INTRODUCTION À UN DOMAINE DE RECHERCHE – SEMI-CLASSICAL ANALYSIS ON NILPOTENT LIE GROUPS AND MAGNETIC SCHRÖDINGER OPERATORS.

LINO BENEDETTO

Contents

1. Introduction 1
2. Harmonic analysis on nilpotent Lie groups 2
2.1. Haar measure and functional spaces 4
2.2. Representations and Fourier transform 5
2.3. Smooth Vectors and infinitesimal representation 6
2.4. Kirillov's theory: a complete decription of the dual 7
3. Semi-classical analysis on a graded nilpotent Lie group 8
3.1. The algebra of symbols $\mathcal{A}_0$ 8
3.2. Semi-classical pseudodifferential operators 9
3.3. Semi-classical measures 10
4. About the magnetic Schrödinger operators 12
4.1. Self-adjoint extensions 12
4.2. Spectral analysis 13
5. Wave packets 13
5.1. Gaussian wave packets 13
5.2. Wave packet dynamics 14
5.3. Non-commutative wave packets 15
5.4. From the nilpotent Lie groups to the magnetic Schrödinger operators 16
References 17

1. Introduction

Nilpotent Lie groups appear naturally in the analysis of manifolds and provide an abstract setting for many notions of Euclidean analysis. They arise as local models in subriemannian or in contact geometry and in the study of hypoellipticity of differential operators, but they are also themselves of interest. In most of these applications, the group comes with a family of dilations compatible with the group structure; thus we will restrict ourselves to the subclass of graded nilpotent Lie groups. It's also worth noting that, from the point of view of general harmonic analysis, these groups and their duals are not only historically important but are still actively studied.

To tackle these problems, the development of a pseudodifferential calculus on these groups has been increasingly needed these last decades. Indeed, the founding idea of the microlocal analysis is to study phenomena simultaneously in standard and Fourier variables, which correspond to position and impulsion variables – the phase space variables of quantum mechanics. The transposition of this point of view from the euclidean space to the setting of Lie groups has been a subject of investigation but was faced with the difficulty of the operator structure of the Fourier transform. In this presentation, we will present the harmonic analysis on general graded nilpotent Lie groups and we will expose the construction of a semi-classical analysis on them.

One field of interest for the author, where nilpotent Lie groups seem to be involved, is the spectral analysis of the magnetic Schrödinger operator and its dynamics. The particular situation where the magnetic field vanishes degenerately on some planar curve is still to be studied. Dynamically, we are interested in how to define appropriate wave packets for the magnetic Schrödinger equation and study their behavior beyond the Erhenfest time.

2. Harmonic analysis on nilpotent Lie groups

A Lie group G is a smooth manifold endowed with smooth mappings

$$G \times G \ni (x, y) \mapsto xy \in G, \quad G \ni x \mapsto x^{-1} \in G$$

and a unit element e satisfying all the algebraic properties of a group's laws. We recall that the tangent space at the unit $\mathrm{T}_e G$, noted $\mathfrak{g}$ from now on, of such a manifold is equipped with a natural Lie algebra structure induced by the commutator bracket of vector fields.

We may indeed identify $\mathfrak{g}$ with the space of left-invariant vector fields via

$$Xf(x) = \frac{d}{dt} f(x\exp(tX))\Big|_{t=0}, \; x \in G.$$

In this presentation, we focus our interest on Lie groups which are *nilpotent*.

Definition 2.1. *A Lie group G is said to be nilpotent if its Lie algebra $\mathfrak{g}$ is nilpotent, meaning its lower central series defined inductively by*

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \quad \mathfrak{g}_{(j)} := \big[\mathfrak{g}, \mathfrak{g}_{(j-1)}\big]$$

terminates at 0 in a finite number of steps. Moreover, if $\mathfrak{g}_{(s+1)} = 0$ and $\mathfrak{g}_{(s)} \neq 0$, G is said to be nilpotent of step s.

For nilpotent Lie groups, the group's law is entirely (not only locally) determined by its Lie algebra.

Theorem 2.2 (Baker-Campbell-Hausdorff formula)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There exists a neighbourhood V of 0 in $\mathfrak{g}$ such that for any $X, Y \in V$, we have*

$$\exp_G(X)\exp_G(Y) = \exp_G\Big(\sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{p,q\in\mathbf{N}_0^n, p_i+q_i>0} \frac{\Big(\sum_{j=1}^n (p_j+q_j)\Big)^{-1}}{p_1!q_1!\dots p_n!q_n!} \times (\mathrm{ad}(X))^{p_1}(\mathrm{ad}(Y))^{q_1}\dots(\mathrm{ad}(X))^{p_n}(\mathrm{ad}(Y))^{q_n-1}Y\Big).$$

The equality holds whenever the sum on the right-hand side is convergent. In particular, when $\mathfrak{g}$ is nilpotent, it is finite.

Example 2.3. If $n_o \in \mathbf{N}$, the *Heisenberg group* $\mathbf{H}_{n_o}$ is the Lie group whose underlying manifold is $\mathbf{R}^{2n_o+1}$ and whose law is

$$h_1h_2 = \left(x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}(x_1y_2 - y_1x_2)\right),$$

for $h_1 = (x_1, y_1, t_1)$ and $h_2 = (x_2, y_2, t_2)$ in $\mathbf{R}^{n_o} \times \mathbf{R}^{n_o} \times \mathbf{R}$. Its Lie algebra $\mathfrak{h}_{n_o}$ is $\mathbf{R}^{2n_o+1}$ equipped with the Lie bracket given by the commutator relations of its canonical basis $\{X_1, ..., X_{n_o}, Y_1, ..., Y_{n_o}, T\}$:

$$\forall j \in \{1, ..., n_o\},\ [X_j, Y_j] = T,$$

and all the other Lie brackets (apart from those obtained by anti-symmetry) are trivial.

Proposition 2.4. *Let G be a connected simply connected nilpotent Lie group. Then, the exponential map $\exp_G : \mathfrak{g} \mapsto G$ is a diffeomorphism. If G is identified with $\mathfrak{g}$ via this map, the group law on G (which is generally not commutative) is provided by the Campbell-Baker-Hausdorff formula, and $(x, y) \mapsto xy$ is a polynomial map.*

Example 2.5. This property (coupled with the Birkhoff embedding theorem) allow us to define a nilpotent Lie group by its nilpotent Lie algebra. This way, we call *Engel group* K_3 the group associated to the Lie algebra $\mathfrak{g} = \mathrm{span}\{V_1, V_2, W, Z\}$ whose non-trivial commutators are

$$[V_1, V_2] = W,\ [V_1, W] = Z.$$

And amongst nilpotent Lie groups, we will investigate a particular subclass of them that allow us to define natural compatible dilations.

Definition 2.6. *A simply connected nilpotent Lie group G is said to be graded if its Lie algebra $\mathfrak{g}$ is equipped with a vector space decomposition*

$$\mathfrak{g} = \bigoplus_{i=1,\dots,n} \mathfrak{g}_i$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$ for all $i, j \in \{1, \dots, n-1\}$. In what follows, $\mathfrak{g}_1$ will be referred as the first stratum of G.

