SOFIC GROUPS AND SURJONCTIVITY

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1. Introduction

Properties of groups such as amenability and residual finiteness have been introduced throughout the 20^{th} century in order to answer paradoxes or give large classes of groups satisfying certain conjectures. The first property, amenability has been first studied by Von Neumann in order to give a better comprehension of the Banach-Tarski paradox, stating that there exists a decomposition of the 3-sphere in a finite number of pieces, such that there are isometries bijectively rearranging the pieces into the union of two copies of the same sphere. The cause has been thus found in the non amenability of subgroups of the special orthogonal group $SO_3(\mathbb{R})$. On the other hand, residual properties of groups have been studied to find topological properties of a path-connected space, given its fundamental group has said properties. Residual finiteness is an important example of such properties of groups. These two properties have common weaker versions that are soficity and hyperlinearity. Hyperlinearity of groups is strongly linked to one of the most important open

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problems in operator algebras, the Connes' embedding conjecture, which states that every von Neumann algebra which is a type II_1-factor on a separable Hilbert space can be embedded into an ultraproduct of the hyperfinite II_1-factor. The conjecture has been proved false by Ji, Natarajan, Vidick, Wright et Yuen in 2020, providing a counter example emanating from quantum information theory. The paper [11] in which the counter-example is presented, is still being reviewed. However, the problem is still open concerning group von Neumann algebras. Soficity is a stronger property than hyperlinearity, introduced by Gromov in [10], but it is unknown whether they are equivalent or not. It measures how well a group can be approximated by finite groups. Sofic groups satisfy several conjectures still open for generic discrete groups. For example, any sofic group Γ is *surjunctive*, meaning that given any finite set S and any continuous and Γ-equivariant map $f : S^G \longrightarrow S^G$ which is injective, then the map is automatically surjective. Gottschalk surjunctivity conjecture states that every discrete group should be surjunctive.

2. Amenability

2.1. **Definition.** In this section we define amenability for a group and an action of a group on a measured space, and the relations between the two. The concept of amenability of a group was first introduced by von Neumann as an answer to Banach-Tarski paradox. He introduced at the same time the notion of amenable actions in [16].

Definition 2.2. Let Γ be a discrete group, we say that Γ is *amenable* if there exists a *left-invariant mean* on Γ, that is to say a finitely additive "probability measure" $m : \ell^\infty(\Gamma) \longrightarrow \mathbb{R}$ such that for all subset $A \subset \Gamma$, and element $g \in \Gamma$, we have

$$m(g \cdot A) = m(A). \tag{2.2.1}$$

Here, probability measure is to be understood as an element of the dual space $m \in \ell^\infty(\Gamma)^*$ such that $m(1) = 1$ and $m(f) \geq 0$ whenever $f \geq 0$, and for a subset $A \subset \Gamma$, $m(A) = m(\mathbb{1}_A)$.

There are several other equivalent definitions summarized in the following proposition. They can all be found in Brown and Ozawa's book [4].

Proposition 2.3. *Let Γ be a discrete group, the following are equivalent*

1. *Γ is amenable.*
2. *Γ satisfies the Følner condition, that is to say, given any $\varepsilon > 0$ and finite subset $E \subset \Gamma$, there exists a finite subset $F \subset \Gamma$ such that*

$$\max_{g\in E} \frac{|g \cdot F \triangle F|}{|F|} < \varepsilon. \tag{2.3.1}$$

3. *the trivial representation τ is* weakly contained *in the left regular representation λ, that is to say there exist unit vectors ξ_i in $\ell^2(\Gamma)$ such that*

$$\|\lambda_g \xi_i - \xi_i\|_2 \to 0, \tag{2.3.2}$$

   for all $g \in \Gamma$.

2.4. **Examples.**

Examples 2.5. The following classes of groups are all amenable :

1. Abelian groups.
2. Finite groups.
3. Groups with subexponential growth.

Example 2.6. An easy example of a non-amenable group is the free group on two generators $\mathbb{F}_2 = \langle a, b\rangle$, and more generally the free groups on more than one generator are non-amenable. To show that we find a *paradoxical decomposition* of $\mathbb{F}_2$. We thus define

$$\begin{aligned} A^+ &= \{x \in \mathbb{F}_2,\, x \text{ starts with } a\},\\ B^+ &= \{x \in \mathbb{F}_2,\, x \text{ starts with } b\},\\ A^- &= \{x \in \mathbb{F}_2,\, x \text{ starts with } a^{-1}\},\\ B^- &= \{x \in \mathbb{F}_2,\, x \text{ starts with } b^{-1}\}. \end{aligned} \tag{2.6.1}$$

Assuming that there exists an invariant mean μ on $\mathbb{F}_2$, and denoting by C the set $\{1, b, b^2, \dots\}$, we have

$$\begin{aligned} \mathbb{F}_2 &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \sqcup C)\\ &= A^+ \sqcup a \cdot A^-\\ &= b^{-1}(B^+ \setminus C) \sqcup (B^- \sqcup C), \end{aligned} \tag{2.6.2}$$

and thus applying μ we get

$$\begin{aligned} 1 = \mu(\mathbb{F}_2) &= \mu(A^+) + \mu(A^-) + \mu(B^+ \setminus C) + \mu(B^- \sqcup C)\\ &= \mu(A^+) + \mu(a \cdot A^-) + \mu(b^{-1}(B^+ \setminus C)) + \mu(B^- \sqcup C)\\ &= 2\mu(\mathbb{F}_2)\\ &= 2. \end{aligned} \tag{2.6.3}$$

That is impossible and therefore $\mathbb{F}_2$ cannot be amenable. As amenability is stable by taking subgroups, then non-abelian free groups are non-amenable as well.

