DERIVED AUTOEQUIVALENCES OF K3 SURFACES

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1 Introduction

Derived categories of coherent sheaves were first considered in the work of Alexander Grothendieck and Jean-Louis Verdier in the early 1960s, following the fundamental works of Verdier in homological algebra, such as axiomatic definition of triangulated categories and their use for localization, allowing the constructive approach to the study derived categories.

Their primary goal was to extend the notion of Serre duality to singular schemes. However with the following fundamental works of Bondal and Orlov (as [2]), the study of derived categories of coherent sheaves on algebraic varieties has become a separate branch of algebraic geometry, allowing to study the geometric properties of the varieties by homological algebra methods.

We start by giving an outlook of the derive geometry in general and then concentrate on the study of derived autoequivalences of complex projective varieties, in particular K3 surfaces. The main reference for this part is [6]. In the second part of the text we give an introduction to Bridgeland stability conditions as a method to study the derived autoequivalences, and formulate Bridgeland theorem as well as Bridgeland conjectures, providing conjectural description of $\mathrm{Aut}(\mathrm{D}^b(X))$ for K3 surfaces.

2 Derived categoried

Given an abelian category $\mathcal{A}$ we define its **bounded derived category** $\mathrm{D}^b(\mathcal{A})$ as a localization of its homotopy category of bounded chain complexes $\mathcal{H}^b(\mathcal{A})$ by the maps that induce isomorphism on cohomology — **quasi-isomorphisms**. A localization of a category by a set of morphisms is defined by a fundamental property and exists all the times, however in the case of a derived category of an abelian category it can be actually constructed using the triangulated structure on $\mathcal{H}^b(\mathcal{A})$. This triangulated structure furthermore induces the triangulated structure on $D^b(\mathcal{A})$. We will further denote the localization functor by $Q_{\mathcal{A}}\colon \mathcal{H}^b(\mathcal{A}) \to D^b(\mathcal{A})$.

Given an exact functor of abelian categories $F\colon \mathcal{A} \to \mathcal{B}$, one can define a functor $\widetilde{F}\colon \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{B})$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H}(\mathcal{A}) & \xrightarrow{\mathcal{H}(F)} & \mathcal{H}(\mathcal{B}) \\ \big\downarrow{\scriptstyle Q_{\mathcal{A}}} & & \big\downarrow{\scriptstyle Q_{\mathcal{B}}} \\ \mathrm{D}^b(\mathcal{A}) & \xrightarrow{\widetilde{F}} & \mathrm{D}^b(\mathcal{B}) \end{array}$$

However for an arbitrary, not necessarily exact F, the functor $\widetilde{F}$ as above does not always exist. When F is right or left exact and $\mathcal{A}$ has enough projective or, correspondingly, injective objects, however, we can still construct an analogue called a **derived functor**.

Denote by $\mathcal{A}_I \subset \mathcal{A}$ the subcategory of all the injective objects in $\mathcal{A}$. We say that $\mathcal{A}$ **has enough injectives** if the composition

$$i\colon\ \mathcal{H}(\mathcal{A}_I) \longrightarrow \mathcal{H}(\mathcal{A}) \longrightarrow \mathrm{D}^b(\mathcal{A})$$

is an equivalence of categories. Then it admits an inverse i^{-1}. For a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ we then define **right derived functor** as the composition

$$RF = Q_{\mathcal{B}} \circ \mathcal{H}(\mathcal{A}) \circ i^{-1}$$

as in the following diagram.

$$\begin{array}{ccccc} \mathcal{H}^b(\mathcal{A}_I) & \longrightarrow & \mathcal{H}^b(\mathcal{A}) & \xrightarrow{\mathcal{H}(F)} & \mathcal{H}^b(\mathcal{B}) \\ & {\scriptstyle i}\searrow \; \nwarrow{\scriptstyle i^{-1}} & \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ & & \mathrm{D}^b(\mathcal{A}) & & \mathrm{D}^b(\mathcal{B}). \end{array}$$

In other words, the equivalence i^{-1} consists of taking the injective resolutions of objects and complexes. Thus to compute a right derived functor of a functor F on a complex $C \in \mathrm{D}^b(\mathcal{A})$ it suffices to take the injective resolution I_C of C and apply F to I_C.

The left derived functor of a right exact functor F is defined similarly using the projective objects.

