Automorphic Forms on Exceptional Groups

Yi Shan

May 14, 2022

Contents

1 Introduction

In harmonic analysis and number theory, an automorphic form is a well-behaved complex-valued function on a topological group which is invariant under the action of a discrete subgroup of the topological group. Automorphic forms can be viewed as a generalization of periodic functions on Euclidean spaces, and was first discovered by Poincaré. Through the Langlands conjectures they play an important role in modern number theory.

Thanks to results of Arthur [Art13], Moeglin-Waldspurger[Wal09], Ngô[Ngô10] and many others, we now have a good understanding of the so-called endoscopic classification of automorphic representations of classical groups over $\mathbb{Q}$, such as orthogonal groups or symplectic groups. In a series of works [CR15][CT20], Chenevier, Renard and Taibi used these classification results to determine, for small n, the number of self-dual, regular, cuspidal and algebraic automorphic representations of GL_n over $\mathbb{Q}$ having a given collection of weights, in the simplest level 1 case. The main problem of the author's PhD project is to study the endoscopic classification of exceptional algebraic grouops, especially the anisotropic semisimple group of type F_4.

In this thesis, we will give a short background on algebraic groups in section 2 and define anisotropic exceptional groups admitting $\mathbb{Z}$-models in section 3, then we talk about automorphic forms in section 4,5 and the connection with classical modular forms in section 6, especially those forms having trivial weight in section 7. After that we state the main conjecture used in endoscopic classification in section 8 and give the motivation and strategy in section 9. Finally in section 10 we introduce some other problems we are interested in.

2 Algebraic Groups

As an analogy of holomorphic modular forms, we choose the domain of an automorphic function to be the adelic points of a linear algebraic group. Due to limited space, we can not give all the details here and readers can turn to [Mil17],[DG70] or [Hum75] as a reference.

Definition 2.1. Let k be a ring. An *linear algebraic group* over k is an affine scheme G of finite type over k with morphisms of k-schemes:

$$m : G \times_{\mathrm{Spec}\, k} G \to G,\ e : \mathrm{Spec}\, k \to G,\ i : G \to G$$

satifying following commutative diagrams:

$$\begin{array}{ccc} G \times_{\mathrm{Spec}\, k} G \times_{\mathrm{Spec}\, k} G & \xrightarrow{\mathrm{id}\times m} & G \times_{\mathrm{Spec}\, k} G \\ \downarrow{\scriptstyle m\times\mathrm{id}} & & \downarrow{\scriptstyle m} \\ G \times_{\mathrm{Spec}\, k} G & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccccc} \mathrm{Spec}\, k \times_{\mathrm{Spec}\, k} G & \xrightarrow{e\times\mathrm{id}} & G \times_{\mathrm{Spec}\, k} G & \xleftarrow{\mathrm{id}\times e} & G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k \\ & \searrow^{\simeq} & \downarrow{\scriptstyle m} & \swarrow^{\simeq} & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(i\times\mathrm{id})\circ\Delta} & G \times_{\mathrm{Spec}\, k} G & \xleftarrow{(\mathrm{id}\times i)\circ\Delta} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ \mathrm{Spec}\, k & \xrightarrow{e} & G & \xleftarrow{e} & \mathrm{Spec}\, k \end{array},$$

where $\Delta : G \to G \times_{\mathrm{Spec}\, k} G$ is the diagonal embedding in the last diagram. We denote the k-algebra representing G by $\mathscr{O}(G)$.

For any k-algebra R, we define the *R-points* of G to be $G(R) := \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, R, G)$.

Given a ring homomorphism $k \to k'$ and an linear algebraic group G over k, we can define its *base change* to k' to be $\mathrm{Spec}(\mathscr{O}(G) \otimes_k k')$.

Example 2.1. The *multiplicative group* $\mathbb{G}_m$ over k is $\operatorname{Spec} k[T, T^{-1}]$. It can be observed that for each k-algebra R, $\mathbb{G}_m(R)$ is the multiplicative group of $R^\times$.

Example 2.2. Over a field k, a *torus* is an algebraic group T such that $T_{\overline{k}}$ is isomorphic to the product of n copies of $\mathbb{G}_{m,\overline{k}}$ for some integer n. If $T \simeq \mathbb{G}_{m,k}^n$, then we call it a *split torus*.

Definition 2.2. Let G be a linear algebraic group over a field k. A torus T contained in G is *maximal* if $T_{\overline{k}}$ is maximal among all the tori contained in $G_{\overline{k}}$. We call G a *split group* if it has a maximal torus that is split over k.

Similarly [1] with abstract groups, we can define notions of *normal subgroups* and *solvable groups*, and then introduce the following notion:

Definition 2.3. When k is a field and $\overline{k}$ is an algebraic closure of k, a linear algebraic group G over k is called *semisimple* if every smooth connected solvable normal subgroup of its base change $G_{\overline{k}}$ is trivial.

Chevalley showed that the semisimple linear algebraic groups over an algebraically closed field are classified up to central isogenies by their Dynkin diagram. In particular the simple groups correspond to the irreducible diagrams. This result is essentially identical to the classification of compact Lie groups.