Example 2.7. More generally, any discrete group containing a subgroup isomorphic to the free group on two generators cannot be amenable. A conjecture by von Neumann stated that the converse should hold, but this has been proved wrong by Olshanskii in 1980, using a group called the *Tarski monster*. Another example is given by free Burnside groups, which are not amenable but cannot contain the free group on two generators as every element as finite order. There are even finitely presented counterexamples.

Amenability is also stable under several common operations on groups.

Proposition 2.8. *The following properties hold.*

1. *Let Γ be an amenable group, then every subgroup of Γ is amenable.*
2. *Let K and H be two amenable groups, then any extension of K by H is also amenable. That is to say if a group Γ is such that there is an exact sequence of the form*

$$0 \longrightarrow H \longrightarrow \Gamma \longrightarrow K \longrightarrow 0, \tag{2.8.1}$$

then Γ is also amenable.

3. *Precisely, if we have an exact sequence of the form 2.8.1, then Γ is amenable if and only if K and H are amenable.*
4. *If $(\Gamma_i)_{i\in I}$ is a diagram of amenable groups, then the inductive limit $\Gamma = \varinjlim \Gamma_i$ is amenable.*

2.9. **Amenable actions.** Being amenable is in fact a property that is strongly linked to how the group acts on measured spaces, as we will now see.

Definition 2.10. Let Γ be a discrete group which acts on a set X. The following conditions are equivalent:

1. There exists a left-Γ-invariant *mean* on X.
2. There is no paradoxical decomposition of X with respect to the action Γ, that is to say there is no partition of X into subsets $A_1, A_2, \dots, A_l$, such that for some elements $g_1, h_1, g_2, h_2, \dots, g_l, h_l \in \Gamma$ the subsets $g_1A_1, h_1A_1, g_2A_2, h_2A_2, \dots, g_lA_l, h_lA_l$ are pairwise disjoint.
3. There exist Følner systems, that is to say for any finite subset $E \subset \Gamma$ and any $\varepsilon > 0$, there is a finite subset $F \subset X$ such that

$$\max_{g\in E} \frac{|g \cdot F \triangle F|}{|F|} < \varepsilon \tag{2.10.1}$$

An action which satisfies the above condition is called and *amenable action.*

Proposition 2.11. *If the group* Γ *is amenable, then any action of* Γ *on a set* X *is amenable.*

Remark 2.12. Every group acts in an amenable way on the set with only one point.

However, comparing the two definitions, we get the following.

Proposition 2.13. *A group* Γ *is amenable if and only if the action of* Γ *on itself by left translation is an amenable action.*

3. Residual finiteness

In this section, we define the notion of residual finiteness of groups, which is a property independent from amenability but will also be generalized by soficity.

3.1. **Definition.**

Definition 3.2. Let Γ be a group. The following conditions are equivalent.

1. For every element $g \in \Gamma$, $g \neq 1$, there is a finite group G and a surjective morphism

$$\varphi : \Gamma \longrightarrow G, \tag{3.2.1}$$

such that $\varphi(g) \neq 1$ in G.

2. For every element $g \in \Gamma$, there is a normal subgroup $N \triangleleft \Gamma$ of finite index, such that $g \notin N$.
3. The intersection of all subgroups of finite index of Γ is reduced to $\{1\}$.
4. The intersection of all normal subgroups of finite index of Γ is reduced to $\{1\}$.
5. The group Γ can be embedded into the direct product of a family of finite groups.

A group satisfying the above conditions is called *residually finite.*

Proof. The equivalence of the first four conditions is straightforward. Given a group Γ satisfying (4), let us denote by $(N_i)_{i\in I}$ the set of normal subgroups of finite index of Γ. Then the quotients Γ/N_i are all finite groups and we can consider the morphism

$$\varphi : \Gamma \longrightarrow G = \prod_{i\in I} \Gamma/N_i, \tag{3.2.2}$$

obtained using the quotient morphism $\varphi_i : \Gamma \to \Gamma/N_i$ on every factor of the right-hand-side product. We just have to show that φ is injective, but if $g \in \Gamma$ is such that $\varphi(g) = 1 = (\bar{1}, \bar{1}, \dots) \in G$, that means that $\varphi_i(g) = \bar{1} \in \Gamma/N_i$ for every $i \in I$, or equivalently that $g \in N_i$ for all $i \in I$. But condition (4) implies then that $g = 1$. To show the other implication, let Γ be a subgroup of a product of nontrivial finite groups,

$$\Gamma \hookrightarrow \prod_{i\in I} G_i. \tag{3.2.3}$$

Then, given $g = (g_i)_{i \in I} \in \Gamma$, such that there is $i_0 \in I$ with $g_{i_0} \neq 1$, we can consider the projection map

$$\psi : \Gamma \twoheadrightarrow G_{i_0}, \tag{3.2.4}$$

which is automatically surjective and sends g to $g_{i_0} \neq 1$ in the finite group G_{i_0}. □

3.3. **Examples.**

Examples 3.4. Of course, finite groups are residually finite. The class of residually finite groups is closed under taking subgroups and direct products. Linear groups $GL_n(\mathbb{Z})$, for $n \in \mathbb{N}$, are residually finite as can be seen using the maps

$$\psi_p : GL_n(\mathbb{Z}) \twoheadrightarrow GL_n\left(\mathbb{Z}/p\mathbb{Z}\right), \tag{3.4.1}$$

for p any prime number. Taking any element g of $GL_n(\mathbb{Z})$, it is enough to consider for example the morphism ψ_p for p larger than the absolute value of any coefficient of g. From the residual finiteness of linear groups, we deduce it for non-abelian free groups. Indeed, the group $SL_2(\mathbb{Z})$ has a subgroup isomorphic to the free group on two generators, namely

$$\mathbb{F}_2 \simeq \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \subset SL_2(\mathbb{Z}). \tag{3.4.2}$$

This shows at the same time that the free groups are residually finite and that the group $SL_2(\mathbb{Z})$ is not amenable.

