STABLE HOMOLOGY AND SCANNING METHOD

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Résumé

The Madsen-Weiss theorem [11] [10] links the stable homology of mapping class groups with the homology of a certain infinite loop space. Its proof makes use of the scanning method, which compares a manifold embedded in $\mathbb{R}^\infty$ with a space of "local images" of the manifold. We will detail a proof of the Barratt-Priddy-Quillen-Segal theorem, which is Madsen-Weiss in dimension 0, in the hopes of adapting this proof in the future to spaces of graphs and trees.

Contents

1. Mapping class groups and the scanning map

Let us introduce the backbone of scanning theory, and further detail this infinite loop space that we mentioned. The idea behind this was introduced by Segal [14],

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McDuff [12], and a little later, Madsen and Tillmann [10]. We follow the approach given by Søren Galatius and Oscar Randal-Williams [3], [4], [6]. An overview of different approaches to this theorem is given by Hatcher [7]

Remark 1.1. To avoid confusion, we will say that a submanifold W of $\mathbb{R}^n$ is topologically closed if it is a closed subet of $\mathbb{R}^n$, and that it is closed if it is closed in the usual manifold-theoretic sense of the term; that is, if it is compact and has no boundary.

Definition 1.2. Let $\Psi(\mathbb{R}^n)$ be the set of pairs (w, W) where W is a smooth topologically closed 2-manifold in $\mathbb{R}^n$ with no boundary and w is an orientation of W.

Definition 1.3. Let X be a smooth manifold. We will say that a map $f : X \to \Psi(\mathbb{R}^n)$ is smooth if

- the graph $\Gamma_f = \{(x, v) \in X \times \mathbb{R}^n | v \in f(x)\}$ is a smooth topologically closed submanifold of $X \times \mathbb{R}^n$,
- the projection $p_f : \Gamma_f \to X$ is a submersion,
- the orientations of $f(x)$ vary continuously, i.e. give a global orientation on the kernel of $Dp_f : T\Gamma_f \to TX$).

We can define a topology on $\Psi(\mathbb{R}^n)$ detailed by Søren Galatius [4]. It satisfies the following property:

Lemma 1.4. *Any smooth map $X \to \Psi(\mathbb{R}^n)$ is continuous. Any continuous map $f : X \to \Psi(\mathbb{R}^n)$ can be perturbed to a smooth map with a perturbation which is constant near any closed set on which f is already smooth.*

Here, a perturbation is defined as a continuous extension of f to $X \times [0, 1[$ which is smooth on $X \times]0, 1[$. Without going into the details of the construction of this topology, let us look at a few properties of it. A neighbourhood basis at $\emptyset \in \Psi(\mathbb{R}^n)$ can be given by

$$\mathcal{U}(K) = \{W \in \Psi(\mathbb{R}^n) | W \cap K = \emptyset\},$$

for K ranging through compact subsets of $\mathbb{R}^n$. Saying that a sequence $(W_i, w_i) \in \Psi(\mathbb{R}^n)$ converges to $\emptyset$ means that for every compact K, there exists an N_K such that for all $i > N$, $W_i \cap K = \emptyset$.

Example 1.5. Let $f : \mathbb{R} \to \Psi(\mathbb{R}^2)$ be the function defined by

$$f(t) = \{t^{-1} \times \mathbb{R}^2\}$$
$$f(0) = \emptyset.$$

The function f is continuous (and even smooth).

Proposition 1.6. *For all $n \geq 3$, $\Psi(\mathbb{R}^n)$ is path-connected.*

Proof. Let $(W, w) \in \Psi(\mathbb{R}^n)$. Let $p \in \mathbb{R}^n \backslash W$. For $t \in [0,1]$, let $W - tp$ be the manifold W translated by the vector $-tp$.Let $(1-t)^{-1}(W - tp)$ be the scaling of that manifold by $(1-t)^{-1}$ (when $t = 0$, we set it to $\emptyset$).
The path

$$t \mapsto (1-t)^{-1}(W \backslash tp)$$

is a continuous path from W to $\emptyset$. □

There is a close relationship between $\Psi(\mathbb{R}^n)$ and surface bundles. Let B_n be the subspace of $\Psi(\mathbb{R}^n)$ defined as follows:

$$B_n = \{W \in \Psi(\mathbb{R}^n) | W \subset]0,1[^n\}.$$

Proposition 1.7. *Let X be a smooth k-dimensional manifold. For $n > 2k + 4$, there is a bijection between $[X, B_n]$ and the set of isomorphism classes of surface bundles $E \to X$.*