3 Derived categories of coherent sheaves

Returning to the geometric setting, given a scheme X over a field k one can consider the k-linear, and thus abelian, category of coherent sheaves on X denoted by $\mathrm{Coh}(X)$. One can now consider a bounded derived category $\mathrm{D}^b(\mathrm{Coh}(X))$ that will later be denoted by $\mathrm{D}^b(X)$.

As mentioned before, the primary motivation for defining this category was the extension of the notion of Serre duality to the case of singular varieties.

For the moment consider a smooth projective variety X and denote by ωX its canonical bundle. Then it is an invertible sheaf, thus the functor $- \otimes \omega_X$ is exact, and therefore is well defined for the derived categories. For an integer n we denote by $[n]$ the n-th power of the shift functor in the triangulated category $\mathrm{D}^b(X)$.

Theorem 3.1 (Serre duality)**.** *Let X be a smooth projective variety of dimension n. Then the functor $S_X = [n] \circ (- \otimes \omega_X)$ is a Serre functor for the category $\mathrm{D}^b(X)$.*

Recall that for a k-linear category k a functor $S\colon \mathcal{A} \to \mathcal{A}$ is called a **Serre functor** if for any two objects A, $B \in \mathcal{A}$ there exists a k-linear isomorphism

$$\eta_{A,B}\colon \mathrm{Hom}(A,B) \to \mathrm{Hom}(B, S(A))^*$$

functorial in A and B.

Recalling that sheaf cohomology is defined as a derived functor of globla sections one can deduce from Theoref 3.1 the classical version of Serre duality: for a sheaf $\mathcal{F}$ on a smooth projective variety of dimension n

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^*)^*. \tag{1}$$

We can define the analogue of the canonical bundle ω_X for the non smooth varieties, however it will no longer be locally free, thus taking the tensor product with it is not an exact functor, and so is not defined for the derived categories. Therefore the tensor product should be replaced by derived tensor product in Theorem 3.1 which can no more be interpreted as (1).

The Serre functor for singular varieties was the primary motivation for the definition of derived categories of coherent sheaves. However soon enough Bondal and Orlov have discovered that most of the geometry of a variety, even all the geometry in many cases, is determined by its derived category of coherent sheaves.

Theorem 3.2 (Bondal, Orlov)**.** *Let X and Y be smooth projective varieties and assume the (anti-)canonical bundle of X is ample. If there exists an exact equivalence $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ then X and Y are isomorphic.*

We therefore are interested in the derived categories of coherent sheaves and morphisms between them as much as we are interested in the varieties. It turned out, again by works of Orlov that all the exact functors between derived categories of coherent sheaves on smooth projective varieties are geometric in the following sense.

Let X and Y be smooth projective varieties and denote the two projections by $q\colon X \times Y \to X$ and $p\colon X \times Y \to Y$. Then for an element $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ one can define a **Fourier–Mukai transform** $\Phi_{\mathcal{P}}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ as follows:

$$\Phi_{\mathcal{P}}(\mathcal{E}) = p_*(q^*(\mathcal{E}) \otimes \mathcal{P}),$$

where the pullback, pushforward and tensor product are taken derived.

The fundamental result of Orlov claims that every functor between derived categories of coherent sheaves is actually a Fourier–Mukai transform, and thus comes from a sheaf or a complex of sheaves on $X \times Y$, allowing many non-discrete geometric methods, such as deformation theory, to be applied to such a seemingly discrete object.

Theorem 3.3 (Orlov). *Let X and Y be two smooth projective varieties and let*

$$F\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

be a fully faithful exact functor. Then there exists an object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}}$.

4 Derived autoequivalences

We now move to the study of exact autoequivalences of derived categories of coherent sheaves of smooth projective varieties. Even though all of those are Fourier–Mukai, it is sometimes highly non-trivial to describe them all.

Let X be a smooth projective variety. We consider the group $\mathrm{Aut}(\mathrm{D}^b(X))$. Note that it always contains the shift functor [1]. Moreover the group $\mathrm{Aut}(X)$ of regular automorphisms of the variety acts as an autoequivalence on its derived category of coherent sheaves. Finally, as we discussed above, given a line bundle $\mathcal{L}$, one can take a tensor product $- \otimes \mathcal{L}$ thus obtaining an exact invertible functor that induces an autoequivalence of $\mathrm{D}^b(X)$.