Example 3.5. An example of a quotient of a residually finite group which is not residually finite is the following quotient of $\mathbb{F}_2$

$$G = \mathbb{F}_2 / \langle a^{-1} b^2 a b^{-3} \rangle. \tag{3.5.1}$$

3.6. **Locally residually finite groups.** The notion of residual finiteness can be weakened by defining a *local* version of it.

Definition 3.7. Let Γ be a group. We say that Γ is *locally residually finite* if every finitely generated subgroup of Γ is residually finite.

Example 3.8. A theorem of Malcev in [12] states that every finitely generated subgroup of $GL_n(\mathbb{R})$ is residually finite, for any $n \in \mathbb{N}$. However, $GL_n(\mathbb{R})$ or even the countable group $GL_n(\mathbb{Q})$ are not residually finite, for example because they contain a copy of the group $\mathbb{Q}^*$ as a subgroup, which is not residually finite because it is divisible and thus any morphism into a finite group must be trivial. They are therefore locally residually finite but not residually finite.

4. Sofic groups

We now introduce the notion of sofic groups, which generalizes both local residual finiteness and amenability. Sofic groups were at first named *initially subamenable groups*, and the name is sometimes still used to describe graphs such as in the following definition 4.1. They were defined by Gromov in [10].

4.1. **Definition.** The first definition of soficity relies on Cayley graphs of groups, so we will start by giving a definition and a few examples of such graphs associated to some groups.

Definition 4.2. Let Γ be a finitely generated group with a set of generators $S = \{s_1, s_2, \dots, s_m\}$. We define the Cayley graph of (Γ, S), as the directed graph with vertices labeled by Γ, and with an edge from g to h if and only if there is i such that $h = s_i g$. As the set S is chosen to be a generating subset, the graph is connected and it comes with a natural distance and we set $B_{(\Gamma,S)}(k)$ the ball of radius k centered on 1.

This is enough to define what a sofic group is.

Definition 4.3. Let (Γ, S) be a finitely generated group together with a generating set. We say that Γ is *sofic* if there exists a sequence of finite directed graphs $\{(V_n, E_n)\}_{n\in\mathbb{N}}$, with edges labeled by elements of S, and subsets $V_n^0 \subset V_n$ such that for any $\delta > 0$, and integer $r \in \mathbb{N}$, there is an integer $n_{r,\delta} > 0$ such that if $m \geq n_{r,\delta}$ then

- For each $v \in V_m^0$, there is a map $\varphi_v : B_{(\Gamma,S)}(r) \to V_m$, which is an isomorphism of directed labeled graphs between $B_{(\Gamma,S)}(r)$ and the r-ball around v in V_m.
- $|V_m^0| \geq (1-\delta)|V_m|$.

To extend to the case of not necessarily finitely generated groups, we turn soficity into a local property as in the case of local residual finiteness.

Definition 4.4. We say that a group Γ is sofic if all of its finitely generated subgroups are sofic in the sense of 4.3.

Example 4.5. As the Cayley graph of a finite group is always a finite directed graph, it is easy to see that any finite group is sofic. Moreover, as soficity is a local property, any locally finite group, in the sense that every finitely generated subgroup is finite, is sofic.

There is a useful equivalent definition, which will bring us closer to the notion of hyperlinearity (and thus to Connes' embedding conjecture), the equivalence between the two definition shows in particular that the choice of the generating set in 4.3 has no influence on the result. The proof can be found in [7].

Proposition 4.6. *A group Γ is sofic if and only if for any $0 < \varepsilon < 1$, and any finite subset $F \subset \Gamma$, there exist an integer n and a function $\psi_n : \Gamma \to S_n$, where S_n is the group of permutations on n elements, such that*

- $|\{k \in \{1, \dots, n\},\ \psi_n(e)\psi_n(f)(k) = \psi_n(ef)(k)\}| \geq (1-\varepsilon)n$ *for every elements $e, f \in F$.*
- $\psi_n(1) = \mathrm{id}$.
- $|k \in \{1, \dots, n\},\ \psi_n(e)(k) = k| \leq \varepsilon n$, *for any $1 \neq e \in F$.*

The class of sofic groups is closed under several classical operations, as developed by Elek and Szabó in [6]

Proposition 4.7. *The following properties hold:*

- *Every subgroup of a sofic group is sofic.*
- *The direct product of sofic groups is sofic.*
- *The inverse limit and direct limit of a diagram of sofic groups are sofic.*
- *Any extension of a sofic group by an amenable group is sofic, that is to say if there is an exact sequence of groups*

$$0 \longrightarrow H \longrightarrow \Gamma \longrightarrow K \longrightarrow 0, \tag{4.7.1}$$

with H a sofic group and K an amenable one, then Γ is sofic.

- *The free product of sofic groups is sofic.*
- *The amalgated free product of sofic groups over amenable subgroups is sofic.*

Remark 4.8. The question of amalgated free products of sofic groups over non-amenable subgroups is an open problem as of today.

4.9. **Almost actions.** There is another equivalent definition of soficity which links it once again to the existence of actions on a space with good properties.