In this paper we focus on *simple* $\mathbb{Q}$*-groups admitting* $\mathbb{Z}$*-models*, where a $\mathbb{Z}$-model of a connected semisimple group G over $\mathbb{Q}$ is a smooth linear algebraic group scheme $\mathscr{G}$ over $\mathbb{Z}$ such that $\mathscr{G}_{\mathbb{Z}/p\mathbb{Z}}$ is semisimple for each p and $\mathscr{G}_{\mathbb{Q}} \simeq G$. There are two special kinds of these groups: split ones and anisotropic ones, here an *anisotropic* group means the $\mathbb{R}$-points is a compact Lie group. For the first kind, we have the following theorem:

Theorem 2.1. *[DG70] Every split connected semisimple group* $\mathbb{Q}$ *has a* $\mathbb{Z}$*-model, and this model is unique up to* $\mathbb{Z}$*-group isomorphism.*

Example 2.3. A basic example is the special linear group SL_n over $\mathbb{Q}$. The automorphism group of a lattice of the $\mathbb{Q}$-vector space gives us a $\mathbb{Z}$-model.

In [Gro96], Gross showed that the simple anisotropic $\mathbb{Q}$-groups admitting $\mathbb{Z}$-models are:

$$\mathrm{B}_{(d-1)/2}: \ \mathrm{Spin}(\mathbb{Q}^d, \frac{1}{2}\sum_{i=1}^{d} x_i^2), \ d \equiv \pm 1 \bmod 8;$$

$$\mathrm{D}_{d/2}: \ \mathrm{Spin}(\mathbb{Q}^d, \frac{1}{2}\sum_{i=1}^{d} x_i^2), \ d \equiv 0 \bmod 8;$$

$$\mathrm{G}_2, \ \mathrm{F}_4, \ \mathrm{E}_8: \ \text{Exceptional groups},$$

where Spin is the double cover of the special orthogonal group.

[1] Actually there are some slight differences but they are not important in this paper so we omit explicit definitions here.

3 Exceptional Groups

Classical groups of Lie types $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ are more familiar for mathematicians, and in this section we will introduce the three anisotropic exceptional $\mathbb{Q}$-groups mentioned in the section above. These groups are related to octonions.

Definition 3.1. As a $\mathbb{Q}$-vector space, *Cayley's definite octonion algebra* $\mathbb{O}$ has the form

$$\mathbb{O} = \mathbb{Q} \oplus \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_7,$$

and the multiplication law is given by:

- $e_i^2 = -1$ for $1 \leq i \leq 7$;
- for any i mod 7, the subalgebra $\mathbb{Q} \oplus \mathbb{Q}e_i \oplus \mathbb{Q}e_{i+1} \oplus \mathbb{Q}e_{i+3}$ is isomorphic to Hamilton's rational quaternion algebra $\mathbb{H} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k$, whose multiplication structure is given by $i^2 = j^2 = -1, ij = -ji = k$;
- If three elements e_r, e_s, e_t do not lie in one of the 7 quaternion algebras described above, then the multiplication is anti-associative: $(e_r \cdot e_s) \cdot e_k = -e_r \cdot (e_s \cdot e_t)$.

By [Spr00, Section 2.3], the algebraic group $\mathrm{G}_2 := \mathrm{Aut}_{\mathbb{O}/\mathbb{Q}}$ associating to each $\mathbb{Q}$-algebra R the automorphism group of the octonion algebra $\mathbb{O} \otimes \mathbb{Q}$ is a 14-dimensional anisotropic semisimple group of type G_2 over $\mathbb{Q}$. The automorphism group of a maximal order in $\mathbb{O}$, which is called *Coxeter's integral octonion* [CS03, Section 9], forms a $\mathbb{Z}$-model of G_2.

Consider the 27-dimensional $\mathbb{Q}$-vector space J consisting of 3×3 Hermitian matrices

$$A = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix}, \ a, b, c \in \mathbb{Q}, \ x, y, z \in \mathbb{O}.$$

On this vector space J, we have a multiplicative structure which is commutative but not associative:

$$A \circ A' := \frac{1}{2}(AA' + A'A).$$

This algebra J has a 2-sided identity $e = \mathrm{diag}(1, 1, 1)$ and is often referred as the *exceptional Jordan algebra*. By [Spr00, Theorem 7.2.1], the automorphism group $\mathrm{F}_4 := \mathrm{Aut}_{(J,\circ)/\mathbb{Q}}$ is a 52-dimensional anisotropic semisimple group of type F_4 over $\mathbb{Q}$. Replacing $\mathbb{O}$ in the definition of J by Coxeter's integral octonion, we can obtain a $\mathbb{Z}$-model of F_4.

The construction of E_8 is more complicated. The ideal is to construct from J a vector space $\mathfrak{g}(J)$ with a $\mathbb{Z}/3\mathbb{Z}$-grading and then give it a Lie bracket, and finally define E_8 to be the identity component of the group $\mathrm{Aut}(\mathfrak{g}(J))$. Readers can find details in [Rum97] or [Yok09].

