INTRODUCTION OF RESEARCH DOMAIN
LIOUVILLE CONFORMAL FIELD THEORY AND RELATED TOPICS

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Abstract. In this note, we will review some recent progresses in the Liouville Conformal field theory within probabilistic framework.

Contents

1. Introduction

Liouville field theory was suggested by Polyakov in 1981 [Po81] to describe the geometry of a canonical random surface whose topology is fixed. To define LFT, physicists use the path integral formalism. Informally it tells us that our Liouville field ϕ will be given in terms of an infinite measure on a suitable functional space. Consider the following space of function from surface $\mathcal{S}$:

$$\Sigma = \{X : \mathcal{S} \to \mathbb{R}\}. \tag{1.1}$$

The Liouville field ϕ is then given by the following formal definition, for any background metric g on $\mathbb{S}^2$,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}} \int_{\Sigma} F(X) e^{-S_L(X,g)} DX, \tag{1.2}$$

where $S_L(X,g)$ is the so-called Liouville action:

$$S(X,g) = \frac{1}{4\pi} \int_{\mathcal{S}} (|\nabla_g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}) g(x) d^2x. \tag{1.3}$$

here R_g is the Ricci curvature and $\mathcal{Z}$ is a formal normalization constant.

The extrema of the Liouville action

$$S(X,g) = \frac{1}{4\pi}\int_{\mathcal{S}}(|\nabla_g X|^2 + QR_g X + 4\pi\mu e^{\gamma X})g(x)d^2x. \tag{1.4}$$

satisfies

$$-2\Delta_g X_{\min} + QR_g + \gamma\mu e^{\gamma X_{\min}} = 0 \tag{1.5}$$

we may get a constant negative curvature

$$R_{e^{\gamma X_{\min}}g} = -\frac{\gamma^2\mu}{2} \tag{1.6}$$

If we simply take $Q = \frac{2}{\gamma}$.

So in the classical theory ($Q = \frac{2}{\gamma}$) the minimum of the Liouville action uniformizes the surface (M,g) and it is therefore natural to look at quantum fluctuations of the uniformized metric $e^{\gamma X_{\min}}g$. This is precisely the meaning of (1.2) in the physical origin. To make sure the conformal invariance in quantum case, we need to choose $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

One should note that the path integral (1.2) diverges for any surface $\mathcal{S}$ of genus 0 or 1 by simply using Gauss-Bonnet formula. For example, in the sphere case, we need to add curvature singularities

$$R_g = \Sigma_{i=1}^n \alpha_i \delta(x - x_i) + \quad bounded \quad term$$

So we need to choose $g(x) \sim \frac{1}{|x-x_i|^{\alpha_i}}$ near x_i with α_i satisfy so called Seiberg bound:

$$\alpha_i < 2 \quad since\ g\ is\ integrable\ and$$

$$\Sigma_{i=1}^n \alpha_i > 4 \quad due\ to\ Gauss-Bonnet\ theorem$$

Now we write the **Liouville correlation function** as

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}}\int_\Sigma F(X)e^{\Sigma_{i=1}^N \alpha_i X(z_i)}e^{-S_L(X,g)}DX, \tag{1.7}$$

where we have chosen N insertion points $z_i \in \mathbb{S}^2$ with weights $\alpha_i \in \mathbb{R}$ (to make sure the path integral converges, the minimal insertions for the sphere case is 3, and for the tori case is 1). By choosing $F = 1$ in the above expression we define the N-point correlation function of LCFT, the most fundamental observable of the theory:

$$\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \rangle_{\mathbb{S}^2,g} = \frac{1}{\mathcal{Z}}\int_\Sigma e^{\Sigma_{i=1}^N \alpha_i X(z_i)}e^{-S_L(X,g)}DX. \tag{1.8}$$

It can be proved that the correlation function satisfies the **Möbius invariance**(3.16) and **Weyl anomaly**(3.17), which imply the Liouville field theory is actually a **Conformal field theory**.

One of the main mission of a conformal field theory is to compute the correlation function of some observables which are vertex operators. In the algebraic formulation of Liouville theory, physicits use a so-called **conformal bootstrap** method. Roughly speaking, it means we may decompose the state of space of the form

$$1 = \Sigma_P\ |P > \times < P|$$

Where $|P>$ form a basis of states and given by primary states (operator algebra) and descendants (acted by Virasoro algebra). However, weather this spectrum is absolutely continuous seems a mystery in many years. In a mathematical rigor, bootstrap means decomposition over spectral of some Hamiltonian. In the Liouville case, this Hamiltonian has a simple form (5.15).
With bootstrap equation (6.3) in [GKRV], we may define the conformal block in a mathematical way but it's explicit form seems unclear. In the seminal work [AGT10], they presented a bridge between 2 dimensional conformal field theory and 4 dimensional supersymmetric Yang-Milles gauge theory, actually this bridge is the instanton moduli space (a kind of Nakajima variety in mathematical language) since the torus action fixed points correspond to a special complete basis in CFT side. Through this bridge, the conformal block can be represented by instanton partition function which has a closed form.

2. Liouville Conformal field Theory on Sphere

First we recall the path integral interpretation of Brownian motion: Consider the space of path $\Sigma = \{\sigma : [0,1] \to \mathbb{R},\ \sigma(0) = 0\}$ and the action $S_{BM} = \frac{1}{2}\int_0^1 |\sigma'(t)|^2 dt$. Then for all suitable F,

$$\mathbb{E}[F((B_s)_{0\leqslant s\leqslant 1})] = \frac{1}{\mathcal{Z}}\int_\Sigma F(\sigma)e^{-S_{BM}(\sigma)}D\sigma,$$

where $D\sigma$ is a formal uniform measure on Σ'.
Let us briefly explain why Brownian motion meets this pathwise construction. Consider Dirichlet Problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = -\lambda u(t) \\ u(0) = u(1) = 0 \quad \|u(t)\|_{L^2[0,1]} = 1 \end{cases} \tag{2.1}$$

Then we get

$$\textit{eigenstates } u_j(t) = \sqrt{2}sin(j\pi t) \ \textit{ with eigenvalues } \lambda_j = j^2\pi^2 \tag{2.2}$$