Corollary 1.8. *Let X be a smooth manifold. There is a natural bijection between $[X, B_\infty]$ and the set of isomorphism classes of surface bundles $E \to X$.*

Remark 1.9.
- Recall now that the classifying space BG of a group G is the quotient of some weakly contractible space EG by the action of G. The space BG is an Eilenberg-MacLane space $K(G, 1)$, which means that all its homotopy groups are trivial, except π_1, which is equal to G.
- A G-principal bundle, for G a topological group, is a fiber bundle $\pi : E \to X$ with a continuous right action of G on P which is free and transitive, such that G preserves the fibers of P, and such that for every $x \in X$ and $y \in P_x$, the map $g \mapsto yg$ defines a homeomorphism between G and P_x.
- Every principal bundle over a compact manifold is isomorphic to a pullback of the bundle $EG \to BG$.

Remark 1.10. Let us note that there is a bijection

$$B_n = \bigsqcup_W \text{emb}(W,]0,1[^n)/Diff(W),$$

where we take the disjoint union over oriented closed surfaces W up to diffeomorphism class. This map is a homeomorphism if the right hand side is endowed with the quotient topology. Moreover, the quotient map

$$\text{emb}(W,]0,1[^n) \to \text{emb}(W,]0,1[^n)/Diff(W)$$

is a principal $Diff(W)$-bundle for all $n \leq \infty$. Since the space $\text{emb}(W,]0,1[^\infty)$ is contractible, it is a model for $EDiff(W)$. A model for $BDiff(W)$ is thus given by the space $\text{emb}(W,]0,1[^n)/Diff(W)$, which is homeomorphic to the path component

of B_∞ consisting of manifolds diffeomorphic to W.
This leads to the following conclusion:

$$B_\infty = \bigsqcup_W BDiff(W).$$

The link between the subspaces $B_n \subset \Psi(\mathbb{R}^n)$ and surface bundles is now clear. Let us delve into why we introduced the space $\Psi(\mathbb{R}^n)$ in the first place.

Definition 1.11. Let $\alpha : B_n \to \Omega^n\Psi(\mathbb{R}^n)$ be the following map:

$$\alpha(W)(v) = \begin{cases} W + v & \text{if } v \in \mathbb{R}^n \\ \emptyset & \text{if } v = \infty. \end{cases}$$

The map α is actually continuous. There is a map

$$\begin{aligned} \Psi(\mathbb{R}^n) &\to \Omega\Psi(\mathbb{R}^{n+1} \\ W &\mapsto (t \mapsto X \times t)). \end{aligned}$$

If we let $n \to \infty$, we obtain a map

$$\alpha : B_\infty \to \Omega^\infty\Psi = \underset{n\to\infty}{colim}\Omega^n\Psi(\mathbb{R}^n).$$

The path components of B_∞ are the $BDiff(W)$ and there is one path component by diffeomorphism type of oriented 2-manifold W. For a certain W, the map α sends $BDiff(W)$ to a path component of $\Omega^\infty\Psi$, and we will denote that path component by $\Omega^\infty_{[W]}\Psi$.

2. Main theorem

Theorem 2.1 (Madsen-Weiss theorem)**.** [11] *If W is a surface of genus g, the restricted map*

$$BDiff(W) \to \Omega^\infty_{[W]}\Psi$$

induces an isomorphism in integral cohomology degree $2g/3$.

2.2. **Homotopy type of $\Psi(\mathbb{R}^n)$.** Let us consider the Grassmannian $Gr_2^+(\mathbb{R}^n)$, which is the set of oriented 2-planes in $\mathbb{R}^n$. Let us consider two vector bundles over that space, the canonical bundle γ_n and its orthogonal complement $\gamma_n^\perp$. A point in the total space of $\gamma_n^\perp$ is a pair (V, v), where V is an oriented 2-plane in $\mathbb{R}^n$, and $v \in V^\perp$. V is an oriented topologically closed 2-manifold. It therefore defines a point $V \in \Psi(\mathbb{R}^n)$. After translation by the vector v, we get a map

$$\begin{aligned} \gamma_n^\perp &\to \Psi(\mathbb{R}^n) \\ (V, v) &\mapsto V + v \end{aligned}$$