Therefore the group $\mathrm{Aut}(\mathrm{D}^b(X))$ always contains the group $\mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$, where $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}(X)$ by the automorphisms.

The following theorem of Bondal and Orlov claims that in many cases there are no autoequivalences other than these.

Theorem 4.1. *Let X be a smooth projective variety with ample (anti-)canonical bundle. Then*

$$\mathrm{Aut}(\mathrm{D}^b(X)) \cong \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)).$$

However in the simplest cases when the condition of the theorem does not hold, the group of autoequivalences becomes highly complicated, as we will see below.

Consider surfaces with a trivial canonical bundle. Those are abelian surfaces and K3 surfaces. A complete description of the group $\mathrm{Aut}(\mathrm{D}^b(X))$ for abelian varieties is due to Mukai and Orlov. From now on we will concentrate on describing derived autoequivalences in the case when X is a K3 surface.

5 Derived autoequivalences of K3 surfaces. First glance

Recall that a smooth complex projective surface is called a **K3 surface** if its canonical bundle is trivial and $H^1(X, \mathcal{O}_X) = 0$. Recall that in this case due to Theorem 3.1 the Serre functor on $\mathrm{D}^b(X)$ is a shift functor [2]. However instead of making the life easier such a simple Serre functor introduces a lot of symmetry in the derived category structure. In particular, the new autoequivalences appear.

For two objects $A, B \in \mathrm{D}^b(X)$ denote by $\mathrm{Hom}^i(A, B)$ or $(A, B)^i$ the vector space $\mathrm{Hom}(A, B[i])$. An object $E \in \mathrm{D}^b(X)$ is called **spherical** if $\dim(A, B)^i = 1$ for $i = 0, 2$ and $\dim(A, B)^i = 0$ otherwise. There are many examples of spherical objects in derived categories of K3 surfaces, such as $\mathcal{O}_X$ and all the line bundles and many others.

Given a spherical object $E \in \mathrm{D}^b(X)$ one can construct a functor $T_E\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ as follows. For am object $C \in \mathrm{D}^b(X)$ consider an evaluation map

$$ev\colon \bigoplus_i \mathrm{Hom}^i(E, C) \otimes E[-i] \to C.$$

Here the tensor product of a vector space and an object of a category denotes the direct sum of the object the dimension of the space times. Denote by $T_E(C)$ the completion of ev to an exact triangle.

Note that T_E is well defined on objects, however this definition does not allow to define it on morphisms, as completion of a commutative square to a morphism of distinguished triangles is not unique. In the geometric setting spherical twists are defined using Fourier–Mukai transforms, and otherwise the honest definition requires dg-enhancements, however here we leave it as it is.

For the case of $D^b(X)$ spherical functors are distinguished autoequivalences for all the spherical objects. Moreover they have very non-trivial interactions between them, that locally look similar to braid groups.

The following should be considered as an example of the relations between spherical twists. One must however note that this is a very special and one of very few cases where these relations are actually understood, moreover the set of all spherical objects is not even close to be described.

We call a set $E_1, \ldots, E_n$ of spherical objects an $\mathbf{A_m}$**-configuration** if

$$\bigoplus_k \dim(E_i, E_j)^k = 1 \text{ for } |i-j| = 1 \text{ and } 0 \text{ for } |i-j| > 1.$$

Proposition 5.1. *Let $E_1, \ldots, E_n$ be an A_m-configuration of spherical objects. Then the group generated by the induced spherical twists T_{E_i} in $\mathrm{Aut}(\mathrm{D}^b(X))$ is isomorphic to the braid group on $n+1$ strands.*

The structure of the entire group $\mathrm{Aut}(\mathrm{D}^b(X))$ remains largely unknown. However, an approach, developed by Thomas Bridgeland in the early 2000, allows us to get close to understanding it and even gives a conjectural description of the entire group. In the remaining part of the text we will give a short introduction to Bridgeland stability conditions, and their use for the study of derived categories of K3 surfaces.

6 Stability conditions on triangulated categories

Stability conditions were first introduced by Thomas Bridgeland in 2002 for a general case of triangulated categories. They provide a way to associate to a triangulated category $\mathcal{D}$ a topological manifold — the space of all stability conditions $\mathrm{Stab}(\mathcal{D})$. They are of interest not only as an invariant of triangulated categories, but as a way to introduce geometric ideas into the homological algebra world. We present the definition here, following Bridgeland's original article (see [3]).