Where $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$, for $\sigma(t) = \Sigma_{j\geqslant 1}\sigma_j u_j$,
(2.3)

$$\int_{\Sigma'} F(\sigma)e^{-S_{BM}(\sigma)}D\sigma = \int F(\Sigma_{j\geqslant 1}\sigma_j u_j)\prod_{j\geqslant 1}(e^{-\lambda_j\sigma_j^2 dx_j}) = \prod_{j\geqslant 1}(\frac{\sqrt{2\pi}}{\sqrt{\lambda_j}})\int F(\Sigma_{j\geqslant 1}\frac{v_j u_j}{\sqrt{\lambda_j}})\prod_{j\geqslant 1}(\frac{e^{-\frac{v_j^2}{2}}}{\sqrt{2\pi}}dv_j)$$

Here $v_j = \sqrt{\lambda_j}\sigma_j$ is standard Gaussian distribution, and we know $B_t := \Sigma_{j\geqslant 1}\frac{v_j\sqrt{2}sin(\pi jt)}{j\pi}$ is just Brownian motion.

3. The probabilistic construction of LCFT

In this note, We will consider the metric $g(x) = \frac{1}{|x|_+^4}$ on Riemann sphere, the curvature of this metric is a measure and is given by $R_g(x)g(x)d^2x = -\Delta \ln g(x)d^2x = 4\nu(d^2x)$ where ν is the uniform measure on the circle of center 0 and radius 1 (normalized such that $\int_{\mathbb{C}}\nu(d^2x) = 2\pi$). Then the normalized **Gaussian free field** X can be defined by solving

Laplace equation as following. Here normalized means it has average 0 with respect to the curvature $\int_{\mathbb{C}} X(x)R_g(x)g(x)d^2x = 0$. Let us consider

$$L^2(\mathbb{S}^2) := \{\varphi \mid \int_{\mathbb{C}} \varphi(x)g(x)d^2x < \infty\}$$

Let $(\varphi_j)_{j\geqslant 1}$ be the eigenvector basis for $-\Delta_g$, i.e.

$$-\frac{1}{g(x)}\Delta_g\varphi_j(x) = \lambda_j\varphi_j(x).$$

normalized to have $L^2(\mathbb{S}^2)$ norm equal to 1: $\int_{\mathbb{C}} \varphi_j(x)^2 g(x)d^2x = 1$. Then every function in $\varphi \in L^2(\mathbb{S}^2)$ can be decomposed in a unique way on the orthonormal basis $(1, (\varphi_j)_{j\geqslant 1})$

$$\varphi = c + \sum_{j\geqslant 1} c_j\varphi_j \tag{3.1}$$

where for all $j \geqslant 1$ $c_j = \int_{\mathbb{C}} \varphi(x)\varphi_j(x)g(x)d^2x{=}(\varphi,\varphi_j)_g$, it is natural to write for a function F defined on $L^2(\mathbb{S}^2)$ that

$$\int_{L^2(\mathbb{S}^2)} F(\varphi)D\varphi = \int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}} F(c + \sum_{j\geqslant 1} c_j\varphi_j)\, dc\prod_{j=1}^{\infty} dc_j \tag{3.2}$$

where dc and each dc_j is the standard Lebesgue measure on $\mathbb{R}$. If φ has decomposition (3.1) then

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla_g\varphi(x)|^2 g(x)\, d^2x = \frac{1}{4\pi}\sum_{j=1}^{\infty} c_j^2\lambda_j$$

hence this leads to the following formal definition

$$\int_{L^2(\mathbb{S}^2)} F(\varphi)e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2}|\nabla_g\varphi(x)|^2 g(x)\, d^2x} D\varphi = \int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}} F(c + \sum_{j\geqslant 1} c_j\varphi_j)\, dc\left(\prod_{j=1}^{\infty} e^{-\frac{c_j^2\lambda_j}{4\pi}} dc_j\right) \tag{3.3}$$

Let us stress that the two previous definitions (3.2) and (3.3) are not meant to be rigorous. However, one can make sense of the previous definition (3.3) using probability theory. First, let us make the change of variable $u_j = \frac{c_j\sqrt{\lambda_j}}{\sqrt{2\pi}}$ in (3.3) which leads to (at the formal level)

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}} F(c{+}\sum_{j\geqslant 1} c_j\varphi_j)dc\left(\prod_{j=1}^{\infty} e^{-\frac{c_j^2\lambda_j}{4\pi}} dc_j\right) = C\int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}} F(c{+}\sqrt{2\pi}\sum_{j\geqslant 1} u_j\frac{\varphi_j}{\sqrt{\lambda_j}})dc\left(\prod_{j=1}^{\infty} e^{-\frac{u_j^2}{2}}\frac{du_j}{\sqrt{2\pi}}\right)$$

where the "constant" C has the following formal definition $C = \prod_{j=1}^{\infty}(2\pi(\lambda_j)^{-1/2})$. This constant can be interpreted as $(\det'(\Delta_g))^{-1/2}$ where $\det'(\Delta_g)$ is the determinant of the Laplacian (it's an interesting topological invariant, see [OPS]).

Now, for any i.i.d. sequence $(\epsilon_j)_{j\geqslant 1}$ of standard centered Gaussian variables the sum $\sqrt{2\pi}\sum_{j\geqslant 1}\epsilon_j\frac{\varphi_j}{\sqrt{\lambda_j}}$ converges in $\mathcal{S}'(\mathbb{C})$, we define the **Gaussian free field X** as this limit; hence this leads to the following **rigorous** definition for any function F defined on $\mathcal{S}'(\mathbb{C})$

$$\int F(\varphi)e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2}|\nabla_g\varphi(x)|^2 g(x)\, d^2x} D\varphi := \int_{\mathbb{R}} \mathbb{E}[F(X + c)]\, dc. \tag{3.4}$$

Let us stress that the GFF X lives in a space of distributions and not a space of functions and in particular not $L^2(\mathbb{S}^2)$.