Recall that the Thom space of a bundle is the one-point compactification of its total space. The manifold $V+v$ has an empty intersection with all compact sets as $|v| \to \infty$, so the map **??** extends to a continuous map

$$\begin{gathered} Th(\gamma_n^{\perp}) \xrightarrow{q} \Psi(\mathbb{R}^n) \\ (V,v) \mapsto V+v \\ \infty \mapsto \emptyset. \end{gathered}$$

Theorem 2.3. *The map q is a weak equivalence.*

This result enables us to relate the space $\Omega^\infty\Psi$ to a more well-known space:

$$\Omega^\infty MTSO(2) = \underset{n\to\infty}{\text{colim}}\Omega^n Th(\gamma_n^{\perp}).$$

Proposition 2.4. *There is a weak equivalence*

$$\Omega^\infty MTSO(2) \to \Omega^\infty\Psi.$$

This theorem has a range of applications, including stable homology computations [2], [5].

3. In dimension 0: the Barratt-Priddy-Quillen-Segal theorem

The Madsen-Weiss theorem for surfaces of dimension 0 is called the Barratt-Priddy-Quillen-Segal theorem, and its proof is much more classical [1], [13]. It is also this proof that we would like to adapt to the context of the classifying spaces of Higman-Thompson groups. We follow the approach given by Alexander Kupers [9] with a slightly different proof.

Theorem 3.1. *We have the following weak equivalence:*

$$\Omega B\big(\bigsqcup_{n\geq 0} B\Sigma_n\big) \simeq QS^0.$$

3.2. **Simplicial and semi-simplicial spaces.** Let us start by some results about simplicial and semi-simplicial spaces.

Definition 3.3. Let Δ be the category of non-empty finite sets and order-preserving maps. A simplicial space is a contravariant functor

$$\Delta^{op} \to Top.$$

A semi-simplicial set is a simplicial set without degeneracies. Instead of considering the category Δ, we consider Δ_{inj} with only injective maps.

Example 3.4. The nerve $N(\mathcal{C})_\bullet$ of a small category $\mathcal{C}$ is the simplicial set with n-simplices

$$N(\mathcal{C})_n := \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1,$$

the set of sequences of composable morphisms of length n, for $n \in \mathbb{N}$. Face maps are given by composing two consecutive morphisms or forgetting one of the morphisms at the endpoints; degeneracy maps are given by inserting an identity morphism.

Definition 3.5 (Geometric realization)**.** The geometric realization of a simplicial space X is given by:

$$|X_\bullet(n)| := \left(\bigsqcup_{p \geq 0} \Delta^p \times I_p(n) \right) / \sim,$$

where $(x, f_* p) \sim (f^* x, p)$, for every morphism $f : [k] \to [l]$ in Δ, $x \in X_l$, $p \in \Delta^k$. The thick geometric realization of a semi-simplicial space X is given by:

$$||X_\bullet(n)|| := \left(\bigsqcup_{p \geq 0} \Delta^p \times I_p(n) \right) / \sim .$$

Example 3.6. The classifying space of a category is the geometric realization of its nerve.

Definition 3.7. Let Δ_+ be the category of possibly empty ordered finite sets and morphisms order-preserving maps, then an augmented simplicial space is a functor $\Delta_+^{op} \to Top$.

This means that an augmented simplicial space is a simplicial space with a map $\epsilon : X_0 \to X_{-1}$ called an augmentation, which coequalizes both face maps $d_0, d_1 : X_1 \to X_0$. This induces a map $|X_\bullet| \to X_{-1}$. The same construction is possible for semi-simplicial objects.

Definition 3.8. A semi-simplicial resolution of a space X is an augmented semi-simplicial object $X_\bullet$ with $X_{-1} = X$ so that $\epsilon : ||X_\bullet|| \to X_{-1} = X$ is a weak equivalence.

3.9. **"Cobordism" category.** Kupers uses the 0-cobordism category to prove the BPQS theorem.

I have worked on a proof that makes use of a similar, albeit slightly different category. The reason for this is that it might be easier to apply this proof to other cases of interest, such as the classifying space of the Higman-Thompson groups.