Given a triangulated category $\mathcal{D}$ we define a **slicing**, which is a way to chose a set of subcategories enumerated by real numbers that, in a sense, generate $\mathcal{D}$.

Definition. A **slicing** $\mathcal{P}(\varphi)$ on a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\varphi)$ for $\varphi \in \mathbb{R}$ such that the following conditions hold:

- For every $P_1 \in \mathcal{P}(\varphi_1)$, $P_2 \in \mathcal{P}(\varphi_2)$ it holds that $\mathrm{Hom}(P_1, P_2) = 0$ if $\varphi_1 > \varphi_2$;
- $\mathcal{P}(\varphi)[1] = \mathcal{P}(\varphi + 1)$;
- For every $0 \neq E \in \mathcal{D}$ there exists a finite sequence of real numbers $\varphi_1 > \varphi_2 > \ldots > \varphi_n$ and a collection of triangles

$$\xymatrix{ E_1 \ar[rr] & & E_2 \ar[r] \ar[dl] & \ldots \ar[r] & E_{n-1} \ar[rr] & & E_n \ar@{=}[r] \ar[dl] & E, \\ A_1 \ar@{=}[u] & A_2 \ar[ul]^{[1]} & & & & A_n \ar[ul]^{[1]} & & } \tag{2}$$

with $A_i \in \mathcal{P}(\varphi_i)$ for all i.

The objects $P \in \mathcal{P}(\varphi)$ are called **semistable** of **phase** φ. It is easy to verify that for any object $E \in \mathcal{D}$ the decomposition (7.1) is unique. It is called a **Harder–Narasimhan filtration** of E and the objects A_i are called its **semistable factors**.

Note that, given a slicing, we can now construct a lot of new slicings by remunerating, in a sense, the categories $\mathcal{P}(\varphi)$ by an order-preserving function $f\colon \mathbb{R} \to \mathbb{R}$. To avoid it, and to gain other useful properties, it turns out natural to add an additional condition to the definition of a slicing — the fact that it behaves well with respect to the triangulated structure.

Definition. A **stability condition** is a pair $\sigma = (Z, \mathcal{P})$, where $\mathcal{P}$ is a slicing and

$$Z\colon K(\mathcal{D}) \to \mathbb{C}$$

is an additive group homomorphism from the Grothendieck group of $\mathcal{D}$ to $\mathbb{C}$ such that for every $P \in \mathcal{P}(\varphi)$ it holds that $Z(P) \in \mathbb{R}_{>0} \cdot \exp(i\pi\varphi)$.

One of the few useful differences of stability conditions from just slicing is that in the case when $(Z, \mathcal{P})$ is a stability condition, the categories $\mathcal{P}(\varphi)$ of semistable objects are not only additive but abelian for all φ. The simple objects in those categories are called **stable**.

7 Topological space of stability conditions

Again following Bridgeland we introduce the structure of topological space on the set of stability conditions, making it into a metric space locally isomorphic to $\mathbb{R}^\infty$.

Given a stability condition $\sigma = (Z, \mathcal{P})$, for an object $E \in \mathcal{D}$ let (7.1) be its Harder–Narasimhan filtration. We then denote by $\varphi^+_\sigma(E)$ and $\varphi^-_\sigma(E)$ the maximal and minimal phases φ_1 and φ_n respectively of its semistable factors. We also denote by $m_\sigma(E)$ the sum $\sum_{1\leqslant i\leqslant n} |Z(A_i)|$.

We can now define a distance on the set $\mathrm{Stab}(\mathcal{D})$ of stability conditions on $\mathcal{D}$.

Definition. Let σ_1, σ_2 be two stability conditions on a triangulated category $\mathcal{D}$. A **distance** between σ_1 and σ_2 is the number $d \in [0, \infty]$ defined as

$$d(\sigma_1, \sigma_2) := \sup_{0\neq E\in\mathcal{D}} (|\varphi^+_{\sigma_1}(E) - \varphi^+_{\sigma_2}(E)|, |\varphi^-_{\sigma_1}(E) - \varphi^-_{\sigma_2}(E)|, |\log \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)}|).$$

One can check that d is indeed a metric, and thus defines a topology on the set $\mathrm{Stab}(\mathcal{D})$.