Proposition 3.1. *The sum* $\sqrt{2\pi}\sum_{j\geqslant 1}\epsilon_j\frac{\varphi_j}{\sqrt{\lambda_j}}$ *converges almost surely in Sobolev space* $H^{-s}(\mathbb{S}^2)$ *for* $s>0$.

Proof. Suppose $f\in H_0^s(\mathbb{S}^2)$, where $||f||_{H_0^s} = \Sigma_{j\geqslant 1}|f_j|^2\lambda_j^s$, $f_j = (f,\phi_j)_g$ then

(3.5)
$$|(X,f)_g| = \sqrt{2\pi}\ |\Sigma_{j=1}^{\infty}\epsilon_j\frac{f_j}{\sqrt{\lambda_j}}| = \sqrt(2\pi)\Sigma_{j=1}^{\infty}|\epsilon_j\lambda_j^{-\frac{1}{2}-\frac{s}{2}}f_j\lambda_j^{\frac{s}{2}}| \leqslant \sqrt{2\pi}\ ||f||_{H_0^s(\mathbb{S}^2)}\ (\Sigma_{j=1}^{\infty}\epsilon_j^2\lambda_j^{-1-s})^{\frac{1}{2}}$$

The random series $\Sigma_{j=1}^{\infty}\epsilon_j^2\lambda_j^{-1-s}$ converges almost surely by noticing that $\lambda_j\sim j$ as j goes to infinity (this asymptotic is called Wely's formula, which can be found in [MP] and the sum of independent variables converges almost surely if it is L^2 convergent. ☐

The GFF from above construction can also be characterized by

$$\mathbb{E}[X(x)X(y)] = ln\frac{1}{|x-y|} + ln|x|_+ + ln|y|_+ \tag{3.6}$$

Since by construction $\int_{\mathbb{C}}X(x)R_g(x)g(x)d^2x = 0$ almost surely and $\int_{\mathbb{C}}R_g(x)g(x)d^2x = 8\pi$, this leads naturally to the following definition

$$\int F(\varphi)e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2}|\nabla_g\varphi(x)|^2g(x)d^2x-\frac{1}{4\pi}\int_{\mathbb{S}^2}R_g(x)\varphi(x)g(x)\,d^2x}D\varphi := \int_{\mathbb{R}}e^{-2Qc}\mathbb{E}[F(X+c)]\,dc \tag{3.7}$$

Let us introduce the **Liouville field**

$$\phi = X(x) + \frac{Q}{2}\ln g + c = c + X(x) - 2Q\ln|x|_+ \tag{3.8}$$

and consider the measure $\mathbb{E}^Q$

$$\mathbb{E}^Q[F(\phi)] = \int_{\mathbb{R}}e^{-2Qc}\mathbb{E}[F(X+\frac{Q}{2}\ln g + c)]\,dc \tag{3.9}$$

The above describes a free field theory $\mu = 0$ and to describe the Liouville theory, we need the **Gaussian Multiplicative chaos**.

Proposition 3.2. *Let* $X_\epsilon = X\star\theta_\epsilon$ *be a modification of* X, *then* $\lim_{\epsilon\to 0}\frac{e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}}{|x|_+^4}dx$ *exists in probability in the space of Radon measure on* $\mathbb{S}^2$ *and the limit does not depend on the mollifier. We note it as* $M_\gamma(dx)$ *and call it Gaussian multiplicative chaos.*

Let ϕ_ϵ be the circle average approximation of ϕ, namely $\phi_\epsilon(x) = \frac{1}{2\pi}\int_0^{2\pi}\phi(x+\epsilon e^{i\theta})d\theta$. We have $\phi_\epsilon = X_\epsilon + \frac{Q}{2}(\ln g)_\epsilon + c$ where $(\ln g)_\epsilon$ denotes the circle average of $\ln g$. We consider the associated **vertex operator**

$$V_{\alpha,\epsilon}(x) = \epsilon^{\frac{\alpha^2}{2}}e^{\alpha\phi_\epsilon(x)}$$

We have for $x\in\mathbb{C}$ that $\mathbb{E}[X_\epsilon(x)^2] = \ln\frac{1}{\epsilon} - \frac{1}{2}\ln g(x) + o(1)$ ($o(1)$ is with respect to ϵ) hence

$$V_{\alpha,\epsilon}(x) = e^{\alpha c}e^{\alpha X_\epsilon(x)-\frac{\alpha^2}{2}\mathbb{E}[X_\epsilon(x)]}g(x)^{\frac{\alpha Q}{2}-\frac{\alpha^2}{4}}(1+o(1)) = |z|_+^{-4\Delta_\alpha}e^{\alpha c}e^{\alpha X_\epsilon(z)-\frac{\alpha^2}{2}\mathbb{E}[X_\epsilon(z)^2]}$$

and Δ_α is called the *conformal weight* of V_α.
For $\alpha = \gamma$, one gets $\frac{\gamma Q}{2} - \frac{\gamma^2}{4} = 1$ and therefore

$$V_{\gamma,\epsilon}(x) = e^{\gamma c}\frac{e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)]}}{|x|_+^4}(1+o(1))$$

hence we get the following convergence in the space of Radon measures

$$V_{\gamma,\epsilon}(x)d^2x \underset{\epsilon\to 0}{\to} e^{\gamma c}M_\gamma(d^2x).$$

Now for F continuous and non negative on $H^s(\hat{\mathbb{C}})$, we use

$$\langle F(\phi)\rangle_{\gamma,\mu} = \int_{\mathbb{R}} e^{-2Qc}\mathbb{E}[F(c+X-2Q\ln|.|_+)e^{-\mu e^{\gamma c}M_\gamma(\mathbb{C})}]\,\mathrm{d}c. \tag{3.10}$$

as probabilistic definition of path integral. Then the n-point correlations can be defined for *real* valued α_i via the following limit

$$\langle\prod_{i=1}^n V_{\alpha_i}(z_i)\rangle_{\gamma,\mu} := \lim_{\epsilon\to 0}\langle\prod_{i=1}^n V_{\alpha_i,\epsilon}(z_i)\rangle_{\gamma,\mu} \tag{3.11}$$