Definition 3.10. Let $\mathcal{C}$ be the category where:

- objects are given by $x_n, n \in \mathbb{N}$, where each x_n is a given unordered configuration of n points in $\mathbb{R}^\infty$. In addition, let I be a compact cube of $\mathbb{R}\infty$ such that the images of all the $x_n, n \in \mathbb{N}$, are contained in I;

- a morphism between x_n and x_n is a collection

$$\{(l,\phi_i)_{1\leq i\leq n}|\begin{cases} l\in\mathbb{N}, \\ \forall i, \phi_i:[0,1]\to I^N\times[0,l], \\ \forall i, \phi_i(0)=x_n^i, \\ \exists\sigma\in\Sigma_n, \forall 1\leq i\leq n, \phi_i(1)=x_n^{\sigma(i)}, \\ \forall i\neq j, \forall t\in[0,1], \phi_i(t)\neq\phi_j(t) \end{cases}\}$$

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The set of morphisms from x_n to x_m for $n\neq m$ is empty.

The identity morphism is given by $l=0$ and composition of $\left(l,(\phi_i)_i\right)$ and $\left(m,(\psi_i)_i\right)$ is given by $\left(l+m,(\psi_i\circ\phi_i)_i\right)$.

Let $\mathcal{C}_N$ be the full subcategory of $\mathcal{C}$ where we only keep objects x_n for $n\leq N$. There is a functor

$$\mathcal{C}\to\bigsqcup\Sigma_n$$
$$x_n\mapsto *_n$$
$$\left(l,(\phi_i)_i\right)\mapsto\sigma \text{ such that } \forall i, \phi_i(1)=x_n^{\sigma_i}$$

Let us show that this functor induces a homotopy equivalence on classifying spaces.

Remark 3.11. We define the classifying space of a category as the geometric realization of its nerve.

Let us call f the topological function we obtain thus. Because it is a function between classifying spaces it suffices to give its image on 0- and 1-simplices to define it altogether.

$$f:B\mathcal{C}\to B\bigsqcup\Sigma_n$$

This time, f admits a left adjoint g. We already know that f is a homotopy equivalence on 0-simplices. In the case of 1-simplices,

$$f^{-1}(\sigma)=\{(l,\phi_i)_{1\leq i\leq n}|\begin{cases} l\in\mathbb{N}, \\ \forall i, \phi_i:[0,1]\to\mathbb{R}^\infty\times[0,l], \\ \forall i, \phi_i(0)=x_n^i, \\ \forall 1\leq i\leq n, \phi_i(1)=x_n^{\sigma(i)}, \\ \forall i\neq j, \forall t\in[0,1], \phi_i(t)\neq\phi_j(t) \end{cases}\}$$

Therefore, g can be defined.

Now that we have replaced the symmetric groups with the classifying space of a category which is easier to visualize, let us move on to the scanning part of the proof.

3.12. **Delooping result.**

Definition 3.13. Let $\Phi(\mathbb{R}^k \times I \times I^{N-k})$ be the collection

$$\{l \in \mathbb{N}, (\phi_i)_{1\leq i\leq n} : [0,l] \to \mathbb{R}^k \times I \times I^{N-k} | \begin{cases} \forall t, \forall i, \phi_i(t) = (\frac{t}{l}, x(t)) \text{ for some } x : I \to \mathbb{R}^k \times I^{N-k} \\ \forall i \neq j, \forall t, \phi(t) \neq \phi_j(t) \\ \exists (0 = t_0 \leq \cdots \leq t_k = l, \exists \sigma \in \Sigma_n \forall i, \forall j, \phi(t_j) = (\frac{t_j}{l}, x_n^i) \end{cases} \}$$

The very definition of this space seems to indicate it is closely linked to the category $\mathcal{C}$, and indeed that is out motive for introducing it. We have the following proposition:

Proposition 3.14. *There is a zigzag of weak equivalences*

$$B\mathcal{C}_N \leftarrow \cdots \to \Phi(\mathbb{R} \times I^N).$$

Proposition 3.15. *For $k > 0$ there is a weak equivalence*

$$\Phi(\mathbb{R}^k \times I \times I^{N-k}) \simeq \Omega\Phi(\mathbb{R} \times \mathbb{R}^k \times I^{N-k}).$$

This proceeds directly from the following result about semi-simplicial spaces:

Lemma 3.16. *If $X_\bullet$ is a semi-Segal space with X_1 path-connected (or group-like), then $X_1 \simeq \Omega||X_\bullet||$.*