Definition 4.10. Let Γ be a group and X be a space endowed with a *mean* μ, and let for any element $g \in \Gamma$, $\varphi(g) : X \to X$ be a bijection. We say that $(\Gamma, X, \mu, \varphi)$ is an *amenable almost-action* if we have that

1. $\varphi(1) = \mathrm{id}_X$.
2. For any $g \in \Gamma$, $\varphi(g)$ preserves the mean, that is for $L \subset X$ we have $\mu(\varphi(g)(L)) = \mu(L)$.
3. For any $g, h \in \Gamma$, there exists a subset $A_{g,h} \subset X$ with $\mu(A_{g,h}) = 1$, such that if $x \in A_{g,h}$ then $\varphi(gh)(x) = \varphi(g)(\varphi(h)(x))$.

We say moreover that the amenable almost-action is *essentially free* if

$$\mu\left(\{x \in X,\, \varphi(g)(x) = x\}\right) = 0, \tag{4.10.1}$$

for all $g \in \Gamma \setminus \{1\}$.

There are several equivalent conditions characterizing an essentially free amenable almost-action.

Proposition 4.11. *Let Γ be a group and X be a set and for every $g \in \Gamma$ let $\varphi(g) : X \to X$ be a bijection. We define for elements $g, h \in \Gamma$ the subsets*

$$B(g,h) = \{x \in X,\, \varphi(g) \circ \varphi(h)(x) \neq \varphi(gh)(x)\}\,. \tag{4.11.1}$$

The following are equivalent

1. *There exists* a mean *μ on X such that $(\Gamma, X, \mu, \varphi)$ is an essentially free amenable almost-action.*
2. *φ is non-paradoxical, that is to say X cannot be written as a union of subsets*

$$X = A_1 \cup A_2 \cup \cdots \cup A_l \cup B_1 \cup \cdots \cup B_m \cup C_1 \cup \cdots \cup C_n, \tag{4.11.2}$$

   such that
   - (a) *There are some elements $g_1, h_1, \ldots, g_l, h_l \in \Gamma$ making the subsets*

     $$\varphi(g_1)A_1, \varphi(h_1)A_1, \ldots, \varphi(g_l)A_l, \varphi(h_l)A_l$$

     pairwise disjoint.
   - (b) *For each B_i, there are elements $a_i, b_i \in \Gamma$ with $B(a_i, b_i) = B_i$.*
   - (c) *For each C_i there is an element $c_i \in \Gamma$ with $C_i = \{x \in X,\, \varphi(c_i)(x) = x\}$.*
3. *For any finite subset $E \subset \Gamma$ and any $\varepsilon > 0$, there exists a finite subset $F \subset X$ such that*
   - (a) $\displaystyle\max_{g \in E} \frac{|g \cdot F \triangle F|}{|F|} < \varepsilon.$
   - (b) $\displaystyle\max_{a,b \in E} \frac{|B(a,b) \triangle F|}{|F|} < \varepsilon.$
   - (c) $\displaystyle\max_{g \in E} \frac{|\{x \in X,\, \varphi(g)(x) = x\} \cap F|}{|F|} < \varepsilon.$

Proof. Assume that (1) holds, and suppose that we have a paradoxical decomposition of X as in (2), then the hypotheses imply that $\mu(B_i)=0$ and $\mu(C_j)=0$ for all i and j. Hence, using the fact that the family $(A_i,B_j,C_k)_{i,j,k}$ covers X, and that the family $(\varphi(g))_{g\in\Gamma}$ is mean-preserving, we get that

$$\begin{aligned} 1\geq \mu\left(\bigcup_{i=1}^{l}\varphi(g_i)A_i\cup\varphi(h_i)A_i\right) &= \sum_{i=1}^{l}\mu(\varphi(g_i)A_i)+\sum_{i=1}^{l}\mu(\varphi(h_i)A_i) \\ &= \sum_{i=1}^{l}\mu(A_i)+\sum_{i=1}^{l}\mu(A_i)\geq 2. \end{aligned} \tag{4.11.3}$$

That is impossible and so there can be no paradoxical decomposition.

Assuming (3), we prove (1) by directly defining the mean on X. For every pair (K,ε), of a finite subset $K\subset\Gamma$ and a real $\varepsilon>0$, we denote by $S_{(K,\varepsilon)}$ the set of finite subsets $F\subset X$ satisfying the conditions of (3). Then, given (K,ε) and (L,δ) we have that

$$S_{(K\cup L,\min(\varepsilon,\delta))}\subset S_{(K,\varepsilon)}\cap S_{(L,\delta)}. \tag{4.11.4}$$

Hence, the pairs (K,ε) form a net, and for each, we can consider the measure

$$\mu_{(K,\varepsilon)}(A)=\frac{|A\cap F|}{|F|}, \tag{4.11.5}$$

for $A\subset X$, where the finite set F is the one obtained with (3). These measures are in the unit ball of $L^\infty(X)'$, seeing X as a discrete set, with counting measure. The unit ball is weak$-*$ compact, and we can consider an accumulation point of the net constructed, which we denote μ. This mean makes (Γ,X,μ,φ) an essentially free amenable almost-action.