Let us here show how to define our dilations when G is graded. There is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we can decompose X as $X = X_1 + \cdots + X_n$ with $X_i \in \mathfrak{g}_i$ for $i \in \{1, \ldots, n\}$, with the previous notations. Then we define

$$\delta_t X := \sum_{i=1}^{n} t^i X_i.$$

This allows us to define the dilation on the Lie group G via the identification by the exponential map:

$$\tilde{\delta}_t := \exp \circ \, \delta_t \, \circ \exp^{-1}$$

To avoid heavy notations, we shall still denote it by δ_t. The dilations δ_t $(t > 0)$ on $\mathfrak{g}$ and G form a one-parameter group of automorphisms of the Lie algebra $\mathfrak{g}$ and of the group G.

2.1. **Haar measure and functional spaces.** The existence of a measure invariant by translation is always true for locally compact topological groups; but in the case of nilpotent Lie groups, the construction of a Haar measure is quite simple.

Proposition 2.7. *Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. If $\mathrm{d}\lambda_{\mathfrak{g}}$ denotes a Lebesgue measure on the vector space $\mathfrak{g}$, then $\mathrm{d}\lambda = \mathrm{d}\lambda_{\mathfrak{g}} \circ \exp^{-1}$ is a bi-invariant Haar measure on G, meaning*

$$\forall f \in L^1(G), \forall x \in G, \int_G f(y) \, \mathrm{d}\lambda(y) = \int_G f(xy) \, \mathrm{d}\lambda(y) = \int_G f(yx) \, \mathrm{d}\lambda(y).$$

From now on, we will simply note $dy = \mathrm{d}\lambda(y)$.

Note that the convolution of two functions f and g on G is given by

$$f * g(x) = \int_g f(xy^{-1}) g(y) \, dy = \int_G f(y) g(y^{-1}x) \, dy$$

and as in the Euclidean case, we define the Lebesgue spaces by

$$\|f\|_{L^q(G)} = \left(\int_G |f(y)|^q \, dy \right)^{\frac{1}{q}},$$

for $q \in [0, +\infty[$ with the usual modification for $q = +\infty$. We note here that the Jacobian of the dilation δ_t is t^Q where

$$Q := \sum_{i=1}^{n} i \dim \mathfrak{g}_i.$$

is called the homogeneous dimension of G.

An other important functional space in the study of a group's harmonic analysis is the space of Schwartz functions.

Proposition 2.8. *Let G be a nilpotent Lie group. We define the Schwartz space $S(G)$ as the set of smooth functions f on G such that for all $\alpha, \beta \in \mathbf{N}^n$ $(n = \dim \mathfrak{g})$, the function $x \mapsto x^\beta X^\alpha f(x)$ belongs to $L^\infty(G)$, where X^α denotes a product of $|\alpha|$ left invariant vector fields forming a basis of $\mathfrak{g}$ and x^β a product of $|\beta|$ coordinate functions on G.*

In the case of a simply connected Lie groupe G, the Schwartz space $S(G)$ can be naturally identified with the Schwartz space $S(\mathfrak{g})$; in particular, it is dense in Lebesgue spaces.

2.2. **Representations and Fourier transform.** In the rest of this paper, we will then restrict ourselves to the study of simply connected graded nilpotent Lie groups: let G be such a group.

Definition 2.9. *A representation π of G on the Hilbert space $\mathcal{H}_\pi$ is is a homomorphism π of G into the group of bounded linear operators on $\mathcal{H}_\pi$ with bounded inverse. A representation is said to be unitary if $\pi(x)$ is unitary for every $x \in G$. Hence a unitary representation π of a group G is a homomorphism $\pi \in Hom(G, \mathrm{U}(\mathcal{H}_\pi))$.*

Definition 2.10. *A representation π of G is said to be irreducible if there is no closed invariant subspace $W \subset \mathcal{H}_\pi$, meaning $\pi(x)W \not\subset W$ for some $x \in G$.*

Definition 2.11. *A representation π of G is strongly continuous (also noted s.c.) if the mapping $\pi : G \mapsto \mathcal{L}(\mathcal{H}_\pi)$ is continuous for the strong operator topology in $\mathcal{L}(\mathcal{H}_\pi)$, that is if the mapping*

$$\begin{cases} G \to \mathcal{H}_\pi \\ x \mapsto \pi(x)v \end{cases}$$

is continuous for all $v \in \mathcal{H}_\pi$.

Definition 2.12. *Two representations π_1 and π_2 are said to be equivalent, or intertwined, if there exists a bounded linear mapping $A : \mathcal{H}_{\pi_1} \mapsto \mathcal{H}_{\pi_2}$ between their representation spaces with a bounded inverse such that the relation*

$$A\pi_1(x) = \pi_2(x)A$$

holds for every $x \in G$. In this case we write $\pi_1 \sim \pi_2$, and $[\pi_1]$ its equivalence class for $\sim$.

It is possible to prove that a representation π is irreducible if its only intertwining operators are homotheties.

The set of all equivalence classes of strongly continuous irreducible unitary representations of G is called the unitary dual of G (or just dual) and is denoted by $\widehat{G}$:

$$\widehat{G} := \{[\pi], \pi \text{ irreducible s.c. unitary representation of } G\}.$$

We can now define the Fourier transform on G: for $f \in L^1(G)$, we define its Fourier coefficient or group Fourier transform at the strongly continuous unitary representation π as

$$\mathcal{F}_G f(\pi) \equiv \widehat{f}(\pi) \equiv \pi(f) := \int_G f(x)\pi(x)^* \, dx.$$

More precisely, we can write

$$\left(\widehat{f}(\pi)v_1, v_2\right)_{\mathcal{H}_\pi} = \int_G f(x) \left(\pi(x)^* v_1, v_2\right)_{\mathcal{H}_\pi} \, dx.$$

This gives a linear mapping $\widehat{f}(\pi) : \mathcal{H}_\pi \mapsto \mathcal{H}_\pi$.

Remark 2.13. We note that the Fourier coefficient $\widehat{f}(\pi)$ depends on the choice of the representation π from its equivalence class. Namely, if $\pi_1 \sim \pi_2$, so that

$$\pi_2(x) = \mathrm{U}^{-1}\pi_1(x)\mathrm{U}$$

for some unitary U and all $x \in G$, then

$$\widehat{f}(\pi_2) = \mathrm{U}^{-1}\widehat{f}(\pi_1)\mathrm{U}.$$

Recalling that the Fourier transform on $\mathbf{R}^n$ maps translations to modulations, here we have an analogous property, namely, if $\pi \in \widehat{G}$, $f \in L^1(G)$ and $x \in G$, then

$$\widehat{f(\cdot x)}(\pi) = \pi(x)\widehat{f}(\pi) \quad \text{and} \quad \widehat{f(x\cdot)}(\pi) = \widehat{f}(\pi)\pi(x)$$

whenever the right hand side makes sense.

With our choice of the definition of the convolution and the Fourier transform, one can readily check that for $f, g \in L^1(G)$, we have:

$$\forall \pi \in \widehat{G},\ \widehat{f * g}(\pi) = \widehat{f}(\pi)\widehat{g}(\pi).$$

2.3. **Smooth Vectors and infinitesimal representation.** The Fourier transform that we just introduced is more complex than its euclidean counterpart: it is now operator-valued. For this reason, for a given irreducible representation π, we are to consider some particular subspaces of $\mathcal{H}_\pi$ in order to extend the Fourier transform to differential operators.