Consider a natural projection $\mathcal{Z}\colon \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ that takes $\sigma = (Z, \mathcal{P})$ to Z. The space on the right is a complex vector space and thus is equipped with a natural topology. It is easy to check that the map $\mathcal{Z}$ is continuous. The following result of Bridgeland shows that for every connected component of $\mathrm{Stab}(\mathcal{D})$ the map $\mathcal{Z}$ behaves as a local homeomorphism on a linear subspace of $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$.

Theorem 7.1 (Bridgeland)**.** *For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there exists a linear subset*

$$V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

such that $\mathcal{Z}(\Sigma) \subset V(\Sigma)$ and $\mathcal{Z}\colon \Sigma \to V(\Sigma)$ is a local homeomorphism.

The theorem shows that $\mathrm{Stab}(\mathcal{D})$ locally behaves as $\mathbb{R}^\infty$. Note that the group of exact autoequivalences $\mathrm{Aut}(\mathcal{D})$ acts on $\mathrm{Stab}(\mathcal{D})$ preserving the distance function.

8 Stability conditions on varieties

We now move on to the geometric case, when the triangulated category $\mathcal{D}$ is a bounded derived category $\mathrm{D}^b(X)$ of coherent sheaves on a smooth complex projective variety X. In this case the space of stability conditions can be reduced without loosing the essential part of its properties to be a finite-dimensional topological manifold.

We begin by defining a bilinear form on the Grothendieck group $K(X)$ of the category $\mathrm{D}^b(X)$. Given E, $F \in \mathrm{D}^b(X)$ an **Euler characteristic** $\chi(E, F)$ is defined as

$$\chi(E, F) = \sum_i (-1)^i \dim(E, F)^i.$$

Due to Serre duality the left and right radicals ${}^{\perp}K(X)$ and $K(X)^{\perp}$ are the same, so the form χ descends to a non-degenerate bilinear form on the **numerical Grothendieck group** $\mathcal{N}(X) = K(X)/K(X)^{\perp}$. Using Riemann–Roch theorem one can prove that $\mathcal{N}(X)$ is finitely generated.

A stability condition $\sigma = (Z, \mathcal{P})$ is called **numerical** if Z factors through $\mathcal{N}(X)$. From now on by a stability condition we mean a numerical stability condition and we denote by $\mathrm{Stab}(X)$ the set of all numerical stability conditions on $\mathrm{D}^b((X))$.

In this case the function $\mathcal{Z}$ defined above factors through $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X), \mathbb{C}) = \mathcal{N}(X) \otimes \mathbb{C}$. We will now denote by $\mathcal{Z}$ the map $\mathrm{Stab}(X) \to \mathcal{N}(X) \otimes \mathbb{C}$, which is a finite dimensional vector space.

In this setting Theorem 7.1 still holds, and thus the connected components of $\mathrm{Stab}(X)$ are all finite dimensional topological manifolds.

9 Sheaves on K3 surfaces

Now let X be a K3 surface. This case is especially well studied. In particular, it is known that, restricted to a distinguished connected component, the map $\mathcal{Z}$ is not only a local homeomorphism but actually a covering map. We will discuss this famous result by Bridgeland (see [4]) together with the following conjectures, and their use to the study of the group of derived autoequivalences of K3 surfaces.

In the case when X is a K3 surface there exists a correctly defined group homomorphism called **Mukai vector** $v \colon \mathcal{N}(X) \to H^*(X, \mathbb{Z})$, defined as

$$v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)},$$

that induces an isomorphism $\mathcal{N}(X) \cong \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$, where $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$ is the Neuron–Severi group of X.

There exists a natural action of the group $\mathrm{Aut}(\mathrm{D}^b(X))$ on the cohomology $H^*(X, \mathbb{Z})$ by Hodge isometries, defined via Fourier–Mukai transform, that coincides with the action of $\mathrm{Aut}(\mathrm{D}^b(X))$ on $\mathcal{N}(X)$ under the Mukai vector isomorphism. The action provides a map $\varphi \colon \mathrm{Aut}(\mathrm{D}^b(X)) \to \mathrm{O}(H^*(X, \mathbb{Z}))$. In 2009 Daniel Huybrechts, Emmanuele Macri and Paolo Stellari have proved that the image of φ is exactly an index two subgroup $\mathrm{O}^+(H^*(X, \mathbb{Z}))$ of $\mathrm{O}(H^*(X, \mathbb{Z}))$ of orientation preserving isometries (see [5]).