4 Level one Automorphic Forms

With the background of algebraic groups, now we can begin to define automorphic forms.

Let $\widehat{\mathbb{Z}}$ be the direct product of all the $\mathbb{Z}_p$. The finite adèle ring is $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ and the adèle ring is $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We endow these rings with the topology of restricted product.

Let G be a semisimple $\mathbb{Q}$-group admitting a $\mathbb{Z}$-model $\mathscr{G}$. By a famous result of Borel and Harish-Chandra [Bor63], there is a nonzero $G(\mathbb{A})$-invariant Radon measure μ of finite volume on the homogeneous space $G(\mathbb{Q})\backslash G(\mathbb{A})$.

Definition 4.1. The space of *level* 1 *square-integrable automorphic forms* of G is the subspace

$$\mathcal{A}^2(G) \subset \mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \mu)$$

consisting of functions invariant under $\mathscr{G}(\widehat{\mathbb{Z}})$ for right translations.

This space is equipped with two representation structures: a unitary representation of $G(\mathbb{R})$ by right translations, and an action of the Hecke algebra $\mathrm{H}(G)$ associated to the pair $(G(\mathbb{A}_f), \mathscr{G}(\widehat{\mathbb{Z}}))$ that commutes with $G(\mathbb{R})$.

We denote by $\Pi(G)$ the set of isomorphism classes of complex representations π of $G(\mathbb{A})$ such that $\pi \simeq \pi_\infty \otimes \pi_f$, where:

(i) π_f is a smooth irreducible complex representation of $G(\mathbb{A}_f)$ and $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})} \neq 0$,

(ii) π_∞ is an irreducible unitary representation of $G(\mathbb{R})$.

The subspace $\mathcal{A}_{\mathrm{disc}}(G) \subset \mathcal{A}^2(G)$ is defined as the closure of the sum of the irreducible closed subspaces for the $G(\mathbb{R})$-action, which is stable by $\mathrm{H}(G)$. By a fundamental result of Harish-Chandra [CM68, Theorem I.1], each irreducible representation of $G(\mathbb{R})$ occuring in $\mathcal{A}_{\mathrm{disc}}(G)$ has finite multiplicity. So we can write:

$$\mathcal{A}_{\mathrm{disc}}(G) = \widehat{\bigoplus}_{\pi \in \Pi(G)} m(\pi) \pi_\infty \otimes \pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}.$$

Definition 4.2. A representation $\pi \in \Pi(G)$ is said to be a *discrete automorphic representation* of G if $m(\pi) \neq 0$. We denote the subset of $\Pi(G)$ consisting of those representations by $\Pi_{\mathrm{disc}}(G)$.

When G is anisotropic, by a result of Borel on the finiteness of $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$ and the Peter-Weyl theorem, one can derives that $\mathcal{A}^2(G) = \mathcal{A}_{\mathrm{disc}}(G)$.

Definition 4.3. A representation $\pi \in \Pi(G)$ is *cuspidal* if it is generated by some $f \in \mathcal{A}^2(G)$ satisfying

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)\mathrm{d}u = 0$$

for every unipotent radical U of every proper parabolic subgroup of G and for almost all $g \in G(\mathbb{A})$. We denote the subset of $\Pi(G)$ consisting of cuspidal representations by $\Pi_{\mathrm{cusp}}(G)$.

Moreover, Gelfand and Piatetski-Shapiro [GGP69] proved that $\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G)$.

5 Algebraic Modular Forms

When G is anisotropic, there is a very simple intepretation of automorphic representations is the so-called *algebraic modular form* in the sense of Gross [Gro99].

Let U be a finite-dimensional unitary representation of the compact Lie group $G(\mathbb{R})$. We denote the finite-dimensional right $\mathrm{H}(G)$-module $\mathrm{Hom}_{G(\mathbb{R})}(U, \mathcal{A}^2(G))$ by $\mathcal{A}_U(G)$.

Definition 5.1. A function $F : G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}}) \to U$ satisfying $F(gx) = gF(x)$ for any $g \in G(\mathbb{Q})$ and $x \in G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$ is called an *algebraic modular form of weight* U for the group G.

We denote the vector space of such functions by $\mathrm{M}_U(G)$. This space naturally inherits a right $\mathrm{H}(G)$-action from $G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$.

Denote the dual unitary representation of U by U^*. For an algebraic modular form F of weight U and an element $u \in U^*$, we denote by $\varphi_F(u)$ the function

$$G(\mathbb{R}) \times G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}}) \to \mathbb{C},\ (h, x) \mapsto \langle u, h^{-1}F(x)\rangle,$$

where $\langle\ ,\ \rangle$ is the *Petersson inner product*. By definition, we have $\varphi_F(u)(gh, gx) = \varphi_F(h, x)$ for any $g \in G(\mathbb{Q})$, thus $\varphi_F(u)$ is a L^2 function on the compact space $G(\mathbb{Q})\backslash G(\mathbb{A})/\mathscr{G}(\widehat{\mathbb{Z}})$. For any $g \in G(\mathbb{R})$,

$$\varphi_F(gu)(h, x) = \langle gu, h^{-1}F(x)\rangle = \langle u, (hg)^{-1}F(x)\rangle = \varphi_F(u)(hg, x),$$

which implies that the map $u \mapsto \varphi_F(u)$ is $G(\mathbb{R})$-equivariant, so $\varphi_F \in \mathcal{A}_{U^*}(G)$ and we get an $\mathrm{H}(G)$-equivariant homomorphism $\mathrm{M}_U(G) \to \mathcal{A}_{U^*}(G)$. It can be easily verified that this is an isomorphism, thus we can turn the problem about $\mathcal{A}_U(G)$ into studying the structure of $\mathrm{M}_{U^*}(G)$.