Definition 2.14. *Let G be a Lie group and let π be a s.c. representation of G on a Hilbert space $\mathcal{H}_\pi$. A vector $v \in \mathcal{H}_\pi$ is said to be smooth or of type $\mathcal{C}^\infty$ if the function*

$$G \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi$$

is of class $\mathcal{C}^\infty$. We denote by $\mathcal{H}_\pi^\infty$ the space of all smooth vectors of π.

Proposition 2.15. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. Then for any $X \in \mathfrak{g}$ and $v \in \mathcal{H}_\pi^\infty$, the limit*

$$\lim_{t\to 0} \frac{1}{t}\left(\pi(\exp_G(tX))v - v)\right)$$

exists in the norm topology of $\mathcal{H}_\pi$ and is denoted by $d\pi(X)v$ or $\pi(X)v$. Each $d\pi(X)$ leaves $\mathcal{H}_\pi^\infty$ invariant, and $d\pi$ is a representation of $\mathfrak{g}$ on $\mathcal{H}_\pi^\infty$ satisfying

$$\forall X, Y \in \mathfrak{g},\ d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X) - d\pi([X,Y]) = 0.$$

Moreover, one can also prove that for $\phi \in S(G)$ and for any $X \in \mathfrak{g}$, viewed as a left-invariant vector field,

$$\forall v \in \mathcal{H}_\pi,\ \pi(\phi)v \in \mathcal{H}_\pi^\infty, \text{ and } d\pi(X)\pi(\phi)v = \pi(X\phi)v.$$

The next proposition reassures us as to how difficult it is to find smooth vectors.

Proposition 2.16. *Let G be a Lie group and let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. Then the subspace $\mathcal{H}_\pi^\infty$ of smooth vectors is dense in $\mathcal{H}_\pi$.*

Among the differential operators, we will be particularly interested in the sublaplacian of our graded group: let us fix a basis $(V_1, \dots, V_m)$ of the first stratum of $\mathfrak{g}$, we consider the operator

$$\Delta_G = V_1^2 + \cdots + V_m^2.$$

The knowledge of its Fourier representation (in particular its spectral decomposition) can give us a lot of information on Δ_G itself.

2.4. **Kirillov's theory: a complete decription of the dual.** In the case of our simply connected nilpotent Lie groups, the method of induced representations for general locally compact groups can be applied readily. Moreover, Kirillov's theory tells us we can obtain all irreducible unitary representations of a nilpotent Lie group G this way. Furthermore the set $\widehat{G}$ admits a nice description in terms of functionals of $\mathfrak{g}$. In our effort to present these results, we need first to introduce some definitions.

There is a natural representation of G acting on $\mathfrak{g}$, called the adjoint representation: we define the inner automorphism $\mathrm{I}_x(y) := xyx^{-1}$ for $x, y \in G$. We have $\mathrm{I}_x : G \to G$ and $\mathrm{I}_{xy} = \mathrm{I}_x \mathrm{I}_y$. Its differential at e gives a linear mapping from $\mathrm{T}_e G$ to $\mathrm{T}_e G$.

Definition 2.17. *With the same notations as before, we define for every $x \in G$*

$$\mathrm{Ad}(x) := (d\mathrm{I}_x)_e : \mathfrak{g} \to \mathfrak{g}.$$

We have

$$\mathrm{Ad}(e) = Id, \ \mathrm{Ad}(xy) = \mathrm{Ad}(x)\mathrm{Ad}(y).$$

The application $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ *is a representation of G on $\mathfrak{g}$, called the adjoint representation.*

Proposition 2.18. *Keeping the previous notations, let's introduce* $\mathrm{ad} : \mathfrak{g} \to \mathcal{L}(\mathfrak{g})$ *the linear mapping defined for all $X, Y \in \mathfrak{g}$ by*

$$\mathrm{ad}(X)Y := [X, Y].$$

Then, we have

$$d(\mathrm{Ad})_e = \mathrm{ad}.$$

The orbit method describes a way to associate to a given linear functional on $\mathfrak{g}$ a collection of unitary irreducible representations of G which are all unitarily equivalent between themselves. Consequently, to any element of the dual $\mathfrak{g}'$, one can associate an equivalence class of unitary irreducible representations. It turns out that any such class is realised in this way. Furthermore, two elements $f_1, f_2 \in \mathfrak{g}'$ lead to the same class if and only if the two elements are in the same orbit under the natural action of G on $\mathfrak{g}'$; this natural action is the so-called co-adjoint representation: since the group G acts on $\mathfrak{g}$ by the adjoint representation Ad, it also acts on its dual $\mathfrak{g}'$ by

$$\text{co-Ad} : G \times \mathfrak{g}' \in (g, f) \mapsto f(\mathrm{Ad}^{-1} g \,\cdot) \in \mathfrak{g}'$$

This gives a one-to-one correspondence between, on the one hand, the dual $\widehat{G}$ of the group, that is, the collection of unitary irreducible representations modulo unitary equivalence, and on the other hand, $\mathfrak{g}'/\text{co-Ad}(G)$, the set of co-adjoint orbits.

The importance of Kirillov's work also reside in the description of how to construct a Plancherel measure for a given nilpotent Lie group.

Theorem 2.19. *Let G be a connected simply connected nilpotent Lie group. The dual $\widehat{G}$ is then equipped with a measure μ called the Plancherel measure satisfying the following property for any $\phi \in S(G)$: the operator $\pi(\phi) \equiv \widehat{\phi}(\pi)$ is trace class, and even Hilbert-Schmidt:*

$$\|\pi(\phi)\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\pi(\phi)\pi(\phi)^*) < +\infty$$

for any strongly continuous unitary irreducible representation π, and this norm depends only on the class of π; the function $\widehat{G} \ni \pi \mapsto \|\pi(\phi)\|_{\mathrm{HS}}^2$ is integrable against μ and

$$\int_G |\phi(x)|^2 \, dx = \int_{\widehat{G}} \|\pi(\phi)\|_{\mathrm{HS}}^2 \, d\mu(\pi).$$

Corollary 2.20. *Let G be a connected simply connected nilpotent Lie group and let μ be the Plancherel measure on $\widehat{G}$. If $f \in S(G)$, then $\pi(x)\pi(f)$ and $\pi(f)\pi(x)$ are trace class for every $x \in G$, the function $\widehat{G} \ni \pi \mapsto \mathrm{Tr}(\pi(x)\pi(f))$ is integrable against μ, and we have*

$$f(x) = \int_{\widehat{G}} \mathrm{Tr}(\pi(x)\pi(f)) \, d\mu(\pi) = \int_{\widehat{G}} \mathrm{Tr}(\pi(f)\pi(x)) \, d\mu(\pi).$$

3. Semi-classical analysis on a graded nilpotent Lie group

As mentioned in the introduction, we are interested in developing a pseudodifferential calculus on nilpotent Lie groups: historically microlocal approach or phase-space analysis, has been first developed in the context of the Heisenberg group and the full program of constructing a pseudodifferential calculs on general graded Lie groups has been achieved by Véronique Fischer and Michael Ruzhansky in. This construction relies on the harmonic analysis of the nilpotent Lie group G that we presented: the role of the frequency space is now played by the dual $\widehat{G}$.

