}$ be a sequence of partitions of $[0,T]$. Consider the sequence of tensor-valued measures

$$\mu_n := \sum_{[t_i^n, t_{i+1}^n]\in\pi_n} \delta(\cdot - t_i^n)\left(x_{t_{i+1}^n} - x_{t_i^n}\right)^{\otimes p}$$

We say $x$ has pth variation along the sequence of partitions $\{\pi_n\}$ if $osc(x,\pi_n) \to 0$ and $\mu_n$ converge weakly to an atomless measure $\mu$. In this case, we write $x \in V_p(\pi)$

and we call $[x]^p : [0,T] \to Sym_p(\mathbb{R}^d)$ defined by $[x]^p(t) = \mu([0,t])$ the pth variation of $x$.

***Remark.*** Suppose $E$ is a vector space and $\mu$ is a vector-valued measure on a locally compact sapce $X$, then the dual space of the measure space is the continuous funcion with compact support and $E^*$ value. We can define the inner product by

$$< f, \mu >= \int_X < f(x), d\mu(x) >$$

Hence here we take the space of symmetric homogenuous polynomial $\mathbb{H}_p[X_1, \cdots, X_d]$ as the dual space of $Sym_p(\mathbb{R}^d)$ and the weak convergence can be reformulized as :

$$\lim_{n\to\infty} < f, \mu_n >= \lim_{n\to\infty} \sum_{[t_i^n, t_{i+1}^n]\in\pi_n} < f(t_i^n), \left(x_{t_{i+1}^n} - x_{t_i^n}\right)^{\otimes p} >=< f, \mu >$$

Hence by the definiton of pth variation for high dimensional paths, we can build following pathwise Ito's formula:

**Theorem 2.19.** *Let $p \in \mathbb{N}$ be an even number. Consider a sequence of partitions $\{\pi_n\}$ and $x \in V_p(\pi) \cap C([0,T], \mathbb{R}^d)$. Then for all $f \in C^p(\mathbb{R}^d, \mathbb{R})$, the limit of compensated Riemannian sums*

$$\int_0^t < \nabla f(x(s)), dx(s) >:= \lim_{n\to\infty} \sum_{[t_i^n, t_{i+1}^n]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} < \nabla^k f(x(t_i^n)), \left(x(t \wedge t_{i+1}^n) - x(t \wedge t_i^n)\right)^{\otimes k} >$$

*exists for every $t \in [0.T]$ and satisfies the pathwise change of variable formula:*

$$f(x(t)) = f(x(0)) + \int_0^t < \nabla f(x(s)), dx(s) > + \frac{1}{p!} \int_0^t < \nabla^p f(x(s)), d[x]^p(s) >$$

***Remark.*** It can be shown that there exists a sequence of partition such that the sample path of Fractional Brownian Motion has pth variation along the sequence. Hence the above pathwise integration shows an Ito-like formula for Fractional Brownian Motions. And hence can be used for differential equation driven by Fractional Brownian Motions.

***Remark.*** This approach gives a natural definition for an integration with irregular paths. And as in the quadratic variation case, we are interested in the situation when the pth variation will be uniquely determined and then this will give the uniqueness of the 'natural' integration with respect to irregular paths. Furthermore, the pth variation is defined on the tensor space, so there isn't an easy expansion for the difference between two sums of two sequences which is a crucial step in the proof of the quadratic variation case. Hence, we will need another way to solve this existence and uniqueness of the pth variation along some sequence of partitions.

## 3. Summary

We are looking for a natural definition of pathwise integration, which can produce as more as possible properties of stochastic integration. The methods that Hans Follmer provided gives the right definition for this pathwise integration when the integrand can be expressed in the form $\nabla f(x_t)$ and it can be extended to more general functions or functionals. Our goal is to see how far this definition can reach and how large the integrand space can be for this kind of integration. And exactly how this can be used to obtain classical stochastic analysis results and how it can extend them to more general cases. Besides, we are also interested in the existence and uniqueness of pth variation along a sequence of partitions, which will need a new approach to solve since it takes values in tensor space.

Furthermore, the model-free approach for the finance hedging strategy catches a lot of attention these days and the pathwise integration can be served as a tool to obtain the pathwise hedging strategies, see[7] for details.

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