Proposition 3.3. *The limit (3.11) exists and is non trivial if and only if the following bounds hold*

$$\sum_{i=1}^n \alpha_i > 2Q, \qquad \alpha_i < Q, \quad \forall i = 1,\dots,n \qquad (\textbf{Seiberg bounds}). \tag{3.12}$$

And the limit (3.11) admits the following representation in terms of the moments of GMC

$$\langle\prod_{i=1}^n V_{\alpha_i}(z_i)\rangle_{\gamma,\mu} = \gamma^{-1}\left(\prod_{1\leqslant j<j'\leqslant n}\frac{1}{|z_j-z_{j'}|^{\alpha_j\alpha_{j'}}}\right)\mu^{-s}\Gamma(s)\mathbb{E}[Z^{-s}] \tag{3.13}$$

where $s = \frac{\sum_{i=1}^n \alpha_i - 2Q}{\gamma}$, Γ *is the standard Gamma function and (recall that* $|x|_+ = \max(|x|,1)$*)*

$$Z = \int_{\mathbb{C}}\left(\prod_{i=1}^n \frac{|x|_+^{\gamma\alpha_i}}{|x-z_i|^{\gamma\alpha_i}}\right)M_\gamma(\mathrm{d}x).$$

Remark 3.4. *If we remove the non trivialness requirement of correlation function, the Seiberg bound can be extened to*

$$-s < \frac{4}{\gamma^2}\wedge\min_{1\leqslant k\leqslant N}\frac{2}{\gamma}(Q-\alpha_k), \qquad \alpha_k < Q, \quad \forall k \tag{3.14}$$

We also introduce the change of variables formula for Liouville field,

Proposition 3.5. *Let* $\psi:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ *be a Möbius map and* $\langle|G(\phi)|\rangle_{\gamma,\mu}<\infty$. *Then*

$$\langle G(\phi\circ\psi + Q\ln|\psi'|)\rangle_{\gamma,\mu} = \langle G(\phi)\rangle_{\gamma,\mu}. \tag{3.15}$$

Finally we state the **KPZ formula** and **Weyl anomaly** which play a fundamental rule in Liouville Conformal field theory,

Proposition 3.6 (KPZ formula)**.**

$$\langle\prod_{k=1}^N V_{\alpha_k}(\psi(z_k))\rangle_{\gamma,\mu} = \prod_{k=1}^N|\psi'(z_k)|^{-2\Delta_{\alpha_k}}\langle\prod_{k=1}^N V_{\alpha_k}(z_k)\rangle_{\gamma,\mu} \tag{3.16}$$

Proposition 3.7 (Weyl anomaly)**.** *Given a metric* $g = e^\varphi\hat{g}$ *conformally equivalent to the spherical metric* $\hat{g}$ *we have,*

$$\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle_{g,\gamma,\mu} = \exp\left(\frac{c_L}{96\pi}\Big(\int_{\mathbb{C}}|\nabla\varphi(x)|^2d^2x + 4\int_{\mathbb{C}}\varphi(x)\hat{g}(x)d^2x\Big)\right)\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle_{\hat{g},\gamma,\mu}, \tag{3.17}$$

where $c_L = 1 + 6Q^2$. *This constant* c_L *is the so-called central charge of the Liouville conformal field theory.*

4. DOZZ formula

In this section, we will give a summary the result of the seminal paper [KRV20], which proves the DOZZ formula in a mathematics rigor. Since the conformal automorphism group of sphere $\cong PSL_2(\mathbb{C})$ is generated by Möbius transformation, then the KPZ formula fixes the three point correlation functions up to a constant:

$$\langle \prod_{k=1}^{3} V_{\alpha_k}(z_k) \rangle_{\gamma,\mu} = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_\gamma(\alpha_1, \alpha_2, \alpha_3) \tag{4.1}$$

The structure constants C_γ in (4.1) can be recovered as the following limit

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \lim_{z_3 \to \infty} |z_3|^{4\Delta_3} \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z_3) \rangle, \tag{4.2}$$

where $\Delta_j := \Delta_{\alpha_j}$. Now we can give a **probabilistic construction** of structure constant:

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = 2\mu^{-s} \gamma^{-1} \Gamma(s) \mathbb{E}(\rho(\alpha_1, \alpha_2, \alpha_3)^{-s}) \tag{4.3}$$

where $s = (\sum_{i=1}^{3} \alpha_i - 2Q)/\gamma$ and

$$\rho(\alpha_1, \alpha_2, \alpha_3) = \int_{\mathbb{C}} \frac{|x|_+^{\gamma(\alpha_1+\alpha_2+\alpha_3)}}{|x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}} M_\gamma(d^2x).$$

In physical literature, Teschner argued that $C(\alpha_1, \alpha_2, \alpha_3)$ should be doubly periodic in each α_i:

$$C(\alpha_1 + \tfrac{\gamma}{2}, \alpha_2, \alpha_3) = -\frac{1}{\pi\mu} \mathcal{D}_\gamma(\tfrac{\gamma}{2}, \alpha_1, \alpha_2, \alpha_3) C(\alpha_1 - \tfrac{\gamma}{2}, \alpha_2, \alpha_3) \tag{4.4}$$

$$C(\alpha_1 + \tfrac{2}{\gamma}, \alpha_2, \alpha_3) = -\frac{1}{\pi\widetilde{\mu}} \mathcal{D}_\gamma(\tfrac{2}{\gamma}, \alpha_1, \alpha_2, \alpha_3) C(\alpha_1 - \tfrac{2}{\gamma}, \alpha_2, \alpha_3) \tag{4.5}$$

with $\widetilde{\mu} = \frac{(\mu\pi l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{\pi l(\frac{4}{\gamma^2})}$ and

$$\mathcal{D}_\gamma(\chi, \alpha_1, \alpha_2, \alpha_3) = \frac{l(-\chi^2) l(\chi\alpha_1) l(\chi\alpha_1 - \chi^2) l(\frac{\chi}{2}(\bar{\alpha} - 2\alpha_1 - \chi))}{l(\frac{\chi}{2}(\bar{\alpha} - \chi - 2Q)) l(\frac{\chi}{2}(\bar{\alpha} - 2\alpha_3 - \chi)) l(\frac{\chi}{2}(\bar{\alpha} - 2\alpha_2 - \chi))} \tag{4.6}$$

where $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and

$$l(x) = \Gamma(x)/\Gamma(1-x). \tag{4.7}$$

He showed equations (4.4), (4.5) have a meromorphic solution C_γ^{DOZZ} by using BPZ equation, crossing symmetry and a mysterious reflection relation:

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1) C(2Q - \alpha_1, \alpha_2, \alpha_3) \tag{4.8}$$

The DOZZ formula is expressed in terms of a special function $\Upsilon_{\frac{\gamma}{2}}(z)$ defined for $0 < \Re(z) < Q$ by the formula

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \Big((\tfrac{Q}{2} - z)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4}) \sinh(\frac{t}{\gamma})} \Big) \frac{dt}{t}. \tag{4.9}$$

The function $\Upsilon_{\frac{\gamma}{2}}$ can be analytically continued to $\mathbb{C}$ by some functional equation. It has no poles in $\mathbb{C}$ and the zeros of $\Upsilon_{\frac{\gamma}{2}}$ are simple (if $\gamma^2 \notin \mathbb{Q}$) and given by the discrete set $(-\frac{\gamma}{2}\mathbb{N} - \frac{2}{\gamma}\mathbb{N}) \cup (Q + \frac{\gamma}{2}\mathbb{N} + \frac{2}{\gamma}\mathbb{N})$. With these notations, the **DOZZ formula** $C_\gamma^{\mathrm{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ is the following expression

$$C_\gamma^{\mathrm{DOZZ}}(\alpha_1,\alpha_2,\alpha_3) = (\pi\,\mu\,l(\tfrac{\gamma^2}{4})\,(\tfrac{\gamma}{2})^{2-\gamma^2/2})^{\frac{2Q-\bar{\alpha}}{\gamma}} \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2Q}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2}-\alpha_3)}. \tag{4.10}$$

The main property of DOZZ formula is that it satisfies the reflection relation:

$$C_\gamma^{\mathrm{DOZZ}}(\alpha_1,\alpha_2,\alpha_3) = R^{\mathrm{DOZZ}}(\alpha_1)C_\gamma^{\mathrm{DOZZ}}(2Q-\alpha_1,\alpha_2,\alpha_3) \tag{4.11}$$

with

$$R^{\mathrm{DOZZ}}(\alpha) = -(\pi\,\mu\,l(\tfrac{\gamma^2}{4}))^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(-\frac{\gamma(Q-\alpha)}{2})}{\Gamma(\frac{\gamma(Q-\alpha)}{2})} \frac{\Gamma(-\frac{2(Q-\alpha)}{\gamma})}{\Gamma(\frac{2(Q-\alpha)}{\gamma})}. \tag{4.12}$$

The probabilistic expression $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$ vanishes identically if some $\alpha_i \geqslant Q$. This is in contradiction with DOZZ since$C_\gamma^{\mathrm{DOZZ}}(\alpha_1,\alpha_2,\alpha_3) = 0$ only on $(-\frac{\gamma}{2}\mathbb{N} - \frac{2}{\gamma}\mathbb{N}) \cup (Q + \frac{\gamma}{2}\mathbb{N} + \frac{2}{\gamma}\mathbb{N})$. To solve this, the probabilistic C_γ is analytic in α_i and the analytic continuation to $\alpha_i > Q$ is nontrivial.

Theorem 4.1. *Let $\alpha_1, \alpha_2, \alpha_3$ satisfy the bounds (3.14) with $N = 3$. The following equality holds*

$$C_\gamma(\alpha_1,\alpha_2,\alpha_3) = C_\gamma^{\mathrm{DOZZ}}(\alpha_1,\alpha_2,\alpha_3).$$

5. THE DYNAMICS OF THE LIOUVILLE FIELD

In this section we explain the dynamics of Liouville field, it inherits the Markov property from GFF and by a kind of Feymann-Kac formula it gives rises to a semigroup generator $\mathbf{H}$ so called the **full Liouville Hamiltonian**.

Given two independent sequences of i.i.d. standard Gaussians $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$, the GFF on the unit circle is the random Fourier series

$$\varphi(\theta) = \sum_{n\neq 0} \varphi_n e^{in\theta} \tag{5.1}$$

where for $n > 0$

$$\varphi_n := \frac{1}{2\sqrt{n}}(x_n + iy_n), \quad \varphi_{-n} := \overline{\varphi}_n. \tag{5.2}$$

The Green kernel is given by:

$$\mathbb{E}[\varphi(\theta)\varphi(\theta')] = \ln\frac{1}{|e^{i\theta} - e^{i\theta'}|}. \tag{5.3}$$

The underlying probability space here is $\Omega_{\mathbb{T}} = (\mathbb{R}^2)^{\mathbb{N}^*}$. It is equipped with the cylinder sigma-algebra $\Sigma_{\mathbb{T}} = \mathcal{B}^{\otimes\mathbb{N}^*}$, where $\mathcal{B}$ stands for the Borel sigma-algebra on $\mathbb{R}^2$ and the product measure

$$\mathbb{P}_{\mathbb{T}} := \bigotimes_{n\geqslant 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2+y_n^2)} \mathrm{d}x_n \mathrm{d}y_n. \tag{5.4}$$