As mentioned above, the Fourier transform being operator-valued, so will be the symbols of semi-classical pseudodifferential operators that we shall now define. One of our aim is to introduce the useful tool of Wigner measures, also called semi-classical measures. In what follows, G will still be a simply connected graded nilpotent Lie group.

3.1. The algebra of symbols $\mathcal{A}_0$.

We denote by $\mathcal{A}_0$ the space of symbols $\sigma = \{\sigma(x,\pi) \mid (x,\pi) \in G \times \widehat{G}\}$ of the form

$$\sigma(x,\pi) = \mathcal{F}_G \kappa_x(\pi) = \int_G \kappa_x(y) \, (\pi(y))^* \, dy$$

where $x \mapsto \kappa_x(\cdot)$ is a smooth and compactly supported function from G to $S(G)$.

As the Fourier transform is injective, it yields a one-to-one correspondence between the symbol σ and the function κ. Moreover, the set $\mathcal{A}_0$ is an algebra for the composition of symbols since if $\sigma_1(x,\pi) = \mathcal{F}\kappa_{1,x}(\pi)$ and $\sigma_2(x,\pi) = \mathcal{F}\kappa_{2,x}(\pi)$ are in $\mathcal{A}_0$, then so is $\sigma_1(x,\pi)\sigma_2(x,\pi) = \mathcal{F}(\kappa_{2,x} * \kappa_{1,x})(\pi)$.

We can introduce two norms on $\mathcal{A}_0$: if $\sigma = \mathcal{F}\kappa_x(\cdot)$ then

$$\|\sigma\|_{\mathcal{A}_0} := \int_G \sup_{x\in G} |\kappa_x(z)|\, dz$$
$$\|\sigma\|_{\mathcal{A}} := \sup_{(x,\pi)\in G\times\widehat{G}} \|\sigma(x,\pi)\|_{\mathcal{L}(L^2(\mathcal{H}_\pi))}$$

From an algebraic viewpoint, they are both submultiplicative on $\mathcal{A}_0$:

$$\|\sigma_1\sigma_2\|_{\mathcal{A}_0} = \int_G \sup_{x\in G} |\kappa_{2,x} * \kappa_{1,x}(z)|\, dz \leq \int_G (\sup_{x_2\in G} |\kappa_{2,x_2}|) * (\sup_{x_1\in G} |\kappa_{1,x_1}|)(z)\, dz \leq \|\sigma_2\|_{\mathcal{A}_0} \|\sigma_1\|_{\mathcal{A}_0}.$$

3.1.1. *Difference operators.* Here we assume that a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ has been fixed. The difference operators Δ^α are related with the coordinate functions $\phi_1, \ldots, \phi_n$ associated with the basis $X_1, \ldots, X_n$, i.e. $x = \mathrm{Exp}(\phi_1(x)X_1 + \cdots + \phi_n(x)X_n)$. We denote by $\Delta_{\phi_1}, \cdots, \Delta_{\phi_n}$ the corresponding difference operators:

$$\left(\Delta_{\phi_j}\sigma\right)(x,\cdot) := \mathcal{F}(\phi_j\kappa_x)(\cdot) \text{ if } \sigma(x,\cdot) = \mathcal{F}(\kappa_x)(\cdot).$$

Then Δ^α is the product of $|\alpha|$ of such difference operators. Note that two difference operators Δ_{ϕ_1} and Δ_{ϕ_2} of that form will commute since $\Delta_{\phi_1}\Delta_{\phi_2} = \Delta_{\phi_1\phi_2}$.

More generally, one can always define a difference operator associated with a function ϕ of polynomial growth (i.e. $|\phi(x)| \leq (1+|x|)^N$ for some $N \in \mathbb{N}$) as the operator defined on $\mathcal{A}_0$ by

$$(\Delta_\phi\sigma)(x,\cdot) := \mathcal{F}(\phi\kappa_x)(\cdot) \text{ if } \sigma(x,\cdot) = \mathcal{F}(\kappa_x)(\cdot).$$

These operators will be needed for the expression of the symbolic calculus of our semi-classical analysis.

3.2. **Semi-classical pseudodifferential operators.** Let $\epsilon > 0$ be a small parameter, the semi-classical parameter that we shall use to weight the oscillations of the functions that we shall consider. Following [6] and [3], we quantify the symbols that we have introduced previously by setting

$$\mathrm{Op}_\epsilon(\sigma)f(x) := \int_{\widehat{G}} \mathrm{Tr}(\pi_x \sigma(x, \epsilon\cdot\pi)\mathcal{F}f(\pi))\, d\mu(\pi), \;\; f \in S(G),$$

where $\epsilon\cdot\pi := \pi \circ \delta_\epsilon$ for $\pi \in \widehat{G}$.

The kernel of the operator $\mathrm{Op}_\epsilon(\sigma)$ is the function

$$G \times G \ni (x,y) \mapsto \kappa_x^\epsilon(y^{-1}x)$$

where $\kappa_x^\epsilon(z) = \epsilon^{-Q}\kappa_x(\delta_{\epsilon^{-1}}z)$ and κ_x is such that $\mathcal{F}(\kappa_x)(\pi) = \sigma(x,\pi)$. For this reason, we call κ_x the convolution kernel of σ.

The norm $\|\cdot\|_{\mathcal{A}_0}$ defined above allows us to bound the action of the symbols in $\mathcal{A}_0$ on $L^2(G)$:

Proposition 3.1. *Let $\sigma \in \mathcal{A}_0$, then $\mathrm{Op}_\epsilon(\sigma)$ is bounded in $L^2(G)$. Moreover, there exists a constant $C > 0$ such that*

$$\forall\sigma \in \mathcal{A}_0, \;\forall\epsilon > 0, \;\|\mathrm{Op}_\epsilon(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C\|\sigma\|_{\mathcal{A}_0}.$$

Proof. We observe that if $f \in S(G)$ then

$$|\mathrm{Op}_\epsilon(\sigma)f(x)| = |\int_G f(y)\kappa^\epsilon_x(y^{-1}x)\,dy| \le \int_G |f(y)| \sup_{x_1\in G}|\kappa^\epsilon_{x_1}(y^{-1}x)|\,dy = |f| * \sup_{x_1\in G}|\kappa^\epsilon_{x_1}(\cdot)|(x)$$

so the Young inequality implies

$$\|\mathrm{Op}_\epsilon(\sigma)f\|_{L^2(G)} \le \|f\|_{L^2(G)} \|\sup_{x_1\in G}|\kappa^\epsilon_{x_1}(\cdot)|\|_{L^1(G)}$$

We recognize this L^1-norm as $\|\sigma\|_{\mathcal{A}_0}$:

$$\|\sup_{x_1\in G}|\kappa^\epsilon_{x_1}(\cdot)|\|_{L^1(G)} = \|\sup_{x_1\in G}|\kappa_{x_1}(\cdot)|\|_{L^1(G)} = \|\sigma\|_{\mathcal{A}_0}.$$

□

3.2.1. *Symbolic calculus.* These operators enjoy a symbolic calculs: let us fix a basis $(V_1, \dots, V_d)$ of the first stratum of $\mathfrak{g}$.