For each element $\sigma \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$, we can choose a representative g_σ. Then we have

$$G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}}) \simeq \bigsqcup_\sigma G(\mathbb{Q})g_\sigma,$$

which gives us an injection:

$$\mathrm{M}_U(G) \to \prod_{\sigma \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})} U, \quad F \mapsto (F(g_\sigma))_\sigma.$$

For each $\tau \in \Gamma_\sigma := g_\sigma \mathscr{G}(\widehat{\mathbb{Z}}) g_\sigma^{-1} \cap G(\mathbb{Q})$, we have $\tau \cdot F(g_\sigma) = F(\tau g_\sigma) = F(g_\sigma)$. So the image $F(g_\sigma)$ lies in U^{Γ_σ} for each σ and the image of the injection above is exactly $\prod_\sigma U^{\Gamma_\sigma}$. Hence the map induces an isomorphism

$$\mathcal{A}_{U^*}(G) \simeq \mathrm{M}_U(G) \simeq \prod_{\sigma \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})} U^{\Gamma_\sigma}.$$

Provided we know all the finite subgroups Γ_σ of $G(\mathbb{R})$, we can compute the dimensions of invariant spaces U^{Γ_σ}:

$$\dim U^{\Gamma_\sigma} = \frac{1}{|\gamma_\sigma|} \sum_{g \in \Gamma_\sigma} \mathrm{Tr}(g|_U) = \frac{1}{|\Gamma_\sigma|} \sum_{c \in \mathrm{Conj}(\Gamma_\sigma)} \mathrm{Tr}(c|_U) \cdot |c|,$$

where $\mathrm{Conj}(\Gamma_\sigma)$ is the set of conjugacy classes of Γ_σ and $|c|$ is the cardinality of a conjugacy class. With the hlep of computer programs like `PARI/GP` [21b] and `GAP` [21a], we can determine the set $\mathrm{Conj}(\Gamma_\sigma)$. For the computation of $\mathrm{Tr}(c|_U)$, we have the following *degenerate Weyl character formula*:

Proposition 5.1. *[CR15, Proposition 2.1] Let G be a connected compact Lie group and T be one maximal torus. Let λ be a dominant weight in the character group $X := X^*(T)$ and t an element in T. We denote by Φ the root system of (G,T) and W its Weyl group. Choose a system of positive roots $\Phi^+ \subset \Phi$ with base Δ and also fix a W-invariant inner product $(\ ,\ )$ on $X \otimes \mathbb{R}$. Denote the connected component $C_G(t)^0$ of the centralizer of t by M. Set $\Phi_M^+ = \Phi(M,T) \cap \Phi^+$ and $W^M = \{w \in W : w^{-1}\Phi_M^+ \subset \Phi^+\}$. Let ρ, ρ_M be the half-sum of the elements of Φ^+, Φ_M^+ respectively. We have:*

$$\mathrm{Tr}|_{V_\lambda}(t) = \frac{\sum_{w\in W^M} \varepsilon(w)\cdot t^{w(\lambda+\rho)-\rho}\cdot \prod_{\alpha\in\Phi_M^+}\frac{(\alpha,w(\lambda+\rho))}{(\alpha,\rho_M)}}{\prod_{\alpha\in\Phi^+\setminus\Phi_M^+}(1-t^{-\alpha})},$$

where $\varepsilon : W \to \{\pm 1\}$ is the signature and t^x denotes $x(t)$ for convenience.

6 From Modular Forms to Automorphic Forms

In this section, we will show the connection between holomorphic modular forms and automorphic forms, as an example showing why we study automorphic forms.

We recall the definition of a classical modular form for $\mathrm{SL}_2(\mathbb{Z})$:

Definition 6.1. Let k be a positive integer and let $\mathfrak{H}$ be the complex upper plane. The space of *weight k modular forms* for $\mathrm{SL}_2(\mathbb{Z})$ is the space $\mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following conditions:

(a) $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathfrak{H}$,

(b) f extends holomorphically to the cusps.

If f additionally vanishes at the cusps we say that f is a *cusp form*. The space of weight k cusp forms is denoted $\mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$.