Therefore, it is the kernel of φ which is of the main interest to us. Denote it by $\mathrm{Aut}_0(\mathrm{D}^b(X))$. By definition, those are exactly the autoequivalences that act identically on $H^*(X, \mathbb{Z})$.

10 Stability conditions on K3 surfaces

Consider all the stability conditions $\sigma \in \mathrm{Stab}(X)$ such that all the skyscraper sheaves are semistable of phase 1 with respect to σ. These stability conditions are called **geometric** as they can also be defined via a geometric construction. It turns out that they all lie in the same, distinguished, connected component of $\mathrm{Stab}(X)$ which we denote by $\mathrm{Stab}^{\dagger}(X)$.

The following theorem provides the main interest to study stability conditions for derived categories of coherent sheaves on K3 surfaces.

Theorem 10.1 (Bridgeland)**.** *Denote by* $\Omega \subset \mathcal{N}(X)$ *the image of* $\mathrm{Stab}^{\dagger}(X)$ *under* $\mathcal{Z}$*. Then the map*

$$\mathcal{Z} \colon \mathrm{Stab}^{\dagger}(X) \to \Omega$$

is a covering and the subgroup $\mathrm{Aut}_0^{\dagger}(\mathrm{D}^b(X))$ *of* $\mathrm{Aut}_0(\mathrm{D}^b(X))$ *that preserves* $\mathrm{Stab}^{\dagger}(X)$ *is the group of deck transformations of* $\mathcal{Z}$*.*

The image $\Omega \subset \mathcal{N}(X) \otimes \mathbb{C}$ can be described in purely linear algebraic terms as follows. Consider the set $\widetilde{\Omega}$ of all vectors $v \in \mathcal{N}(X) \otimes \mathbb{C}$ such that their real and imaginary parts generate a 2-plane in $\mathcal{N}(X) \otimes \mathbb{R}$ on which the form χ is positive. Let $\widetilde{\Omega}^+$ be one of the two connected components of $\widetilde{\Omega}$. Denote by Δ the set of all vectors $v \in \mathcal{N}(X) \otimes \mathbb{C}$ such that $\chi(v, v) = -2$. Then

$$\Omega = \widetilde{\Omega}^+ \setminus \bigcup_{\delta \in \Delta} \delta^{\perp}.$$

11 Back to the derived autoequivalences of K3 surfaces

Given a K3 surface X we want to study the group of its derived autoequivalences $\mathrm{Aut}(\mathrm{D}^b(X))$. As we mentioned above, we are interested in $\mathrm{Aut}_0(\mathrm{D}^b(X))$ of the autoequivalences acting identically on $H^*(X, \mathbb{Z})$.

Bridgeland's theorem gives hopes to describing the group $\mathrm{Aut}_0(\mathrm{D}^b(X))$. Indeed, if the whole group $\mathrm{Aut}_0(\mathrm{D}^b(X))$ preserves the distinguished connected component of the space of Bridgeland stability conditions, then it is described as a space of deck transformations of a given topological covering. If, moreover, the space of stability conditions is simply connected the desired group would be described as a fundamental group of Ω, which is very explicit.

Conjecture. The group $\mathrm{Aut}_0(\mathrm{D}^b(X))$ preserves the distinguished connected component $\mathrm{Stab}^\dagger(X)$ of the space of stability conditions on X and the space $\mathrm{Stab}^\dagger(X)$ is simply connected. Therefore

$$\mathrm{Aut}_0(\mathrm{D}^b(X)) \cong \pi_1(\Omega).$$

The conjecture is proved in case when Picard rank of the surface X is equal to 1 and in the non-algebraic case of Picard rank 0. For the Picard rank 1 this gives the following.

Theorem 11.1 (Bayer, Bridgeland, see [1])**.** *The group* $\mathrm{Aut}_0(\mathrm{D}^b(X))$ *of a K3 surface* X *of Picard rank* 1 *is a free group on generators enumerated by vector bundles on* X*.*

For the higher Picard ranks one may even hope that the distinguished connected component of the space of stability conditions is not only simply connected, but contractible, and that it is not only preserved by autoequivalences, but is the only connected component.

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