Proposition 3.2. *Let $\sigma \in \mathcal{A}_0$. Then in $\mathcal{L}(L^2(G))$,*

$$\mathrm{Op}_\epsilon(\sigma)^* = \mathrm{Op}_\epsilon(\sigma^*) - \epsilon\,\mathrm{Op}_\epsilon(V \cdot \Delta_v \sigma) + O(\epsilon^2)$$

Let $\sigma_1, \sigma_2 \in \mathcal{A}_0$. Then in $\mathcal{L}(L^2(G))$,

$$\mathrm{Op}_\epsilon(\sigma_1) \circ \mathrm{Op}_\epsilon(\sigma_2) = \mathrm{Op}_\epsilon(\sigma_1\sigma_2) - \epsilon\,\mathrm{Op}_\epsilon(\Delta_v\sigma_1 \cdot V\sigma_2) + O(\epsilon^2).$$

Remark 3.3. The difference operators Δ_{v_j} play the role of $-i\partial_{\xi_j}$ in the Euclidean setting. We can check that the symbolic calculus formulae are exactly the same as in the case of Kohn-Nirenberg quantization in the Euclidean setting.

3.3. **Semi-classical measures.** Semi-classical measures are the adaptation of microlocal defect measures in a context where a scale of oscillations is specified. This scale which is the semi-classical scale ϵ is prescribed by the sequence of family of functions to study or by the parameters of a given problem. However, we need to adapt our approach to the Lie group setting and we do so by considering the algebra $\mathcal{A}$ which is the closure of $\mathcal{A}_0$ for the norm $\|\cdot\|_{\mathcal{A}}$ introduced in section 3.1.

The next definition aims to introduce the right objects for our setting.

Definition 3.4. *Let Z be a complete separable metric space, and let $\xi \mapsto \mathcal{H}_\xi$ a measurable field of complex Hilbert spaces of Z.*

- *The set $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi\in Z})$ is the set of pairs (γ, Γ) where γ is a positive Radon measure on Z and $\Gamma = \{\Gamma(\xi) \in \mathcal{L}(\mathcal{H}_\xi) : \xi \in Z\}$ is a measurable field of trace-class operators such that for all compact set $K \subset Z$,*

$$\int_K \mathrm{Tr}\,|\Gamma(\xi)|\,d\gamma(\xi) < +\infty.$$

- *Two pairs (γ, Γ) and (γ', Γ') in $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi\in Z})$ are equivalent when there exists a measurable function $f : Z \to \mathbb{C} \setminus \{0\}$ such that*

$$d\gamma'(\xi) = f(\xi)d\gamma(\xi) \text{ and } \Gamma'(\xi) = \frac{1}{f(\xi)}\Gamma(\xi)$$

for γ-almost every $\xi \in Z$. The equivalence class of (γ, Γ) is denoted by $\Gamma d\gamma$.

- *A pair (γ, Γ) in $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ is positive when $\Gamma(\xi) \geq 0$ for γ-almost all $\xi \in Z$. In this case, we may write $\Gamma d\gamma \geq 0$ or $(\gamma, \Gamma) \in \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi \in Z})$.*

We will use the short-hands

$$\mathcal{M}_1^+(G \times \widehat{G}) = \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi \in Z}) \text{ when } Z = \{(x, \pi) \in G \times \widehat{G}\}, \text{ and } \mathcal{H}_{x,\pi} = \mathcal{H}_\pi.$$

With this concept in mind, the states of the C^*-algebra $\mathcal{A}$, i.e its functionals ℓ verifying $\ell(\sigma^*\sigma) \geq 0$ pour tout $\sigma \in \mathcal{A}$, can be described as follows:

Proposition 3.5. *If ℓ is a state of $\mathcal{A}$, then there exists $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$, unique up to its equivalence class, satisfying*

$$\int_{G \times \widehat{G}} \mathrm{Tr}\left(\Gamma(x, \lambda)\right) d\gamma(x, \lambda) = 1, \tag{1}$$

and

$$\forall \sigma \in \mathcal{A} \qquad \ell(\sigma) = \int_{G \times \widehat{G}} \mathrm{Tr}\left(\sigma(x, \lambda)\Gamma(x, \lambda)\right) d\gamma(x, \lambda). \tag{2}$$

Conversely, if a pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ satisfies (1)*, then the linear form ℓ defined via* (2) *is a state of $\mathcal{A}$.*

We have now all we need to define our Wigner measure: we associate to a bounded family $(u^\epsilon)_{\epsilon > 0}$ of $L^2(G)$ the quantities

$$\ell_\epsilon(\sigma) = \left(\mathrm{Op}_\epsilon(\sigma) u^\epsilon, u^\epsilon\right)_{L^2(G)}, \ \sigma \in \mathcal{A}_0,$$

the limits of which are characterized by an element of $\mathcal{M}_1^+(G \times \widehat{G})$.

Theorem 3.6. *Let $(u^\epsilon)_{\epsilon > 0}$ be a bounded family of $L^2(G)$. There exist a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ in $(0, +\infty)$ with $\epsilon_k \xrightarrow[k \to \infty]{+\infty} 0$ and a pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ such that we have*

$$\forall \sigma \in \mathcal{A}_0, \ \left(\mathrm{Op}_{\epsilon_k}(\sigma) u^{\epsilon_k}, u^{\epsilon_k}\right)_{L^2(G)} \xrightarrow[k \to \infty]{+\infty} \int_{G \times \widehat{G}} \mathrm{Tr}\left(\sigma(x, \lambda)\Gamma(x, \lambda)\right) d\gamma(x, \lambda).$$

Given the sequence $(\epsilon_k)_{k \in \mathbb{N}}$, the pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ is unique up to equivalence in $\mathcal{M}_1^+(G \times \widehat{G})$ and satisfies

$$\int_{G \times \widehat{G}} \mathrm{Tr}\left(\Gamma(x, \lambda)\right) d\gamma(x, \lambda) \leq \limsup_{\epsilon > 0} \|u^\epsilon\|^2_{L^2(G)}.$$

Any equivalence class $\Gamma d\gamma$ satisfying to Theorem 3.6 for some subsequence $(\epsilon_k)_{k \in \mathbb{N}}$ is called a *semi-classical measure* of the family $(u^\epsilon)_{\epsilon > 0}$.