For a function $f \in \mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, we can transfer it to a function φ_f on $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) : \det g > 0\}$:

$$\varphi_f(g) := \det(g)^{k/2}(ci+d)^{-k} f(\frac{ai+b}{ci+d}),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It can be easily verified that φ_f factors through $\mathrm{PGL}_2(\mathbb{R})^+$, which is the image of $\mathrm{GL}_2(\mathbb{R})^+$ in $\mathrm{PGL}_2(\mathbb{R})$, and φ_f is invariant under the right translation of $\mathrm{PGL}_2(\mathbb{Z})^+$, the image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{PGL}_2(\mathbb{Z})$. When f is a cusp form, this φ_f is also a square-integrable function on $\mathrm{PGL}_2(\mathbb{R})^+/\,\mathrm{PGL}_2(\mathbb{Z})^+$.

We can derive the following homeomorphism from the strong approximation theorem for PGL_2:

$$\mathrm{PGL}_2(\mathbb{Q})\backslash \mathrm{PGL}_2(\mathbb{A})/ \mathrm{PGL}_2(\widehat{\mathbb{Z}}) \simeq \mathrm{PGL}_2(\mathbb{R})^+/ \mathrm{PGL}_2(\mathbb{Z})^+.$$

Hence we can view φ_f as an automorphic form in $\mathcal{A}^2(\mathrm{PGL}_2)$. Moreover this automorphic form also lies in $\mathcal{A}_{\mathrm{cusp}}(\mathrm{PGL}_2)$.

Notice that we only use the condition (a) in the definition of modular forms and we also did not use the fact that $i \in \mathfrak{H}$ is fixed by $\mathrm{SO}_2(\mathbb{R})$. In fact, the space $\mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to a subspace of $\mathcal{A}_{\mathrm{cusp}}(\mathrm{PGL}_2)$ consisting of automorphic forms satisfying some equivariant property for $\mathrm{SO}_2(\mathbb{R})$ and that are eigenforms for some differential operator. In general we have a similar result for *Siegel modular forms* [CL19, Section 4.5.3] and this gives an intepretation of automorphic representations of symplectic groups.

7 Classification of $\mathbb{Z}$-models

When we take $U = \mathbb{C}$ to be the trivial representation, the space $\mathcal{A}_C(G) \simeq \mathrm{M}_{\mathbb{C}}(G)$ is isomorphic to $\prod_\sigma \mathbb{C}$, which is the space spanned by characteristic functions of images of $G(\mathbb{Q})g_\sigma\mathscr{G}(\widehat{\mathbb{Z}})$ in $G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$. The dimension of this space is the number of non-equivalent $\mathbb{Z}$-models of G in the same genus of $\mathscr{G}$, where two $\mathbb{Z}$-models $\mathscr{G}, \mathscr{G}'$ are said to be in the same *genus* if $\mathscr{G}(\widehat{\mathbb{Z}})$ and $\mathscr{G}'(\widehat{\mathbb{Z}})$ are conjugate in the group $G(\mathbb{A}_f)$ and they are equivalent if they are conjugate by $G(\mathbb{Q})$. We denote this dimension by $d_0(G)$.

However, even in this simplest case, the dimension $d_0(G)$ is not so easy to get as imagined. This is because we need to enumerate all the $\mathbb{Z}$-models in the same genus of $\mathscr{G}$. For classical groups $\mathrm{Spin}(\mathbb{Q}^d, \frac{1}{2}\sum x_i^2)$ of type B or D, this leads to the classification of (even) unimodular lattices of rank d, which is a very classical problem. The classification is due to Louis J.Mordell in dimension 8, to Ernst Witt in dimension 16, and to Hans-Volker Niemeier in dimension 24 [Nie73]. In [CL19] Chenevier and Lannes give a new method on this problem.

A powerful tool in the classification of $\mathbb{Z}$-models is the mass formula:

Theorem 7.1. *[Gro96, Proposition 2.2] Let G be an anisotropic semisimple $\mathbb{Q}$-group admitting a $\mathbb{Z}$-model $\mathscr{G}$ and let $d_1, d_2, \ldots, d_r$ be the degrees of the invariants for the Weyl group W of G, acting on the symmetric algebra of the reflection representation. We have the formula*

$$\sum_{\mathscr{G}'} \frac{1}{|\mathscr{G}'(\mathbb{Z})|} = \prod_{i=1}^{r} \frac{1}{2}\zeta(1-d_i),$$

where $\mathscr{G}'$ runs over non-equivalent $\mathbb{Z}$-models in the genus containing $\mathscr{G}$ and ζ is Riemann's zeta function.

The strategy is each time a new $\mathbb{Z}$-model $\mathscr{G}'$ is founded we add $1/|\mathscr{G}'(\mathbb{Z})|$ to the partial sum appearing in the left-hand side of the mass formula, and the classfication terminates when this quantity reachs the right-hand side.