The last part is proving (2) from (3). By contradiction, assume that there is a finite set $K\subset X$ and an $\varepsilon>0$ such that no finite subset $F\subset X$ satisfies the conditions of (3). We can assume that $1\in K$ and that K is symmetric in the sense that if $k\in K$, then $k^{-1}\in K$. Denoting K^r the set of elements that can be written as the products of r elements of K, we have the following:

$$\bigcup_{g\in K^r}\varphi(g)^{-1}\left(\bigcup_{s,t\in K}B(s,t)\cup\bigcup_{u\in K}\{x\in X,\varphi(u)(x)=x\}\right)\subset\bigcup_{a,b\in K^{r+1}}B(a,b)\cup\bigcup_{c\in K^{2r+1}}\{x\in X,\varphi(c)(x)=x\}. \tag{4.11.6}$$

the verification of this assertion can be found in [8]. Now, we define the subsets

$$A_r=X\setminus\left(\bigcup_{a,b\in K^{r+1}}B(a,b)\cup\bigcup_{c\in K^{2r+1}}\{x\in X,\varphi(c)(x)=x\}\right), \tag{4.11.7}$$

and we take an integer p such that $(1+\varepsilon)^p>2$. The assumption says that the sets A_r are nonempty for any $r\geq 1$. We consider the bipartite graph G with vertex sets A_p and X, with the condition that (a,x) is an edge if and only if there is $t\in K^p$ such that $\varphi(t)(a)=x$. for any finite subset $L\subset A_p$, the number of neighbors in X must be at least equal to $2|L|$. To see this, we define $L_0=L$ and recursively

$$\begin{aligned} L_s &= \{x\in X,\,\exists g\in K,\,x\in\varphi(g)(L_{s-1})\} \\ &= \{x\in X,\,\exists g\in K^s,\,x\in\varphi(g)(L)\}. \end{aligned} \tag{4.11.8}$$

The inclusion 4.11.6 shows that we have

$$L_s \cap \left(\bigcup_{s,t \in K} B(s,t) \cup \bigcup_{u \in K} \{x \in X, \varphi(u)(x) = x\} \right) = \emptyset. \tag{4.11.9}$$

Hence, $|L_s| \geq (1+\varepsilon)|L_{s-1}| \geq (1+\varepsilon)^s |L|$ for all s. From this, we know that the set of neighbors of L in G, which is L_p, contains at least $2|L|$ elements. We can then use Hall's (2,1)-matching theorem to G, which gives two functions $m_1 : A \to X$ and $m_2 : A \to X$ with disjoint images. Then for $s, t \in K^p$, with $s \neq t$, we set

$$Q_{s,t} = \{a \in A_p,\, m_1(a) = \varphi(s)(a),\, m_2(a) = \varphi(t)(a)\}\,. \tag{4.11.10}$$

the construction of the maps m_1 and m_2 implies that

$$A_p = \bigcup_{s \neq t \in K^p} Q_{s,t}, \tag{4.11.11}$$

and the sets

$$\{\varphi(s)\,(Q_{s,t})\,, \varphi(t)\,(Q_{s,t})\}_{s \neq t \in K^p}\,, \tag{4.11.12}$$

are disjoint and hence we obtain a cover of X as described in (2). The proof is then complete. □

From this proposition, we get the following corollary. It is the reason why Elek and Szabó introduced the notions of essentially free amenable almost-actions in [8].

Corollary 4.12. *Let Γ be a discrete group, then Γ is sofic if and only if it admits an essentially free amenable almost-action.*

Proof. Our goal is to see how condition (3) of 4.11 is equivalent to the group being sofic. If the group has an essentially free amenable almost-action $(\Gamma, X, \mu, \varphi)$, then for any $\varepsilon > 0$ and finite set $E \in \Gamma$, we define the functions from Γ to S_n considering n given by $|F|$ where F is the finite subset of X given by (3). We set $\psi_n : \Gamma \to S_n$ to be equal to φ on $F \cup \{1\}$ and to the identity outside this set. The hypotheses of (3) exactly state that the functions satisfy the conditions for Γ to be sofic. To show the other implication, we assume that Γ is sofic, as in proposition 4.6. We write Γ as an increasing union of finite sets $(E_n)_{n\geq 1}$,

$$\Gamma = \bigcup_{n \geq 1} E_n. \tag{4.12.1}$$

For every $n \geq 1$, we choose a corresponding finite set F_n together with a map

$$\psi_n : \Gamma \to S(F_n), \tag{4.12.2}$$

where $S(F_n)$ is the set of permutations of the finite set F_n, such that

- $|\{k \in F_n,\, \psi_n(g)\psi_n(h)(k) = \psi_n(gh)(k)\}| \geq \left(1 - \dfrac{1}{n}\right)|F_n|$ for every elements $g, h \in E_n$.
- $\psi_n(1) = \mathrm{id}_{F_n}$.
- $|k \in F_n,\, \psi_n(g)(k) = k| \leq \dfrac{1}{n}|F_n|$, for any $1 \neq g \in E_n$.

The set on which Γ will have an almost-action is then

$$X = \bigsqcup_{n \geq 1} F_n, \tag{4.12.3}$$

together with the map

$$\psi : \Gamma \to S(X), \tag{4.12.4}$$

such that

$$\begin{aligned} \psi(g) : X &\to X \\ x &\mapsto \psi_n(g)(x) \text{ if } x \in F_n. \end{aligned} \tag{4.12.5}$$

Note that with our first assumptions, we have $\psi(1) = \mathrm{id}_X$. There is still to define the mean on X making this an essentially free amenable almost-action. For every $n \geq 1$, we define

$$\begin{aligned} \nu_n(g) : \mathcal{P}(X) &\to [0,1] \\ A &\mapsto \frac{|A \cap F_n|}{|F_n|}. \end{aligned} \tag{4.12.6}$$