4. About the magnetic Schrödinger operators

What is the magnetic Laplacian? Let Ω be a $\mathcal{C}^1$ domain in $\mathbf{R}^n$ and $\mathbf{A} = (A_1, \dots, A_n)$ a smooth vector potential on $\overline{\Omega}$. We define the 1-form

$$\omega_{\mathbf{A}} = \sum_{k=1}^{n} A_k \, \mathrm{d}x_k.$$

The exterior derivative of $\omega_{\mathbf{A}}$ is

$$\sigma_{\mathbf{B}} = \mathrm{d}\omega_{\mathbf{A}} = \sum_{1 \le k < l \le n} B_{kl} \, \mathrm{d}x_k \wedge \mathrm{d}x_l$$

with $B_{kl} = \partial_k A_l - \partial_l A_k$. In dimension two, which will be our main focus in the following sections, there is only one coefficient $\mathbf{B} = B_{12} = \partial_{x_1} A_2 - \partial_{x_2} A_1$. Our interest reside in the study of some self-adjoint realizations of the *magnetic Schrödinger operator*

$$\mathcal{L}_{\mathbf{A},\Omega} = \sum_{k=1}^{n} (-ih\partial_{x_k} + A_k)^2$$

where $h > 0$ is a parameter (related to the Planck constant). We will also make use of the following notation

$$\nabla_{\mathbf{A}} := \nabla + i\mathbf{A}.$$

4.1. Self-adjoint extensions.

One of our first interest is the analysis of the bottom of the spectrum of $\mathcal{L}_{\mathbf{A},\Omega}$. In the case of a bounded open set Ω, we can consider the Dirichlet or the Neumann realizations. We will develop here the Neumann one.

To this end, we look at the Friedrich extension of the quadratic form

$$\mathcal{C}^\infty(\overline{\Omega}; \mathbb{C}) \ni u \mapsto Q_{\mathbf{A},\Omega}(u) := \int_\Omega |\nabla_{\mathbf{A}} u(x)|^2 + |u(x)|^2 \, dx.$$

In our case, where Ω is smooth, we choose the form domain of our operator to be

$$\mathcal{V}^N(\Omega) = H^1(\Omega).$$

We can extend our quadratic form $Q_{\mathbf{A},\Omega}(u)$ to the space $\mathcal{V}^N(\Omega)$ and associate to it a self-adjoint operator, denoted $\mathcal{L}_{\mathbf{A},\Omega} + Id$, in the classical way: denoting by $q_{\mathbf{A},\Omega}$ the coercive sesquilinear form associated to $Q_{\mathbf{A},\Omega}$, the domain of the operator is defined as the subspace in $\mathcal{V}^N(\Omega)$ of the u's such that $\mathcal{V}^N(\Omega) \ni v \mapsto q_{\mathbf{A},\Omega}(u, v)$ admits a continuous extension to $L^2(\Omega)$. Our domain Ω being regular, the domain of the operator can be characterized as

$$\mathrm{Dom}(\mathcal{L}_{\mathbf{A},\Omega}) = \{u \in H^2(\Omega) \,|\, \nu \cdot (-i\nabla + \mathbf{A})u = 0 \text{ on } \partial\Omega\},$$

where $\nu(x)$ is the unit normal vector to $\partial\Omega$ at x. This characterization involves the Green–Riemann formula and a regularity result for the magnetic Laplacian.

In the case of $\Omega = \mathbb{R}^n$, we don't have to make a distinction between the Dirichlet and the Neumann realizations.

Theorem 4.1. *The operator $(-i\nabla + \mathbf{A})^2$ defined on $\mathcal{C}_0^\infty(\mathbb{R}^n)$, where $\mathbf{A}$ is in $\mathcal{C}^1(\mathbb{R}^n)$, is essentially self-adjoint.*

4.2. **Spectral analysis.** In the case of a bounded domain, since the inclusion $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact by Rellich's theorem, our operator $\mathcal{L}_{\mathbf{A},\Omega}$ has a compact resolvent and its spectrum consists of a nondecreasing sequence of eigenvalues: its spectrum is entirely discrete.

However for $\Omega = \mathbb{R}^n$, its resolvent is no more compact and the essential part of its spectrum has no more reason to be empty. But the Persson's theorem gives us an information about it either way.

Theorem 4.2 (Persson)**.** *Let $\mathbf{A} \in \mathcal{C}^1(\mathbb{R}^n)$ a magnetic potential vector and let $\mathcal{L}_{\mathbf{A},\mathbb{R}^n}$ the corresponding self-adjoint, semibounded Schrödinger operator. Then the bottom of the essential spectrum is given by*

$$\inf \sigma_{ess}(\mathcal{L}_{\mathbf{A},\mathbb{R}^n}) = \Sigma(\mathcal{L}_{\mathbf{A},\mathbb{R}^n})$$

where

$$\Sigma(\mathcal{L}_{\mathbf{A},\mathbb{R}^n}) := \sup_{K\subset\mathbb{R}^n} \left[\inf_{\|\phi\|=1} \{\langle \phi, \mathcal{L}_{\mathbf{A},\mathbb{R}^n}\phi\rangle \,|\, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus K)\} \right].$$

where the supremum is over all compact sets $K \subset \mathbb{R}^n$.

5. Wave packets

The concept of wave packets, also called coherent states, is a major topic in mathematical physics. They are initial data in the Schrödinger equations for which the evolution is concentrated on the classical trajectories. Historically, Erwin Schrödinger derived them as a minimum uncertainty Gaussian wave packet in 1926, searching for solutions of the Schrödinger equation that satisfy the correspondence principle. These first endeavors have been mathematically abstracted and the prevalent view nowadays is the group theoretic construction of wave packets introduced by Gilmore and Perelomov. We recall here important results concerning the gaussian wave packets and expose what seems to be a generalization of them to our nilpotent Lie groups.

5.1. Gaussian wave packets.

Proposition 5.1. *Let $z = (q,p) \in \mathbb{R}^{2d}$. The Gaussian wave packet*

$$g_z^\epsilon(x) = (\pi\epsilon)^{-d/4}\exp\left(-\frac{1}{2\epsilon}|x-q| + \frac{i}{\epsilon}p\cdot(x-q)\right), x \in \mathbb{R}^d,$$

is normalized, $\|g_z^\epsilon\|_{L^2(\mathbb{R}^d)} = 1$, and centered in z, in the sense that $\langle g_z^\epsilon, \mathrm{Op}_\epsilon(x) g_z^\epsilon\rangle = q$ and $\langle g_z^\epsilon, \mathrm{Op}_\epsilon(\xi) g_z^\epsilon\rangle = p$. The second moments are ϵ-standardised,

$$\langle g_z^\epsilon, \mathrm{Op}_\epsilon((x-q)\otimes(x-q)) g_z^\epsilon\rangle = \frac{\epsilon}{2}\mathrm{Id},\ \ \langle g_z^\epsilon, \mathrm{Op}_\epsilon((\xi-p)\otimes(\xi-p)) g_z^\epsilon\rangle = \frac{\epsilon}{2}\mathrm{Id}.$$

Definition 5.2. *For any function $a : \mathbb{R}^d \mapsto \mathbb{C}$ of Schwartz class and any point $z = (q,p) \in \mathbb{R}^{2d}$, we define its wave packet transform*

$$a_z^\epsilon(x) = \epsilon^{-d/4} a(\frac{x-q}{\sqrt{\epsilon}}) e^{i(x-q)\cdot\frac{p}{\epsilon}},\ x \in \mathbb{R}^d.$$

Our notation for the Gaussian wave packets is consistent with the above definition. For the amplitude function $g(x) = \pi^{-d/4}\exp(-\frac{1}{2}|x|^2)$, $x \in \mathbb{R}^d$, we obtain the standard Gaussian wave packet g_z^ϵ centred in z. For a more general width matrix $\Gamma \in S_+(d)$, the Siegel half-space, and amplitude $g^\Gamma(x) = \pi^{-d/4}c_\Gamma\exp(-\frac{1}{2}x \cdot \Gamma x)$, we obtain the thawed Gaussian wave packet $g_z^{\epsilon,\Gamma}$. Regardless of the choice of amplitude function, arbitrary square integrable functions can be represented as continuous superpositions of wave packets.