Example 7.1. We give an example here to explain how this strategy works. Consider the classical groups $G = \mathrm{Spin}(\mathbb{Q}^{16}, \frac{1}{2}\sum x_i^2)$ of type D_8. We can construct two non-isometric even unimodular lattices $\mathrm{E}_8 \oplus \mathrm{E}_8$ and D_{16}^+ [Gro96, Section 3], which give rise to two $\mathbb{Z}$-models $\mathrm{Spin}(\mathrm{E}_8 \oplus \mathrm{E}_8)$ and $\mathrm{Spin}(\mathrm{D}_{16}^+)$ of G. The right-hand side of the mass formula for G equals to

$$\frac{691}{2^{30} \cdot 3^6 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13},$$

which equals exactly to $\frac{1}{|\mathrm{Spin}(\mathrm{E}_8 \oplus \mathrm{E}_8)|} + \frac{1}{|\mathrm{Spin}(\mathrm{D}_{16}^+)|}$. This shows that we have already enumerated all the $\mathbb{Z}$-models in a genus.

For exceptional groups, Gross proved that $d_0(\mathrm{G}_2) = 1$ and $d_0(\mathrm{F}_4) = 2$ in [Gro96] via some results in [Con85] whose proofs are hard to find, and in the author's master thesis we give another proof without using these results. In these two cases, since we know the $\mathbb{Z}$-models explicitly, we can compute $\dim \mathcal{A}_U(G)$ by the method mentioned in the previous section.

The number $d_0(\mathrm{E}_8)$ is still mysterious. By a rough estimate

$$d_0(\mathrm{E}_8) \geq \sum_{\mathscr{G}'} \frac{1}{|\mathscr{G}'(\mathbb{Z})|} = \prod_{i=1}^{8} \frac{1}{2}\zeta(1 - d_i) \approx 13934.5,$$

there are at least 13935 non-equivalent $\mathbb{Z}$-models in a given genus.

8 The Arthur-Langlands Conjecture

In this section we are going to state the Arthur-Langlands conjecture, which can be viewed as a global version of the famous *local Langlands correspondence* and is a special case of *Langlands' functoriality principle*. It predicts a correspondence between discrete automorphic representations and some admissible homomorphisms from a conjectural group analogous to the Galois group or Weil group to the dual group of G.

Given a representation $\pi \in \Pi(G)$, the *Satake isomorphism* [Gro98] and the infinitesimal character [Dix77, Section 7] associate to π a collection of conjugacy classes:

$$c(\pi) = (c_\infty(\pi), c_2(\pi), c_3(\pi), \cdots),$$

where $c_p(\pi)$ is a conjugacy class in the complex dual group $\widehat{G}(\mathbb{C})$ of G and $c_\infty(\pi)$ is a conjugacy class in the Lie algebra $\widehat{g}$ of $\widehat{G}(\mathbb{C})$. In general, such a collection of conjugacy classes determines a finite number of representations $\pi \in \Pi(G)$, but when $G(\mathbb{R})$ is compact it determines π uniquely.

Example 8.1. Here we give an simple example: the group PGL_2, with dual group $\mathrm{SL}_{2,\mathbb{C}}$. Let $f = \sum_{n\geq 1} a_n q^n$, $q = e^{2\pi i z}$ be a cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. As explained in [Ser93], we have an operator T_p on $\mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ for each prime p, which is called the pth *Hecke operator*. These operators commute with each other and decompose $\mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ into eigenforms of these operators. Suppose that f is an eigenform and let π be the automorphic representation generated by the associated φ_f as we stated in section 6. The infinitesimal character $c_\infty(\pi)$, viewed as an element in $\mathfrak{sl}_{2,\mathbb{C}}$ is the semisimple conjugacy class with eigenvalues $\pm(k-1)/2$, and the Satake parameter $c_p(\pi)$ satisfies

$$p^{(k-1)/2}\mathrm{Tr}(c_p(\pi)) = a_p.$$

In order to define the other side of the conjecture, we assume the existence of the conjectural *Langlands group of* $\mathbb{Z}$ [CR15, Appendix B], which is denoted by $\mathcal{L}_{\mathbb{Z}}$. This is a locally compact Hausdorff topological group satisfying several axioms.

Definition 8.1. A *discrete global Arthur parameter* of G is a $\widehat{G}(\mathbb{C})$-conjugacy class of continuous group homomorphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$$

such that $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic and the centralizer C_ψ of $\operatorname{Im}\psi$ in $\widehat{G}(\mathbb{C})$ is finite modulo the center of $\widehat{G}(C)$.

We denote the set of discrete global Arthur parameters by $\Psi_{\mathrm{disc}}(G)$. In parallel with what has been done for $\Pi(G)$, we can also associate to $\psi \in \Psi_{\mathrm{disc}}(G)$ a collection of conjugacy classes

$$c(\psi) = (c_\infty(\psi), c_2(\psi), c_3(\psi), \cdots).$$

This conjugacy class also can not determine the class of ψ uniquely in general.