Proof. Let us prove proposition 3.15 It suffices to link $\Phi(\mathbb{R}^{k+1} \times I^{N-k})$ to a semi-Segal space $X_\bullet$ such that $X_1 \simeq \Phi(\mathbb{R}^k \times I^{N-k+1})$. We will take $X_\bullet$ to be the semi-simplicial space with the space of p-simplices X_p defined as the subspace of $\Phi(\mathbb{R} \times \mathbb{R}^k \times I^{N-k} \times \mathbb{N}^p$ consisting of $(lfrm[o]--, \dots, l_p) \in \mathbb{N}^p$, $\Big((\phi_i)_i, x_{n,0}, \dots, x_{n,p}\Big)$ such that

$$\exists (t_0, \dots, t_p) \in \mathbb{R}^{p+1} | \forall i, t_{i+1} - t_i = l_{i+1} \text{ and } \phi_i(t_j) = x_{n,j}^i.$$

This is a semi-Segal space with $X_1 \simeq \Phi(\mathbb{R}^k \times I^{N-k})$.
We have a map $\epsilon : ||X_\bullet|| \to \Phi(\mathbb{R}^{k+1} \times I^{N-k})$ and we would like to show that it is a weak equivalence, for which purpose we will introduce an intermediary space $X'_\bullet$.
The space of p-simplices of $X'_\bullet$ is the subspace of $\Phi(\mathbb{R}^{k+1} \times I^{N-k-1}) \times C_n(I^\infty)^{p+1} \times \mathbb{N}^p$ consisting of $(l_1, \dots, l_p) \in \mathbb{N}^p$, $\Big((\phi_i) \text{ and } y_{0,n}, \dots, y_{p,n}\Big)$ configurations of spaces such that

$$\exists (t_0, \dots, t_p) \in \mathbb{R}^{p+1} | \forall i, t_{i+1} - t_i = l_{i+1} \text{ and } \phi_i(t_j) \neq y_{n,j}^i.$$

We therefore obtain the following factorization

$$\begin{array}{ccc} ||X_\bullet|| & \longrightarrow & ||X'_\bullet|| \\ & \searrow \quad \swarrow & \\ & \Phi(\mathbb{R}^{k+1} \times I^{N-k}). & \end{array}$$

The top map and the bottom right map are equivalences, so the bottom left one is as well. □

Remark 3.17. To take it a step further, it is even possible to give maps $\Phi(\mathbb{R}^k \times I^{N-k+1}) \to \Omega\Phi(\mathbb{R}^{k+1} \times I^{N-k})$ homotopic to the ones obtained above but also compatible with the inclusions $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}$.
This map is obtained by considering $[-\infty, \infty]$ as the domain of the loop space functor and defining its adjoint by:

$$\begin{aligned} &[-\infty, \infty] \times \Phi(\mathbb{R}^k \times I^{N-k+1}) \to \Phi(\mathbb{R}^{k+1} \times I^{n-k}) \\ &(t, (\phi_i)_i) \mapsto ((\phi_i + t \cdot e_{k+1})_i) \end{aligned}$$

We get the following sequence of weak equivalences

$$\Phi(\mathbb{R} \times I^N) \xrightarrow{\simeq} \Omega\Phi(\mathbb{R}^2 \times I^{N-1}) \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \Omega^N\Phi(\mathbb{R} \times \mathbb{R}^N).$$

3.18. **Space of local images.** We would now like to understand the last space in the chain above, $\Phi(\mathbb{R} \times \mathbb{R}^N)$.

Proposition 3.19. *We have the following weak equivalence:*

$$\Phi(\mathbb{R} \times \mathbb{R}^N) \simeq S^N.$$

Proof. There is a much simpler definition for $\Phi(\mathbb{R} \times \mathbb{R}^N)$:

$$\Phi(\mathbb{R} \times \mathbb{R}^N) = \{\phi : I \to D^N \times I | \phi(t) = (x(t), t)\} \bigsqcup \emptyset,$$

where the topology is such that when ϕ approaches ∂D^N, it approaches the emptyset.
Let us define the following function

$$\begin{aligned} f : \Phi(\mathbb{R} \times \mathbb{R}^N) &\to D^N \bigsqcup * \\ \phi &\mapsto \phi(x) \in D^N \\ \emptyset &\mapsto * \end{aligned}$$