Proposition 5.3. *Let $a : \mathbb{R}^d \mapsto \mathbb{C}$ be a function of Schwartz class and unit L^2-norm, $\|a\|_{L^2(\mathbb{R}^d)} = 1$. Let $f \in L^2(\mathbb{R}^d)$. The wave packet transform satisfies the following properties:*

- $\|a_z^\epsilon\|_{L^2(\mathbb{R}^d)} = 1$ *for all* $z \in \mathbb{R}^d$.
- *The mapping* $\mathbb{R}^{2d} \mapsto [0, \infty[$, $z \mapsto |\langle a_z^\epsilon, f\rangle|^2$ *is of Schwartz class.*
- *Inversion formula:*
$$f = (2\pi\epsilon)^{-d} \int_{\mathbb{R}^{2d}} \langle a_z^\epsilon, f\rangle a_z^\epsilon \, dz.$$
- *Preservation of norm:*
$$\|f\|^2_{L^2(\mathbb{R}^d)} = (2\pi\epsilon)^{-d} \int_{\mathbb{R}^{2d}} |\langle a_z^\epsilon, f\rangle|^2 \, dz.$$
- *Combination with Weyl quantized operators: For all symbols* $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ *and* $z \in \mathbb{R}^{2d}$,
$$\mathrm{Op}_\epsilon(b)a_z^\epsilon = (\mathrm{op}_1(b_{\epsilon,z}a))_z^\epsilon$$
where $b_{\epsilon,z}(w) = b(z + \sqrt{\epsilon}w)$, $w \in \mathbb{R}^{2d}$.

The inner product with a wave packet transform has various names in the literature: Fourier–Bros–Iagolnitzer (FBI) transform, Bargmann transform, Gabor transform, depending on the context. Typically, the transform is considered for Gaussian amplitude functions $a(x) = \pi^{-d/4}\exp(-\frac{1}{2}|x|^2)$.

5.2. **Wave packet dynamics.** We now consider a general time-dependent Schrödinger equation
$$i\epsilon\partial_t\Psi^\epsilon(t) = \mathrm{Op}_\epsilon(h)\Psi^\epsilon(t), \;\; \Psi^\epsilon(0) = \Psi_0^\epsilon,$$
where the Hamiltonian operator is the Weyl-quantized pseudo-differential operator $\mathrm{Op}_\epsilon(h)$ of a smooth, real-valued symbol $h \in \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R})$, that is of subquadratic growth, that is,
$$\forall\alpha \in \mathbb{N}^{2d}, \; |\alpha| \le 2, \; \exists C_\alpha > 0, \forall z \in \mathbb{R}^{2d} : \; |\partial^\alpha h(z)| \le C_\alpha.$$
Equipped with an appropriate domain, such operators are self-adjoint and define a unitary evolution $U_\epsilon(t) = \exp(-i\mathrm{Op}_\epsilon(h)t/\epsilon)$, $t \in \mathbb{R}$, that in turn gives the solution of the Schrödinger equation,
$$\Psi^\epsilon(t) = U_\epsilon(t)\Psi_0^\epsilon$$
for arbitrary square integrable initial data $\Psi_0^\epsilon \in L^2(\mathbb{R}^d)$. We now consider special initial data, namely an initial wave function $\Psi_0^\epsilon = (a_0)_{z_0}^\epsilon$ that is a semi-classical wave packet,

with a normalised amplitude function $a_0 \in S(\mathbb{R}^d)$ such that $\|a_0\|_{L^2(\mathbb{R}^d)} = 1$ and phase space centre $z_0 = (q_0, p_0) \in \mathbb{R}^{2d}$.

Theorem 5.4 (Wave packets dynamics for general Hamiltonian)**.** *Let $h \in \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R})$ be a smooth real-valued symbol of subquadratic growth. Let $a_0 \in S(\mathbb{R}^d)$ be a normalised Schwartz class function, $\|a_0\|_{L^2(\mathbb{R}^d)} = 1$, and $z_0 \in \mathbb{R}^d$. Consider the solution Ψ^ϵ to the time-dependent Schrödinger equation with hamiltonian $\mathrm{Op}_\epsilon(h)$ and initial value $\Psi_0^\epsilon = (a_0)_{z_0}^\epsilon$. We build the approximate solution*

$$\Psi^\epsilon_{app}(t) = e^{iS(t)/\epsilon} a(t)^\epsilon_{z(t)},$$

where

$$\begin{aligned}
&\dot S(t) = p(t)\cdot \dot q(t) - h(z(t)),\ \ S(0) = 0,\\
&\dot z(t) = \nabla^\perp h(z(t)),\ \ z(0) = z0,\\
&i\partial_t a(t) = \frac{1}{2}\mathrm{Op}_1(w\cdot \nabla^2 h(z(t))w)a(t),\ \ a(0) = a_0.
\end{aligned}$$

Then, the approximation error satisfies

$$\sup_{0\le t\le T} \|\Psi^\epsilon(t) - \Psi^\epsilon_{app}(t)\|_{L^2(\mathbb{R}^d)} = O_T(\sqrt{\epsilon})$$

for all finite times $T \in \mathbb{R}$. In particular, if the Hamiltonian symbol h is a polynomial of degree less than 2, then the approximation is exact.

5.3. **Non-commutative wave packets.** Given $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and $a \in S(\mathbb{R}^d)$, we consider the family indexed by ϵ of functions

$$u^\epsilon_{\mathrm{eucl}}(x) = \epsilon^{-d/4} a(\frac{x-x_0}{\sqrt{\epsilon}}) e^{i\frac{\xi_0}{\epsilon}\cdot(x-x_0)},\ \ x \in \mathbb{R}^d.$$

The oscillation along ξ_0 is forced by the term en $e^{i\frac{\xi_0}{\epsilon}\cdot(x-x_0)}$ and the concentration on x_0 is performed at the scale $\sqrt{\epsilon}$ for symmetry reasons: the ϵ-Fourier transform of $u^\epsilon_{\mathrm{eucl}}$, $\epsilon^{-d/2}\hat u^\epsilon_{\mathrm{eucl}}(\xi/\epsilon)$, presents a concentration on ξ_0 at the scale $\sqrt{\epsilon}$. The regularity of these wave packets make them a flexible tool.

We introduce here a generalization of these wave packets to the non-commutative setting of Lie groups and nilmanifolds which is strongly inspired by [6]. We can do so by mimicking the structure of the euclidean wave packet. In what follows, G will designate a graded nilpotent Lie group.