Conjecture 8.1. *(Arthur-Langlands conjecture) For any $\pi \in \Pi_{\mathrm{disc}}(G)$, there is a discrete Arthur parameter ψ_π of G such that $c(\psi) = c(\pi)$. Conversely, if ψ is a discrete global Arthur parameter of G and $\Pi(\psi)$ is the finite set consisting of $\pi \in \Pi(G)$ such that $c(\psi) = c(\pi)$, then there is a formula for $\sum_{\pi\in\Pi(\psi)} m(\pi)$, where $m(\pi)$ is the multiplicity of π in $\mathcal{A}_{\mathrm{disc}}(G)$.*

The first part of this conjecture can be described in terms of elements in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$: let $r : \widehat{G}(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ be a representation, then the representation $r \circ \psi$ of $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C})$ can be written as

$$\bigoplus_{i=1}^{k} r_i \otimes \mathrm{Sym}^{d_i-1}\,\mathrm{St}$$

for some irreducible representations r_i of dimension n_i of $\mathcal{L}_{\mathbb{Z}}$ and certain integers $d_i \geq 1$, where St denotes the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. From one axiom that $\mathcal{L}_{\mathbb{Z}}$ satisfies, each $r_i : \mathcal{L}_{\mathbb{Z}} \to \mathrm{SL}_{n_i}(\mathbb{C})$ corresponds to a unique cuspidal representation π_i of PGL_{n_i}. Then we have the following intepretation:

Conjecture 8.2. *For any $\pi \in \Pi_{\mathrm{disc}}(G)$ and every representation $r : \widehat{G}(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$, there exists a collection of triples $(n_i, \pi_i, d_i), 1 \leq i \leq k$ with $d_i, n_i \geq 1$ satisfying $n = \sum_i n_i d_i$ and $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i})$ such that the following equalities hold:*

$$r(c_v(\pi)) = \bigoplus_i c_v(\pi_i) \otimes \mathrm{Sym}^{d_i-1}(e_v),\ \ v = p \text{ or } \infty,$$

where

$$e_p = \begin{pmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}),\ \ e_\infty = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \in \mathfrak{sl}_{2,\mathbb{C}}.$$

Moreover, this collection is unique up to permutation.

Arthur proved the conjecture above in [Art13] for classical groups G[2] and the standard representation r of $\widehat{G}$.

For the second part of this conjecture, [CL19] gives an explicit multiplicity formula for classical groups, and the general recipe can be found in [Art89].

[2]Here a classical group means that $G_{\mathbb{C}}$ is isomorphic to Sp_{2g} for $g \geq 1$ or SO_m for $m \neq 2$ over $\mathbb{C}$.

9 Endoscopic Classification

Combining the conjectures stated in the previous section and the work of Chenevier-Renard-Taibi [CR15][CT20] on $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, we can study what are the Arthur parameters corresponding to automorphic representations appearing in $\mathcal{A}_U(G)$ for a given weight U. For classical groups with smaller ranks and G_2, this was done in [CR15]. We are now doing this conjectural classification for semisimple anisotropic group F_4.

One particular goal of us is to find a stable tempered representation $\pi \in \mathcal{A}_{\mathrm{disc}}(\mathrm{F}_4)$, which means the associated collection of triples as in Conjecture 8.2 satisfies $k = 1, d_1 = 1$.

What makes the problem simpler is that for anisotropic group F_4, a collection of conjugacy classes $c(\pi)$ or $c(\psi)$ determines π and ψ both uniquely. So our strategy is to enumerate all the possible Arthur parameters ψ that are not stable tempered and can be ψ_π associated to some $\pi \in \mathcal{A}_U(\mathrm{F}_4)$ and then use Arthur's multiplicity formula to compute $m(\pi_\psi)$ for the corresponding representation. Since we can also compute the dimension of $\mathcal{A}_U(\mathrm{F}_4)$, the difference of this dimension and the sum of $m(\pi_\psi)$ mentioned above is the sum of multiplicities of stable tempered representations.

The motivation of this problem is from some standard conjectures like *Fontaine-Mazur conjecture*. According to these conjectures, finding a stable tempered representation for F_4 is equivalent to finding a geometric l-adic Galois representations with conductor 1 of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in an exceptional algebraic group of type F_4 and Zariski-dense image. Potential concrete applications of this Galois representation include new cases of the inverse Galois problem, or indications about where to find the first F_4 motive over $\mathbb{Z}$.

10 Further problems

As we mentioned several times, the Langlands group $\mathcal{L}_{\mathbb{Z}}$ is conjectural and its intepretations (Conjecture 8.2 and Arthur's multiplicity formula) still remain open for exceptional algebraic groups. This means our endoscopic classification for F_4 is conditional, so we hope to use some other technologies like *excpetional theta correspondences* to make at least some parts of the classification to be unconditional.

Also except three anisotropic exceptional groups $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_8$, we still have two exceptional Lie types E_6 and E_7. There is no reductive group $\mathscr{G}$ over $\mathbb{Z}$ such that $\mathscr{G}(\mathbb{R})$ is a real form of E_6 having discrete series, so we only care about the case E_7. This case seems very difficult since we have to work with the split gruop, but ideas from recent work by Chenevier and Taibi could help.

To end this thesis, we mention a last question. It is about exploring level one automorphic forms for the group G_2, but over real quadratic fields instead of $\mathbb{Q}$. One motivation is to study an open problem asked by Philippe Gille: does there exist an number field K and two non-isomorphic octonion algebras over its ring of integers $\mathscr{O}_K$ whose underlying quadratic spaces are isometric?