Every ν_n is a probability measure on X and therefore it can be seen as an element of the unit ball $B_1((\ell^\infty(X))^*)$, which is compact for the weak$-^*$ topology. We can therefore consider an accumulation point m. Let us show that this mean makes (Γ, X, ψ, m) an essentially free amenable almost-action. First, given $g \in \Gamma$ we have for every $n \geq 1$ and $A \in \mathcal{P}(X)$

$$\begin{aligned} \nu_n(\psi(g)A) =& \frac{|\psi(g)A \cap F_n|}{|F_n|} \\ =& \frac{|A \cap \psi(g)^{-1}F_n|}{|F_n|} \\ =& \frac{|A \cap \psi_n(g)^{-1}F_n|}{|F_n|} \\ =& \frac{|A \cap F_n|}{|F_n|} \\ =& \nu_n(A), \end{aligned} \tag{4.12.7}$$

taking the limit we have that for every subset $A \subset X$, $m(\psi(g)A) = m(A)$. Then, given $g, h \in \Gamma$, recall the definition of the set $B(g,h)$:

$$\begin{aligned} B(g,h) =& \{x \in X,\ \psi(gh)(x) \neq \psi(g)\psi(h)(x)\} \\ =& \bigsqcup_{n \geq 1} \{x \in F_n, \psi_n(gh)(x) \neq \psi_n(g)\psi_n(h)(x)\} \\ =& \bigsqcup_{n \geq 1} B_n(g,h), \end{aligned} \tag{4.12.8}$$

where $B_n(g,h)$ is the set $B(g,h) \cap F_n$. The assumption on the sequence $(F_n)_{n \geq 1}$ gives for n large enough (so that E_n contains g and h)

$$\nu_n(B(g,h)) = \frac{|B_n(g,h)|}{|F_n|} \leq \frac{1}{n}. \tag{4.12.9}$$

Taking the limit, we get that $m(B(g,h)) = 0$ and therefore we have an amenable almost-action. Lastly there is to check that it is essentially free. Given $g \in \Gamma$, $g \neq 1$, we have that there is n_0 with $g \in E_{n_0}$, and for $n \geq n_0$, as

$$\{x \in X,\ \psi(g)(x) \neq x\} = \bigsqcup_{n \geq 1} \{x \in F_n, \psi_n(g)(x) \neq x\}, \tag{4.12.10}$$

we have

$$\nu_n\left(\{x \in X,\, \psi(g)(x) \neq x\}\right) = \frac{|\{x \in F_n, \psi_n(g)(x) \neq x\}|}{|F_n|} \geq 1 - \frac{1}{n}. \tag{4.12.11}$$

Therefore, taking the limit we get that $m\left(\{x \in X,\, \psi(g)(x) \neq x\}\right) = 1$, that is the amenable almost-action is essentially free. ☐

Corollary 4.13. *Let* Γ *be a group. If* Γ *is (residually) amenable, then it is sofic.*

Proof. If the group Γ is amenable, then its action on itself is free and amenable, it is then an essentially free amenable almost-action, and Γ is therefore sofic. ☐

On the other hand, it is really easy to see that soficity is a generalization of residual finiteness.

Theorem 4.14. *Let* Γ *be a group. If* Γ *is residually finite, then it is sofic.*

Proof. This is just a consequence of the fact residually finite groups are embeddable into a product of finite groups, which is sofic, and that any subgroup of a sofic group is also sofic. ☐

We give one last result, involving Kahzdan's property (T), which is often viewed as an opposite of amenability, in the sense that any group having both property must be finite.

Proposition 4.15. *Let* Γ *be a discrete group having Kahzdan's property* (T)*, then* Γ *has an essentially free amenable action if and only if it is residually finite.*

This property doesn't answer the question about soficity (that is asking Γ to have an essentially free amenable *almost*-action instead of an action), which is still an open problem.

5. Surjunctivity

5.1. **Definition and main conjecture.** The idea behind surjunctivity is to generalize an easy property which is that any function mapping a finite group to itself which is injective must also be surjective. This bring the following definition.

Definition 5.2. Let Γ be a group, we say that Γ is *surjunctive* if given any finite set S and any continuous Γ-equivariant map $f : S^\Gamma \to S^\Gamma$,we have that if f is injective, then f must also be surjective.

Remark 5.3. Using this definition, the property we generalize is equivalent to the fact that the trivial group $\{1\}$ is surjonctive. It also implies that every finite group is surjunctive.

This notion has been introduced by Gottschalk in 1973 in [9], when he observed there were no known example of a group which is not surjunctive. He stated the following conjecture.

Conjecture 5.4. *Let* Γ *be any group. Then* Γ *is a surjunctive group.*

This conjecture is still open, and linked to the existence of non-sofic groups, as we have the following theorem proved first by Gromov in [10], before the name *sofic* was coined.

Theorem 5.5. *Let* Γ *be a sofic group. Then* Γ *is surjunctive.*

Thus, an example of a non-surjunctive group would necessarily be non-sofic, but there are no example of non-sofic group as of today. We will give the proof in the case of the group $\mathbb{Z}$, introducing quickly a notion of entropy which can be extended to amenable, and then sofic groups and used in the general case. Entropy for sofic groups was introduced by Bowen in [1] as a generalization of the entropy for amenable actions.