Here $\mathbb{P}_{\mathbb{T}}$ is supported on $H^s(\mathbb{T})$ for any $s<0$. First, we introduce the **reflection property**, which gives a representation of Liouville correlation functions on $L^2(\mathbb{R}\times\Omega_{\mathbb{T}})$.
For $B\subset\hat{\mathbb{C}}$ Borel set, let $\mathcal{A}_B$ be the sigma-algebra in $H^s(\hat{\mathbb{C}})$ generated by the functions $g\mapsto\langle g,c+f\rangle$ with $f\in C_0^\infty(B)$ and $c\in\mathbb{R}$ ($\langle\cdot,\cdot\rangle$ stands for duality bracket) and let $\mathcal{F}_B$ denote the $\mathcal{A}_B$-measurable complex valued functions $F:H^s(\hat{\mathbb{C}})\to\mathbb{R}$. Let $\theta:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be the reflection at the equator:

$$\theta(z)=1/\bar{z} \tag{5.5}$$

and extend θ to $F:W^s(\hat{\mathbb{C}})\to\mathbb{C}$ by

$$(\Theta F)(g)=F(g\circ\theta). \tag{5.6}$$

Let $\mathbb{D}=\{|z|<1\}$ be the unit disk. Recall that $\phi=c+X-2Q\ln|.|_+$ and let $\mathcal{F}^2_{\mathbb{D}}$ denote the subset of $\mathcal{F}_{\mathbb{D}}$ made up of those F such that $\langle|F(c+X)\Theta F(c+X)|\rangle_{\gamma,\mu}<\infty$. For $F,G\in\mathcal{F}^2_{\mathbb{D}}$ we define

$$(F,G)_{\mathbb{D}}:=\langle\Theta F(c+X)\overline{G(c+X)}\rangle_{\gamma,\mu}. \tag{5.7}$$

Reflection positivity is the following statement:

Proposition 5.1. *The sesquilinear form* (5.7) *is non-negative: for all $F\in\mathcal{F}^2_{\mathbb{D}}$*

$$(F,F)_{\mathbb{D}}\geqslant 0.$$

Now we introduce a decomposition of full plane gaussian free field: the GFF on the Riemann sphere X decomposes as the sum of three independent variables

$$X=P(\varphi)+X_{\mathbb{D}}+X_{\mathbb{D}^c} \tag{5.8}$$

where P is the harmonic extension of the circle GFF φ defined on $\Omega_{\mathbb{T}}$ and $X_{\mathbb{D}},X_{\mathbb{D}^c}$ are two independent GFFs defined on $\mathbb{D}$ and $\mathbb{D}^c$ with Dirichlet boundary conditions. The Dirichlet GFF $X_{\mathbb{D}^c}$ on the complement $\mathbb{D}^c$ of $\mathbb{D}$ can be constructed as $X_{\mathbb{D}^c}(e^{t+i\cdot})\overset{law}{=}X_{\mathbb{D}}(e^{-t+i\cdot})$, $t\geqslant 0$. The canonical Hilbert space $\mathcal{H}_{\mathbb{D}}$ of LCFT is then defined as the completion of $\mathcal{F}^2_{\mathbb{D}}/\mathcal{N}_0$, where $\mathcal{N}_0$ is the null space $\mathcal{N}_0=\{F\in\mathcal{F}_{\mathbb{D}}\,|\,(F,F)_{\mathbb{D}}=0\}$, with respect to the sesquilinear form (5.7). Now, we construct a map

$$U:\mathcal{F}_{\mathbb{D}}\to L^2(\mathbb{R}\times\Omega_{\mathbb{T}}) \tag{5.9}$$

$$(UF)(c,\varphi):=e^{-Qc}\mathbb{E}_\varphi[F(c+X)e^{-\mu e^{\gamma c}M_\gamma(\mathbb{D})}] \tag{5.10}$$

where $X=X_{\mathbb{D}}+P\varphi$ and M_γ is its GMC measure. which descends to a unitary map from $\mathcal{H}_{\mathbb{D}}$ onto $L^2(\mathbb{R}\times\Omega_{\mathbb{T}})$, denoted also by U.

Then by independence of $X_{\mathbb{D}}$ and $X_{\mathbb{D}^c}$, we have $(F,G)_{\mathbb{D}}=\langle UF\,|\,UG\rangle_2$. So we can represent the four point correlation function as:

Proposition 5.2. *Let*

$$U_{\alpha,\beta}(z_1,z_2):=\lim_{\epsilon\to 0}U(V_{\alpha,\epsilon}(z_1)V_{\beta,\epsilon}(z_2)),$$

Then we have

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(z')V_{\alpha_4}(\infty)\rangle_{\gamma,\mu}=|z'|^{-4\Delta_{\alpha_3}}\langle U_{\alpha_1,\alpha_2}(0,z)\,|\,U_{\alpha_4,\alpha_3}(0,\frac{1}{z'})\rangle_2 \tag{5.11}$$

The dilation $z \in \mathbb{C} \to s_q(z) = qz$ maps $\mathbb{D}$ to itself for $|q| \leqslant 1$ and it extends to a map on distributions $X \in W^s(\hat{\mathbb{C}})$ by $X \to X \circ s_q$. We then define for $F \in \mathcal{F}_{\mathbb{D}}$

$$(S_q F)(X) = F(c + X \circ s_q + Q \log |q|). \tag{5.12}$$

We can use möbius invariance (3.15) to show that S_q extends to $\mathcal{H}_{\mathbb{D}}$ and defines a strongly continuous contraction semigroup

$$S_q S_{q'} = S_{qq'} \tag{5.13}$$

with the operator norm $\|S_q\| \leqslant 1$. Taking $q = e^{-t}$ with $t \geqslant 0$ the Hille-Yosida theorem gives a semigroup generator:

$$U S_{e^{-t}} U^{-1} = e^{-t\mathbf{H}*} \tag{5.14}$$

The main theorem is:

Proposition 5.3. *Consider the full Liouville Hamiltonian,*

$$\mathbf{H} := -\frac{1}{2}\partial_c^2 + \frac{1}{2}Q^2 + \mathbf{P} + \mu e^{\gamma c} V \tag{5.15}$$

where V is a positive potential whose expression is

$$V = \int_0^{2\pi} e^{\gamma \varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\varphi(\theta)^2]} \mathrm{d}\theta. \tag{5.16}$$

and $\mathbf{P} = \sum_{n=1}^{\infty} n((-\partial_{x_n} + x_n)\partial_{x_n} + (-\partial_{y_n} + y_n)\partial_{y_n})$. Then if $\gamma \in (0, \sqrt{2})$, $\mathbf{H}$ admits a Friedrichs extension, and $\mathbf{H} = \mathbf{H}*$.