The concentration in space is performed by use of dilations: for $a \in \mathcal{C}_0^\infty(G)$, we associate

$$a_\epsilon(x) = a(\delta_{\epsilon^{-1/2}}(x)),\ \ x \in G.$$

The oscillations are forced by using coefficients of the representations, in the spirit of [52]: with $\pi \in \widehat{G}$, Φ_1, Φ_2 smooth vectors in the space of representations, i.e. in $S(\mathbb{R}^d)$ (with d depending on π), we associate the oscillating term

$$e_\epsilon(x) = (\pi^\epsilon_x \Phi_1, \Phi_2)\,,\ \ \pi^\epsilon = \pi\circ\delta_{\epsilon^{-1}}.$$

Proposition 5.5. *Let $\Phi_1, \Phi_2 \in S(\mathbb{R}^d)$, $a \in \mathcal{C}_0^\infty(G)$, $x_0 \in G$, $\pi_0 \in \widehat{G}$. Then, there exists $\epsilon_0 > 0$ such that the family $(v_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ defined by*

$$v_\epsilon(x) = \epsilon^{-\kappa}(e_\epsilon a_\epsilon)(x_0^{-1}x)$$

is a bounded ϵ-oscillating family in $L^2(G)$ for a well-chosen $\kappa > 0$ depending on Q and π, with bounded ϵ-derivatives and momenta:

$$\forall k \in \mathbb{N},\ \exists C_k > 0,\ \forall \epsilon \in (0, \epsilon_0), \|(-\epsilon^2\Delta_G)^{k/2}v_\epsilon\|_{L^2(G)} \leq C_k.$$

In the following, we shall say that the family v_ϵ is a wave packet on G with cores (x_0, π_0), profile a and harmonics (Φ_1, Φ_2), and write

$$\forall \epsilon \in (0, \epsilon_0), v_\epsilon = WP^\epsilon_{x_0,\pi_0}(a, \Phi_1, \Phi_2).$$

We are interested here in generalizing to our Lie group setting the properties already known for the euclidean wave packets. In particular, can the theorem 5.4 be appropriately restated for our newly-defined wave packets? We present here a positive result in the special case of the Heisenberg group.

We consider again the Schrödinger equation with this time the sublplacian of $\mathbb{H}$:

$$i\epsilon^2\partial_t\Psi^\epsilon(t) = \epsilon^2\Delta_{\mathbb{H}}\Psi^\epsilon(t),\ \Psi^\epsilon(0) = \Psi^\epsilon_0,$$

As in the euclidean case, we are endowed with a Schrödinger propagator but remark here the new scaling in time with the presence of ϵ^2 in front of the time derivative: the next result present a stability result beyond the Erhenfest time.

Proposition 5.6. *Let $\Psi^\epsilon(t)$ be the solution of the precedent equation with initial data of the form*

$$\Psi^\epsilon_0 = WP^\epsilon_{x_0,\lambda_0}(a, h_0, h_0)$$

where $(x_0, \lambda_0) \in \mathbb{H} \times (\mathfrak{z}^* \setminus 0)$, $a \in S(\mathbb{H})$ and h_0 is the first Hermite function. Then, there exists a map $(t, x) \mapsto a(t, x)$ in $\mathcal{C}^1(\mathbb{R}, S(\mathbb{H}))$ such that for all $k \in \mathbb{N}$,

$$\Psi^\epsilon(t, x) = WP^\epsilon_{x(t),\lambda_0}(a(t, \cdot), h_0, h_0) + O(\sqrt{\epsilon})$$

in Σ^ϵ_k (see (4.8) for definition), with

$$x(t) = \exp_{\mathbb{H}}\left(\frac{t}{2}Z\right)x_0.$$

5.4. **From the nilpotent Lie groups to the magnetic Schrödinger operators.** We present here a result similar in spirit to those of Rotschild and Stein in their paper. In the case of the simple potential vector $\mathbf{A} = (y, 0)$, properties of the magnetic laplacian and its Schrödinger propagator can be deduced from those of the Heisenberg sublaplacian.

Indeed, we can define a family $\left(\mathrm{U}^{\lambda,h}\right)_{\lambda,h}$ of surjectives bounded operators from $L^2(\mathbf{H})$ to $L^2(\mathbb{R}^2)$ by the formulae

$$\mathrm{U}^{\lambda,h}\varphi(x, y) = \int_{\mathbb{R}} e^{i\frac{\lambda s}{h}}\theta(s)\varphi(x, y, s)\, ds$$

where θ is taken in $\mathcal{C}_0^\infty(\mathbb{R})$. This family of operators is uniformly bounded by $\|\theta\|_{L^2(\mathbb{R})}$.

We recall here the magnetic operator and the Heisenberg sublaplacian:

$$-\mathcal{L}_{\mathbf{A}}^{\lambda,h} = \left(\frac{h}{i}\partial_x + \lambda y\right)^2 - h^2\partial_y^2$$
$$\mathbf{L} = -\left(\partial_x - y\partial_s\right)^2 - \partial_y^2$$

Proposition 5.7. *For $\lambda \in \mathbb{R}$ and $h > 0$, the magnetic laplacian $-\mathcal{L}_{\mathbf{A}}^{\lambda,h}$ and the sublaplacian $h^2\mathbf{L}$ are related by the following relation*

$$\forall \varphi \in S(\mathbb{H}),\ h^2 \mathrm{U}^{\lambda,h}\mathbf{L}\,\varphi = -\mathcal{L}_{\mathbf{A}}^{\lambda,h}\mathrm{U}^{\lambda,h}\varphi + hR^h(y\varphi(y)) + O(h^2)$$

where R^h is an operator similar to $\mathrm{U}^{\lambda,h}$ and bounded (uniformly in h), and the constant in the O depends of θ and the semi-norms of φ in $S(\mathbb{H})$.

The non-commutative wave packets introduced before give then a stable class of initial data under the magnetic Schrödinger laplacian propagator.

It is the author's goal to further study magnetic Schrödinger laplacians in the light of their connection with nilpotent Lie group's sublaplacians.

References

[1] L. Corwin and F. P. Greenleaf. Representations of Nilpotent Lie Groups and Their Applications: Part 1, Basic Theory and Examples, 1990.

[2] H. Bahouri, C. Fermanian Kammerer and I. Gallagher. Phase space analysis on the Heisenberg group, *Astérisque*, 2012.

[3] V. Fischer and M. Ruzansky. Quantization on nilpotent Lie groups, *Progress in Mathematics*, Birkhäuser Basel, 2016.

[4] C. Fermanian Kammerer and V. Fischer. Quantum evolution and sub-Laplacian operators on groups of Heisenberg type, *Journal of Spectral Theory*, 2021.

[5] Fermanian Kammerer, Clotilde and Letrouit, Cyril. Observability and Controllability for the Schrödinger Equation on Quotients of Groups of Heisenberg Type, *Journal de l'École polytechnique - Mathématiques*, 2021.

[6] C. Fermanian Kammerer and V. Fischer. Semi-classical analysis on H-type groups, *Science China Mathematics*, 2019.

[7] C. Fermanian Kammerer and V. Fischer. Defect measures on graded lie groups, 2017.

[8] Montgomery, Richard. Hearing the Zero Locus of a Magnetic Field, 1994.

[9] Raymond, Nicolas. Bound States of the Magnetic Schrödinger Operator, 2015.