Proposition 5.6. *The group $\mathbb{Z}$ is surjunctive.*

Proof. Let A be a finite set and $\varphi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ a continuous $\mathbb{Z}$-equivariant map. The fact that it is equivariant is equivalent to the fact that it commutes to the shift on $\mathbb{Z}$, which we denote σ, that is to say for $f \in A^{\mathbb{Z}}$, and $k \in \mathbb{Z}$

$$(\sigma f)(k) = f(k+1). \tag{5.6.1}$$

We define the entropy of any closed subset X of $A^{\mathbb{Z}}$, which is also invariant under the shift σ. We first set for such an X, for every $n \geq 1$

$$X_n = \{f_{|\{0,\dots,n-1\}},\ f \in X\} \subset A^{\{0,\dots,n-1\}}, \tag{5.6.2}$$

and then

$$h(X) = \inf_{n\geq 1} \frac{1}{n} \log |X_n|. \tag{5.6.3}$$

As the sequence $(|X_n|)_{n\geq 1}$ is sub-multiplicative in the sense that if $m, n \geq 1$, then

$$|X_{n+m}| \leq |X_n||X_m|,$$

it converges towards $h(X)$. Moreover, we have that $h(X) \leq \log |A|$, as every X_n is a subset of $A^{\{0,\dots,n-1\}}$, and we have equality if and only if X is equal to the whole set $A^{\mathbb{Z}}$. To see this, assume that $X \subsetneq A^{\mathbb{Z}}$, then as it is closed, there is n_0 such that $X_{n_0} \subsetneq A^{\{0,\dots,n_0-1\}}$ and finally

$$h(X) \leq \frac{1}{n_0} \log |X_{n_0}| \leq \frac{1}{n_0} \log (|A|^{n_0} - 1) < \log |A|. \tag{5.6.4}$$

Going back to the map φ, we denote $X = \varphi(A^{\mathbb{Z}})$, as it is injective we can consider its inverse map $\psi : X \to A^{\mathbb{Z}}$ which is continuous and commutes with the shift σ. From this, ψ is given by its restriction to a finite map ψ_0, with domain contained in a set of the form $[-N; N]$ for a large enough N. From this we deduce that $|X_{n+2N}| \geq |A|^n$, for all n, because ψ is bijective and any element of $A^{\{0,\dots,n-1\}}$ must be obtained from a map in X_n shifted at most by $-N$ or $+N$. Therefore we must have $h(X) \geq \log |A|$ and finally $h(X) = \log |A|$, which leads to $X = A^{\mathbb{Z}}$ and the surjunctivity of the group $\mathbb{Z}$. □

The extension to the case of amenable groups works in the same way by replacing the sets $\{0, \dots, n-1\}$ by a Følner sequence. Actually, the first proof of the surjunctivity of amenable groups is a result by Ceccherini-Silberstein, Machi and Scarabotti in [5] where they proved a stronger conjecture, namely the *Garden of Eden* conjecture for these groups.

6. OTHER OPEN PROBLEMS

6.1. Properties implied by soficity.

Several other conjectures have been proved true for sofic groups but not in the more general case. Here are several open questions about groups, some being answered when considering the class sofic groups.

Examples 6.2. Let Γ be a group, and $\mathbb{K}$ be an integral domain then the following propositions are conjectured.

1. If $\mathbb{K}$ is a skew field, and $a, b \in \mathbb{K}\Gamma$, if $ab = 1$, then $ba = 1$ too.
2. If Γ is torsion-free, and $a, b \in \mathbb{K}\Gamma$, if $ab = 0$, then $a = 0$ or $b = 0$.
3. If Γ is torsion-free, and $a \in \mathbb{K}\Gamma$, if there is $n \in \mathbb{N}$ such that $a^n = 0$, then $a = 0$.

The first conjecture has been proved to be true if Γ is sofic.

6.3. **Hyperlinearity.** There is a generalization of soficity, named *hyperlinearity*, which can be either defined at the level of the Von Neumann algebra of the group, or defining another form of embedding using unitary groups rather than symmetric groups in a way generalizing the definition 4.3. Every sofic group is hyperlinear, and therefore there are of course no known example of a group which is not hyperlinear. The study of hyperlinearity of groups is linked to one of the most important conjecture in operator algebras, Connes' embedding conjecture. Radulescu was the first to consider the particular case of von Neumann algebras coming from groups to give examples in favor of the conjecture, in [15]. We will not go into the details of what follows but we indicate nonetheless the possible extensions of soficity in the framework of operator algebras.

We first give a definition of hyperlinearity which does not rely on ultraproducts (which is the case for the usual definition), the different definitions are explained by Pestov in [14]. In order to do so, we first need to define a metric on the unitary groups $U(n)$, for $n \in \mathbb{N}$.

Definition 6.4. We define the normalized *Hilbert-Schmidt* distance on the unitary group of rank n, $U(n)$,

$$U(n) = \{u \in M_n(\mathbb{C}),\, u^*u = uu^* = \mathrm{Id}\}\,, \tag{6.4.1}$$

as follows:

$$d_{HS}(u,v) = \|u-v\|_2 = \sqrt{\frac{1}{n}\sum_{1\le i,j\le n} |u_{ij}-v_{ij}|^2} = \frac{1}{\sqrt{n}}\sqrt{\mathrm{Tr}((u-v)^*(u-v))}. \tag{6.4.2}$$

It is a bi-invariant metric, and we denote $U(n)_2$ the group $U(n)$ endowed with this metric.

We can now give a definition of hyperlinearity.

Definition 6.5. Let Γ be a group, we say that Γ is *hyperlinear* if for any finite subset $F \subset \Gamma$ and $\varepsilon > 0$, there is an integer $n \in \mathbb{N}$ and a map $\varphi : F \to U(n)_2$, such that

1. For any $g, h \in F$ with $gh \in F$, we have $d_{HS}(\varphi(g)\varphi(h), \varphi(gh)) < \varepsilon$.
2. For any $g, h \in F$ with $g \neq h$, we have $d_{HS}(\varphi(g), \varphi(h)) > \sqrt{2} - \varepsilon$.