Remark 5.4. *V is a non-trivial measure if and only if $\gamma \in (0, \sqrt{2})$.*

We also have a so called Feymann-Kac formula for $e^{-t\mathbf{H}*}$,

Proposition 5.5.

$$e^{-t\mathbf{H}*} = e^{-\frac{Q^2 t}{2}} \mathbb{E}_{\varphi}\big[f(c + B_t, \varphi_t) e^{-\mu \int_0^t e^{\gamma(c+B_s)} V(\varphi_s) \mathrm{d}s}\big] \tag{5.17}$$

Remark 5.6. *We also define the U_0 for the free case $\mu = 0$ as:*

$$(U_0 F)(c, \varphi) := e^{-Qc} \mathbb{E}_{\varphi}[F(c + X)]. \tag{5.18}$$

and it corresponds to the semigroup generator $\mathbf{H}^0 = -\frac{1}{2}\partial_c^2 + \frac{1}{2}Q^2 + \mathbf{P}$.

6. CONFORMAL BOOTSTRAP

Theorem 6.1. *Let $\gamma \in (0, \sqrt{2})$. the spectrum of $\mathbf{H}$ is absolutely continuous and given by the half-line $[\frac{Q^2}{2}, \infty)$. Each $E \in [\frac{Q^2}{2}, \infty)$ is of finite multiplicity (in the sense of absolutely continuous spectrum) and there is a family of generalized eigenstates $\Psi_{Q+iP,\nu,\widetilde{\nu}} \in \cap_{\varepsilon > 0} e^{-\varepsilon c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ labeled by $P \in \mathbb{R}^+$ and $\nu, \widetilde{\nu}$ belong to the set of Young diagram $\mathcal{T}$ such that*

$$\mathbf{H}\Psi_{Q+iP,\nu,\widetilde{\nu}} = \Big(\frac{Q^2}{2} + \frac{P^2}{2} + |\nu| + |\widetilde{\nu}|\Big)\Psi_{Q+iP,\nu,\widetilde{\nu}}.$$

Moreover $\Psi_{Q+iP,\nu,\widetilde{\nu}}$ *is a family diagonalizing* $\mathbf{H}$ *in the sense that for each* $u_1, u_2 \in e^{\delta c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ *for some* $\delta > 0$

(6.1)

$$\langle u_1 \,|\, u_2 \rangle_2 = \lim_{\substack{N\to\infty \\ L\to\infty}} \sum_{\substack{\nu,\widetilde{\nu},\nu',\widetilde{\nu}' \in \mathcal{T}, \\ |\nu|+|\widetilde{\nu}| \leqslant N, \\ |\nu'|=|\nu|, |\widetilde{\nu}'|=|\widetilde{\nu}|}} \int_0^L \langle u_1 \,|\, \Psi_{Q+iP,\nu',\widetilde{\nu}'} \rangle_2 \langle \Psi_{Q+iP,\nu,\widetilde{\nu}} \,|\, u_2 \rangle_2 F^{-1}_{Q+iP}(\nu,\nu') F^{-1}_{Q+iP}(\widetilde{\nu},\widetilde{\nu}') \, \mathrm{d}P$$

Here F^{-1}_{Q+iP} *is called* **Schapovalov matrix element** *comes form the non-orthogonal property of* $\Psi_{Q+iP,\nu,\widetilde{\nu}}$*.*

The eigenstates $\Psi_{\alpha,\nu,\widetilde{\nu}}$ is essential for mathematical treatment of conformal bootstrap since the construction of $\Psi_{Q+iP,\nu,\widetilde{\nu}}$ shows it is analytic in a large region called W_λ which contains both spectrum line $Q + i\mathbb{R}^+$ and very negative real numbers. When α is negative enough we can deduce a probabilistic representation of $\Psi_{Q+iP,\nu,\widetilde{\nu}}$ by GMC and Feymann-Kac formula(5.5)and prove the following property first in the negative real number region and then analytically continues back to the spectrum line:

Proposition 6.2 (Ward identities)**.** *For all* P *such that* $\alpha = Q + iP$ *belongs to* W_λ*, thus in particular for* $P > 0$*, the scalar product* $\langle \Psi_{Q+iP,\nu,\widetilde{\nu}} \,|\, U_{\alpha_1,\alpha_2}(0,z) \rangle_2$ *is explicitly given by the following expression*

$$\langle \Psi_{Q+iP,\nu,\widetilde{\nu}} \,|\, U_{\alpha_1,\alpha_2}(0,z) \rangle_2 = v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \widetilde{\nu}) v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) \times \frac{1}{2} C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1, \alpha_2, Q+iP) \bar{z}^{|\nu|} z^{|\widetilde{\nu}|} |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} \tag{6.2}$$

where $v(\Delta, \Delta', \Delta'', \nu) \coloneqq \prod_{j=1}^{k} (\nu_j \Delta' - \Delta + \Delta'' + \sum_{u<j} \nu_u)$.

With all ingredients above, we come to the so-called bootstrap formula,

Theorem 6.3. *Let* $\gamma \in (0, \sqrt{2})$ *and* $\alpha_i < Q$ *for all* $i \in [\![1, 4]\!]$*. Then the following identity holds for* $\alpha_1 + \alpha_2 > Q$ *and* $\alpha_3 + \alpha_4 > Q$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\gamma,\mu} = \frac{1}{8\pi} \int_0^\infty C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1, \alpha_2, Q-iP) C^{\mathrm{DOZZ}}_{\gamma,\mu}(Q+iP, \alpha_3, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 \mathrm{d}P \tag{6.3}$$

Where $\mathcal{F}_P(z)$ *is called* **conformal block** *whose coefficients(as power series of z)* β_n *are given by*

$$\beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_4}) = \sum_{\nu,\nu' \in \mathcal{T}_n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) F^{-1}_{Q+iP}(\nu, \nu') v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+iP}, \nu').$$

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7. Some Future Projects

7.1. **4-point sphere conformal block and Nekrasov partition function.** This part is mainly adapted from [AGT10].