References

[21a] *GAP – Groups, Algorithms, and Programming, Version 4.11.1*. The GAP Group. 2021. URL: https://www.gap-system.org.

[21b] *PARI/GP version 2.13.2*. available from http://pari.math.u-bordeaux.fr/. The PARI Group. Univ. Bordeaux, 2021.

[Art13] James Arthur. *The Endoscopic classification of representations orthogonal and symplectic groups*. Vol. 61. American Mathematical Soc., 2013.

[Art89] James Arthur. "Unipotent automorphic representations : conjectures". In: *Orbites unipotentes et représentations - II. Groupes p-adiques et réels*. Astérisque 171-172. Société mathématique de France, 1989.

[Bor63] Armand Borel. "Some finiteness properties of adele groups over number fields". In: *Publications Mathématiques de l'IHÉS* 16 (1963), pp. 5–30.

[CL19] Gaëtan Chenevier and Jean Lannes. *Automorphic Forms and Even Unimodular Lattices*. Vol. 69. Ergebnisse der Mathematik und ihrer Grenzgebiete. la version publiée est une traduction anglaise, et corrigée, de la première version déposée. Springer Verlag, 2019, 428 p.

[CM68] H. Chandra and J.J.M. Mars. *Automorphic Forms on Semisimple Lie Groups*. Lecture Notes in Mathematics. Springer, 1968.

[Con85] J.H. Conway. *Atlas of finite groups : maximal subgroups and ordinary characters for simple groups*. Oxford University Press Oxford, New York, 1985.

[CR15] G. Chenevier and D. Renard. *Level one algebraic cusp forms of classical groups of small rank*. Vol. 237. Memoirs of the A.M.S. 1121. American Math. Soc., 2015, 128 pp.

[CS03] J.H. Conway and D.A. Smith. *On Quaternions and Octonions*. 2003. DOI: https://doi.org/10.1201/9781439864180.

[CT20] Gaëtan Chenevier and Olivier Taïbi. "Discrete series multiplicities for classical groups over $\mathbf{Z}$ and level 1 algebraic cusp forms". In: *Publications mathématiques de l'IHÉS* 131 (2020), pp. 261–323.

[DG70] M. Demazure and A. Grothendieck. *Schémas en Groupes I,II,III*. Lecture Notes in Math 151,152,153. Springer-Verlag, New York, 1970.

[Dix77] Jacques. Dixmier. *Enveloping algebras / Jacques Dixmier ; [translated by Minerva Translations, ltd., London]*. English. North-Holland Pub. Co. ; sole distributors for the U.S.A. and Canada, American Elsevier Pub. Co Amsterdam : New York, 1977, xvi, 375 p. ISBN: 0444110771.

[GGP69] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro. *Representation theory and automorphic function. Translated from the Russian by K. A. Hirsch*. English. 1969.

[Gro96] B. Gross. "Groups over Z". In: *Invent.math.124* (1996).

[Gro98] Benedict H. Gross. "On the Satake isomorphism". In: *Galois Representations in Arithmetic Algebraic Geometry*. Ed. by A. J. Scholl and R. L.Editors Taylor. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998, pp. 223–238. DOI: 10.1017/CBO9780511662010.006.

[Gro99] Benedict H. Gross. "Algebraic modular forms". In: *Israel Journal of Mathematics* 113.1 (1999), pp. 61–93.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Graduate texts in mathematics. New York Berlin Heidelberg: Springer-Verlag, 1975. ISBN: 0-387-90108-6.

[Mil17] J. S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge University Press, 2017. DOI: 10.1017/9781316711736.

[Ngô10] Bảo Châu Ngô. "Le lemme fondamental pour les algèbres de Lie". fr. In: *Publications Mathématiques de l'IHÉS* 111 (2010), pp. 1–169. DOI: 10.1007/s10240-010-0026-7.

[Nie73] Hans-Volker Niemeier. "Definite quadratische formen der dimension 24 und diskriminante 1". In: *Journal of Number Theory* 5.2 (1973), pp. 142–178. ISSN: 0022-314X. DOI: https://doi.org/10.1016/0022-314X(73)90068-1. URL: https://www.sciencedirect.com/science/article/pii/0022314X73900681.

[Rum97] Karl Rumelhart. "Minimal representations of exceptional p-adic groups /". In: *Representation Theory of the American Mathematical Society* 1 (June 1997). DOI: 10.1090/S1088-4165-97-00009-5.

[Ser93] Jean-Pierre Serre. *A course in arithmetic*. Springer, 1993.

[Spr00] A. Springer. *Octonions, Jordan Algebras and Exceptional Groups*. Springer, Berlin, Heidelberg, 2000.

[Wal09] Jean-Loup Waldspurger. "Endoscopie et changement de caractéristique : intégrales orbitales pondérées". fr. In: *Annales de l'Institut Fourier* 59.5 (2009), pp. 1753–1818. DOI: 10.5802/aif.2476.

[Yok09] Ichiro Yokota. "Exceptional Lie groups". In: *arXiv: Differential Geometry* (2009).