There are no known example of a group which is not hyperlinear. This leads to the following conjecture, which is as we will see a special case of Connes' embedding conjecture.

Conjecture 6.6. *Every group is hyperlinear.*

This is enhanced by the fact that the class of hyperlinear groups contains the class of sofic groups, this has first been proved by Elek and Szabó in [8]. In this article in particular they also prove that an important conjecture, namely the Determinant Conjecture holds in the case of sofic groups.

Theorem 6.7. *Let Γ be a group. If Γ is sofic, then Γ is hyperlinear.*

Proof. The only thing to prove, using definition 4.6, is that we can embed the symmetric group S_n into the group $U(n)$ in such a way that the condition for soficity makes the conditions for hyperlinearity be satisfied. We set

$$\begin{aligned} f_n : S_n &\to U(n)_2 \\ \sigma &\to M_\sigma = (\delta_{\sigma(i),j})_{1\le i,j\le n}. \end{aligned} \tag{6.7.1}$$

The map is well-defined, it is a morphism and the number of fixed point of a permutation σ is equal to the non-normalized trace $\mathrm{Tr}(M_\sigma)$. Assume now that Γ is a given sofic group. Let $F \subset \Gamma$

be a finite subset and $\varepsilon > 0$, soficity gives maps $\psi_n : \Gamma \to S_n$ as in 4.6. We consider the maps $\varphi_n = f_n \circ \psi_n$. Then for $g, h \in F$, we have

$$\begin{aligned} d_{HS}(\varphi_n(g)\varphi_n(h), \varphi(gh)) =& d_{HS}(M_{\psi_n(g)}M_{\psi_n(h)}, M_{\psi_n(gh)}) \\ =& d_{HS}(M_{\psi_n(g)\psi_n(h)}, M_{\psi_n(gh)}) \\ =& \frac{1}{\sqrt{n}}\sqrt{\mathrm{Tr}((M_{\psi_n(g)\psi_n(h)} - M_{\psi_n(gh)})^*(M_{\psi_n(g)\psi_n(h)} - M_{\psi_n(gh)}))} \\ =& \sqrt{\frac{2}{n}}\sqrt{\mathrm{Tr}\left(M_{\psi_n(g)\psi_n(h)\psi_n(gh)^{-1}}\right)} \\ \leq& \sqrt{2\varepsilon}. \end{aligned} \tag{6.7.2}$$

That gives (1), changing the ε a bit. Now, if we have $g, h \in \Gamma$ with $g \neq h$, then

$$\begin{aligned} d_{HS}(\varphi_n(g), \varphi_n(h)) =& d_{HS}(M_{\psi_n(g)}, M_{\psi_n(h)}) \\ =& \sqrt{\frac{2}{n}}\sqrt{\mathrm{Tr}\left(M_{\psi_n(g)\psi_n(h)^{-1}}\right)} \\ \geq& \sqrt{2}(1-\varepsilon). \end{aligned} \tag{6.7.3}$$

from this we get (2) and the fact that Γ is hyperlinear. □

However, to see how hyperlinearity is linked to Conne's embedding conjecture, we need to talk a bit about ultraproducts and von Neumann algebras. The conjecture is the following. Several properties and equivalent conjectures are stated in Narutaka Ozawa's review [13].

Conjecture 6.8. *Any separable II_1-factor is embeddable into an ultraproduct R^ω of the hyperfinite II_1-factor R.*

Definition 6.9. We say that a group Γ is *hyperlinear* if its von Neumann algebra $L(\Gamma)$ is embeddable into an ultraproduct of the hyperfinite II_1-factor.

The conjecture 6.8 has been proved wrong in 2021 but the article [11] is yet to be published. However the case of von Neumann algebras coming from groups 6.6 is still open.

6.10. **Summary.** To conclude this section, we will give a diagram which summarizes what has been proved for now in terms of classifications of groups.

$$\begin{tikzcd} Residually finite \arrow[dr, phantom, "\subsetneq" sloped] & & Hyperlinear \arrow[dr, phantom, "\subseteq" sloped] & \\ & Sofic \arrow[ur, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\subseteq" sloped] & & Groups. \\ Amenable \arrow[ur, phantom, "\subsetneq" sloped] & & Surjunctive \arrow[ur, phantom, "\subseteq" sloped] & \end{tikzcd} \tag{6.10.1}$$

Every subset which is not marked as strict has no known counterexample and is possibly an equality.

6.11. **What about quantum groups?** The other extension of hyperlinearity of groups is to consider hyperlinearity generalized to the case of quantum groups. A compact quantum group can be defined as a unital C^*-algebra endowed with a comultiplication satisfying properties making the algebra the C^*-algebra of continuous functions from a group to $\mathbb{C}$, in the case where it is commutative. This construction allows to define a coherent representation theory and von Neumann algebra, still agreeing with the usual ones when the algebra comes from a classical group. Thus we can extend definition 6.9 to the case of (compact) quantum groups and ask whether conjecture 6.6 holds in this framework. This is developed by Brannan, Collins and Vergnioux in [3]. Moreover, residual finiteness admits a generalization to the case of quantum groups too, and still implies hyperlinearity. A way to prove that a quantum group is residually finite is to show that it is *topologically generated* by a family of residually finite quantum subgroups, and proofs of hyperlinearity have been done this way by Brannan, Chirvasitu and Freslon in [2] for the case of the quantum permutation and quantum reflection groups.

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