This map is smooth and admits a homotopy inverse:

$$\begin{aligned} g : D^N \bigsqcup * &\to \Phi(\mathbb{R} \times \mathbb{R}^N) \\ * &\mapsto \emptyset \\ x \in D^N &\mapsto \phi : t \mapsto (x, t) \end{aligned}$$

It is easy to check that f and g are homotopy inverses, which proves the proposition. □

3.20. **Conclusion.** Using propositions above, we have the following result:

$$B\mathcal{C}_N \simeq \bigsqcup_{n\leq N} B\Sigma_n \simeq \Omega^N S^N.$$

Let us recall the notation

$$QS^0 = \Omega^\infty \mathbb{S} = \operatorname*{colim}_{N\to\infty} \Omega^N S^N.$$

The final result needed to prove Barratt-Priddy-Quillen-Segal is the group-completion theorem [15]:

Theorem 3.21 (Group-completion theorem)**.** *Let M be a topological monoid and let BM be its classifying space. Let us consider that the localization of the Pontryagin ring $H_*(M)$ at its multiplicative subset $\pi_0 M$, $H_*(M)[(\pi_0 M)^{-1}]$ can be constructed by right fractions. Then the map $M \to \Omega BM$ induces an isomorphism*

$$H_*(M)[(\pi_0 M)^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

Putting all of this together, we get:

$$\Omega B(\bigsqcup B\Sigma_n) \simeq QS^0.$$

4. Research questions

4.1. **Defining Higman-Thompson groups.** Let us consider groups other than the symmetric groups, such as the family of Higman-Thompson groups [8], $(V_n)_{n\in\mathbb{N}}$, $(F_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$.
Thompson introduced three groups in 1965, F, T, and V, which possess interesting properties. The group F is the group of linear piecewise homeomorphisms of the unit interval that preserve orientation, and such that the points of non-derivability are dyadic rationals (implying that all the slopes are powers of 2).
The groups T and V are defined in a similar way, but elements of T are allowed to have at most one discontinuity, and elements of V are allowed a finite number of discontinuities.

Remark 4.2. [8] The group V is an infinite group that admits a finite presentation and contains all the finite groups.

We can generalize these groups to define families of Higman-Thompson groups. The formal definition of $(V_{n,r})$ is the following:

Definition 4.3. A Cantor algebra of type n is a set X with a bijection $X^n \overset{\cong}{=} X$. For $n \geq 2$ and $r \geq 1$, $V_{n,r}$ is the automorphism group of the free Cantor algebra $C_n[r]$ of type n on r generators.

It is also possible to define Higman-Thompson groups in a similar fashion as we have defined F, T and V above, using trees.

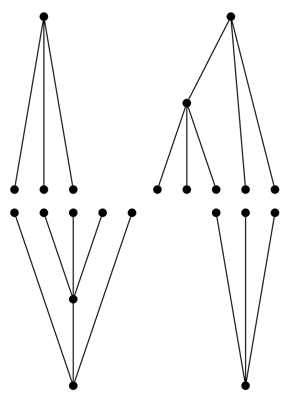
Definition 4.4. An n-ary tree is a tree where

- the leaves have degree 1;
- the root has degree n;
- every other vertex has degree $n+1$.

A (n,r)-forest is a collection of r many n-ary trees.

Definition 4.5. A paired (n,r)-forest diagram is a triple (F_-,ρ,F_+) consisting of two (n,r)-forests both with l leaves for some integer l, and ρ a permutation of l elements.

Example 4.6. A paired (3,2)-forest diagram:



Definition 4.7. The Higman-Thompson group $V_{n,r}$ is the group of equivalence classes of (n,r)-forest diagrams with the multiplication induced by stacking of trees.

Analogously, we can define $F_{n,r}$ and $T_{n,r}$ where only certain permutations are allowed in the definition of paired forest diagrams.

4.8. **Stable homology of these groups.** The stable homology of the family (V_n) has been computed by Szymik and Wahl [16].

Theorem 4.9 (Szymik-Wahl). *Let $n \geq 2$. The stabilization homomorphisms induce isomorphisms*

$$s_r : H_d(V_{n,r}; M) \to H_d(V_{n,r+1}; M),$$

in homology in all dimensions $d \geq 0$, for all $r \geq 1$, and for all $H_1(V_{n,\infty})$-modules M.

Can a proof be given using a similar scanning method?

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