Nekrasov partition function arises in the study of four-dimensional supersymmetric gauge theories [Nek02], that is theories described by a connection on some vector bundle and that rely on a path integral formulation. The Lagrangian of this theory enjoys some invariance under local transformations, encoded by Lie groups. In his seminal work [Nek02], Nekrasov considered a deformation of the Lagrangian of its model quantified two deformation parameters ϵ_1, ϵ_2 (corresponding to a rotation in the space-time $\mathbb{R}^4$). These deformations break the symmetry of the model and this crack is measured using Nekrasov partition function. This partition function factorizes into three parts, the one relevant in the context of the AGT conjecture being the instanton part and which represents the contribution coming from the moduli space of instanton. In the case of $U(2)$ instantons the path integral provides an explicit expression for the partition function which takes the form:

$$Z_{inst}(a; m_0, m_1; \widetilde{m_0}, \widetilde{m_1}; q|\epsilon_1, \epsilon_2) = 1 + \sum_{N \geqslant 1} q^N Z_N\left(a, m|\epsilon_1, \epsilon_2\right) \tag{7.1}$$

where the $Z_N\left(a, m|\epsilon_1, \epsilon_2\right)$ are defined through integral expressions ([Nek02, Equation (3.25)]) and which represents integrals over the instantons with topological number N. By using the theory of torus action on Nakajima varities, these integral can be explicitly computed and enjoys a nice combinatorial form ([NO03, Equation (6.3)]), which is central in the AGT correspondence:

$$Z_N\left(a, m|\epsilon_1, \epsilon_2\right) = \sum_{|\vec{\lambda}|=N} \frac{Z_{bif}(\alpha_2|i(Q-\alpha_1), \varnothing; P, \vec{\lambda})}{Z_{bif}(0|P, \vec{\lambda}; P, \vec{\lambda})} \frac{Z_{bif}(\alpha_3|P, \vec{\lambda}; i(Q-\alpha_4), \varnothing)}{Z_{bif}(0|i(Q-\alpha_4), \varnothing; i(Q-\alpha_4), \varnothing)} \tag{7.2}$$

where the sum ranges over pairs of Young tableaux with total size N. In the expression of we have considered:

- the *deformation parameters* $\epsilon_{1,2}$
- the *multiplet of massless vectors* $a = (a_1)$
- the *masses of the adjoint multiplet* $m = (m_0, m_1)$ and $\widetilde{m_0}$, $\widetilde{m_1}$
- the *instanton parameter* $q = e^{2i\pi\tau}$ with τ the *microscopic coupling.*

The sum ranges over pairs of Young diagrams $\vec{\lambda} = (\lambda_1, \lambda_2)$ with total weight N and the quantities denoted by Z_{bif} are explicit:

$$Z_{bif}(\alpha|P, \vec{\lambda}; P', \vec{\nu}) = \prod_{i,j=1}^{2} \prod_{t \in \lambda^{(i)}} \left(Q - E_{\lambda_i, \nu_j}(P_i - P'_j|t) - \alpha\right) \prod_{s \in \nu^{(j)}} \left(E_{\nu_j, \lambda_i}(P'_j - P_i|s) - \alpha\right) \tag{7.3}$$

The AGT conjecture adresses this issue by relating Virasoro conformal blocks to the Nekrasov partition function. Indeed, both theories - gauge and conformal theories - are somehow related to the *moduli space of instantons*, via a path integral for the $\mathcal{N} = 2$ supersymmetric gauge theory and through the action of the symmetry algebra of a CFT on its equivariant cohomology. The case we consider here is the one studied by the authors in [AGT10] and corresponds to $U(2)$ instantons on $\mathbb{C}^2$, *i.e.* $\mathcal{N} = 2$ $U(2)$ supersymmetric gauge theory and a CFT which corresponds to the tensor product of Liouville theory and a free boson. Indeed, we can decompose $U(2) = U(1) \oplus SU(2)$, and then the term $U(1)$ would come from the free boson while the $SU(2)$ part corresponds to a puncture in Liouville theory via the S-duality. This motivates the following identity, which is the formulation of the main conjecture in [AGT10] for Liouville theory on the sphere:

$$Z_{inst}(a; m_0, m_1; \widetilde{m_0}, \widetilde{m_1}; q|\epsilon_1, \epsilon_2) = (1-z)^{2\alpha_2(Q-\alpha_3)} \times \mathcal{F}_p(z) \tag{7.4}$$

where the Nekrasov partition function on the left corresponds to the $U(2)$ theory with four flavours and the prefactor $(1-z)^{2\alpha_2(Q-\alpha_3)}$ corresponds to the contribution of the group $U(1)$. Parameters on the left and right-hand sides are related by:

- $\epsilon_1 = \frac{\gamma}{2}$ and $\epsilon_2 = \frac{2}{\gamma}$
- $q = z \in \mathbb{D}$
- $\widetilde{m}_0 = \alpha_1$, $m_0 = \alpha_2$, $m_1 = \alpha_3$, $\widetilde{m}_1 = \alpha_4$
- $a = iP$

In [AFLT], they produce an orthogonal special basis by bootstrap method, with the strategy of [GKRV], we can make this proof rigorously. In the future, we hope to construct this basis directly from probabilistic method.

7.2. Convergence of the conformal block.

As we mentioned above, the 4-point sphere conformal block can be formally written as a power series in variable z. In [GKRV], they proved that the 4-point sphere conformal block is convergent when $|z| < 1$. We can use the conformal bootstrap method to derive an expression for n-point conformal block recursively, we hope to prove the convergence property in the future.

7.3. Extend to other topologies.

In this note, we mainly focus on the construction of conformal block on sphere by probabilistic method. It's also possible to construct this on a complex tori. For higher genus surface, we may get a new surface by gluing two lower genus surfaces M, N, noted as $M\#N$ which called connected sum. Then we hope to understand the relation between conformal block of $M\#N$ and of M, N in a path integral manner.

7.4. Imaginary Liouville theory.

The main assumption of probabilistic Liouville theory is $\gamma \in (0,2)$. If we take $\gamma = i\beta$ $\beta \in \mathbb{R}$, it's not hard to construct the complex Gaussian Multiplicative chaos, actually we can prove it's a nontrivial measure when $\beta^2 < 2$, but how to construct the Liouville correlation function is still an open problem. Indeed, if we go back to formula (3.13), we need to deal with the case when s is imaginary in the right side. The imaginary Liouville theory is interesting since it's related to critical FK model, CLE, etc...

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